INCOMPRESSIBLE SURFACES IN RANK ONE LOCALLY SYMMETRIC SPACES

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Abstract. We show that cocompact lattices in rank one simple Lie groups of non-compact type distinct from $SO(2m,1)$ $(m \geq 1)$ contain surface subgroups.

1. Introduction

In recent seminal work, Jeremy Kahn and Vladimir Markovic showed that every cocompact lattice in $SO(3,1)$ contains infinitely many surface subgroups, i.e. subgroups which are isomorphic to the fundamental group of a closed surface [KM12]. Moreover, as a subgroup of $SO(3,1)$, each of these groups is quasi-Fuchsian, and every round circle in the ideal boundary of hyperbolic three-space is the limit in the Hausdorff topology of a sequence of limit sets of such groups.

In view of a conjecture of Gromov that every one-ended hyperbolic group contains a surface subgroup, it seems desirable to extend the result of Kahn and Markovic to a larger class of groups. The goal of this paper is to undertake such an extension to cocompact lattices in simple Lie groups of rank one which are distinct from $SO(2m,1)$ for $m \geq 1$.

Theorem 1. *Let G be a simple rank one Lie group of non-compact type distinct from $SO(2m,1)$ for some $m \geq 1$ and let $\Gamma < G$ be a cocompact lattice. Then Γ contains surface subgroups.*

Since by Selberg's lemma every finitely generated subgroup of a linear group contains a torsion-free subgroup of finite index (see [Ra94] for a proof), the theorem is an immediate consequence of the statement that every closed locally symmetric manifold M of negative curvature which is different from a hyperbolic manifold of even dimension contains closed incompressible immersed surfaces.

The proof of this fact uses the strategy developed by Kahn and Markovic. Namely, the surfaces will be glued from immersed incompressible pants with geodesic boundary. These pants are viewed as topological objects, but they have geometric realizations as piecewise ruled surfaces which are close to totally geodesic pants. Closeness in this sense can be quantified, and this allows to establish a glueing condition for such pants which results in an incompressible surface.

This differential geometric viewpoint is the main novelty of this work. It also results in a significant simplification of the original construction for closed hyperbolic three-manifolds.

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The glueing condition can be expressed as a system of linear glueing equations, and the task is then to find a non-negative solution. For manifolds of higher dimension, this turns out to be much more difficult than for hyperbolic three-manifolds, and this is also the place where our argument is not valid for even dimensional hyperbolic manifolds. We refer to Section 7 for a more detailed discussion of this difficulty.

We do not formulate an approximation result along the lines of the work of Kahn and Markovic since we are not aware of any interesting application, although suitable versions of such an approximation result hold as well.

The paper is completely self-contained. Some of our arguments are valid for arbitrary closed negatively curved manifolds with generic metrics, but we do not know whether the beautiful result of Kahn and Markovic holds true in this setting.

In Section 2 we collect some background and tools used later on, and we give a more detailed overview of the argument.

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2. Setup and tools

This section has four parts. In the first part we give a short outline of the proof, enhancing its similiarities and differences to the argument of Kahn and Markovic. The second part summarizes those properties of rank one symmetric spaces which are used later on. In the third part we collect some dynamical properties of the frame flow on a closed rank locally symmetric space. In part 4 we give a simple (and well known to the experts but hard to find in the literature) criterion for incompressibility of a map from a surface of higher genus into a closed nonpositively curved manifold.

2.1. **Structure of the proof and outline of the paper.** To prove Theorem 1 from the introduction, we construct closed surfaces S with piecewise smooth locally $\mathrm{CAT}(-1/2)$-metrics and piecewise smooth isometric immersions $f : S \to M$. The surfaces are constructed in such a way that they admit a distinguished *pants decomposition* $\mathcal{P}$ which is mapped by f to a collection of closed geodesics in M. Such a pants decomposition consists of a collection of pairwise disjoint simple closed geodesics which decompose S into *pairs of pants*, i.e. three-holed spheres. The image under f of each pair of pants is a piecewise smooth isometrically immersed surface in M which is geometrically close to a totally geodesic totally real hyperbolic subsurface.

To show that these surfaces are incompressible we associate to each component of the thin part of such a pair of pants an immersed totally geodesic totally real hyperbolic plane in M. We then lift these immersed hyperbolic planes to the universal covering $\tilde{M}$ of M and use these chains of planes to establish a sufficient condition for incompressibility which relies on the simple characterization formulated in Proposition 2.4. The construction is carried out in Sections 5 and 6, and it is valid for symmetric spaces of higher rank as well, where we work with totally geodesic hyperbolic planes of constant curvature -1 (which therefore intersect a given maximal flat in at most one geodesic).

The construction of the basic building blocks of the surfaces, the pairs of pants, is carried out in Section 4 and uses the strategy from [KM12]. In our approach, it is essential to equip these pairs of pants with a geometric structure and to keep track of the monodromy defined by parallel transport of frames along boundary geodesics.

Glueing the pants to closed surfaces which satisfy the condition for incompressibility established in Section 6 amounts to solving a glueing equation. For this we use ideas from [KM12]: project the Lebesgue measure on suitable frame bundles to the sphere bundles over closed geodesics and use these projected measures to attach pants along their boundary geodesics in such a way that the glueing equation is fulfilled.

The projected measures are not so easy to control. If the monodromy of a closed geodesic has a single fixed real line (which may happen in even dimensional hyperbolic manifolds) then it is not clear whether the projected measure is symmetric enough to construct a solution of the glueing equation, and this difficulty is the reason why our proof of Theorem 1 fails for even dimensional real hyperbolic manifolds. We refer to the short section 8 for a more detailed discussion of this difficulty.

Section 3 contains some distance estimates in universal coverings of surfaces with CAT(-1)-metric which are equipped with a pants decomposition with specific properties. The surfaces we construct will be equipped with pants decompositions with precisely these properties.

The results in this work are formulated so that they can easily be used in more general situations than the one we consider here.

For a nice exposition of the work of Kahn and Markovic we refer the reader to [B12]. An alternative approach to incompressibility of immersed surfaces in hyperbolic 3-manifolds can be found in [S12].

2.2. **The geometry of rank one symmetric spaces.** A rank-one symmetric space is a simply connected Riemannian manifold $\tilde{M}$ of negative sectional curvature with a transitive action of one of the simple Lie groups

$$G = SO(n,1), SU(n,1), Sp(n,1), F_4^{-20}$$

by orientation preserving isometries. More precisely, $\tilde{M} = G/K$ where $K < G$ is a maximal compact subgroup and the Riemannian metric on $\tilde{M}$ is induced from the Killing form on G. We always assume that the maximum of the sectional curvature of $\tilde{M}$ equals -1. The geometry of these symmetric spaces can be described as follows.

The symmetric space $SO(n,1)/SO(n)$ is the *hyperbolic n-space* $\mathbf{H}^n$ of constant curvature -1. The group $SO(n,1)$ can naturally be identified with the $SO(n)$-principal bundle over $\tilde{M}$ of all orthonormal frames in the tangent bundle of $\tilde{M}$. Using the standard embedding $SO(n-1) \to SO(n)$, the quotient space $SO(n,1)/SO(n-1)$ is the unit tangent bundle of $\tilde{M}$.

The symmetric space $SU(n,1)/S(U(n)U(1))$ is *complex hyperbolic n-space* $\mathbb{C}\mathbf{H}^n$ which is Hermitean symmetric. The group $SU(n,1)$ can be identified with the principal bundle of unitary frames over $\tilde{M}$.

For any given point $p \in \mathbb{C}\mathbf{H}^n$, the stabilizer of p in $SU(n,1)$ acts transitively on the unit sphere $T_p^1\mathbb{C}\mathbf{H}^n$ in the tangent space of $\mathbb{C}\mathbf{H}^n$ at p. This action is just the standard action of $S(U(n)U(1))$ on the unit sphere in $\mathbb{C}^n$. The stabilizer in $S(U(n)U(1))$ of a unit tangent vector $X \in T^1\mathbb{C}\mathbf{H}^n$ equals the group $S(U(n-1)U(1))$ which acts simply transitively on the space of unitary frames in the orthogonal complement of the span of X and JX where J denotes the complex structure of $\mathbb{C}\mathbf{H}^n$, viewed as an automorphism of the tangent bundle.

Each real line in the tangent space at p is tangent to a unique complex one-dimensional submanifold of $\mathbb{C}\mathbf{H}^n$ which is a totally geodesic embedded hyperbolic plane of constant curvature -4. A two-plane spanned by a vector $0 \neq X$ and a vector Y which is orthogonal to the span of X and JX is *totally real* and tangent to a unique totally geodesic embedded hyperbolic plane of constant curvature -1. The set of all oriented totally real planes containing the fixed vector X can naturally be identified with a sphere of dimension $2n-3$. Parallel transport along a geodesic γ commutes with the complex structure and hence it preserves the sphere bundle over γ corresponding to unit tangents which span together with γ' a totally real plane. We refer to the monograph [Go99] for more information on complex hyperbolic space.

The symmetric space $Sp(n,1)/Sp(n)Sp(1)$ is *quaternionic hyperbolic n-space* $\mathbb{H}\mathbf{H}^n$. There is a two-sphere of complex structures on $\mathbb{H}\mathbf{H}^n$. Each unit tangent vector $v \in T_p\mathbb{H}\mathbf{H}^n$ spans a unique quaternionic line in $T_p\mathbb{H}\mathbf{H}^n$, and this line is tangent to a unique totally geodesic embedded real hyperbolic 4-space of constant curvature -4. A two-plane spanned by a vector X and a vector which is orthogonal to the quaternionic line determined by X is tangent to a unique totally geodesic hyperbolic plane of constant curvature -1. As before, we call such a plane *totally real*. The set of all oriented totally real planes through X can naturally be identified with a sphere of dimension $4n-5$. The group $Sp(n,1)$ can be viewed as the principal bundle of orthonormal quaternionic frames over $\tilde{M}$. Parallel transport along a geodesic γ commutes with the quaternionic structure and hence it preserves the sphere bundle over γ corresponding to unit vectors which span together with γ' a totally real plane.

The *Cayley plane* $\mathrm{Ca}\mathbf{H}^2 = F_4^{-20}/\mathrm{Spin}(9)$ has a similiar geometric description. The isotropy representation of the group $\mathrm{Spin}(9)$ is the spin representation. The action of $\mathrm{Spin}(9)$ on the unit sphere $T_p^1\mathrm{Ca}\mathbf{H}^2 = S^{15}$ in the tangent space at the point p is transitive. The stabilizer in $\mathrm{Spin}(9)$ of a unit vector X is the subgroup $\mathrm{Spin}(7)$ which acts transitively on the unit sphere S^7 in the orthogonal complement of the span of X over the Cayley numbers. The unit tangent vector X determines a unique totally geodesic embedded hyperbolic 8-space of constant curvature -4. A unit tangent vector orthogonal to the tangent space of this hyperbolic space spans with X the tangent space of a totally real hyperbolic plane of constant curvature -1. The group F_4^{-20} is a $\mathrm{Spin}(7)$-principal bundle over the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$.

The monograph [W11] (see in particular Section 8.2) contains all information summarized above, and we refer to it for a more detailed discussion.

Write $\tilde{M} = G/K$ and let $T^1\tilde{M}$ be the unit tangent bundle of $\tilde{M}$. For a unit tangent vector $v \in T^1\tilde{M}$ with foot-point p let

$$v_{\mathbb{K}}^{\perp}$$

be the $\mathbb{K}$-orthogonal complement of v in $T_p\tilde{M}$. Here we put $\mathbb{K} = \mathbb{R}$ if $G = SO(n,1)$, $\mathbb{K} = \mathbb{C}$ if $G = SU(n,1)$, $\mathbb{K} = \mathbb{H}$ if $G = Sp(n,1)$, and $\mathbb{K} = \mathbb{O}$ if $G = F_4^{-20}$. A unit tangent vector $w \in T_p^1 \tilde{M}$ orthogonal to v is contained in $v_{\mathbb{K}}^{\perp}$ if and only if the plane in $T_p\tilde{M}$ spanned by v and w is *real*, i.e. its curvature equals -1.

The orientation of $\tilde{M}$ induces an orientation on $v_{\mathbb{K}}^{\perp}$. In the case $G = SO(n,1)$ this orientation is determined by the requirement that for every positive basis $X_2, \dots, X_n$ of $v_{\mathbb{K}}^{\perp}$ the basis $v, X_2, \dots, X_n$ of T_pM is positive. If $G = SU(n,1)$ (or $G = Sp(n,1), F_4^{-20}$) we use the fact that the complex structure (or the quaternionic structure or the Cayley structure) defines an orientation on the $\mathbb{K}$-lines in $T\tilde{M}$.

We need some information on parallel transport of $\mathbb{K}$-orthonormal frames along loops in totally geodesic hyperbolic planes $H \subset \tilde{M}$ of curvature -1, i.e. totally real planes. Recall that parallel transport along a loop $\gamma : [0,1] \to \tilde{M}$ is a $\mathbb{K}$-linear isometry of $T_{\gamma(0)}\tilde{M}$.

For a linear subspace $L \subset T\tilde{M}$ denote by $L \otimes \mathbb{K}$ the $\mathbb{K}$-linear subspace of $T\tilde{M}$ spanned by L.

Lemma 2.1. *Let $\gamma : [0,1] \to H \subset \tilde{M}$ be a smooth loop. Parallel transport along γ induces the identity on the orthogonal complement of $T_{\gamma(0)}H \otimes \mathbb{K}$.*

Proof. The real plane H is totally geodesic in $\tilde{M}$ and therefore the tangent plane of H is invariant under parallel transport along loops γ in H.

Assume for the moment that $G = SO(n,1)$. Then for every unit vector $z \in T_{\gamma(0)}^1\tilde{M}$ which is orthogonal to $T_{\gamma(0)}H$, there is a unique totally geodesic hyperbolic 3-space $Q \supset H$ embedded in $\tilde{M}$ whose tangent space at $\gamma(0)$ equals the span of $T_{\gamma(0)}H$ and z. Since Q is totally geodesic, parallel transport along smooth curves in Q preserves the tangent space of Q. Moreover, it preserves the orientation of a frame. Thus parallel transport along γ maps z to itself. This shows the claim in the case that $G = SO(n,1)$.

If $G = SU(n,1)$ then observe that the complex structure J is invariant under parallel transport. On the other hand, the above reasoning shows that the restriction of parallel transport along $\gamma \subset H$ to the orthogonal complement of $TH \otimes \mathbb{C}$ is the identity. If $G = Sp(n,1)$ then the same argument applies to the quaternionic span $T_{\gamma(0)}H \otimes \mathbb{H}$ of $T_{\gamma(0)}H$. The statement is empty for $G = F_4^{-20}$. ☐

2.3. **Dynamics of the frame flow.** As in Subsection 2.2, consider a rank one symmetric space $\tilde{M} = G/K$. The *geodesic flow* Φ^t acts on the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$: The *frame flow* Ψ^t is the lift of Φ^t to a flow on the bundle of $\mathbb{K}$-frames which is defined by

$$\Psi^t(v, F) = (\Phi^t v, \|F).$$

Here $\|F$ denotes parallel transport of the frame F along the projection of the flow line $t \to \Phi^t v$ of the geodesic flow to a geodesic in $\tilde{M}$. The frame flow is well defined since the Riemannian metric and the complex (or quaternionic or Cayley) structure are parallel.

Let $\Gamma < G$ be any torsion free cocompact lattice. The existence of such a lattice was established by Borel [Bo63]. Then $M = \Gamma\backslash\tilde{M}$ is a compact locally symmetric space. The geodesic flow descends to a flow on the unit tangent bundle T^1M of M again denoted by Φ^t.

Denote by

$$P : \mathcal{F} \to T^1M$$

the bundle of $\mathbb{K}$-frames over T^1M. Thus we have $\mathcal{F} = \Gamma\backslash G$. The frame flow descends to a flow on on the bundle $\mathcal{F}$ which we denote again by Ψ^t.

The Riemannian metric on M naturally defines Riemannian metrics on T^1M and on the $\mathbb{K}$-frame bundle $\mathcal{F}$ as follows. The Levi Civita connection of the Riemanian metric on M can be viewed as a connection on the principal bundle $\mathcal{F} \to M$. This connection is a K-invariant subbundle $\mathcal{H}$ of the tangent bundle of $\mathcal{F}$ or of T^1M which is complementary to the tangent bundle of the fibers. Define these bundles to be orthogonal, equip $\mathcal{H}$ with the metric induced fron the metric on M and equip the fibres with the standard metric.

The metric on $\mathcal{F}$ determines a Borel probability measure λ on $\mathcal{F}$ in the Lebesgue measure class which is invariant under the flow Ψ^t. The measure λ is called *exponentially mixing* for the flow Ψ^t if the following holds true. For some $k \geq 2$ equip the space of smooth functions on $\mathcal{F}$ with the C^k-norm $\| \|$ with respect to the Riemannian metric. Then there is a number $\kappa > 0$ so that for any two functions f, g of class C^k we have

$$| \int (f \circ \Psi^t) g d\lambda - \int f d\lambda \int g d\lambda | \leq e^{-\kappa t} \|f\| \, \|g\| / \kappa.$$

The following was worked out by Moore [M85] as a consequence of some a classical important results in representation theory.

Theorem 2.2. *The frame flow Ψ^t is exponentially mixing for the measure λ.*

Proof. As explained on p. 176-178 of [M85], exponential mixing for the geodesic flow on M is a consequence of a spectral gap for the Laplacian on M. As M is compact, such a spectral gap is automatic.

Namely, existence of a spectral gap guarantees that some tensor product of the representation of the Lie group G on the Hilbert space $L_0(\Gamma\backslash G, \lambda)$ of square integrable functions on $\Gamma\backslash G$ with zero mean is tempered [M85]. This is a property of the matrix coefficients of the representation (see [CHH88] for more detailed information).

The frame flow on M is just given by the right action of a one-parameter subgroup of G on $\Gamma\backslash G$ and hence exponential mixing again follows from the fact that some tensor product of the representation of G on $L_0(\Gamma\backslash G, \lambda)$ is tempered (see [CHH88, M85] for more details). ☐

Remark 2.3. For closed manifolds of negative curvature which are not locally symmetric, exponential mixing of the geodesic flow was established in [Li04]. For the frame flow, we do not expect exponential mixing in general, see however [BG80].

2.4. **A criterion for incompressibilty.** In this subsection we consider a closed Riemannian manifold M of nonpositive sectional curvature with universal covering $\tilde{M}$.

Let S be a closed oriented surface of genus $g \geq 2$ equipped with a locally CAT(-1) geodesic metric. We say that a continuous map $f : S \to M$ is *incompressible* if $f_* : \pi_1(S) \to \pi_1(M)$ is injective, and we use this terminology also in the case that S is a compact surface with boundary.

The fundamental group $\pi_1(S)$ of S acts on the universal covering $\tilde{S}$ of S as a group of isometries. Since S, M are $K(\pi, 1)$ spaces, there is an f_*-equivariant map $F : \tilde{S} \to \tilde{M}$ which projects to f. Here $f_* : \pi_1(S) \to \pi_1(M)$ is the homomorphism induced by f. We call such a map F a *canonical lift* of f.

Proposition 2.4. *Let S be a closed oriented surface with a locally* $\mathrm{CAT}(-1)$ *geodesic metric, and let $f : S \to M$ be a Lipschitz map with canonical lift $F : \tilde{S} \to \tilde{M}$. The following are equivalent.*

(1) *f is incompressible.*
(2) *F is proper.*

Proof. To show (1) $\Rightarrow$ (2) assume that f_* is injective. Let $K \subset \tilde{S}$ be a compact fundamental domain for the action of $\pi_1(S)$. Then $F(K) \subset \tilde{M}$ is compact. If $B \subset \tilde{M}$ is any compact set, then the set

$$A = \{\psi \in \pi_1(M) \mid \psi(F(K)) \cap B \neq \emptyset\}$$

is finite, and by equivariance,

$$F^{-1}(B) \subset \cup\{\gamma(K) \mid f_*\gamma \in A\}.$$

Since f_* is injective and A is finite, this means that the closed set $F^{-1}(B) \subset \tilde{S}$ is contained in finitely many translates of K and hence $F^{-1}(B)$ is compact. Therefore F is proper as claimed.

To show the implication (2) $\Rightarrow$ (1) assume that F is proper. Assume to the contrary that there is an element $0 \neq \alpha \in \pi_1(S)$ with $f_*\alpha = 0$. Represent α by a closed geodesic in S, again denoted by α. The loop $f(\alpha) \subset M$ is contractible. Then $f(\alpha)$ lifts to a compact loop β in $\tilde{M}$. The loop β is the image under F of a lift $\tilde{\alpha}$ of α to S. As $\tilde{\alpha}$ is unbounded, $F^{-1}(\beta) \supset \tilde{\alpha}$ is not compact. The proposition follows. ☐

3. Spaced laminations

The goal of this section is to establish some distance estimates on closed surfaces of genus $g \geq 2$ equipped with a locally $\mathrm{CAT}(-1)$ geodesic metric with some specific properties.

The immersed surfaces in closed rank one locally symmetric spaces we are going to construct in Section 7 will have all these properties. They are glued from smooth pieces with geodesic boundary, where the smooth pieces are equipped with a smooth metric of Gauss curvature at most $-1/2$. We begin with showing that the metric on such a surface is locally $\mathrm{CAT}(-1/2)$. Lemma 3.1 below is certainly known to the experts. As we were not able to find a reference in the literature, we give a proof. For ease of exposition, we rescale and work with locally $\mathrm{Cat}(-1)$-metrics.

Lemma 3.1. *Let S be a surface equipped with a length metric d, with (perhaps empty) geodesic boundary. Assume that S contains a compact embedded geodesic graph Q such that the restriction of the metric d to each component of $S - Q$ is a smooth Riemannian metric of curvature at most -1. Assume moreover that at each vertex of Q the cone angle is not smaller than 2π. Then d is locally* $\mathrm{CAT}(-1)$.

Proof. It suffices to show that every point $x \in S$ has a convex neighborhood $U(x)$ so that the triangle comparison property holds for triangles with vertices in $U(x)$ (see Proposition II.1.7 of [BH99]).

This follows from standard comparison if x is an interior point of a component of $S - Q$ where the metric is smooth. Let x be an interior point of an edge ζ of Q. Assume that ζ separates an open contractible neighborhood U of x into halfplanes W_1, W_2 with smooth metric and geodesic boundary.

Let $y_i \in W_i$ and consider a geodesic triangle T with vertices x, y_1, y_2 and edges of minimal length. Since the edge ζ of the graph Q is a geodesic for the metric d and the metric in $S - Q$ is of curvature at most -1, the side of T connecting y_1 to y_2 intersects ζ in a single point y provided that the distance between y_1, y_2 is sufficiently small. The triangle T_i with vertices x, y, y_i $(i = 1, 2)$ satisfies the angle comparison property. In particular, the Aleksandrov angle of T_i at y is not bigger than the comparison angle in the hyperbolic plane $\mathbf{H}^2$.

For $i = 1, 2$ let $\overline{T_i}$ be a comparison triangle in $\mathbf{H}^2$ whose side lengths coincide with the side lengths of T_i. Assume that $\overline{T_1} \cap \overline{T_2}$ is a common side of $\overline{T_1}, \overline{T_2}$ of length $d(x, y)$. By the discussion in the previous paragraph, the angle sum of the geodesic quadrangle $\overline{T_1} \cup \overline{T_2}$ at the point $\hat{y} \in \overline{T_1} \cap \overline{T_2}$ corresponding to the common vertex y of T_1 and T_2 is not smaller than π. Hyperbolic trigonometry now shows that in a comparison triangle $\overline{T} \subset \mathbf{H}^2$ for T, the distance between the points $\bar{x}, \bar{y} \in \overline{T}$ corresponding to the points $x, y \in T$ is not smaller than the distance between x and y.

Using comparison for the Riemannian triangles T_1, T_2 with triangles in $\mathbf{H}^2$, we conclude that the distance between the vertex $\bar{x}$ of $\overline{T}$ corresponding to x and a point on the opposite opposite side of $\bar{T}$ is not smaller than the distance between x and the corresponding point on the side of T opposite to x. With the same argument, this estimate also holds true for distances between the other vertices and points on the opposite sides.

Proposition II.1.7 of [BH99] now shows that the path metric d on S is locally CAT(-1) on the complement of the vertex set of Q.

Using the condition on cone angles, the same argument also holds true near a vertex of the graph. This implies the lemma. □

As a consequence, a closed curve on a surface with the properties in Lemma 3.1 has a unique geodesic representative in its free homotopy class. Moreover, local geodesics lift to curves in the universal covering which realize the distance between their endpoints.

The surfaces we construct will be glued from pairs of pants with specific properties. In the remainder of this section we discuss those properties that are used to establish incompressibility.

Let P_0 be such a pair of pants with geodesic boundary. Any two boundary geodesics are connected by a unique shortest geodesic arc. Such a geodesic arc is called a *seam* of P_0. These seams decompose P_0 into hexagons with geodesic boundary. The angles in the sense of Aleksandrov of these hexagons are at least $\pi/2$. The endpoints of the seams define two distinguished points on each boundary geodesic of P_0. We call these points the *feet* of the pair of pants.

In the cases we are interested in, these hexagons are all right angled since the restriction of the metric to a pair of pants is smooth in a neighborhood of the seams. In the remainder of this section we will use this assumption to faciliate the notations, although it is nowhere used in the arguments.

Each component α of the pants decomposition is contained in the boundary of two (not necessarily distinct) pairs of pants P_1, P_2. These pairs of pants are glued with an orientation reversing isometry along α. The feet of P_1 on α need not coincide with the feet of P_2 on α. We define the *shear* of the component α of $\mathcal{P}$ to be the pair of distances on α between the feet of P_1, P_2 which are determined by the orientations of α as boundary components of P_1, P_2.

More precisely, choose an endpoint $x \in \alpha$ of a seam of P_1 on α. The orientation of P_1 defines an orientation of α. The oriented distance between x and a seam of P_2 is the distance along α between x and the first point y on α which is a seam of P_2. This distance equals the oriented distance between y and a seam of P_1 provided that the second seam of P_1 does not lie between x and y along the oriented subarc of α connecting x to y. Since the latter property holds true in the situations we are interested in we will always assume in the sequel that this is the case.

If the metric on S is smooth and of constant curvature then the seams decompose each boundary geodesic into two arcs of equal length and the two shear parameters coincide. However, this need not be the case if the curvature is non-constant. We say that the shear of the pants curve α is *contained in an interval* $[a, b]$ only if both shear parameters in the pair are contained in this interval. In this vain, the following definition (which is motivated by the work of Kahn and Markovic [KM12]) is natural for hyperbolic surfaces but harder to accomplish for surfaces with arbitrary locally CAT(-1) geodesic metrics.

Definition 3.2. For some $\delta \in (0, 1/4)$, $R > 10$, a pants decomposition $\mathcal{P}$ of S is (R, δ)-*tight* if the following holds true,

- The lengths of the pants curves are contained in the interval $[R - \delta, R + \delta]$.
- The seams of a pair of pants decompose the boundary geodesics into two subarcs whose lengths are contained in the interval $[R/2 - \delta, R/2 + \delta]$.
- The shear of each component of $\mathcal{P}$ is contained in the interval $[1 - \delta, 1 + \delta]$.

An (R, δ)-tight pants decomposition of S lifts to a discrete $\pi_1(S)$-invariant geodesic lamination μ on the universal covering $\tilde{S}$ of S. Figure A shows a geometric model for the geodesic lamination defined by an (R, δ)-tight pants decomposition.

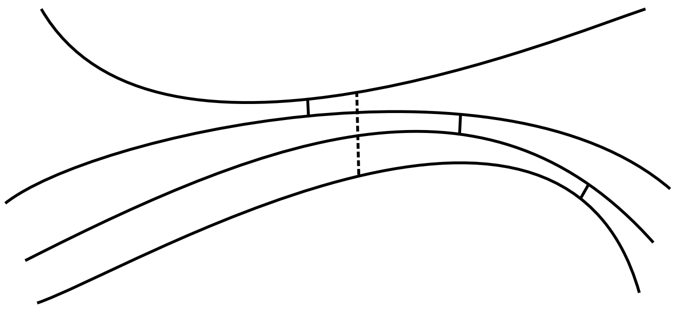

Figure A

We will need some additional geometric information on the pants decompositions we are going to use. Namely, for a number $C > 0$ call an (R, δ)-tight pants decomposition $\mathcal{P}$ of S *centrally* C-*thick* if the following holds true. Let $\beta : [0, s] \to S$ be a geodesic arc with endpoints on $\mathcal{P}$. Assume that β intersects some pants curve α of $\mathcal{P}$ at a point whose distance along α is at least $R/4 - 2$ from the endpoint of a seam on α. Then the length of β is at least C.

The following technical lemma uses the property described in this definition in an essential way. For its formulation, let $\mathcal{P}$ be an (R, δ)-tight pants decomposition of a surface S equipped with a locally CAT(-1)-metric. For each pants curve α and every $x \in \alpha$ define $\tau(x)$ to be the maximum of one and the distance of x along

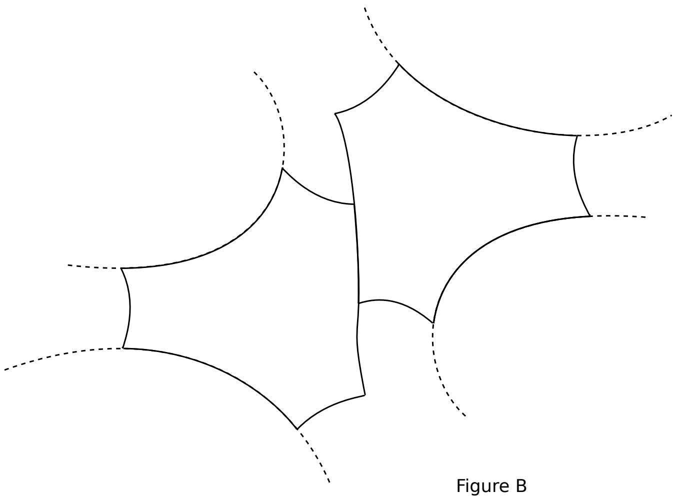

Figure B

α to the endpoint of a seam of $\mathcal{P}$ on α. We then can view τ as a function on $\mathcal{P}$ with values in the interval $[1, R/2+\delta]$. Let $b > 1$. Let $\zeta : [0,T] \to S$ be any geodesic segment which is transverse to $\mathcal{P}$ and intersects $\mathcal{P}$ in the points $\zeta(t_i)$ $(1 \leq i \leq m)$. Define

$$f(\zeta) = \sum_{\zeta(t_i)\in\mathcal{P}} \frac{1}{\tau(\zeta(t_i))^b}.$$

The estimate in the following lemma (which is a variant of a construction in [KM12]) is used in the proof of Lemma 5.7 which is the main technical tool of this work.

Lemma 3.3. *For every $C > 0$ there is a number $\chi > 0$ with the following property. Let $R > 10$, $\delta < 1/10$ and let S be a closed oriented surface of genus $g \geq 2$ equipped with a locally $\mathrm{CAT}(-1)$-metric and an (R,δ)-tight centrally C-thick pants decomposition. Then $f(\zeta) < \chi$ for every geodesic arc ζ on S of length smaller than $\min\{1/4, C\}$ which is transverse to P.*

Proof. Let $\tilde{S}$ be the universal covering of S. The pants decomposition $\mathcal{P}$ lifts to a geodesic lamination μ on $\tilde{S}$. The lifts of the seams of $\mathcal{P}$ decompose the complementary components of μ into right angled hexagons. These hexagons define a tesselation of $\tilde{S}$ which is invariant under the action of $\pi_1(S)$. Call a lift to $\tilde{S}$ of a seam of $\mathcal{P}$ a *seam* of μ. Let $\tau : \mu \to [0,\infty)$ be the function which associates to a point x on a leaf α of μ the maximum of one and the minimal distance between x and an endpoint on α of some seam.

Let $\zeta : [0,a] \to \tilde{S}$ be a geodesic arc parametrized by arc length which is transverse to μ and such that $\zeta(0) \in \mu, \zeta(a) \in \mu$. Assume that the length a of ζ does not exceed $\min\{1/4, C\}$.

Let $0 = t_0 < t_1 < \cdots < t_m = a$ be the consecutive intersection points of ζ with μ. Assume for the moment that $\zeta[t_0,t_1]$ does not intersect a seam. This is equivalent to stating that $\zeta(t_0,t_1)$ is contained in the interior of a hexagon H_0 of the $\pi_1(S)$-invariant tesselation and cuts H_0 into a quadrangle Q_0 and a hexagon $H_0 - Q_0$. The side ξ of Q_0 opposite to $\zeta[t_0,t_1]$ is a seam. The quadrangle Q_0 has two right angles at the endpoints of ξ. The sides of Q_0 adjacent to $\zeta[t_0, t_1]$ are subarcs of leaves of μ.

Consider first the case that with respect to the orientation of $\tilde{S}$, the orientation of ζ defines the boundary orientation of Q_0. Then the side ξ of Q_0 is to the left of ζ. Since the length of ζ is smaller than C, by the definition of an (R, δ)-tight centrally C-thick pants decomposition, we have $\tau(\zeta(t_0)) < R/4 - 2$.

Now there are two possibilities. In the first case, $\tau(\zeta(t_0)) > 1$. Since $\delta < 1/10$ and since the length of ζ does not exceed 1/4, the value $\tau(\zeta(t_1))$ is the distance between $\zeta(t_1)$ and the side ξ of the quadrangle Q_0. Moreover, we have $\tau(\zeta(t_1)) \geq \tau(\zeta(t_0)) + 1/2$. In the second case, we have $\tau(\zeta(t_0)) = 1$. This means that the distance along the leaves of μ between $\zeta(t_0)$ and a seam on the leaf of μ containing $\zeta(t_0)$ is at most one. The argument from the first case now shows that $\tau(\zeta(t_3)) \geq \tau(\zeta(t_2)) + 1/2$.

Proceeding inductively and using the definition of a centrally thick pants decomposition and the assumption on the length of ζ, we conclude that

$$f(\zeta) \leq \sum_{i=0}^{m} \frac{1}{(\tau(\zeta(t_0)) + i/2)^b} + 2$$

where m is the smallest integer larger than $R/2$. This shows the lemma for geodesic arcs ζ with the following property. The arc ζ does not intersect a seam and moreover, with respect to a fixed orientation of ζ, all seams which are closest to ζ along the leaves of μ crossed through by ζ are to the left of ζ as described above, i.e. the quadrangles defined by the intersection of ζ with the interiors of the hexagons from the tesselation lie to the left of ζ.

If with respect to a fixed orientation of ζ, some of the quadrangles defined by the intersection of ζ with the interiors of the hexagons from the tesselation lie to the right of ζ and some others lie to the left, then we can decompose ζ into two disjoint subarcs to which the above discussion can be applied. The same holds true if ζ intersects a seam. The lemma follows. ☐

Remark 3.4. Lemma 3.3 is also valid for an arbitrary discrete geodesic lamination on a simply connected CAT(-1)-surface so that there is a system of shortest distance arcs between neighboring leaves of the lamination with the properties described in the definition of a tight pants decomposition.

Hyperbolic trigonometry implies that (R, δ)-tight pants decompositions of hyperbolic surfaces are centrally C_0-thick for a universal number $C_0 > 0$. We formulate this as a lemma.

Lemma 3.5. *There are numbers $C_0 > 0, R_0 > 0, \delta_0 > 0$ such that for all $R > R_0, \delta < \delta_0$, every (R, δ)-tight pants decomposition of a surface of constant curvature -1 is centrally C_0-thick.*

Proof. Let S be a hyperbolic surface equipped with an (R, δ)-tight pants decomposition $\mathcal{P}$. The universal covering of S is the hyperbolic plane $\mathbf{H}^2$.

Each component X_0 of $S - \mathcal{P}$ is a union of two isometric right angled hyperbolic hexagons which are obtained by cutting X_0 open along the seams. The length of a long side of such a hexagon equals half the length of the component of $\mathcal{P}$ containing it, i.e. it equals $R/2$ up to an additive error of at most $\delta/2$. Hyperbolic trigonometry (Theorem 2.4.1 of [B92]) shows that the length of a side which corresponds to a seam is comparable to $e^{-R/4}$.

Let $T\subset \mathbf{H}^2$ be an ideal hyperbolic triangle, i.e. a geodesic triangle with vertices on the ideal boundary $\partial\mathbf{H}^2$ of $\mathbf{H}^2$, and let γ be one of the sides of T. The shortest distance projection into γ of the ideal vertex of T opposite to γ is a distinguished point x on γ. There is a number $c>0$ such that the distance between any point on γ whose distance to x is at most three and a side of T distinct from γ is at least c.

The group $PSL(2,\mathbb{R})$ of orientation preserving isometries of $\mathbf{H}^2$ acts transitively on oriented ideal triangles. This implies that as $R\to\infty$ and $\delta\to 0$, a right angled hyperbolic hexagon H with three pairwise non-adjacent sides of length within $[(R-\delta)/2,(R+\delta)/2]$ converges up to the action of $PSL(2,\mathbb{R})$ in the Hausdorff topology for closed subsets of the closed unit disc $\mathbf{H}^2\cup\partial\mathbf{H}^2$ to an ideal triangle. As a consequence, there are numbers $R_0>0,\delta_0>0$ such that for $R\geq R_0$ and $\delta<\delta_0$ an (R,δ)-tight pants decomposition of a hyperbolic surface is centrally $c/2$-thick. □

4. Constructing geometrically controlled pants

In this section we construct topological versions of the pairs of pants which form the basic building blocks for our surfaces. More precisely, for a given closed rank one locally symmetric manifold M, we construct maps from a fixed pair of pants into M which map the boundary circles of the pair of pants to closed geodesics in M.

The underlying principle for this construction is very general and applies to any closed negatively curved manifold with a generic metric (so that the frame flow is topologically mixing). As we will need to obtain a good geometric control for these pairs of pants we will however only work in rank one locally symmetric manifolds. Control of these geometric invariants in the construction is the only part of the argument in this section which is not taken from [KM12].

We begin with a geometric construction in the hyperbolic plane $\mathbf{H}^2$. Define a *tripod* in $\mathbf{H}^2$ to be an ordered triple (v_1,v_2,v_3) of unit tangent vectors over a fixed point x which mutually enclose an angle of $2\pi/3$. The tripod defines an oriented ideal hyperbolic triangle T whose endpoints in the ideal boundary $\partial\mathbf{H}^2$ of $\mathbf{H}^2$ are the endpoints of the geodesic rays γ_{v_i} with initial velocity $v_i=\gamma_{v_i}'(0)$. The orientation of T is defined by the cyclic order of the vertices v_i.

We call the basepoint x of the tripod the *center* of the triangle T. The oriented boundary of T is denoted by ∂T. Note that T is preserved by the cyclic subgroup Λ of $PSL(2,\mathbb{R})$ of order 3 which fixes the point x and which acts by rotation with angle $2\pi/3$ in the tangent plane of $\mathbf{H}^2$ at x.

For $R>1$ let $H_R\subset T$ be the intersection of T with the half-planes containing x whose boundaries are the geodesics through $\gamma_{v_i}(R)$ which are perpendicular to γ_{v_i}. Then H_R is a Λ-invariant oriented hyperbolic hexagon which is not right-angled. Three sides of H_R are contained in the sides of the ideal triangle T, and these sides are called the *long sides.* The length of each long side equals

$$L(R)=2R+t(R)$$

where $t(R)\in(-\infty,0)$ is uniformly bounded in norm (recall that we require $R>1$. The number $t(R)$ can explicitly be computed using the formulas of hyperbolic trigonometry, see [B92]). The length of the *short sides* of H_R (i.e. the three sides which intersect the geodesics γ_{v_i}) does not exceed $\kappa_0 e^{-R}$ where $\kappa_0>0$ is a universal constant. We call the footpoint x of the tripod the *center* of the hexagon H_R (see Figure C).

Resuming the notations from Section 2, let G be a simple rank one Lie group of non-compact type and let $\Gamma < G$ be a torsion free cocompact lattice. Let $K < G$ be a maximal compact subgroup, let $\tilde{M} = G/K$ be the corresponding symmetric space and let $M = \Gamma\backslash\tilde{M}$ be the locally symmetric space defined by Γ. We always assume that M is equipped with the locally symmetric metric whose upper curvature bound is -1. By perhaps passing to a subgroup of Γ of index 2 we may assume that M is oriented.

Define a *real tripod* in $T\tilde{M}$ (or TM) to be an ordered triple (v_1, v_2, v_3) of three unit tangent vectors contained in the same real plane $V \subset T\tilde{M}$ (or $V \subset TM$) which mutually enclose an angle of $2\pi/3$. The cyclic order of the tripod defines an orientation of V.

A real plane $V \subset T\tilde{M}$ is tangent to a unique oriented totally geodesic embedded hyperbolic plane $H \subset \tilde{M}$. Thus a real tripod $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ in $T\tilde{M}$ defines an oriented real ideal triangle T in the hyperbolic plane $H \subset \tilde{M}$ containing the tripod in its tangent plane. The group G acts transitively on these oriented real ideal triangles.

For each $R > 1$, a real tripod $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ in $T\tilde{M}$ determines an oriented hexagon

$$H_R(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$$

in the totally geodesic hyperbolic plane $H \subset \tilde{M}$ tangent to the tripod. The hexagon $H_R(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ is isometric to the hexagon $H_R \subset \mathbf{H}^2$ described above. We use the terminology which was introduced above for hexagons in $\mathbf{H}^2$ also for hexagons in $\tilde{M}$ defined by real tripods.

Define a *framed real tripod* in $T\tilde{M}$ to be a pair of the form $((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3), F)$ where $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ is a real tripod contained in a real plane $V \subset T\tilde{M}$ and where F is a positive $\mathbb{K}$-orthonormal frame in the orthogonal complement of the $\mathbb{K}$-span $V \otimes \mathbb{K}$ of V. Here the orientation of V is determined by the tripod. The group G acts on framed real tripods and therefore we can also consider framed real tripods in TM. In fact, the action of G on such framed real tripods is simply transitive, so framed real tripods in $T\tilde{M}$ can be viewed as points in G. However, we will use the specific geometric meaning of framed real tripods, moreover the geometric discussion is valid in any closed oriented negatively curved manifold.

A framed real tripod $((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3), F)$ in $T\tilde{M}$ determines for each $R > 1$ a *framed hexagon* $(H_R(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3), F)$. Namely, for $i = 1, 2, 3$, the frame F determines frames

$$F_i \to \gamma'_{\tilde{v}_i}(R) \to \gamma_{\tilde{v}_i}(R)$$

in the fibre of the bundle $\mathcal{F}$ at the point $\gamma'_{\tilde{v}_i}(R)$ as follows. Choose the first vector of the frame F_i to be the oriented normal of $\gamma'_{\tilde{v}_i}(R)$ in the oriented hyperbolic plane $H \subset \tilde{M}$ defined by the tripod. The remaining ordered vectors of the frame are obtained by parallel transport of the frame F along $\gamma_{\tilde{v}_i}$.

By Lemma 2.1, for all i, j the complement of the first vector of the frame F_j can also be obtained from the complement of the first vector of the frame F_i by parallel transport along the boundary of $H_{R_i}(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$. Moreover, the first vector of F_i is uniquely determined by the tripod $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ and the size parameter R.

Each real tripod $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ in $T\tilde{M}$ projects to a real tripod (v_1, v_2, v_3) in TM, and the hexagon $H_R(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \subset \tilde{M}$ projects to a totally geodesic immersed hexagon $H_R(v_1, v_2, v_3)$ in M which is uniquely determined by the tripod (v_1, v_2, v_3) and the size parameter R. Since the stabilizer in $\Gamma = \pi_1(M)$ of the hyperbolic plane H tangent to $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ may be non-trivial, the hexagon $H_R(v_1, v_2, v_3)$ may have non-transverse self-intersections. However, the projection $H_R(\tilde v_1,\tilde v_2,\tilde v_3)\to H_R(v_1,v_2,v_3)$ is an isometric immersion. The geodesics γ_{v_i} with initial velocity v_i will be called the *center geodesics* of the hexagon $H_R(v_1,v_2,v_3)$. Their initial segments of length R are contained in $H_R(v_1,v_2,v_3)$.

For a number $\epsilon>0$ define an element $A\in SO(m)$ ($m=\dim(\tilde M)$) to be *ϵ-close to the identity* if for every unit vector v the angle between v and Av is at most ϵ.

The following definition is taken from the beginning of Section 4 of [KM12]. For its formulation, we say that the angle between two planes $E_1,E_2\subset T_yM$ in the tangent space of M at some point y is at most ϵ if for each vector $0\neq v$ in E_i there is a vector $0\neq v'\in E_{i+1}$ with $\angle(v,v')<\epsilon$ (indices are taken modulo two).

Definition 4.1. For $R>0,\delta>0,\kappa>0$ call the framed tripods $((v_1,v_2,v_3),E)$ and $((w_1,w_2,w_3),F)$ in TM *(R,δ,κ)-well connected* if for each $i=1,2,3$ there is a geodesic segment α_i connecting $\gamma_{v_i}(R)$ to $\gamma_{w_{-i}}(R)$ (indices are taken modulo three) so that the following holds true.

(1) The length of α_i is contained in the interval $[4R-\delta/4,4R+\delta/4]$.
(2) The breaking angles of the concatenation
$$\gamma_{w_{-i}}^{-1}\circ\alpha_i\circ\gamma_{v_i}$$
at the points $\gamma_{v_i}(R),\gamma_{w_{-i}}(R)$ are not bigger than $e^{-\kappa R}/\delta$.
(3) Let E_i (or F_i) be the frame over $\gamma_{v_i}(R)$ (or over $\gamma_{w_{-i}}(R)$) defined by the framed tripod as above. Let $\hat E_i$ be the parallel transport of E_i along α_i. Then the element of $SO(m)$ ($m=\dim(M)$) which transforms the frame $\hat E_i$ to the frame over $\gamma_{w_{-i}}(R)$ which is obtained from F_i by replacing the first vector by its negative is δ-close to the identity.

The geodesics α_i are called *good connections* for the tripods.

By property (2), the angle between $-\gamma'_{w_{-i}}(R)$ and the parallel transport of $\gamma'_{v_i}(R)$ along α_i does not exceed $2e^{-\kappa R}/\delta$. The third requirement implies that the angle between the following two real planes in $T_{\gamma_{w_{-i}}(R)}M$ is at most δ:

- The tangent plane of the totally geodesic hyperbolic plane containing the hexagon $H_R(w_1,w_2,w_3)$.
- The image under parallel transport along α_i of the tangent plane of the hyperbolic plane containing the hexagon $H_R(v_1,v_2,v_3)$.

Remark 4.2. For the purpose of this work, the constant $\kappa>0$ plays no role- it is geared at treating the case of locally symmetric spaces of higher rank. We will only work with $(R,\delta,1)$-well connected tripods which we call (R,δ)-well connected in the sequel.

Recall that there is a natural Riemannian metric on the bundle $\mathcal F$ characterized by the property that the projection $\mathcal F\to T^1M$ is a Riemannian submersion with homogeneous fibre isometric to $SO(n-1)$ (or $SU(n-1)$ or $Sp(n-1)$ or Spin(7)). We next give a criterion for framed tripods to be (R,δ)-well connected.

Let $((v_1,v_2,v_3),E),((w_1,w_2,w_3),F)$ be two framed tripods in TM and let $R>10,\delta\in(0,1/4)$. The framed tripods define frames $V_i\in\mathcal F,W_i\in\mathcal F$ over the tangent vectors $\gamma'_{v_i}(2R),\gamma'_{w_i}(2R)$.

Lemma 4.3. *Assume that for each i there is a frame V_i' contained in the $\delta/2$-neighborhood of V_i with the following property. Let $PV_i'\in T^1M$ be the base vector* *of the frame. Then the frame obtained from $\Psi^{2r}(V_i')$ by replacing the base vector $\Phi^{2R}(PV_i')$ as well as the first vector of $\Psi^{2R}(V_i')$ by their negatives is contained in the $\delta/2$-neighborhood of W_{-i}. Then the framed tripods together with the frames V_i' determine an (R,δ)-well connected pair of framed tripods provided $\delta > 0$ is sufficiently small.*

Proof. Using the notation from the lemma, construct a piecewise geodesic $\hat{\alpha}_i$ connecting $\gamma_{v_i}(R)$ to $\gamma_{w_{-i}}(R)$ as a concatenation of the following geodesic arcs.

- $\gamma_{v_i}[R, 2R]$,
- an arc of length at most $\delta/2$ connecting $\gamma_{v_i}(2R)$ to $\beta_i(0)$,
- the geodesic β_i,
- an arc of length at most $\delta/2$ connecting $\beta_i(2R)$ to $\gamma_{w_{-i}}(2R)$,
- the inverse of $\gamma_{w_{-i}}[R, 2R]$.

Let α_i be the geodesic in M which is homotopic to $\hat{\alpha}_i$ with fixed endpoints. The length of α_i is contained in the interval $[4R-\delta, 4R+\delta]$. We claim that the angle between α_i and γ_{v_i}, $\gamma_{w_{-i}}^{-1}$ at the endpoints $\gamma_{v_i}(R), \gamma_{w_{-i}}(R)$ is at most κe^{-R} where $\kappa > 0$ does not depend on R.

To this end choose lifts of the geodesics $\gamma_{v_i}, \gamma_{w_{-i}}, \beta_i$ to $\tilde{M}$, say geodesics

$$\tilde{\gamma}_{v_i}, \tilde{\gamma}_{w_{-i}}, \tilde{\beta}_i,$$

so that the distance between the tangents $\tilde{\gamma}_{v_i}'(2R), \tilde{\beta}_i'(0)$ and between the tangents $\tilde{\beta}_i'(2R)$, $-\tilde{\gamma}_{w_{-i}}'(2R)$ is at most δ. Let ξ_i be the geodesic connecting $\tilde{\gamma}_{v_i}(2R)$ to $\tilde{\beta}_i(R)$. Hyperbolic trigonometry [B92] and comparison [CE75] shows that the angle between $\tilde{\gamma}_{v_i}'(2R)$ and the tangent of ξ_i at $\tilde{\gamma}_{v_i}(2R)$ is at most $c_0\delta$ where $c_0 > 0$ is a universal constant. Moreover, the angle between $\tilde{\beta}_i'$ and ξ_i' at $\tilde{\beta}_i(R)$ is at most $c_0 e^{-R}$.

Let ζ_i be the geodesic connecting $\tilde{\gamma}_{v_i}(R)$ to $\tilde{\beta}_i(R)$. Use comparison for the geodesic triangle with vertices $\tilde{\gamma}_{v_i}(R), \tilde{\gamma}_{v_i}(2R), \tilde{\beta}_i(R)$ to conclude that the angle at $\tilde{\gamma}_{v_i}(R)$ between $\tilde{\gamma}_{v_i}'$ and the tangent of ζ_i is at most $c_1 e^{-R}$ where again, $c_1 \geq c_0$ is a universal constant. The angle at $\tilde{\beta}_i(R)$ between $\tilde{\beta}_i'$ and the tangent of ζ_i does not exceed $c_1 e^{-R}$ as well.

Apply this reasoning to the geodesics $\tilde{\gamma}_{w_{-i}}$ and the inverse of $\beta_i[R, 2R]$ to control the tangents at the endpoints of the geodesic η_i connecting $\tilde{\gamma}_{w_{-i}}(R)$ to $\tilde{\beta}_i(R)$. We find that the angle at $\tilde{\beta}_i(R)$ between ζ_i and the inverse of η_i does not exceed $c_2 e^{-R}$ for a universal constant $c_2 > 0$. Thus by triangle comparison, the angle at $\tilde{\gamma}_{v_i}(R)$ between $\tilde{\gamma}_{v_i}'$ and the tangent of the geodesic connecting $\tilde{\gamma}_{v_i}(R)$ to $\tilde{\gamma}_{w_{-i}}(R)$ is a most $c_2 e^{-R}$. The above claim now follows from this and symmetry.

The statement about the parallel transport is derived in the same way. □

Lemma 4.3 is the method for the construction of the building blocks for our surfaces, namely pairs of pants immersed in M. In the remainder of this section we explain why it gives rise to pairs of pants. We also collect some first properties of these pairs of pants which will be used to get some geometric control as R, δ vary. In Section 5 and Section 6 we will determine suitable sizes for R, δ using this a-priori geometric control to establish a sufficient condition for incompressibility of surfaces glued from pants.

Let $((v_1, v_2, v_3), E)$ and $((w_1, w_2, w_3), F)$ be (R, δ)-well connected framed tripods in TM with foot-points $p, q \in M$. Let as before $H_R(v_1, v_2, v_3)$ and $H_R(w_1, w_2, w_3)$

be the totally geodesic immersed hyperbolic hexagons defined by these tripods. We use the notations as in the definition of well connected tripods.

For each i let β_i be the geodesic arc in the hexagon $H_R(v_1, v_2, v_3)$ which connects the point $\gamma_{v_i}(R)$ to the point $\gamma_{v_{i+1}}(R)$ and define in the same way a geodesic arc η_i in $H_R(w_1, w_2, w_3)$ connecting $\gamma_{w_i}(R)$ to $\gamma_{w_{i+1}}(R)$ as shown in Figure C.

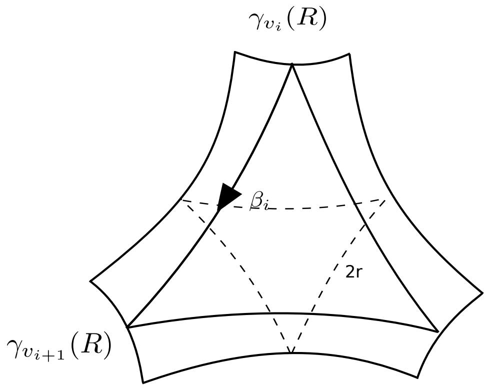

Figure C

There is a number $q > \delta$ not depending on R such that the length $L(R)$ of these geodesics is contained in the interval $[2R - q, 2R]$. Hyperbolic trigonometry shows that the angle at $\gamma_{v_i}(R)$ of the triangle with vertices $p, \gamma_{v_i}(R), \gamma_{v_{i+1}}(R)$ is not bigger than $\kappa_1 e^{-R}$ where $\kappa_1 > 0$ is a universal constant.

Thus for each $i \in \{1, 2, 3\}$ the concatenation

$$\hat{\gamma}_i = \alpha_i^{-1} \circ \eta_i^{-1} \circ \alpha_{i+1} \circ \beta_i$$

(read from right to left and indices are taken modulo three) is a piecewise geodesic loop with 4 breakpoints of breaking angle at most $\kappa_2 e^{-R}$ where $\kappa_2 > \kappa_1$ is a universal constant. The lengths of the geodesic segments which form these piecewise geodesic loops are at least $2R - \kappa_3$ where $\kappa_3 > 0$ is a universal constant. The piecewise geodesic $\hat{\gamma}_i$ inherits from the boundary orientation of the oriented hexagons in the construction a natural orientation.

Standard comparison implies that for sufficiently large R the piecewise geodesic loop $\hat{\gamma}_i$ is freely homotopic to a closed geodesic γ_i in M. The Hausdorff distance between the tangent line of $\hat{\gamma}_i$ and the tangent line of γ_i is at most $\kappa_4 e^{-R}$ where once again, $\kappa_4 > 0$ does not depend on R. By increasing R we may assume that $\kappa_4 e^{-R} < \delta/4$. Then the lengths $\ell(\gamma_i)$ of the geodesics γ_i satisfy

$$\ell(\gamma_i) \in [2L(R) + 8R - \delta, 2L(R) + 8R + \delta].$$

The geodesics γ_i $(i = 1, 2, 3)$ are pairwise distinct, and there is an oriented pair of pants P and an incompressible map $f_P : P \to M$ which maps the boundary geodesics of P onto the three geodesics $\gamma_1, \gamma_2, \gamma_3$.

The homotopy class of the map f as well as the orientation of P are determined by the tripods (v_1, v_2, v_3), (w_1, w_2, w_3) and the good connections α_i. Note however that the tripods are not determined by the homotopy class of f. Following [KM12] we call $f(P)$ an (R, δ)-*skew pants*, and we identify two such skew pants if they are defined by homotopic maps.

Let γ be a boundary geodesic of a skew pants. Then γ is the quotient of a geodesic line $\tilde{\gamma}$ in $\tilde{M}$ under a hyperbolic isometry. Such an isometry is determined by its translation length (which is the length of γ) and a rotational part which is an element in $SO(n-1)$ for $G = SO(n,1)$, an element in $SU(n-1)$ for $G = SU(n,1)$, an element in $Sp(n-1)$ for $G = Sp(n,1)$ and an element in $\mathrm{Spin}(7)$ for $G = F_4^{-20}$. We call this rotational part the *monodromy* of γ. As before, there is a notion of being ϵ-close to the identity for such a monodromy map. The following proposition is immediate from the above construction and from uniform continuity of parallel transport along piecewise geodesics.

Proposition 4.4. *There is a number $\chi > 0$ with the following property. If γ is a boundary curve of an (R,δ)-skew-pants then the monodromy of γ is $\chi\delta$-close to the identity.*

5. Twisted bands

Consider again a rank one symmetric space $\tilde{M}$ of curvature contained in the interval $[-4,-1]$ and dimension at least three. If the curvature of $\tilde{M}$ is constant then we require that this constant equals -1. The goal is to introduce a geometric model for the thin parts of the pairs of pants constructed in Section 4 in a compact quotient $M = \Gamma\backslash\tilde{M}$ of $\tilde{M}$. Such a model is a twisted ruled band as defined below. The geometric realization of an (R,δ)-skew pants will consist of three ruled surfaces which are exponentially close such twisted ruled bands. These ruled surfaces are attached to two ruled geodesic triangles which are exponentially close to the center triangle of an ideal immersed hyperbolic triangle as defined in the next paragraph.

Let $\mathbf{H}^2$ be the hyperbolic plane and let $T \subset \mathbf{H}^2$ be an ideal hyperbolic triangle. The projection of an ideal vertex of T to the opposite side γ is a special point on γ. The three special points on the three sides of T are the vertices of an equilateral hyperbolic triangle $T_0 \subset T$ which we call the *center triangle*. Let $2r > 0$ be the length of the sides of this triangle. This length does not depend on T. The number r will be used throughout the rest of this section.

Let $R \geq 10$ and let $\gamma : [-R,R] \to \tilde{M}$ be any geodesic arc of length $2R$. Let

$$V \to \gamma$$

be the subbundle of the restriction of $T\tilde{M}$ to γ whose fibre at $\gamma(t)$ equals the $\mathbb{K}$-orthogonal complement $(\gamma'(t))^{\perp}_{\mathbb{K}}$ of $\gamma'(t)$. The bundle V is invariant under parallel transport along γ.

Let $w_{-R} \in V_{\gamma(-R)}, w_R \in V_{\gamma(R)}$ be unit vectors. Let $\delta \in [0,\pi/4]$ and assume that the non-oriented angle between w_R and the parallel transport of w_{-R} along γ equals δ. This does not depend on the orientation of γ.

Let ν_{-R} and ν_R be the geodesic connecting $\exp(-rw_{-R})$ to $\exp(rw_{-R})$ and connecting $\exp(-rw_R)$ to $\exp(rw_R)$, respectively. We assume that the geodesics ν_{-R}, ν_R are parametrized by arc length on $[-r,r]$. Then

$$\nu_{-R}(0) = \gamma(-R),\ \nu_R(0) = \gamma(R).$$

Moreover, ν_{-R}, ν_R meet γ orthogonally at $\gamma(-R), \gamma(R)$.

Let $\ell > R$ be such that the distance between $\nu_{-R}(r)$ and $\nu_R(r)$ equals 2ℓ. For each $s \in [-r,r]$ connect $\nu_{-R}(s)$ to $\nu_R(s)$ by a geodesic α_s parametrized proportional to arc length on $[-\ell,\ell]$. Up to parametrization, we have $\alpha_0 = \gamma$. The map

$$\alpha : [-r,r] \times [-\ell,\ell] \to \tilde{M}$$

defined by $\alpha(s,t)=\alpha_s(t)$ is an embedding. Its image is a ruled surface E with induced orientation and oriented boundary $\alpha_r^{-1}\circ\nu_R\circ\alpha_{-r}\circ\nu_{-R}^{-1}$ (read from right to left). We call this surface a *δ-twisted ruled band of size* $2R$ with central geodesic γ, and we call α_{-r},α_r the *long sides* of the band. We also say that E is a ruled band of twisting number δ. The map α is called the *standard parametrization* of the twisted ruled band E.

Let $x=\gamma(0)$ be the midpoint of γ and let $\hat{w}_{-R},\hat{w}_R\in T_x\tilde{M}$ be the images of w_{-R},w_R under parallel transport along γ. The angle between $\hat{w}_{-R},\hat{w}_R$ equals δ. Let $\hat{w}$ be the midpoint between $\hat{w}_{-R},\hat{w}_R$ in the fibre V_x of V over x, i.e. the midpoint of the unique shortest arc in the unit sphere in V_x connecting $\hat{w}_{-R},\hat{w}_R$. Let $\hat{\nu}$ be the geodesic in $\tilde{M}$ through x which is tangent to $\hat{w}$. The geodesic $\hat{\nu}$ is orthogonal to γ. There is a unique totally geodesic hyperbolic plane $H\subset\tilde{M}$ of curvature -1 containing both γ and $\hat{\nu}$.

Lemma 5.1. *A subsegment of $\hat{\nu}$ is the unique geodesic arc in $\tilde{M}$ which is orthogonal to both α_{-r},α_r, and it is contained in the ruled surface E.*

Proof. We begin with showing the lemma in the case $G=SO(n,1)$. Then there is an isometric involution Ψ of $\tilde{M}=\mathbf{H}^n$ which fixes $\hat{\nu}$ pointwise and whose differential acts as a reflection in the orthogonal complement of the tangent line of $\hat{\nu}$. The isometry Ψ preserves the geodesic γ and exchanges its endpoints. Moreover, we have

$$d\Psi(\hat{w}_{-R})=\hat{w}_R.$$

Since isometries commute with parallel transport, this implies that $d\Psi(w_{-R})=w_R$. As a consequence, Ψ preserves the ruled surface E and acts as a reflection on the sides α_{-r},α_r. Since the fixed point set of Ψ equals $\hat{\nu}$, there is a subarc ν of $\hat{\nu}$ which is contained in E. This subarc is the shortest geodesic between α_{-r} and α_r, and it meets the geodesics α_{-r},α_r orthogonally at its endpoints.

Next consider the case that $\tilde{M}=\mathbb{C}\mathbf{H}^2$ equals the complex hyperbolic plane. With respect to a suitable choice of complex coordinates in the unit ball in $\mathbb{C}^2$, complex conjugation is an anti-holomorphic isometry Θ of $\mathbb{C}\mathbf{H}^2$ which fixes the hyperbolic plane H containing γ and $\hat{\nu}$ pointwise and acts as a reflection in the normal bundle of H. Thus we have

$$d\Theta(\hat{w}_{-R})=\hat{w}_R.$$

Let σ be the geodesic symmetry at $\gamma(0)$. Then $\sigma\circ\Theta$ preserves both γ and $\hat{\nu}$, and it exchanges the endpoints of γ. Moreover, we have $d(\sigma\circ\Theta)(\hat{w}_{-R})=-\hat{w}_R$. Thus $\sigma\circ\Theta$ exchanges the geodesics ν_{-R} and ν_R. In particular, it preserves the δ-twisted ruled band E. As before, this implies the statement of the lemma.

If more generally $G=SU(n,1)$ for $n\geq 3$ then the real hyperbolic plane H is contained in a unique totally geodesic complex hyperbolic plane $V=\mathbb{C}\mathbf{H}^2\subset\tilde{M}$. The anti-holomorphic involution of V which fixes H pointwise and acts as a reflection in the normal bundle of H in V can be extended to an anti-holomorphic isometry Θ of $\tilde{M}$ which fixes the point $\gamma(0)$ and maps the orthogonal projection of $\hat{w}_{-R}$ into $T_{\gamma(0)}V^\perp\subset T_{\gamma(0)}\tilde{M}$ to its negative, i.e. to the orthogonal projection of $\hat{w}_R$. The argument for the case $\tilde{M}=\mathbb{C}\mathbf{H}^2$ applies and yields the statement of the lemma in this case as well.

The case $G=Sp(n,1),F_4^{-20}$ is completely analogous to the case $G=SU(n,1)$ and will be omitted. ☐

In the sequel we call the subsegment ν_0 of the geodesic $\hat{\nu}$ as in Lemma 5.1 whose endpoints are contained in the two long sides of E the *seam* of E. This is consistent with the terminology used in Section 3.

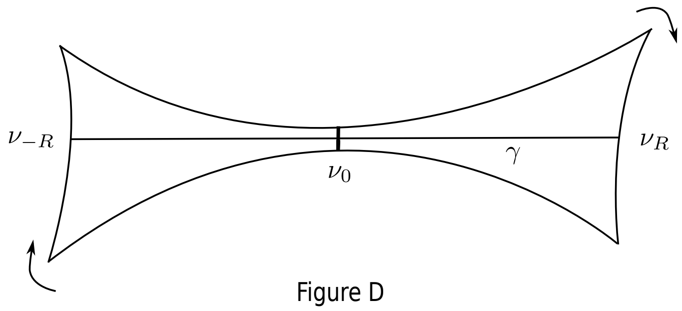

Figure D

Remark 5.2. The proof of Lemma 5.1 also implies the following.

(1) For a δ-twisted ruled band E, there is a unique totally geodesic real hyperbolic plane $H(E) \subset \tilde{M}$ containing both the central geodesic γ and the seam ν_0 of E.
(2) A δ-twisted ruled band E consists of two isometric copies of a ruled quadrangle Q. One side of Q is the seam ν_0 of E, with adjacent right angles. The sides adjacent to ν_0 have the same length, and the length of the opposite side ξ of Q (which is a short side of E) is $2r$. The ruling consists of geodesic segments connecting the side ν_0 to ξ. The length of the side ν_0 is contained in the interval $[c^{-1}e^{-R}, ce^{-R}]$ for a universal constant $c > 0$. Note that this estimate also holds true if the curvature of $\tilde{M}$ is not constant.

Let

$$\mathcal{V} \to \tilde{M}$$

be the bundle of oriented 2-planes in $T\tilde{M}$. Its fibre over a point $x \in \tilde{M}$ is the Grassmannian of all oriented two-dimensional linear subspaces of $T_x\tilde{M}$. The symmetric Riemannian metric of $\tilde{M}$ naturally induces a Riemannian metric on $\mathcal{V}$ so that $\mathcal{V} \to \tilde{M}$ is a Riemannian submersion, with fibre isometric to a compact symmetric space. Denote by $d_{\mathcal{V}}$ the induced distance function on $\mathcal{V}$.

If $H \subset \tilde{M}$ is an oriented totally geodesic real hyperbolic plane, then the oriented tangent bundle

$$\mathcal{T}(H)$$

of H is naturally a totally geodesic submanifold of $\mathcal{V}$. The projection $\mathcal{T}(H) \to H$ is an isometry. Moreover, there is a unique shortest distance projection

$$\Pi_H : \tilde{M} \to H.$$

Let $E \subset \tilde{M}$ be a δ-twisted ruled band of size $2R$, with central geodesic γ and long sides α_{-r}, α_r. By Remark 5.2, there is a unique real hyperbolic plane $H(E) \subset \tilde{M}$ which contains γ and the seam ν_0 of E. Note that for $\delta = 0$ the band E is contained in $H(E)$.

We next use the shortest distance projection $\Pi_{H(E)}$ to investigate the geometry of a twisted ruled band E. To this end note that for every oriented long side β of a δ-twisted ruled band E there is a natural oriented plane field

$$V(\beta, E) \to \beta$$

whose fibre at a point $\beta(t)$ is spanned by $\beta'(t)$ and the parallel transport along β of the tangent of the seam ν_0 at the midpoint $\beta(0)$ of β, oriented as the inner normal of the band. Note that if $\tilde{M}$ is a real hyperbolic space then this plane field is tangent to a totally geodesic hyperbolic plane embedded in $\tilde{M}$, but this need not be the case in general. We have

Lemma 5.3. *There is a number $C_1 > 0$ with the following property. Let $R > 10, \delta \in [0, \pi/4]$ and let E be a δ-twisted ruled band of size $2R$. Let β be a long side of E and let $y \in E$ be a point of distance t to the seam of E; then*

$$d_{\mathcal{V}}(T_{\Pi_{H(E)}(y)}H(E), V(\beta, E)) \le C_1 e^{t-R}.$$

Proof. Let $\alpha : [-r, r] \times [-\ell, \ell] \to \tilde{M}$ be the standard parametrization of the δ-twisted ruled band E. Then $\alpha(0, \ell)$ is contained in the central geodesic $\gamma \subset H(E)$ of E. By the definition of a δ-twisted ruled band of size $2R$, for $s \in [-r, r]$ we have

$$d(\alpha(s, \ell), \Pi_{H(E)}(\alpha(s, \ell))) \le C_2 \delta |s| \tag{1}$$

where $C_2 > 0$ is a universal constant. Namely, the central geodesic γ of E is contained in the hyperbolic plane $H(E)$. The point $\alpha(s, \ell)$ can be obtained from the endpoint $\alpha(0, \ell)$ of γ by a geodesic of length $|s| \le r$ which makes an angle $\delta/2$ to the tangent plane of $H(E)$.

Let $\beta_s \subset H(E)$ be the geodesic connecting $\alpha(s, 0)$ to $\Pi_{H(E)}(\alpha(s, \ell))$. We assume that β_s is parametrized proportional to arc length on $[0, \ell]$. Since the curvature of $\tilde{M}$ is bounded from above by -1, comparison shows that for $0 \le t \le \ell$ we have

$$d(\alpha(s, t), \beta_s(t)) \le C_3 \delta |s| e^{t-R} \tag{2}$$

for a universal constant $C_3 > 0$.

Parallel transport of tangent planes along geodesics in $\tilde{M}$ defines horizontal geodesics in the bundle $\mathcal{V}$. From the estimate (2) we therefore obtain that for $t \in [0, \ell]$ we have

$$d_{\mathcal{V}}(V(\alpha_r, E)(\alpha(r, t)), T_{\beta_r(t)}H(E)) \le C_4 \delta e^{t-R}. \tag{3}$$

The geodesics β_{-r}, β_r and the seam ν_0 of E define three sides of a geodesic quadrangle Q in the hyperbolic plane $H(E)$. The length of the sides β_{-r}, β_r equals ℓ up to an error of size at most δ. Since the projection $\Pi_{H(E)}$ is distance non-increasing, the length of the side of Q opposite to ν_0 is at most $2r$. Thus inequality (1) implies that if $y \in E$ is at distance t from the seam, then there is some $z \in \beta_r$ such that $d(z, \Pi_{H(E)}(y)) \le C_5 e^{t-R}$.

As the map $z \in H(E) \to T_z H(E)$ is an isometric embedding of $H(E)$ into $\mathcal{V}$, we then have

$$d_{\mathcal{V}}(T_z H(E), T_{\Pi_{H(E)}(y)}H(E)) \le C_5 e^{t-R}$$

as well. Together with the estimate (3), this shows the lemma. □

Remark 5.4. Lemma 5.3 immediately extends to symmetric spaces X of higher rank as follows. Let $H \subset X$ be a totally geodesic embedded plane of constant curvature -1. For $\delta > 0$ define a δ-twisted ruled band in X by rotating the small sides of an embedded band $\hat{E}$ in H about the central geodesic of $\hat{E}$ by an angle δ as described above. Then

$$d_{\mathcal{V}}(T_{\Pi_{H(E)}(y)}H(E), V(\beta, E)) \le \max\{\delta, C_1 e^{t-R}\}.$$

The next lemma compares distances in twisted ruled bands E with the distance of their projections to $H(E)$. To this end denote for a δ-twisted ruled band E by d_E the intrinsic path metric on E. Note that E is a smoothly embedded submanifold of $\tilde{M}$, in particular its tangent plane is defined everywhere.

Lemma 5.5. *For every $\epsilon > 0$ there is a number $\delta_1 = \delta_1(\epsilon) > 0$ with the following property. Let $R > 10$, $\delta \le \delta_1$ and let E be a δ-twisted ruled band of size $2R$. Then for all $x, y \in E$ we have*

$$d(\Pi_{H(E)}(x), \Pi_{H(E)}(y)) \ge d_E(x, y)(1 + \epsilon)^{-1}.$$

Proof. Let $\epsilon > 0$, let $\delta > 0$ and let E be a δ-twisted ruled band of size $2R > 10$. By the discussion in the proof of Lemma 5.3 (or by a standard compactness argument), for sufficiently small δ the distance in $\mathcal{V}$ between a tangent plane T_yE of E and the tangent bundle $\mathcal{T}H(E)$ of $H(E)$ is at most ϵ.

As a consequence, for sufficiently small δ the restriction of the projection $\Pi_{H(E)}$ to E is a diffeomorphism onto its image which moreover is bilipschitz with bilipschitz constant at most $1 + \epsilon$. □

In the following definition, the oriented distance of two points on the boundary of an oriented surface is taken with respect to the induced boundary orientation. The definition is a variant of a definition from [KM12].

Definition 5.6. For numbers $\sigma_1, \sigma_2 \in [0, 1/4]$, two oriented twisted ruled bands E_1, E_2 are called (σ_1, σ_2)-*well attached* along a common boundary geodesic β if the following holds.

- The orientations of β induced by the orientations of E_1, E_2 are opposite.
- Let $x_i \in \beta$ be the endpoint on β of the seam ν_i of E_i $(i = 1, 2)$. The oriented distance along β between x_1, x_2 is contained in the interval $[1 - \sigma_1, 1 + \sigma_1]$.
- Let $v_i \in T^1_{x_i}\tilde{M}$ be the oriented tangent of ν_i at x_i $(i = 1, 2)$; then the angle between v_2 and the parallel transport of $-v_1$ along β is at most σ_2.

Note that in view of Lemma 3.3, the first two properties control the intrinsic geometry of the attached bands. The third property is used to relate the intrinsic geometry of the attached bands to the extrinsic geometry of the ambient manifold.

Fix a number $b > 1$. For numbers $m > 10, \delta > 0$ define an (R, δ)-*admissible chain* of twisted ruled bands to be a surface of the form $E = E_1 \cup \cdots \cup E_m$ where E_i is an oriented $\delta_i < \delta$-twisted band of size $2R_i$ for some $R_i \in [R - \delta, R + \delta]$ and where E_i is (δ, R^{-b})-well attached to E_{i-1} along a boundary geodesic which is disjoint from E_{i-2}.

In what follows, whenever we estimate distances in a fibre bundle over $\tilde{M}$, then these distances are taken with respect to the natural Riemannian metric on the bundle which is induced from the Riemannian metric on $\tilde{M}$. The distance in $\tilde{M}$ will simply be denoted by d.

δ-twisted ruled bands with their intrinsic metric are isometrically immersed smooth submanifolds of $\tilde{M}$ which are C^2-close to totally geodesic embedded hyperbolic planes. Thus the intrinsic curvature is close to -1, and by making δ smaller we may assume that this metric is CAT$(-1/2)$. A rescaled version of Lemma 3.1 then shows that the intrinsic path metric on any (R, δ)-admissible chain of ruled bands is locally CAT$(-1/2)$. In particular, any two points are connected by a unique geodesic.

Our next goal is to control the geometry of (R,δ)-admissible chains $E_1\cup\cdots\cup E_m$ of twisted ruled bands by comparing distances for the intrinsic path metric with distances in $\tilde M$.

The constant $C_0>0$ in the formulation of the following lemma is the constant from Lemma 3.5. Up to changing δ_2, the number $\min\{1/4,C_0/2\}$ can be replaced by any other positive constant.

Lemma 5.7. *For every $\epsilon>0$ there are numbers $\delta_2=\delta_2(\epsilon)>0$, $R_2=R_2(\epsilon)>10$ with the following property. Let $R>R_2$ and let $E=E_1\cup\cdots\cup E_m$ be an (R,δ_2)-admissible chain of ruled bands. For $i,j\leq m$ let $y\in\partial E_i, z\in\partial E_j$ be points which are connected by a geodesic ζ in E for the intrinsic metric of length $\ell(\zeta)\leq\min\{1/4,C_0/2\}$. Assume that ζ does not meet a short side of any band in the chain. Then the length of ζ is at most $(1+\epsilon)d(y,z)$.*

Proof. For a number $R>10$, a number $\delta\in[0,1/10]$ and some $m\geq 1$ let $E=E_1\cup\cdots\cup E_m$ be an (R,δ)-admissible chain.

Let $1\leq i<j\leq m$ and let $x\in\partial E_i, y\in\partial E_j$. Assume that the geodesic ζ on E connecting x to y does not intersect a small side of any of the bands which make up E and that its length does not exceed $\min\{1/4,C_0/2\}$.

For $k\leq m$ let
$$\beta_k=E_{k-1}\cap E_k.$$
For $i<\ell<j$ let $u_\ell=\zeta\cap\beta_\ell$. Let τ_ℓ be the distance between u_ℓ and the seam of the band E_ℓ. By assumption, for every $z\in\zeta\cap E_\ell$ the distance between z and the seam of E_ℓ is contained in the interval $[\tau_\ell-1/4,\tau_\ell+1/4]$.

By Lemma 3.1, the intrisinc path metric on E is locally CAT$(-1/2)$. Thus we can use Remark 3.4 and deduce from Lemma 3.3 and its proof that up to subdividing ζ into two disjoint segments and reversing the orientation of one of these segments as well as reversing the numbering of the bands in the chain, we may assume that $\tau_{\ell+1}>\tau_\ell$ for all ℓ. Moreover, we have $j-i\leq 4R$.

We now claim that there is a number $\chi_0>0$ with the following property. For every $i<k<j$ and every $z_k\in\zeta\cap E_k$ we have

$$(4)\qquad d_{\mathcal V}(T_{\Pi_{H(E_k)}(z_k)}H(E_k),T_{\Pi_{H(E_i)}(z_k)}H(E_i))\leq\chi_0\sum_{\ell=i}^{k}\max\{e^{\tau_\ell-R},R^{-b}\}.$$

This estimate holds true for every $\delta\in[0,1/10]$.

We proceed by induction on $k-i$. The claim for $k=i$ is trivial- in fact, every number $\chi_0>0$ will do. Thus assume that the statement holds true for $k-i<n\leq j-i$ where $n\geq 1$.

Lemma 5.3 shows that there is some $u\in\beta_k$ such that

$$(5)\qquad d_{\mathcal V}(T_{\Pi_{H(E_k)}(z_k)}H(E_k),V(\beta_k,E_k)(u))\leq C_1e^{\tau_k-R}.$$

Now $V(\beta_k,E_k)(u)$ is obtained from the span of β_k' and the tangent of the seam ν_k of E_k by parallel transport along β_k. Similarly, $V(\beta_k,E_{k-1})(u)$ is obtained from the span of β_k' and the tangent of the seam ν_{k-1} by parallel transport along β_k. In particular, by the definition of well attached bands,

$$(6)\qquad d_{\mathcal V}(V(\beta_k,E_{k-1})(u),V(\beta_k,E_k)(u))\leq R^{-b}.$$

Thus from the estimate (5) we conclude that

$$(7)\qquad d_{\mathcal V}(T_{\Pi_{H(E_k)}(z_k)}H(E_k),V(\beta_k,E_{k-1})(u))\leq C_2\max\{e^{\tau_k-R},R^{-b}\}.$$

Lemma 5.5 implies that up to replacing C_1 by $2C_1$, the intrinsic distance in E between z_k and u is at most $C_1 e^{\tau_k - R}$. Since the intrinsic path metric on E is Cat$(-1/2)$ and since the projections $\Pi_{H(E_i)}$ are distance non-increasing, the estimate (7) yields that for the proof of inequality (4), it suffices to show that

$$d_{\mathcal{V}}(V(\beta_k, E_{k-1}(u)), T_{\Pi_{H(E_i)(u)}} H(E_i)) \leq \chi_0 \sum_{\ell=i}^{k-1} \max\{e^{\tau_\ell - R}, R^{-b}\}.$$

Lemma 5.3 allows to replace $V(\beta_k, E_{k-1}(u))$ by $T_{\Pi_{H(E_{k-1})(u)}} H(E_{k-1})$. The estimate (4) now follows from the induction hypothesis provided that the constant $\chi_0 > 0$ is sufficiently large (in particular, it has to be chosen larger than $2C_2$).

For $\rho > 0$ there is a number $r(\rho) > 0$ with the following property. Let $H_1, H_2 \subset \tilde{M}$ be two totally geodesic real hyperbolic planes. Assume that $x \in H_1$ and that $d_{\mathcal{V}}(T_x H_1, T_{\Pi_{H_2}(x)} H_2) < r(\rho)$; then the restriction of the projection Π_{H_2} to the ball of radius one about x in H_1 is a $(1+\rho)$-bilipschitz diffeomorphism onto its image. Moreover, for every $y \in H_1$ with $d(x,y) \leq 1$ we have $d(y, \Pi_{H_2}(y)) \leq \rho$.

Let $\epsilon > 0$. By inequality (4), by Lemma 5.5, Lemma 3.5 and by Lemma 3.3 and its proof, there are numbers $p > 0$, $R_2 > 0$ with the following property. Let E be an (R, δ)-admissible chain for some $R \geq R_2$ and some $\delta < 1/10$. Let $\zeta : [0, a] \to E$ be any geodesic arc of length $a \leq \min\{1/4, C_0/2\}$ as above which connects a point $x \in \partial E_i$ to a point $y \in \partial E_j$. Assume that $j - i \geq p$. Then there are numbers $i_0 \in [i, i+p], j_0 \in [j-p, j]$ with the following property.

Let $0 \leq s \leq t \leq a$ be such that $\zeta(s) \in \beta_{i_0}, \zeta(t) \in \beta_{j_0}$; then

$$d_{\mathcal{V}}(T_{\Pi_{H(E_{i_0})}(\zeta(s))} H(E_{i_0}), T_{\Pi_{H(E_{j_0})}(\zeta(s))} H(E_{j_0})) \leq r(\epsilon/2)/3. \tag{8}$$

As before, this estimate holds true for all $\delta \in [0, 1/10]$.

With this number $p > 0$, it follows from the estimate (4), Lemma 5.5 and its proof and the definition of an admissible chain that there is a number $\sigma = \sigma(\epsilon, p) < 1/10$ with the following property.

Let $R > R_2$ and let $E = E_1 \cup \cdots \cup E_p$ be any (R, σ)-admissible chain of twisted ruled bands. Let $x \in E$; then for $1 \leq i \leq j \leq p$ we have

$$d_{\mathcal{V}}(T_{\Pi_{H(E_i)}(x)} H(E_i), T_{\Pi_{H(E_j)}(x)} H(E_j)) \leq r(\epsilon/2)/3. \tag{9}$$

The point is here that the number p is fixed, and that the number σ can be chosen arbitrarily small.

Now let $\delta_1(\epsilon) > 0$ be as in Lemma 5.5 and let $\delta_2 = \min\{\sigma, \delta_1(\epsilon)\}$. Let $R > R - 2$, $\delta < \delta_2$, let $m > 0$ be arbitrary and let $E = E_1 \cup \cdots \cup E_m$ be an (R, δ)-admissible chain. Let $1 \leq i < j \leq m$ and let $y \in E_i, z \in E_j$ be such that $d(y, z) \leq \min\{1/4, C_0/2\}$ as before and that moreover the geodesic in E connecting x to y does not meet a small side of a band in the chain. Choose $i \leq i_0 < j_0 \leq j$ with $i_0 - i \leq p, j - j_0 \leq p$ as above.

Let $\Pi = \Pi_{H_{i_0}}$. By the choice of δ, the restriction of the projection Π to the ball of radius one about y in E is injective. Thus the geodesic η in $H(E_{i_0})$ connecting $\Pi(y)$ to $\Pi(z)$ crosses through the lines $\hat{\beta}_\ell = \Pi(\beta_\ell)$ where $i < \ell < j$ in increasing order.

Assume that η is parametrized by arc length on an interval $[0, c]$. Let $t_\ell \geq 0$ be such that $\eta(t_\ell) = \eta \cap \hat{\beta}_\ell$. Let $\zeta_\ell \in E$ be the preimage of $\eta(t_\ell)$ under the map $\Pi|E$. Using again the choice of δ, the length q_ℓ of a shortest geodesic in $H(E_\ell)$ connecting $\Pi_{H(E_\ell)}(\zeta_{\ell-1})$ to $\Pi_{H(E_\ell)}(\zeta_\ell)$ does not exceed $(1+\epsilon)(t_\ell - t_{\ell-1})$. As $\delta < \delta_1(\epsilon)$, Lemma

5.5 shows that the length of a shortest geodesic in E_ℓ connecting $\zeta_{\ell-1}$ to ζ_ℓ is not bigger than $q_\ell(1+\epsilon)$. Summing over ℓ implies the lemma. □

Remark 5.8. Lemma 5.7 is valid without change for chains of δ-twisted ruled bands of size R in higher rank symmetric spaces provided that these bands are constructed as in Remark 5.4 and are $(\delta, e^{-\kappa R})$-well attached for some $\kappa > 0$.

The thin parts of (R,δ)-skew pants are not δ-twisted ruled bands, but they are exponentially close to such bands in a sense we now specify. Namely, define an *approximate δ-twisted ruled band of size R* to be a ruled surface E with the following property.

Fix a number $\kappa \in (0,1)$ whose precise value will be determined later. We require that there is a δ-twisted ruled band E^0 of size R, with short sides ν^0_{-R}, ν^0_R, and there are geodesics ν_{-R}, ν_R parametrized proportional to arc length on $[-r,r]$ with

$$d(\nu_i(-r), \nu_i^0(-r)) \le e^{-\kappa R}, d(\nu_i(r), \nu_i^0(r)) \le e^{-\kappa R}\ (i = -R, R)$$

such that E is obtained by connecting for each $s \in [-r,r]$ the points $\nu_{-R}(s)$ and $\nu_R(s)$ by a geodesic. We call the geodesics α_{-r}, α_r connecting the endpoints of ν_{-R}, ν_R the *long sides* of the ruled band E, and we call the shortest geodesic in $\tilde{M}$ connecting the two long sides of E the *seam* of the band.

Let $\alpha_0 : [-r,r] \times [-\ell,\ell] \to E_0$ be the standard parametrization of the twisted ruled band E_0 as described in the beginning of this section and let $\alpha : [-r,r] \to [-\ell,\ell] \to E$ be the parametrization of E defined by requiring that $t \to \alpha(s,t)$ is the geodesic connecting $\nu_{-R}(s)$ to $\nu_R(s)$ parametrized proportional to arc length on $[-R,R]$. Hyperbolicity and comparison shows that

$$d(\alpha_0(s,t), \alpha(s,t)) \le e^{-\kappa R}$$

for all s,t.

The notion of an (R,δ)-admissible chain is also defined for approximate δ-twisted ruled bands. As in Lemma 5.7 we conclude

Corollary 5.9. *For every $\epsilon > 0$ there is a number $\delta_3 = \delta_3(\epsilon) > 0$ with the following property. Let $m > 10$ and let $E = E_1 \cup \dots \cup E_m$ be an (m,δ_3)-admissible chain of approximate ruled bands. Let x,y be points which are connected by a geodesic ζ in E for the intrinsic metric. Assume that ζ does not meet the short side of any band in the chain. Then the length of ζ is at most $(1+\epsilon)d(x,y)$.*

6. Surfaces glued from skew pants

In this section we glue skew-pants to surfaces and investigate their geometry. We continue to use the assumptions and notations from Section 5 and Section 4.

Let $P \subset M$ be a skew pants defined by an (R,δ)-well connected pair of framed tripods $x = ((v_1,v_2,v_3),E)$, $y = ((w_1,w_2,w_3),F)$ with footpoints p,q. For the remainder of the section, only the real planes defined by the tripods (v_1,v_2,v_3) and (w_1,w_2,w_3) are relevant, so we drop the information on the frames E,F.

From the (R,δ)-well connected tripods x,y we construct a ruled surface in M in the homotopy class of P as follows.

Let α be a boundary geodesic of P. It contains in its $\kappa_4 e^{-R}$-neighborhood a long side of each of the immersed hexagons $H_R(v_1,v_2,v_3)$ and $H_R(w_1,w_2,w_3)$. Here $\kappa_4 > 0$ is as in Section 4. In particular, there is a geodesic arc $\xi_{p,\alpha} : [0,s] \to M$ connecting $p = \xi_{p,\alpha}(0)$ to a point $\xi_{p,\alpha}(s)$ on α which meets α orthogonally at $\xi_{p,\alpha}(s)$

and which is determined by the homotopy type of P as follows. Lift the hexagon $H_R(v_1, v_2, v_3)$ locally isometrically to a totally geodesic embedded hexagon $\tilde{H}$ in $\tilde{M}$, and lift α to a geodesic line $\tilde{\alpha}$ whose $\kappa_4 e^{-R}$-neighborhood contains a long side of $\tilde{H}$. Let $\tilde{\xi}_{p,\alpha}$ be the shortest geodesic connecting the center of the hexagon $\tilde{H}$ to $\tilde{\alpha}$ and let $\xi_{p,\alpha}$ be the projection of $\tilde{\xi}_{p,\alpha}$ to M.

Let $(\xi_{p,\alpha}, \xi_{p,\beta}, \xi_{p,\gamma})$ be the triple of these geodesic arcs in M connecting the footpoint p of the tripod (v_1, v_2, v_3) to the three boundary geodesics α, β, γ of the skew pants. The geodesic arcs in M which are homotopic to $\xi_{p,i} \circ \xi_{p,j}^{-1}$ with fixed endpoints $(i \neq j \in \{\alpha, \beta, \gamma\})$ define a geodesic triangle $\partial\Sigma$. Since the $\kappa_4 e^{-R}$-neighborhoods of the boundary geodesics of the skew-pants P contain the long sides of the hexagon $H_R(v_1, v_2, v_3)$, by convexity the side lengths of $\partial\Sigma$ are within $2\kappa_4 e^{-R}$ of the number $2r$ used in the definition of a δ-twisted band of size R in Section 5.

Choose a vertex z of $\partial\Sigma$ and connect this vertex to each point on the opposite side by a geodesic arc whose homotopy class is determined by the homotopy classes of the two sides of $\partial\Sigma$ which are incident on z. This defines a ruled surface $\Sigma \subset M$ with boundary $\partial\Sigma$ (which however depends on the choice of a vertex of $\partial\Sigma$). We call such a ruled surface a *center triangle* for the skew-pants. Thus each pair of well connected framed tripods defines a skew-pants together with the choice of two center triangles. By convexity and the fact that the hexagon $H_R(v_1, v_2, v_3)$ is totally geodesic, such a ruled triangle is contained in the $\kappa_4 e^{-R}$-neighborhood of $H_R(v_1, v_2, v_3)$.

Up to modifying the skew-pants P by a homotopy with fixed boundary, we may assume that the center triangles are embedded in P. Then the complement of these center triangles in P consists of three rectangles. The boundary of each such rectangle is composed of four sides which are geodesic segments in M. Two sides are sides of a center triangle, the other two sides are geodesic subarcs of the boundary geodesics of P. We call the sides contained in the center triangle the *short sides* of the rectangle, the other two sides are called the *long sides*.

Parametrize the short sides of such a rectangle Q proportional to arc length on $[-r, r]$. Use this parametrization to construct a ruled surface with boundary Q and with ruling containing the long sides in the boundary of P. By construction, there is a number $\nu < \kappa_4\delta$ such that this ruled surface is an approximate ν-twisted ruled band of size $6R - 2\tau + \chi$ for a number $\chi \in [-\delta, \delta]$, where $\tau > 0$ is the distance of the center of an equilateral triangle of side length 2τ in the hyperbolic plane to each of its sides. The long sides of these bands are subsegments of the boundary geodesics of P.

The three ruled bands are glued to the center triangles along the short sides of their boundary. The union of the three ruled bands and the two center triangles defines a piecewise ruled surface which is a pair of pants with geodesic boundary. We call such a piecewise ruled surface a (R, δ)-*geometric skew-pants*, or simply a geometric skew pants if we do not have to specify the size parameters (R, δ).

A geometric skew pants P is a pair of pants with a piecewise smooth Riemannian metric with geodesic boundary. This piecewise smooth metric defines a path metric on P. Lemma 3.1 shows that this path metric is locally CAT(-1). Note that an (R, δ)-skew pants P can be equipped with a structure of a geometric (R, δ)-skew pants, but such a structure is not unique.

An (R,δ)-skew pants P has three boundary geodesics. Each pair of such geodesics is connected by a shortest geodesic arc in the homotopy class defined by the skew-pants. These geodesic arcs are called the *seams* of P.

Each boundary geodesic contains an endpoint of precisely two seams, and these endpoints decompose the boundary geodesic into two subarcs of roughly the same length. By Lemma 5.1 and comparison, the angle between the direction of a seam at a point in a boundary geodesic γ and a direction in the $\mathbb{K}$-orthogonal complement of γ' is exponentially small in R.

Recall that by convention, a skew-pants is oriented and hence each of its boundary geodesics is oriented as well. Following Section 5, we can now define

Definition 6.1. For a number $\sigma > 0$, two skew-pants P, P' are *σ-well attached* along a common boundary geodesic β if the following holds true.

- The orientations of β as a boundary geodesic of P and P' are opposite.
- Let x be an endpoint of a seam of P on β. Then there is an endpoint y of a seam of P' on β whose oriented distance to x is contained in the interval $[1-\sigma, 1+\sigma]$.
- Let v_1 be the direction of the seam of P at x and let v_2 be the direction of the seam of P' at y; then the angle between v_2 and the image of $-v_1$ under parallel transport along the oriented subarc of β connecting x to y is at most σ.

As before, let $b > 1$ be a fixed number. Recall from Section 5 the definition of an admissible chain for this number $b > 1$. We are now ready to show

Proposition 6.2. *There are numbers $\delta_4 \in (0, \pi/4], R_4 > 10$ with the following property. Let $R > R_4$ and let $S \subset M$ be a piecewise immersed closed surface composed of finitely many (R, δ_4)-skew pants which are R^{-b}-well attached along their common boundary geodesics. Then S is incompressible.*

Proof. Let $\delta \in (0, \pi/4]$, let $R > 10$ and let $S \subset M$ be an immersed surface which is composed of finitely many (R,δ)-skew pants. Equip each of these skew-pants with a structure of a geometric (R,δ)-skew pants. By construction and Lemma 3.1, the length metric on S defined by the piecewise ruled pairs of pants is locally CAT(-1). Thus this metric lifts to a CAT(-1)-metric on the universal covering $\tilde{S}$ of S.

The (R,δ)-skew pants define a geodesic pants decomposition $\mathcal{P}$ of S. Let $\sigma > 0$. By Lemma 5.5 and Lemma 3.5, for sufficiently large R and sufficiently small δ, say for all $\delta \leq \hat{\delta}$, this pants decomposition is (R', σ)-tight for some $R' > 0$ (here the constant R' is determined by R and the definition of (R,δ)-skew pants) and centrally $C_0/2$-thick where $C_0 > 0$ is as in Lemma 3.5.

For $\rho > 0$ define the *ρ-thick part* of a skew-pants P to be the set of all points x so that the open metric ball of radius ρ about x with respect to the intrinsic path metric does not intersect the boundary of the skew-pants. By Lemma 3.3, there is a number $\rho_0 > 0$ not depending on R, δ so that any geodesic arc ζ on S which is not contained in an admissible chain of twisted bands meets the ρ_0-thick part of some skew-pants (here as before, we assume that $R > 10$ is sufficiently large and that $\delta > 0$ is sufficiently small).

Now recall that a skew-pants is a union of 5 ruled surfaces with geodesic boundary which are close to being totally geodesic immersed in M. By construction, a geodesic arc ζ which intersects the ρ_0-thick part of a skew-pants crosses through at most two of the boundary arcs of such a surface. As the angle with which two of these ruled surfaces meet at a common boundary geodesic tends to zero with δ, we conclude that for a given fixed $\epsilon > 0$ the following holds true. Let ζ be a geodesic arc of length ρ_0 on S which intersects the ρ_0-thick part of a skew-pants. Then for the length of the lift of ζ to $\tilde{M}$ is at most $(1+\epsilon)$-times the distance in $\tilde{M}$ between its endpoints provided that the number $\delta > 0$ used in the construction is sufficientlly small.

Thus by Proposition 2.4, Corollary 5.9 and the definition of a geometric skew-pants, for the proof of the proposition it now suffices to show that for every $\rho > 0$ there is a number $\epsilon = \epsilon(\rho) > 0$ with the following property. Let $\gamma : \mathbb{R} \to \tilde{M}$ be a piecewise smooth curve. Assume that for every subarc $\gamma[t, t+\rho]$ of γ of length ρ we have $d(\gamma(t), \gamma(t+\rho)) \geq \rho/(1+\epsilon)$; then γ is an L-quasi-geodesic in $\tilde{M}$ for a number $L > 1$ only depending on ρ and ϵ.

However, the existence of such a number $\epsilon > 0$ follows from hyperbolicity. Namely, for $\rho > 0$ let $\gamma : \mathbb{R} \to \tilde{M}$ be a piecewise smooth curve as in the previous paragraph, and let $\hat{\gamma}$ be the piecewise geodesic in $\tilde{M}$ such that for all $m \in \mathbb{Z}$ we have

- $\hat{\gamma}(m\rho/2) = \gamma(m\rho/2)$ and
- $\hat{\gamma}[m\rho/2, (m+1)\rho/2]$ is a geodesic parametrized proportional to arc length.

For each m let $\alpha(m)$ be the breaking angle of the segments of $\hat{\gamma}$ which come together at $\hat{\gamma}(m\rho/2)$. By this we mean that $\pi - \alpha(m)$ is the angle at $\hat{\gamma}(m\rho/2)$ of the triangle in $\tilde{M}$ with vertices $\hat{\gamma}((m-1)\rho/2), \hat{\gamma}(m\rho/2), \hat{\gamma}((m+1)\rho/2))$.

By the assumption on γ, the lengths of the sides adjacent to $\hat{\gamma}(m\rho/2)$ of this triangle are contained in the interval $[\rho/2(1+\epsilon), \rho]$, and the length of the opposite side is at least $(1+\epsilon)^{-1}$ times the sum of the lengths of the adjacent sides. By angle comparison, for any number $\delta > 0$ there is some $\epsilon = \epsilon(\delta) < 1/2$ such that for this ϵ, the breaking angles $\alpha(m)$ do not exceed δ.

On the other hand, by hyperbolicity, there is a number $\delta = \delta(\rho) > 0$ and a number $L > 1$ with the following property. Let $\gamma : \mathbb{R} \to \tilde{M}$ be a piecewise geodesic composed of geodesic segments of length at least $\rho/4$. If the breaking angles of γ at the breakpoints do not exceed δ then γ is an L-quasi-geodesic.

Together this shows the existence of a number $\epsilon = \epsilon(\delta(\rho))$ as required above and completes the proof of the proposition. □

7. The glueing equation

In this section we show that for $G \neq SO(2m, 1)$ $(m \geq 1)$ it is possible to construct a closed immersed surface S in $M = \Gamma\backslash G/K$ which is composed of (R, δ_4)-skew-pants for some $R > R_4$ in such a way that the assumptions in Proposition 6.2 are satisfied. Proposition 6.2 then implies that the surface S is incompressible in M.

It is only in this section that we fully use the assumption that $M = \Gamma\backslash G/K$ for a simple rank one Lie group G and a cocompact torsion free lattice $\Gamma < G$. First we use controlled rate of mixing for the frame flow on M to construct sufficiently well distributed (R, δ)-skew pants with the method from Lemma 4.3. To specify the idea of good distribution of these pants we equip them with a weight function constructed from the Lebesgue measure on a suitably chosen bundle over the universal covering $\tilde{M}$ of M. The fact that this measure is invariant under the action of the entire group G is essential for the argument. We also use a property which only holds for simple rank one Lie-groups of non-compact type different from $SO(2m, 1)$.

Namely, let $K_0 < G$ be the compact stabilizer of a unit vector in $T^1\tilde{M}$ (which is a compact subgroup of the special orthogonal group $SO(\ell-1)$ where $\ell > 0$ is the real dimenison of M). Then the component of the identity of the centralizer of A in K_0 contains $-\mathrm{Id} \in K_0 < SO(\ell-1)$. This property does not hold for $SO(2m,1)$. We refer to [B12] for comments prior to this work why this property is useful for the construction of incompressible surfaces.

We begin with a general observation about the construction of incompressible surfaces, not necessarily in locally symmetric manifolds. To this end fix a number $\delta < \delta_4$ and a number $R > R_4$ as in Proposition 6.2. We allow to decrease δ and increase R throughout the construction.

Let $\mathcal{P}(R,\delta)$ be the collection of all oriented (R,δ)-skew pants in M. The boundary of each such skew pants consists of a triple of closed geodesics whose lengths are contained in the interval $[8R + 2L(R) - \delta, 8R + 2L(R) + \delta]$ (compare Section 4), with properties as specified in the previous sections. Since for every $k > 0$ there are only finitely many closed geodesics in M of length at most k, the set $\mathcal{P}(R,\delta)$ is finite. If P is a geometric skew pants defining a skew-pants in $\mathcal{P}(R,\delta)$ then we write $P \in \mathcal{P}(R,\delta)$ although by definition, a skew pants in $\mathcal{P}(R,\delta)$ is not equipped with a preferred geometric structure.

For $b > 1$ and $\delta < \delta_4$ as in Proposition 6.2 define a graph $\mathcal{G}(R,\delta)$ whose vertex set is the set $\mathcal{P}(R,\delta)$ and where two such vertices P_1, P_2 are connected by an edge if the following two properties hold true.

(1) P_1, P_2 have precisely one cuff γ in common.
(2) P_1, P_2 are R^{-b}-well attached along γ.

We label the edge in $\mathcal{G}(R,\delta)$ connecting the vertices P_1 and P_2 with the common cuff $\gamma \subset P_1 \cap P_2$ (here γ is viewed as an unoriented geodesic). Note that the requirement (2) above does not give any restriction on the homotopy class of a geodesic arc with endpoints on γ which determines the homotopy classes of the two boundary geodesics of P_i distinct from γ $(i = 1, 2)$.

For each vertex P of $\mathcal{G}(R,\delta)$, the edges of $\mathcal{G}(R,\delta)$ incident on P are labeled with three distinct labels $1, 2, 3$ corresponding to the three distinct cuffs of P. Let $\mathcal{E}_i(P)$ be the set of edges with label i $(i = 1, 2, 3)$.

Definition 7.1. An *admissible weight function* f assigns a real valued weight to each edge of $\mathcal{G}(R,\delta)$. These weights satisfy the following *glueing equations*: For each vertex P of $\mathcal{G}(R,\delta)$, there are three glueing equations (one being a consequence of the other two)

$$\sum_{e\in\mathcal{E}_i(P)} f(e) = \sum_{e\in\mathcal{E}_{i+1}(P)} f(e).$$

Here the index i is taken modulo three. We call an admissible weight function a *solution to the glueing equation.*

Lemma 7.2. *If there is a non-negative non-trivial admissible weight function then there is a non-negative non-trivial integral admissible weight function.*

Proof. Since the coefficients of the glueing equations are integral, each gluing equation cuts out a rational hyperplane in the space of all weight functions on the set $\mathcal{E}$ of edges of $\mathcal{G}(R,\delta)$. Thus if there is a non-negative non-trivial admissible weight function, then there is a non-negative non-trivial admissible weight function with rational weights, and such a function can be multiplied with an integer to yield a non-negative integral admissible weight function. This shows the lemma. □

Proposition 7.3. *Each non-negative non-trivial integral admissible weight function for $\mathcal{G}(R,\delta)$ defines an incompressible surface in M.*

Proof. Let f be a non-negative non-trivial integral admissible weight function for $\mathcal{G}(R,\delta)$. Let $v_1,\dots,v_k$ be those vertices of $\mathcal{G}(R,\delta)$ which are adjacent to edges with positive weight. For each such vertex v let

$$\ell(v)=\sum_{e\in\mathcal{E}_1(v)} f(e).$$

Choose $\ell(v)$ copies of the skew-pants P_v corresponding to v. For each cuff γ_i of P_v connect these copies to copies of the skew-pants with the same cuff γ_i as prescribed by the weight: if the edge e connects v to v' then attach $f(e)$ copies of P_v to $f(e)$ copies of $P_{v'}$. By the glueing equation, this can be done in such a way that each cuff of each of the skew pants P_v is glued to precisely one cuff of a neighboring pants, and orientations of these skew pants match. As the consequence, the union of these skew-pants defines the homotopy class of a closed oriented surface S in M. By Proposition 6.2, this surface is incompressible. ☐

We are left with showing the existence of a non-negative non-trivial solution to the glueing equation. This is the most subtle part of the construction, and it is accomplished using ideas from [KM12].

Let λ be the normalized Lebesgue measure (of volume one) on the bundle $\mathcal{F}\to T^1M\to M$ of orthonormal $\mathbb{K}$-frames. Recall that $\mathcal{F}$ is an $SO(n-1)$-principal bundle (or $SU(n-1)$-principal bundle or $Sp(n-1)$-principal bundle or Spin(7)-principal bundle) over the smooth closed manifold T^1M. The measure λ lifts to a G-invariant Radon measure $\tilde{\lambda}$ on the bundle $\tilde{\mathcal{F}}\to T^1\tilde{M}\to\tilde{M}$ of orthonormal $\mathbb{K}$-frames in $T\tilde{M}$. The group G acts simply transitively on $\tilde{\mathcal{F}}$.

By Lemma 4.3, we can construct (R,δ)-skew pants by connecting framed tripods with arcs obtained from orbit segments for the frame flow which begin and end uniformly near the tripods. To make the idea of being uniformly near quantitiative we first construct for each frame $F\in\mathcal{F}$ a neighborhood in an G-equivariant way. To this end let now $\delta<\delta_4/2$. Choose a point $\tilde{z}\in\tilde{\mathcal{F}}$ and a smooth function $f_{\tilde{z}}:\tilde{\mathcal{F}}\to[0,\infty)$ which is supported in the δ-neighborhood of $\tilde{z}$. We assume that

$$\int f_{\tilde{z}}\,d\tilde{\lambda}=1.$$

For $\tilde{u}\in\tilde{\mathcal{F}}$ let $\psi\in G$ be an isometry which maps $\tilde{u}$ to $\tilde{z}$ and define $f_{\tilde{u}}=f_{\tilde{z}}\circ\psi$. Via the projection

$$Q:\tilde{\mathcal{F}}\to\mathcal{F},$$

the functions $f_{\tilde{u}}$ project to functions f_u on the frame bundle $\mathcal{F}\to T^1M\to M$ which are defined as follows. For $u\in\mathcal{F}$ choose some $\tilde{u}$ with $q(\tilde{u})=u$ and put $f_u(v)=\sum_{Q(\tilde{v})=v} f_{\tilde{u}}(\tilde{v})$. Note that this does not depend on any choices made.

Let

$$\mathcal{FT}\to M$$

be the bundle of framed tripods over M (see Section 4 for the definition of a framed tripod). Recall that for $R>1$ and $i=1,2,3$ a framed tripod $((v_1,v_2,v_3),F)$ defines a frame F_i in the fibre of $\mathcal{F}$ over $\Phi^R v_i$. For a framed tripod $z=((v_1,v_2,v_3),F)\in$

$\mathcal{FT}$ and a number $R > 0$ define a function $b_{z,R}$ on the space $\mathcal{F}^3$ of triples of points in $\mathcal{F}$ by

$$b_{z,R}(z_1, z_2, z_3) = \prod_{i=1}^{3} f_{(\Phi^R v_i, F_i)}(z_i)$$

(here $(\Phi^R v_i, F_i)$ is a point in the bundle $\mathcal{F}$ whose basepoint in T^1M is the vector $\Phi^R v_i$).

The involution

$$\mathcal{A} : \mathcal{F} \to \mathcal{F}$$

which replaces the base vector of the frame and the first vector in the fibre by its negative preserves the normalized Lebesgue measure λ.

Denote by $\underline{\Psi^t}$ the product frame flow on $\mathcal{F}^3$. Then for suffciently large $R > 0$, a five-tuple of points $(x, y, u_1, u_2, u_3) \in \mathcal{FT}^2 \times \mathcal{F}^3$ which consists of a pair of framed tripods $(x, y) \in \mathcal{FT}^2$ and some $(u_1, u_2, u_3) \in \mathcal{F}^3$ with

$$b_{x,R}(u_1, u_2, u_3) b_{y,R}(\mathcal{A}^3 \underline{\Psi^R}(u_3, u_2, u_1)) > 0$$

determines an (R, δ)-skew pants. Namely, the framed tripod x determines the frames F_1, F_2, F_3. For each i, the frame u_i is contained in the δ-neighborhood of F_i, and its image under the map $\mathcal{A}\Psi^R$ is contained in the δ-neighborhood of the frame determined by the framed tripod y (with the order of the vectors permuted so that the glueing reverses orientation). Lemma 4.3 now shows that (x, y, u_1, u_2, u_3) defines an (R, δ)-skew-pants.

Our next goal is to observe that the (R, δ)-skew pants constructed in this way abound. To this end we use exponential mixing with respect to the Lebesgue measure of the frame flow on the bundle $\mathcal{F}$. and we let λ^3 be the product measure on $\mathcal{F}^3$. The volume of λ^3 equals one. Let $\sigma(u_1, u_2, u_3) = (u_3, u_2, u_1)$.

Lemma 7.4. *There is a number $\kappa > 0$ such that for any two framed tripods x, y we have*

$$\int b_{x,R}(u) b_{y,R}(\sigma \mathcal{A}^3 \underline{\Psi^R} u) d\lambda^3(u) \geq 1 - e^{-\kappa R}/\kappa.$$

Proof. Since by Theorem 2.2 the frame flow is exponentially mixing and the functions f_z are fixed, there is a number $\kappa_0 > 0$ such that for all frames $y, z \in \mathcal{F}$ and all $R \geq 0$ we have

$$\int f_y(\mathcal{A}\Psi^R v) f_z(v) d\lambda(v) \geq 1 - e^{-\kappa_0 R}/\kappa_0.$$

Taking a triple of frames and multiplying the result shows the lemma. □

Remark 7.5. Lemma 7.4 is the only part of the argument which uses controlled decay of correlation for the frame flow on $\mathcal{F}$. In fact, it is immediate from our discussion that polynomial mixing with exponent at least two is sufficient for the proof of the main theorem from the introduction.

A tripod (v_1, v_2, v_3) is determined by the unit tangent vector v_1 and the oriented normal of v_1 in the oriented real plane defined by the tripod. Thus there is a natural bundle isomorphism from the bundle of framed tripods onto the bundle $\mathcal{F}$. The symmetry of order three which cyclically permutes the vectors in the tripod induces a symmetry of order three in the bundle $\mathcal{F}$ which preserves the Lebesgue measure.

Let $x = ((v_1, v_2, v_3), F), y = ((w_1, w_2, w_3), E)$ be any two framed tripods. These framed tripods define two functions $b_{x,R}$ and $b_{y,R}$ on $\mathcal{F}^3$. With the notations from Proposition 6.2, let $R > R_4$ be sufficiently large that $e^{-\kappa R}/\kappa < R^{-b}$. Let μ be the measure on $\mathcal{FT}^2 \times \mathcal{F}^3$ defined by

$$d\mu(x, y, F_1, F_2, F_3)$$
$$= b_{x,R}(F_1, F_2, F_3) b_{y,R}(\mathcal{A}\Psi^{2R}F_3, \mathcal{A}\Psi^{2R}F_2, \mathcal{A}\Psi^{2R}F_1) d\lambda(x) d\lambda(y) d\lambda^3(F_1, F_2, F_3).$$

By Lemma 7.4 and Fubini's theorem, the total volume of μ is contained in the interval $[1 - e^{-\kappa R}/\kappa, 1]$. Moreover, μ is invariant under the natural action of the cyclic group Λ of order three which acts as a group of rotations on the well connected tripods and as a cyclic group of permutations on the frames. By the above discussion, every point $z \in \mathrm{supp}(\mu)$ determines a geometric skew-pants $P(z)$. Forgetting the geometric structure of $P(z)$ determines a natural map

$$\hat{P} : \mathrm{supp}(\mu) \to \mathcal{P}(R, \delta).$$

Let

$$\mathcal{S} \to T^1M$$

be the bundle over T^1M whose fibre at a point $v \in T^1M$ equals the unit sphere in $v_{\mathbb{K}}^{\perp}$, i.e. the sphere of all vectors $w \in T^1M$ which are orthogonal to the $\mathbb{K}$-line spanned by v. We construct a push-forward of the measure μ to $\mathcal{S}$ as follows.

Let $z \in \mathrm{supp}(\mu)$. Then $z = (x, y, F_1, F_2, F_3)$ where $x, y \in \mathcal{FT}$ and where $F_i \in \mathcal{F}$. The two framed tripods x, y define two oriented ideal triangles contained in an immersed totally geodesic hyperbolic plane in M. Let T_x, T_y be the center triangles of these oriented ideal triangles. The side length of T_x, T_y is $2r$. The vertices of T_x, T_y depend smoothly on x, y, and the orientation of T_x, T_y determines a cyclic order of the vertices of T_x, T_y.

The order of the components of the point z determines an order of the boundary components of the skew-pants $\hat{P}(z)$. More precisely, the tripods x, y and the first two frames F_1, F_2 in the triple of frames from z determine two geodesic arcs which connect the two footpoints of the tripods, and the concatenation of these arcs is freely homotopic to a boundary geodesic α of the skew-pants defined by z. As α is a boundary geodesic of the oriented pair of pants $\hat{P}(z)$, it is oriented.

Let (u_x^1, u_x^2, u_x^3) be the ordered triple of vertices of the triangle T_x, and let (u_y^1, u_y^2, u_y^3) be the ordered triple of vertices of the triangle T_y. The order of these vertices is chosen in such a way that they define the orientation of T_x, T_y and that moreover the oriented geodesic arc connecting u_x^1 to u_x^2 (or connecting u_y^1 to u_y^2) crosses through the first connecting arc for the tripods in the triple (up to a homotopy which moves points at most a distance $\kappa_4 e^{-R}$ where $\kappa_4 > 0$ is as in Section 4). Thus the geodesic segment in the homotopy class determined by $\hat{P}(z)$ which connects u_x^2 to u_y^1 is contained in the $\kappa_4 e^{-R}$-neighborhood of the boundary geodesic α of $\hat{P}(z)$.

The points u_x^1, u_y^2 depend smoothly on z and hence the same holds true for the geodesic segment $\beta(z)$ which connects u_x^1 to u_y^2 and which is contained in the homotopy class determined by the skew-pants. The geodesic segment $\beta(z)$ is contained in the $\kappa_4 e^{-R}$-neighborhood of a boundary geodesic of $\hat{P}(z)$ distinct from α.

There is a unique geodesic arc η in M which connects the closed geodesic α and the geodesic arc β and which is the shortest arc with this property in the homotopy class relative to α, β determined by the skew-pants $\hat{P}(z)$. The initial velocity η' of η is a unit tangent vector with foot-point on α which is orthogonal to α'. The angle between η' and a direction which is orthogonal to $\alpha' \otimes \mathbb{K}$ is exponentially small in R.

Write

$$\mathcal{S}_\alpha = \mathcal{S}|\alpha'$$

and define

$$\mathcal{O}(z) \in \mathcal{S}_\alpha$$

to be the projection of η' into $\mathcal{S}_\alpha$. Recall that this makes sense since α is oriented. In particular, the footpoint of $\mathcal{O}(z)$ is the endpoint of the geodesic arc η on α. This construction defines a map $\mathcal{O} : \mathrm{supp}(\mu) \to \mathcal{S}$ whose image is contained in the union of the restriction of $\mathcal{S}$ to finitely many closed geodesics in M. Let $\mathcal{S}_\mu = \cup_\alpha \mathcal{S}_\alpha$ be the union of these finitely many sphere bundles containing $\mathcal{O}(\mathrm{supp}(\mu))$.

View $\mathcal{S}_\mu$ as a smooth (disconnected) manifold. The restriction of the map $\mathcal{O}$ to the interior of $\mathrm{supp}(\mu)$ (which is a disconnected smooth manifold as well) is smooth, moreover it is easily seen to be open (as a map into $\mathcal{S}_\mu$). As a consequence, the restriction of the push-forward $\mathcal{O}_*(\mu)$ of μ to a component $\mathcal{S}_\alpha$ of $\mathcal{S}_\mu$ (where as before, α is a closed geodesic in M) is contained in the Lebesgue measure class.

For each point $z = (x, y, F_1, F_2, F_3) \in \mathrm{supp}(\mu)$, the point $\mathcal{O}(z)$ is uniquely determined by x, y, F_1, F_2. Namely, the geodesics α and β used for the construction of $\mathcal{O}$ only depend on these data.

The involution ι of $\mathcal{FT}^2$ which exchanges the tripods x and y and reverses the orders of the vectors in the tripods (hence reversing the orientation of T_x, T_y) preserves the Lebesgue measure. Since the frame flow Ψ^t preseves the Lebesgue measure, it follows from the choice of the functions $b_{x,R}$ and the definition of the measure μ that the involution on $\mathcal{FT}^2 \times \mathcal{F}^3$ which maps a point (x, y, F_1, F_2, F_3) to $(\iota(x, y), \mathcal{A}\Psi^R(F_3), \mathcal{A}\Psi^R(F_2), \mathcal{A}\Psi^r(F_3))$ preserves μ. Thus for every oriented closed geodesic α we have

$$\mathcal{O}_*(\mu)(\mathcal{S}_{\alpha^{-1}}) = \mathcal{O}_*(\mu)(\mathcal{S}_\alpha).$$

For a closed geodesic α in the support of $\mathcal{O}(\mu)$ let

$$\mu_\alpha = \mathcal{O}_*(\mu)|\mathcal{S}_\alpha / \mathcal{O}_*(\mu)(\mathcal{S}_\alpha)$$

be the normalization of $\mathcal{O}_*(\mu)$ on $\mathcal{S}_\alpha$. Our next goal is to investigate the measures μ_α. To this end define for a closed oriented geodesic α in M a fibre bundle map

$$\rho_\alpha : \mathcal{S}_\alpha \to \mathcal{S}_{\alpha^{-1}}$$

by requiring that ρ_α maps a point in a fibre of $\mathcal{S}_\alpha$ to its negative, viewed as a point in a fibre of $\mathcal{S}_{\alpha^{-1}}$.

Lemma 7.6. *For every oriented closed geodesic α in M, the measures $\mu_{\alpha^{-1}}$ and $(\rho_\alpha)_*\mu_\alpha$ are absolutely continuous, with Radon Nikodym derivative in the interval $[1 - e^{-\kappa R}/\kappa, (1 - e^{-\kappa R}/\kappa)^{-1}]$.*

Proof. By the definition of well connected framed tripods, the following holds true.

Let $z = (x, y, F_1, F_2, F_3) \in \mathrm{supp}(\mu)$ and assume that $\mathcal{O}(z) \in \mathcal{S}_\alpha$. The point z determines a geodesic arc η connecting the footpoint of the tripod x to the footpoint of y which is homotopic with fixed endpoints to a geodesic in the geometric skew-pants $P(z)$ determined by z. The geodesic η defines the good connection between the first two frames in the well connected tripods x, y.

Choose a lift $\tilde{\alpha}$ of the geodesic α to $\tilde{M}$. The tripods x, y admit lifts $\tilde{x}, \tilde{y}$ to tripods in $\tilde{M}$ in such a way that a long side of each of the two totally geodesic hexagons $H_R(\tilde{x}), H_R(\tilde{y}) \subset \tilde{M}$ is contained in the $\kappa_4 e^{-R}$-neighborhood of $\tilde{\alpha}$. We also require that the footpoints of the tripods $\tilde{x}, \tilde{y}$ are connected by a lift $\tilde{\eta}$ of the geodesic arc η. These lifts then determine lifts $\tilde{F}_1, \tilde{F}_2$ of the frames F_1, F_2. They also determine a lift $\tilde{p}$ of the footpoint p of $\mathcal{O}(z)$.

Let σ be the geodesic reflection about $\tilde{p}$. Then $d\sigma(\tilde{x}), d\sigma(\tilde{y})$ is a pair of tripods in $\tilde{M}$, and $d\sigma(\tilde{F}_1), d\sigma(\tilde{F}_2)$ are frames. The projection $\eta(x, y, F_1, F_2)$ to M of the quadruple $(d\sigma(\tilde{x}), d\sigma(\tilde{y}), d\sigma(\tilde{F}_1), d\sigma(\tilde{F}_2))$ determines the point $\rho_\alpha(\mathcal{O}(z))$ on $\mathcal{S}_{\alpha^{-1}}$. As a consequence, for every choice of a frame $\hat{F}_3$ so that $\hat{z} = (\eta(x, y, F_1, F_2), \hat{F}_3) \in \operatorname{supp}(\mu)$ we have $\mathcal{O}(\hat{z}) = \rho_\alpha(\mathcal{O}(z))$.

As the reflection σ is an isometry and hence it acts as a bundle automorphism on the bundle of framed tripods over $\tilde{M}$ and on the bundle of frames in $T\tilde{M}$ preserving the Lebesgue measure, Fubini's theorem and Lemma 7.4 implies that the map ρ_α is absolutely continuous with respect to the measure μ_α and the measure $\mu_{\alpha^{-1}}$, with Radon Nikodym derivative contained in the interval $[1 - e^{-\kappa R}/\kappa, (1 - e^{-\kappa R}/\kappa)^{-1}]$. This shows the lemma. □

Let again α be a closed geodesic in M and for $t \geq 0$ let $B_\alpha^t : \mathcal{S}_\alpha \to \mathcal{S}_\alpha$ be the map induced by parallel transport of distance t. The map B_α^t in turn is the projection of a map $B_{\tilde{\alpha}}^t$ which is defined as follows. Let $\tilde{\alpha}$ be a lift of α to $\tilde{M}$, and let $B_{\tilde{\alpha}}^t$ be parallel transport of distance t along $\tilde{\alpha}$. Then $B_{\tilde{\alpha}}^t$ is the restriction of a bundle automorphism of $T\tilde{M}$ defined by an isometry of $\tilde{M}$ which preserves $\tilde{\alpha}$ and acts on $\tilde{\alpha}$ as a translation. As in Lemma 7.6, we use this fact to conclude

Lemma 7.7. *The measures μ_α and $(B_\alpha^t)_*\mu_\alpha$ are absolutely continuous, with Radon Nikodym derivative contained in the interval*

$$[1 - e^{-\kappa R}/\kappa, (1 - e^{-\kappa R}/\kappa)^{-1}].$$

Recall that for a closed geodesic α in M the monodromy of α is defined. This monodromy is an isometry contained in the intropy group of the tangent of α and hence it is an element $A \in SO(n-1)$ (or $A \in SU(n-1), A \in Sp(n-1), A \in \mathrm{Spin}(7)$). For a given point $p \in \alpha$, it has a natural representative as an isometry of the $\mathbb{K}$-orthogonal complement of α' in T_pM.

The following observation is completely analogous to Lemma 7.6 and Lemma 7.7. For its formulation, note that an isometry of the $\mathbb{K}$-orthogonal complement of α' in T_pM which commutes with the monodromy of α determines a bundle automorphism of $\mathcal{S}_\alpha$ commuting with parallel transport.

Lemma 7.8. *Let $U \in SO(n-1)$ (or $U \in SU(n-1), U \in Sp(n-1), U \in \mathrm{Spin}(7)$) be an isometry of the $\mathbb{K}$-orthogonal complement of α' in T_pM which commutes with the monodromy of α. Then the measures μ_α and $\mu_\alpha \circ U$ are absolutely continuous, with Radon Nikodym derivative contained in the interval* $[1 - e^{-\kappa R}/\kappa, (1 - e^{-\kappa R}/\kappa)^{-1}]$.

The bundle $\mathcal{S}_\alpha$ is a standard sphere bundle over the circle. There is a natural Riemannian metric for this bundle which restricts to the round metric on each fibre. The length of the base equals the length $\ell(\alpha)$ of α. Let d be the distance function on $\mathcal{S}_\alpha$ induced by this metric. The maps $\rho_\alpha : \mathcal{S}_\alpha \to \mathcal{S}_{\alpha^{-1}}$ and B_α^t are isometries for these metrics. Write $B_\alpha = B_\alpha^1$.

Proposition 7.9. *If $G \neq SO(2m,1)$ for some $m \geq 1$ then there is a number $\theta > 0$ not depending on α, and there is a homeomorphism $\psi_\alpha : \mathcal{S}_\alpha \to \mathcal{S}_\alpha$ with*

$$(\rho_\alpha \circ B_\alpha \circ \psi_\alpha)_* \mu_\alpha = \mu_{\alpha^{-1}}$$

and $d(x, \psi_\alpha(x)) \leq \theta e^{-\kappa R}$ for all $x \in \mathcal{S}_\alpha$.

Proof. We observed before that the measures μ_α are contained in the Lebesgue measure class. Denote by ω the standard volume form on the smooth oriented manifold $\mathcal{S}_\alpha$; then we may assume that

$$(\rho_\alpha \circ B_\alpha)_*^{-1}(\mu_{\alpha^{-1}}) = g\omega,\ \mu_\alpha = f\omega$$

for continuous positive functions f, g with

$$\int f d\omega = \int g d\omega = 1.$$

Our goal is to show that there is a homeomorphism ψ_α of $\mathcal{S}_\alpha$ which satisfies $d(x, \psi_\alpha(x)) \leq \theta e^{-\kappa R}$ for some $\theta > 0$ and such that $\psi_\alpha^*(g\omega) = f\omega$.

Write $q = 1 - e^{-\kappa R}/\kappa$. By Lemma 7.7, the function f is invariant under parallel transport up to a multiplicative factor of at most q^{-1}. This implies the following.

Choose a parametrization of α by arc length on the interval $[0,\ell]$. Let $\pi : \mathcal{S}_\alpha \to \alpha$ be the natural projection, let ω_s be the standard volume form on $\pi^{-1}(s)$ and let $f_0 : [0,\ell] \to (0,\infty)$ be the function obtained by

$$f_0(s) = \int_{\pi^{-1}(s)} f d\omega_s.$$

Then

$$\int_a^b f_0 dt \in [q(b-a)/\ell, q^{-1}(b-a)/\ell]$$

for all $a < b$, and $\int_0^\ell f_0 dt = 1$. Therefore if we define $\chi(t) = \ell \int_0^t f_0 ds$ then $\chi : [0,\ell] \to [0,\ell]$ is a homeomorphism which moves points a distance at most $1-q$. Moreover, the measure $\chi_*(\ell f_0 dt)$ is the standard Lebesgue measure dt on the base $[0,\ell]$.

Lift the homeomorphism χ to a homeomorphism $\Psi : \mathcal{S}_\alpha \to \mathcal{S}_\alpha$ defined by

$$\Psi(v) = \|_{\alpha[\pi(v), \chi(\pi(v))]} v.$$

Then the fibres of the bundle $\pi : \mathcal{S}_\alpha \to \alpha$ have volume one for the volume form $\Psi_*(\ell f \omega)$. This implies that via moving fibres of $\mathcal{S}_\alpha$ with parallel transport and renormalization, it suffices to show the lemma under the additional assumption that each of the fibre integrals of f and g equals one.

Let $m = \dim(M) - \mathrm{rk}(\mathbb{K})$ where $\mathrm{rk}(\mathbb{K})$ is the rank of $\mathbb{K}$ as an $\mathbb{R}$-vector space. Let A be the monodromy of α. Then A is an element of the orthogonal group which fixes γ'. If $G = SO(n,1)$ then there are no further constraints, and we have $A \in SO(n-1) = SO(m)$. In the case $G = SU(n,1)$ the element A also fixes the image of γ' under the complex structure and we have $A \in SU(n-1) < SO(2n-2) = SO(m)$. Similarly, if $G = Sp(n,1)$ then A fixes the quaternionic line spanned by γ' although perhaps not pointwise, and we can view A as an element in $SO(4n-4)SO(4) < SO(m)SO(4)$. Finally if $G = F_4^{-20}$ then A fixes the Cayley line spanned by γ' and we can view A as an element in $SO(8)SO(8) = SO(m)SO(8)$.

We first consider the case that the component of the identity $C(A) < SO(n-1)$ (or $C(A) < SU(n-1) < S(U(n)U(1)), C(A) < Sp(n-1), C(A) < \mathrm{Spin}(9)$) of the centralizer of A in $SO(n-1)$ (or in $SU(n-1), Sp(n-1), \mathrm{Spin}(9)$) acts transitively on S^{m-1}, viewed as a fibre of the bundle $\mathcal{S}_\alpha$. Observe that this holds true if M is a hyperbolic 3-manifold.

In this case Lemma 7.8 shows that there is a function β with values in the interval $[q, q^{-1}]$ such that $f\beta$ is a positive constant function.

Let ω_0 be the smooth normalized volume form of the round metric on the round sphere S^{m-1}. For a number $\rho \in (0, 1/4)$ consider for the moment an arbitrary continuous function $h : S^{m-1} \to [1-\rho, 1+\rho]$ with the property that $\int (h-1) d\omega_0 = 0$. The function $h - 1$ is bounded in norm by ρ. Let Δ be the Laplacian of the round metric on S^{m-1}. Then there is a unique function $\varphi : S^{m-1} \to \mathbb{R}$ such that

$$\Delta(\varphi) = h - 1 \text{ and } \int \varphi \, d\omega_0 = 0.$$

Let $*$ be the Hodge star operator of the round metric on S^{m-1}. Schauder theory shows that the $(m-2)$-form $\eta = *d*(\varphi\omega_0)$ is bounded in norm by a constant multiple of ρ.

Let

$$\nu_t = (1-t)\omega_0 + th\omega_0.$$

Then for each t the norm of the vector field X_t defined by

$$\iota_{X_t}\nu_t = -\eta$$

is bounded from above by a constant multiple of ρ. Let Λ be the time-one map of the flow of the time dependent vector field X_t. There is a number $\theta > 0$ such that $d(x, \Lambda x) \leq \theta\rho$ for all $x \in S^{m-1}$. On the other hand, we have $\Lambda^*(h\omega_0) = \omega_0$.

We now apply this construction to the restrictions of the function f to the fibres of $\mathcal{S}_\alpha \to \alpha$. These restrictions depend continuously on the fibre. As all functions and forms in the above construction depend continuously on the function h with respect to the C^0-topology, the fibrewise defined homeomorphisms which transform the volume form ω_s on the fibre $\pi^{-1}(s)$ to the volume form $f\omega_s$ determine a fibre preserving homeomorphism $\Lambda_f : \mathcal{S}_\alpha \to \mathcal{S}_\alpha$, and there is similarly a homeomorphism Λ_g. Then

$$\psi_\alpha = \Lambda_f^{-1} \circ \Lambda_g$$

(read from right to left) is a map with the properties stated in the proposition. This concludes the proof of the proposition in the case that the component $C(A)$ of the identity of the centralizer of the monodromy A of α acts transitively on the fibres of $\mathcal{S}_\alpha \to \alpha$.

The general case is similar. By the assumption $G \neq SO(2k, 1)$, the dimension $m-1$ of the fibre of the sphere bundle $\mathcal{S}_\alpha$ is odd. The group $C(A)$ can be described as the group of all isometries of a fixed fibre of S^{m-1} which preserve the generalized eigenspaces of the monodromy A (and the complex structure for $G = SU(n, 1)$ or the quaternionic structure for $G = Sp(n, 1)$). As $m - 1$ is odd, $C(A)$ contains the element $-\mathrm{Id}$.

For $v \in \mathcal{S}_\alpha$ the orbit $C(A)(v)$ of v under the group $C(A)$ (which is viewed as a group of isometries of the fibre of $\mathcal{S}_\alpha$ containing v) is a smooth submanifold of S^{m-1} which contains with w the antipode $-w$. This submanifold is preserved by the monodromy A of α. Thus if $v \in \mathcal{S}_\alpha$ is a vector with footpoint $\alpha(0)$ (for a parametrization of α by arc length as before) then

$$\mathcal{C} = \cup_{t\in[0,\ell]} \|_{\alpha[0,t]} C(A)(v)$$

is a fibre bundle over α with smooth fibre which is invariant under the antipodal bundle involution.

Multiply the natural volume form $\omega_{\mathcal{C}}$ on $\mathcal{C}$ with the functions f, g. As $\mathcal{C}$ is invariant under the fibrewise antipodal map and under parallel transport, the total integrals of f and g on $\mathcal{C}$ coincide. Since $C(A)$ acts transitively on the fibres of $\mathcal{C}$, Lemma 7.8 shows that the restrictions to $\mathcal{C}$ of the functions f, g have values in an interval of the form $[c(1-e^{-\kappa R}/\kappa), c(1-e^{-\kappa R}/\kappa)^{-1}]$ for some $c > 0$. As a consequence, the argument for the case that the action of $C(A)$ on the fibres of $\mathcal{S}_\alpha$ is transitive can be applied to the manifold $\mathcal{C}$ and yields a homeomorphism of $\mathcal{C}$ with the properties required in the lemma (where we may have to adjust the constant θ to take into account the various geometries of the manifolds $C(A)(v)$).

Now the orbits of $C(A)$ form a compact family of manifolds, and the functions f, g are globally defined and continuous. Thus carrying out this construction separately on each of the fibre bundles constructed from the orbits of $C(A)$ yields a homeomorphism ψ_α of $\mathcal{S}_\alpha$ as claimed. □

For $x \in \mathrm{supp}(\mu)$ let as before $P(x)$ be the geometric skew-pants defined by x. We have (compare [KM12] for the case $\tilde{M} = \mathbf{H}^3$)

Lemma 7.10. *For sufficiently small $\delta < \delta_4$ as in the definition of the measure μ, the following holds true. Let $x, y \in \mathrm{supp}(\mu)$; if $\mathcal{O}(x) \in \mathcal{S}_\alpha$ and if $\mathcal{O}(y) = (\rho_\alpha \circ B_\alpha \circ \psi_\alpha)(x)$ where ψ_α is as in Proposition 7.9, then $\hat{P}(y)$ is well attached to $\hat{P}(x)$ along α.*

Proof. Let $\tilde{\alpha}$ be a lift of α to $\tilde{M}$. Then α is the quotient of $\tilde{\alpha}$ by a loxodromic isometry $\Lambda \in G$. The translation length of Λ equals the length of α. The rotational part of Λ is the monodromy $A \in SO(n-1)$ (or $A \in SU(n-1)$ or $A \in Sp(n-1)$) of α.

By Proposition 4.4, for a number $\epsilon < \pi/4$ depending on δ, the monodromy A of α is ϵ-close to the identity. In particular, there is a unique root $\Lambda^{1/2}$ of Λ whose rotational part is ϵ-close to the identity. The map $\Lambda^{1/2}$ acts as an involution on the bundle $\mathcal{S}_\alpha$.

Each skew-pants $P \in \mathcal{P}(R, \delta)$ which contains α in its boundary has two seams β_1, β_2 with endpoints on α. Let v_i be the unit tangent vector of β_i on α $(i = 1, 2)$. We claim that the distance in $\mathcal{S}_\alpha$ between $\Lambda^{1/2}(v_1)$ and v_2 is at most $\rho e^{-\zeta R}$ where $\rho > 0, \zeta > 0$ are universal constants.

To this end note that the seams $\beta_1, \beta_2, \beta_3$ of P decompose P into two right angled hexagons H_1, H_2 with geodesic sides in M. Equip P with the structure of a geometric skew pants. For this geometric structure there are numbers $\delta_1 < \delta, \delta_2 < \delta, \delta_3 < \delta$, and there are approximate $\delta_1, \delta_2, \delta_3$-twisted ruled bands B_1, B_2, B_3 of size roughly $6R - 2\tau$ which are (locally) embedded in P (see Section 6). The bands B_1, B_2, B_3 are separated by the seams $\beta_1, \beta_2, \beta_3$ of P into half-bands B_i^j $(i = 1, 2, 3, j = 1, 2)$. Up to an error which is exponentially small in R, the hexagon H_j is composed of B_1^j, B_2^j, B_3^j and a triangle which is exponentially close to an equilateral triangle of side length $2r$ in a totally geodesic immersed hyperbolic plane in M. The pairs of twisting angles of the half-bands B_i^1, B_j^2 contained in H_1, H_2 coincide. But this means that the hexagons H_1, H_2 are isometric up to an error which is exponentially small in R.

Now if $x \in \mathrm{supp}(\mu)$, if $P(x) = P$ and if $\mathcal{O}(x) \in \mathcal{S}_\alpha$ then up to exchanging v_1 and v_2, the vector $\mathcal{O}(x)$ is at distance at most a constant times $e^{-\kappa R}/\kappa$ from v_1. In particular, if x, y are as in the lemma, then the approximate twisted ruled bands of $P(x), P(y)$ which pass through the foot points of $\mathcal{O}(x), \mathcal{O}(y)$ are well attached along a subarc of γ.

By the above, the second pair of approximate twisted ruled band of $P(x), P(y)$ is well attached along a subarc of γ as well. Together this completes the proof of the lemma. $\square$

Each point $z \in \mathrm{supp}(\mu)$ defines a skew-pants $\hat{P}(z) \in \mathcal{P}(R, \delta)$. The set $\mathcal{P}(R, \delta)$ is finite. For $P \in \mathcal{P}(R, \delta)$ define

$$h(P) = \mu\{z \mid \hat{P}(z) = P\}.$$

Then $P \mapsto h(P)$ is a non-negative weight function on the set $\mathcal{P}$ of all skew pants. This weight function is invariant under the involution $\mathcal{J}$ of $\mathcal{P}(R, \delta)$ which reverses the orientation.

For $P \in \mathcal{P}(R, \delta)$ let $\chi_P : \mathrm{supp}(\mu) \to [0, 1]$ be the function defined by $\chi_P(z) = 1$ if the skew-pants $P(z)$ defined by x equals P, and let $\chi_P(z) = 0$ otherwise. Then we have

$$h(P) = \int \chi_P d\mu.$$

If γ is a cuff of P then the weighted measure $\chi_P \mu$ projects via the map $\mathcal{O}$ to a weighted measure $\chi_{P,\gamma}\mu_\gamma$ on $\mathcal{S}_\gamma$. Since the measure μ is invariant under the map which exchanges the two tripods in a point in $\mathcal{FT}^2 \times \mathcal{F}^3$ and permutes the frames in $\mathcal{F}^3$, the total mass of the measure

$$\chi_{P,\gamma}\mu_\gamma$$

does not depend on the choice of the boundary geodesic γ of P.

Let $\psi_\gamma : \mathcal{S}_\gamma \to \mathcal{S}_\gamma$ be as in Proposition 7.9. We may assume that $\psi_{\gamma^{-1}} = \psi_\gamma^{-1}$ where we identify $\mathcal{S}_\gamma$ with $\mathcal{S}_{\gamma^{-1}}$ with the obvious homeomorphism.

Define

$$h(P, P') = \int \chi_{P',\gamma^{-1}}(\rho_\gamma \circ B_\gamma \circ \psi_\gamma(x))\chi_{P,\gamma}(x) d\mu_\gamma(x).$$

By construction and Lemma 7.10, if $h(P, P') > 0$ then the pants P, P' are well attached along γ. In particular, P, P' define an edge in the graph $\mathcal{G}(R, \delta)$.

The function $h(P, P')$ can be viewed as a non-negative weight function on the edges of $\mathcal{G}(P, \delta)$. Since $\psi_{\gamma^{-1}} = \psi_\gamma^{-1}$, by Proposition 7.9 this weight function is symmetric: We have

$$h(P, P') = h(P', P)$$

for all P, P'. Moreover, clearly

$$\sum_{P'} h(P, P') = h(P)$$

which is equivalent to stating that this weight function is admissible.

Theorem 1 is now a consequence of Proposition 7.3 and Lemma 7.2.

8. Concluding remarks

The proof of Proposition 7.9 is the only part of the argument which is not valid for the groups $G = SO(2m, 1)$ for $m \geq 2$ (with $SO(2, 1)$ not relevant for the purpose of this work, see however [KM11]).

Namely, if $G = SO(2m,1)$ for some $m \geq 2$ then the monodromy of any closed geodesic α has a fixed unit vector. If the eigenspace for the monodromy transformation with respect to the eigenvalue one is bigger than one then the component of the identity of the subgroup of $SO(2m-1)$ of all elements which commute with the monodromy transformation contains $-\mathrm{Id}$. In this case the argument in the proof of Proposition 7.9 is valid. However, if the dimension of the eigenspace for the eigenvalue one equals one then we can not use this argument. Call such a periodic geodesic α with this property *generic*.

For a generic closed geodesic α, the bundle $\mathcal{S}_\alpha$ contains a sphere subbundle Σ_α whose fibre is a sphere of dimension $2m-3$. It is the sphere subbundle of the orthogonal complement of the one-dimensional eigenspace of the monodromy transformation for the eigenvalue one. This sphere subbundle is invariant under parallel transport.

Choose a parametrization of α and invariant orientations of $\mathcal{S}_\alpha, \Sigma_\alpha$. For all t and all $s \in [-\pi/2, \pi/2]$ the set $\Sigma_\alpha^s(t)$ of vectors in the fibre $\mathcal{S}_\alpha(t)$ whose oriented distance to $\Sigma_\alpha(t) = \Sigma_\alpha^0(t)$ equals s defines a decomposition of $\mathcal{S}_\alpha(t)$ which is parametrized on $[-\pi/2, \pi/2]$. For each s the set $\cup_t \Sigma_\alpha^s(t)$ is invariant under parallel transport. In the glueing construction, we have to match a point in $\cup_t \Sigma_\alpha^s(t)$ with a point which is exponentially close to $\cup_t \Sigma_\alpha^{-s}(t)$. Now if the measure of $\cup_t \cup_{s\leq 0} \Sigma_\alpha^s(t)$ is bigger than the measure of $\cup_t \cup_{s\geq 0} \Sigma_\alpha^s(t)$ and if the measure of a small neighborhood of $\cup_t \Sigma_\alpha^0(t)$ is exponentially small, then we can not match pants as required in the condition for incompressibility.

In spite of this difficulty, we believe that Theorem 1 holds true for even dimensional hyperbolic manifolds. We also conjecture that it is true for closed locally symmetric manifolds of the form $M = \Gamma\backslash G/K$ where G is a semisimple Lie group with finite center, without compact factors and without factors locally isomorphic to $SL(2,\mathbb{R})$, and where $\Gamma < G$ is a cocompact irreducible lattice.

The Kahn-Markovic argument does not seem to generalize in an easy way to rank one locally symmetric manifolds of finite volume. However, Theorem 1 is known for non-compact finite volume hyperbolic 3-manifolds. We refer to [BC14] for a recent proof and references to related ealier results.

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