GEOMETRY OF THE COMPLEX OF CURVES AND OF TEICHMÜLLER SPACE

URSULA HAMENSTÄDT

ABSTRACT. Using train tracks on a nonexceptional oriented surface S of finite type in a systematic way we give a proof that the complex of curves $\mathcal{C}(S)$ of S is a hyperbolic geodesic metric space. We also discuss the relation between the geometry of the complex of curves and the geometry of Teichmüller space.

CONTENTS

1. INTRODUCTION

Consider a compact oriented surface S of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. Such a surface is called *of finite type*. We assume that S is *non-exceptional*, i.e. that $3g - 3 + m \geq 2$; this rules out a sphere with at most four punctures and a torus with at most one puncture.

In [11], Harvey associates to such a surface the following simplicial complex.

Definition 1.1. The *complex of curves* $\mathcal{C}(S)$ for the surface S is the simplicial complex whose vertices are the free homotopy classes of essential simple closed curves on S and whose simplices are spanned by collections of such curves which can be realized disjointly.

Here we mean by an *essential* simple closed curve a simple closed curve which is not contractible nor homotopic into a puncture. Since $3g - 3 + m$ is the number of curves in a *pants decomposition* of S, i.e. a maximal collection of disjoint mutually not freely homotopic essential simple closed curves which decompose S into $2g - 2 + m$ open subsurfaces homeomorphic to a thrice punctured sphere, the dimension of $\mathcal{C}(S)$ equals $3g - 4 + m$.

In the sequel we restrict our attention to the one-skeleton of the complex of curves which is usually called the *curve graph*; by abuse of notation, we denote it again by $\mathcal{C}(S)$. Since $3g - 3 + m \geq 2$ by assumption, $\mathcal{C}(S)$ is a nontrivial graph which moreover is connected [11]. However, this graph is locally infinite. Namely, for every simple closed curve α on S the surface $S - \alpha$ which we obtain by cutting S

open along α contains at least one connected component of Euler characteristic at most -2, and such a component contains infinitely many distinct free homotopy classes of simple closed curves which viewed as curves in S are disjoint from α.

Providing each edge in $\mathcal{C}(S)$ with the standard euclidean metric of diameter 1 equips the curve graph with the structure of a geodesic metric space. Since $\mathcal{C}(S)$ is not locally finite, this metric space $(\mathcal{C}(S), d)$ is not locally compact. Masur and Minsky [19] showed that nevertheless its geometry can be understood quite explicitly. Namely, $\mathcal{C}(S)$ is hyperbolic of infinite diameter. Here for some $\delta > 0$ a geodesic metric space is called δ-*hyperbolic in the sense of Gromov* if it satisfies the δ-*thin triangle condition*: For every geodesic triangle with sides a, b, c the side c is contained in the δ-neighborhood of $a \cup b$. Later Bowditch [2] gave a simplified proof of the result of Masur and Minsky which can also be used to compute explicit bounds for the hyperbolicity constant δ.

Since the Euler characteristic of S is negative, the surface S admits a complete hyperbolic metric of finite volume. The group of diffeomorphisms of S which are isotopic to the identity acts on the space of such metrics. The quotient space under this action is the *Teichmüller space* $\mathcal{T}_{g,m}$ for S of all *marked* isometry classes of complete hyperbolic metrics on S of finite volume, or, equivalently, the space of all marked complex structures on S of finite type. The Teichmüller space can be equipped with a natural topology, and with this topology it is homeomorphic to $\mathbb{R}^{6g-6+2m}$. The *mapping class group* $\mathcal{M}_{g,m}$ of all isotopy classes of orientation preserving diffeomorphisms of S acts properly discontinuously as a group of diffeomorphisms of Teichmüller space preserving a complete Finsler metric, the so-called *Teichmüller metric*. The quotient orbifold is the *moduli space* $\mathrm{Mod}(S)$ of S of all isometry classes of complete hyperbolic metrics of finite volume on S (for all this see [12]).

The significance of the curve graph for the geometry of Teichmüller space comes from the obvious fact that the mapping class group acts on $\mathcal{C}(S)$ as a group of simplicial isometries. Even more is true: If S is not a twice punctured torus or a closed surface of genus 2, then the *extended mapping class group* of isotopy classes of *all* diffeomorphisms of S coincides precisely with the group of simplicial isometries of $\mathcal{C}(S)$; for a closed surface of genus 2, the group of simplicial isometries of $\mathcal{C}(S)$ is the quotient of the extended mapping class group under the hyperelliptic involution which acts trivially on $\mathcal{C}(S)$ (see [13] for an overview on this and related results). Moreover, there is a natural map $\Psi : \mathcal{T}_{g,m} \to \mathcal{C}(S)$ which is *coarsely* $\mathcal{M}_{g,m}$-*equivariant* and *coarsely Lipschitz* with respect to the Teichmüller metric on $\mathcal{T}_{g,m}$. By this we mean that there is a number $a > 1$ such that $d(\Psi(\phi h), \phi(\Psi h)) \leq a$ for all $h \in \mathcal{T}_{g,m}$ and all $\phi \in \mathcal{M}_{g,m}$ and that moreover $d(\Psi h, \Psi h') \leq a d_T(h, h') + a$ for all $h, h' \in \mathcal{T}_{g,m}$ where d_T denotes the distance function on $\mathcal{T}_{g,m}$ induced by the Teichmüller metric (see Section 4).

As a consequence, the geometry of $\mathcal{C}(S)$ is related to the large-scale geometry of the Teichmüller space and the mapping class group. We discuss this relation in Section 4. In Section 3 we give a proof of the hyperbolicity of the curve graph using *train tracks* and splitting sequences of train tracks in a consistent way as the main tool. Section 2 introduces train tracks, geodesic laminations and quadratic differentials and summarizes some of their properties.

2. Train tracks and geodesic laminations

Let S be a nonexceptional surface of finite type and choose a complete hyperbolic metric on S of finite volume. With respect to this metric, every essential free homotopy class of loops can be represented by a closed geodesic which is unique up to parametrization. This geodesic is *simple*, i.e. without self-intersection, if and only if the free homotopy class has a simple representative (see [4]). In other words, there is a one-to-one correspondence between vertices of the curve graph and simple closed geodesics on S. Moreover, there is a fixed *compact* subset S_0 of S containing all simple closed geodesics.

The *Hausdorff distance* between two closed bounded subsets A, B of a metric space X is defined to be the infimum of all numbers $\epsilon > 0$ such that A is contained in the ϵ-neighborhood of B and B is contained in the ϵ-neighborhood of A. This defines indeed a distance and hence a topology on the space of closed bounded subsets of X; this topology is called the *Hausdorff topology*. If X is compact then the space of closed subsets of X is compact as well. In particular, for the distance on S induced by a complete hyperbolic metric of finite volume, the space of closed subsets of the compact set $S_0 \subset S$ is compact with respect to the Hausdorff topology.

The collection of all simple closed geodesics on S is *not* a closed set with respect to the Hausdorff topology, but a point in its closure can be described as follows.

Definition 2.1. A *geodesic lamination* for a complete hyperbolic structure of finite volume on S is a compact subset of S which is foliated into simple geodesics.

Thus every simple closed geodesic is a geodesic lamination which consists of a single leaf. The space of geodesic laminations on S equipped with the Hausdorff topology is compact, and it contains the closure of the set of simple closed geodesics as a *proper* subset. Note that every lamination in this closure is necessarily connected. To describe the structure of the space of geodesic laminations more explicitly we introduce some more terminology.

Definition 2.2. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ. A geodesic lamination λ is *maximal* if all its complementary components are ideal triangles or once punctured monogons. A geodesic lamination is called *complete* if it is maximal and can be approximated in the Hausdorff topology for compact subsets of S by simple closed geodesics.

As an example, a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves. Every minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed geodesics [5]. Moreover, a minimal geodesic lamination λ is a *sublamination* of a complete geodesic lamination [9], i.e. there is a complete geodesic lamination which contains λ as a closed subset. In particular, every simple closed geodesic on S is a sublamination of a complete geodesic lamination. *Every* geodesic lamination λ is a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of λ either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components [1, 5, 24]. This means that the set of accumulation points of an isolated half-leaf of λ is a minimal component of λ.

Geodesic laminations which are disjoint unions of minimal components can be equipped with the following additional structure.

Definition 2.3. A *measured geodesic lamination* is a geodesic lamination together with a translation invariant *transverse measure*.

A transverse measure for a geodesic lamination λ assigns to every smooth compact arc c on S with endpoints in the complement of λ and which intersects λ transversely a finite Borel measure on c supported in $c \cap \lambda$. These measures transform in the natural way under homotopies of c by smooth arcs transverse to λ which move the endpoints of the arc c within fixed complementary components of λ. The support of the measure is the smallest sublamination ν of λ such that the measure on any such arc c which does not intersect ν is trivial. This support is necessarily a union of minimal components of λ. An example for a measured geodesic lamination is a *weighted simple closed geodesic* which consists of a simple closed geodesic α and a positive weight $a > 0$. The measure disposed on a transverse arc c is then the sum of the Dirac masses on the intersection points between c and α multiplied with the weight a.

The space $\mathcal{ML}$ of measured geodesic laminations on S can naturally be equipped with the *weak*-topology*. This topology restricts to the weak*-topology on the space of measures on a given arc c which is transverse to each lamination from an open subset of lamination space. The natural action of the group $(0, \infty)$ by scaling is continuous with respect to this topology, and the quotient is the space $\mathcal{PML}$ of *projective measured geodesic laminations*. This space is homeomorphic to a sphere of dimension $6g - 7 + 2m$ (see [5, 6, 25]).

The *intersection number* $i(\gamma, \delta)$ between two simple closed curves $\gamma, \delta \in \mathcal{C}(S)$ equals the minimal number of intersection points between representatives of the free homotopy classes of γ, δ. This intersection function extends to a continuous pairing $i : \mathcal{ML} \times \mathcal{ML} \to [0, \infty)$, called the *intersection form*.

Measured geodesic laminations are intimately related to more classical objects associated to Riemann surfaces, namely *holomorphic quadratic differentials*. A holomorphic quadratic differential q on a Riemann surface S assigns to each complex coordinate z an expression of the form $q(z)dz^2$ where $q(z)$ is a holomorphic function on the domain of the coordinate system, and $q(z)(dz/dw)^2 = q(w)$ for overlapping coordinates z, w. We require that q has at most a simple pole at each puncture of S. If q does not vanish identically, then its zeros are isolated and independent of the choice of a complex coordinate. If $p \in S$ is not a zero for q then there is a coordinate z near p, unique up to multiplication with ± 1, such that p corresponds to the origin and that $q(z) \equiv 1$. Writing $z = x + iy$ for this coordinate, the euclidean metric $dx^2 + dy^2$ is uniquely determined by q. The arcs parallel to the x-axis (or y-axis, respectively) define a foliation $\mathcal{F}_h$ (or $\mathcal{F}_v$) on the set of regular points of q called the *horizontal* (or *vertical*) foliation. The vertical length $|dy|$ defines a *transverse measure* for the horizontal foliation, and the horizontal length $|dx|$ defines a transverse measure for the vertical foliation. The foliations $\mathcal{F}_h, \mathcal{F}_v$ have singularities of the same type at the zeros of q and at the punctures of S (see [27] for more on quadratic differentials and measured foliations).

There is a one-to-one correspondence between measured geodesic laminations and (equivalence classes of) measured foliations on S (see [16] for a precise statement). The pair of measured foliations defined by a quadratic differential q corresponds under this identification to a pair of measured geodesic laminations $\lambda \neq \mu \in \mathcal{ML}$ which *jointly fill up* S. This means that for every $\eta \in \mathcal{ML}$ we have $i(\lambda, \eta) + i(\mu, \eta) > 0$.

Vice versa, every pair of measured geodesic laminations $\lambda \neq \mu \in \mathcal{ML}$ which jointly fill up S defines a unique complex structure of finite type on S together with a holomorphic quadratic differential $q(\lambda, \mu)$ (see [14] and the references given there) whose *area*, i.e. the area of the singular euclidean metric defined by $q(\lambda, \mu)$, equals $i(\lambda, \mu)$. If α, β are *simple multi-curves* on S, which means that α and β consist of collections $c = c_1 \cup \cdots \cup c_\ell \subset \mathcal{C}(S)$ of free homotopy classes of simple closed curves which can be realized disjointly, and if α, β jointly fill up S, then for all $a > 0, b > 0$ the quadratic differential $q(a\alpha, b\beta)$ defined by the measured geodesic laminations $a\alpha, b\beta$ can explicitly be constructed as follows. Choose smooth representatives of α, β, again denoted by α, β, which intersect transversely in precisely $i(\alpha, \beta)$ points; for example, the geodesic representatives of α, β with respect to any complete hyperbolic metric on S of finite volume have this property. For each intersection point between α, β choose a closed rectangle in S with piecewise smooth boundary containing this point in its interior and which does not contain any other intersection point between α, β. We allow that some of the vertices of such a rectangle are punctures of S. These rectangles can be chosen in such a way that they provide S with the structure of a cubical complex: The boundary of each component D of $S - \alpha - \beta$ is a polygon with an even number of sides which are subarcs of α, β in alternating order. If D does not contain a puncture, then its boundary has at least four sides. Thus we can construct the rectangles in such a way that their union is all of S and that the intersection between any two distinct such rectangles either is a common side or a common vertex. Each rectangle from this cubical complex has two sides which are "parallel" to α and two sides "parallel" to β (see Section 4 of [14] for a detailed discussion of this construction).

Equip each rectangle with an euclidean metric such that the sides parallel to α are of length b, the sides parallel to β are of length a and such that the metrics on two rectangles coincide on a common boundary arc. These metrics define a piecewise euclidean metric on S with a singularity of cone angle $k\pi \geq 3\pi$ in the interior of each disc component of $S - \alpha - \beta$ whose boundary consists of $2k \geq 6$ sides. The metric also has a singularity of cone angle π at each puncture of S which is contained in a punctured disc component with two sides. Since there are precisely $i(\alpha, \beta)$ rectangles, the area of this singular euclidean metric on S equals $abi(\alpha, \beta)$. The line segments of this singular euclidean metric which are parallel to α and β define singular foliations $\mathcal{F}_\alpha, \mathcal{F}_\beta$ on S with transverse measures induced by the singular metric. The metric defines a complex structure on S and a quadratic differential $q(a\alpha, b\beta)$ which is holomorphic for this structure and whose horizontal and vertical foliations are just $\mathcal{F}_\alpha, \mathcal{F}_\beta$, with transverse measures determined by the weights a and b. The assignment which associates to $a > 0$ the Riemann surface structure determined by $q(a\alpha, \beta)$ is up to parametrization the geodesic in the Teichmüller space with respect to the *Teichmüller metric* whose cotangent bundle contains the differentials $q(a\alpha, \beta)$ (compare [12, 14]).

A *geodesic segment* for a quadratic differential q is a path in S not containing any singularities in its interior and which is a geodesic in the local euclidean structure defined by q. A closed geodesic is composed of a finite number of such geodesic segments which meet at singular points of q and make an angle at least π on either side. Every essential closed curve c on S is freely homotopic to a closed geodesic with respect to q, and the length of such a geodesic η is the infimum of the q-lengths of any curve freely homotopic to η (compare [26] for a detailed discussion of the

technical difficulties caused by the punctures of S) and will be called the *q-length* of our closed curve c. If $q = q(\lambda, \nu)$ for $\lambda, \nu \in \mathcal{ML}$ then this q-length is bounded from above by $2i(\lambda, c) + 2i(\mu, c)$ (see [26]).

Thurston invented a way to understand the structure of the space of geodesic laminations by squeezing almost parallel strands of such a lamination to a simple arc and analyzing the resulting graph. The structure of such a graph is as follows.

Definition 2.4. A *train track* on the surface S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. In particular, the branches which are incident on a fixed switch are divided into "incoming" and "outgoing" branches according to their inward pointing tangent at the switch. Each closed curve component of τ has a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with $0, 1$ or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. A train track is called *generic* if all switches are at most trivalent.

In the sequel we only consider generic train tracks. For such a train track τ, every complementary component is a bordered subsurface of S whose boundary consists of a finite number of arcs of class C^1 which come together at a finite number of *cusps*. Moreover, for every switch of τ there is precisely one complementary component containing the switch in its closure which has a cusp at the switch. A detailed account on train tracks can be found in [25] and [23].

A geodesic lamination or a train track λ is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is isotopic to the identity and maps λ to τ in such a way that the restriction of its differential dF to every tangent line of λ is non-singular. Note that this makes sense since a train track has a tangent line everywhere.

If c is a simple closed curve carried by τ with carrying map $F : c \to \tau$ then c defines a *counting measure* μ_c on τ. This counting measure is the non-negative weight function on the branches of τ which associates to an open branch b of τ the number of connected components of $F^{-1}(b)$. A counting measure is an example for a *transverse measure* on τ which is defined to be a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every switch s of τ, the sum of the weights over all incoming branches at s is required to coincide with the sum of the weights over all outgoing branches at s. The set $V(\tau)$ of all transverse measures on τ is a closed convex cone in a linear space and hence topologically it is a closed cell. More generally, every measured geodesic lamination λ on S which is carried by τ via a carrying map $F : \lambda \to \tau$ defines a transverse measure on τ by assigning to a branch b the total mass of the pre-image of b under F; the resulting weight function is independent of the particular choice of F. Moreover, every transverse measure for τ can be obtained in this way (see [25]).

Definition 2.5. A train track is called *recurrent* if it admits a transverse measure which is positive on every branch. A train track τ is called *transversely recurrent* if every branch b of τ is intersected by an embedded simple closed curve $c = c(b) \subset S$ which intersects τ transversely and is such that $S - \tau - c$ does not

contain an embedded *bigon*, i.e. a disc with two corners at the boundary. A recurrent and transversely recurrent train track is called *birecurrent*. A generic transversely recurrent train track which carries a complete geodesic lamination is called *complete*.

For every recurrent train track τ, measures which are positive on every branch define the interior of the convex cone $V(\tau)$ of all transverse measures. A complete train track is birecurrent [9].

A half-branch $\tilde{b}$ in a generic train track τ incident on a switch v is called *large* if the switch v is trivalent and if every arc $\rho : (-\epsilon, \epsilon) \to \tau$ of class C^1 which passes through v meets the interior of $\tilde{b}$. A branch b in τ is called *large* if each of its two half-branches is large; in this case b is necessarily incident on two distinct switches (for all this, see [25]).

There is a simple way to modify a transversely recurrent train track τ to another transversely recurrent train track. Namely, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure A below. The split τ' of a train track τ is carried by τ. If τ is complete and if the complete geodesic lamination λ is carried by τ, then for every large branch e of τ there is a unique choice of a right or left split of τ at e with the property that the split track τ' carries λ, and τ' is complete. In particular, a complete train track τ can always be split at any large branch e to a complete train track τ'; however there may be a choice of a right or left split at e such that the resulting train track is not complete any more (compare p.120 in [25]).

Figure A
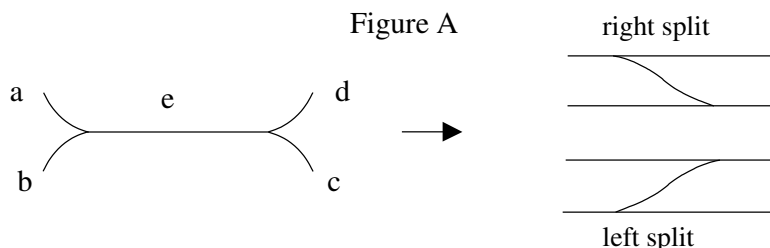
In the sequel we denote by $\mathcal{TT}$ the collection of all isotopy classes of complete train tracks on S. A sequence $(\tau_i) \subset \mathcal{TT}$ of complete train tracks is called a *splitting sequence* if τ_{i+1} can be obtained from τ_i by a single split at some large branch e.

3. Hyperbolicity of the complex of curves

In this section we present a proof of hyperbolicity of the curve graph using the main strategy of Masur and Minsky [19] and Bowditch [2] in a modified form. The first step consists in guessing a family of uniform *quasi-geodesics* in the curve graph connecting any two points. Here a p-quasi-geodesic for some $p > 1$ is a curve $c : [a, b] \to \mathcal{C}(S)$ which satisfies

$$d(c(s), c(t))/p - p \leq |s - t| \leq pd(c(s), c(t)) + p \quad \text{for all } s, t \in [a, b].$$

Note that a quasi-geodesic does not have to be continuous. In a hyperbolic geodesic metric space, every p-quasi-geodesic is contained in a fixed tubular neighborhood of any geodesic joining the same endpoints, so the δ-thin triangle condition also holds for triangles whose sides are uniform quasi-geodesics [3]. As a consequence,

for every triangle in a hyperbolic geodesic metric space with uniform quasi-geodesic
sides there is a "midpoint" whose distance to each side of the triangle is bounded
from above by a universal constant. The second step of the proof consists in finding
such a midpoint for triangles whose sides are curves of the distinguished curve
family. This is then used in a third step to establish the δ-thin triangle condition
for the distinguished family of curves and derive from this hyperbolicity of $\mathcal{C}(S)$.
By abuse of notation, in the sequel we simply write $\alpha \in \mathcal{C}(S)$ if α is a free homotopy
class of an essential simple closed curve on S, i.e. if α is a vertex of $\mathcal{C}(S)$.

We begin with defining a map from the set $\mathcal{TT}$ of complete train tracks on S
into $\mathcal{C}(S)$. For this we call a transverse measure μ for a complete train track τ a
vertex cycle [19] if μ spans an extreme ray in the convex cone $V(\tau)$ of all transverse
measures on τ.

Up to scaling, every vertex cycle μ is a counting measure of a simple closed curve
c which is carried by τ [19]. Namely, the switch conditions are a family of linear
equations with integer coefficients for the transverse measures on τ. Thus an ex-
treme ray is spanned by a nonnegative *rational* solution which can be scaled to
a nonnegative integral solution. From every integral transverse measure μ for τ
we can construct a unique *simple weighted multi-curve*, i.e. a simple multi-curve
together with a family of weights for each of its components, which is carried by
τ and whose counting measure coincides with μ as follows. For each branch b of
τ draw $\mu(b)$ disjoint arcs parallel to b. By the switch condition, the endpoints of
these arcs can be connected near the switches in a unique way so that the resulting
family of arcs does not have self-intersections. Let c be the simple multi-curve con-
sisting of the free homotopy classes of the connected components of the resulting
curve $\tilde{c}$. To each such homotopy class associate the number of components of $\tilde{c}$
in this class as a weight. The resulting simple weighted multi-curve is carried by
τ, and its counting measure equals μ. Thus if there are at least two components
of $\tilde{c}$ which are not freely homotopic then the weighted counting measures of these
components determine a decomposition of μ into transverse measures for τ which
are not multiples of μ. This is impossible if μ is a vertex cycle. Hence c consists of
a single component and up to scaling, μ is the counting measure of a simple closed
curve on S.

A simple closed curve which is carried by τ, with carrying map $F : c \to \tau$, defines
a vertex cycle for τ only if $F(c)$ passes through every branch of τ at most twice,
with different orientation (Lemma 2.2 of [7]). In particular, the counting measure
μ_c of a simple closed curve c which defines a vertex cycle for τ satisfies $\mu_c(b) \leq 2$
for every branch b of τ.

In the sequel we mean by a vertex cycle of a complete train track τ an *integral*
transverse measure on τ which is the counting measure of a simple closed curve
c on S carried by τ and which spans an extreme ray of $V(\tau)$; we also use the
notion vertex cycle for the simple closed curve c. Since the number of branches of
a complete train track on S only depends on the topological type of S, the number
of vertex cycles for a complete train track on S is bounded by a universal constant
(see [19] and [7]).

The following observation of Penner and Harer [25] is essential for all what follows.
Denote by $\mathcal{MC}(S)$ the space of all simple multi-curves on S. Let $P = \cup_{i=1}^{3g-3+m} \gamma_i \in$
$\mathcal{MC}(S)$ be a pants decomposition for S, i.e. a simple multi-curve with the maximal
number of components. Then there is a special family of complete train tracks with

the property that each pants curve γ_i admits a closed neighborhood A diffeomorphic to an annulus and such that $\tau \cap A$ is diffeomorphic to a *standard twist connector* depicted in Figure B. Such a train track clearly carries each pants curve from the

<div align="center">Figure B</div>
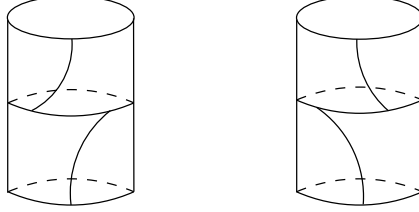
pants decomposition P as a vertex cycle; we call it *adapted* to P. For every complete geodesic lamination λ there is a train track τ adapted to P which carries λ ([25], see also [9],[7]).

Since every simple multi-curve is a subset of a pants decomposition of S, we can conclude.

Lemma 3.1 ([10]). *For every pair $(\alpha, \beta) \in \mathcal{MC}(S) \times \mathcal{MC}(S)$ there is a splitting sequence $(\tau_i)_{0 \leq i \leq m} \subset \mathcal{TT}$ of complete train tracks with the property that τ_0 is adapted to a pants decomposition $P_\alpha \supset \alpha$ and that each component of β is a vertex cycle for τ_m.*

We call a splitting sequence as in the lemma an $\alpha \to \beta$-splitting sequence. Note that such a sequence is by no means unique.

The distance in $\mathcal{C}(S)$ between two simple closed curves α, β is bounded from above by $i(\alpha, \beta) + 1$ (Lemma 1.1 of [2] and Lemma 2.1 of [19]). In particular, there is a number $D_0 > 0$ with the following property. Let $\tau, \tau' \in \mathcal{TT}$ and assume that τ' is obtained from τ by at most one split. Then the distance in $\mathcal{C}(S)$ between any vertex cycle of τ and any vertex cycle of τ' is at most D_0 (see [19] and the discussion following Corollary 2.3 in [7]).

Define a map $\Phi : \mathcal{TT} \to \mathcal{C}(S)$ by assigning to a train track $\tau \in \mathcal{TT}$ a vertex cycle $\Phi(\tau)$ for τ. By our above discussion, for any two choices Φ, Φ' of such a map we have $d(\Phi(\tau), \Phi'(\tau)) \leq D_0$ for all $\tau \in \mathcal{TT}$. Images under the map Φ of splitting sequences then define a family of curves in $\mathcal{C}(S)$ which connect any pair of points in a D_0-dense subset of $\mathcal{C}(S) \times \mathcal{C}(S)$, equipped with the product metric. As a consequence, we can use such images of splitting sequences as our guesses for uniform quasi-geodesics. It turns out that up to parametrization, these curves are indeed p-quasi-geodesics in $\mathcal{C}(S)$ for a universal number $p > 0$ only depending on the topological type of the surface S ([20], see also [7]).

To explain this fact we use the following construction of Bowditch [2]. For multi-curves $\alpha, \beta \in \mathcal{MC}(S)$ which jointly fill up S, i.e. which cut S into components which are homeomorphic to discs and once punctured discs, and for a number $a > 0$ let $q(a\alpha, \beta/ai(\alpha, \beta))$ be the area one quadratic differential whose horizontal foliation corresponds to the measured geodesic lamination $a\alpha$ and whose vertical measured foliation corresponds to the measured geodesic lamination $\beta/ai(\alpha, \beta)$. For $r > 0$ define

$$L_a(\alpha, \beta, r) = \{\gamma \in \mathcal{C}(S) \mid \max\{ai(\gamma, \alpha), i(\gamma, \beta)/ai(\alpha, \beta)\} \leq r\}.$$

Then $L_a(\alpha, \beta, r)$ is contained in the set of all all simple closed curves on S whose $q(a\alpha, \beta/ai(\alpha, \beta))$-length does not exceed $2r$. Note that we have $L_a(\alpha, \beta, r) = L_{1/ai(\alpha,\beta)}(\beta, \alpha, r)$ for all $r > 0$, moreover $\alpha' \in L_a(\alpha, \beta, r)$ for every component α' of α and every sufficiently large $a > 0$, and $\beta' \in L_a(\alpha, \beta, r)$ for every component β' of β and every sufficiently small $a > 0$. Thus for fixed $r > 0$ we can think of a suitably chosen assignment which associates to a number $s > 0$ a point in $L_s(\alpha, \beta, r)$ as a curve in $\mathcal{C}(S)$ connecting a component of β to a component of α (provided, of course, that the sets $L_s(\alpha, \beta, r)$ are non-empty). Lemma 2.5 of [7] links such curves to splitting sequences.

Lemma 3.2 ([7]). *There is a number $k_0 \geq 1$ with the following property. Let P be a pants decomposition of S, let $\alpha \in \mathcal{MC}(S)$ be such that α and P jointly fill up S and let $(\tau_i)_{0 \leq i \leq m} \subset \mathcal{TT}$ be a $P \to \alpha$-splitting sequence. Then there is a non-decreasing surjective function $\kappa : (0, \infty) \to \{0, \ldots, m\}$ such that $\kappa(s) = 0$ for all sufficiently small $s > 0, \kappa(s) = m$ for all sufficiently large $s > 0$ and that for all $s \in (0, \infty)$ there is a vertex cycle of $\tau_{\kappa(s)}$ which is contained in $L_s(\alpha, P, k_0)$.*

Since for every multi-curve $\alpha \in \mathcal{MC}(S)$ and every pants decomposition P of S there is a $P \to \alpha$-splitting sequence, we conclude that for every $k \geq k_0$ and every $s > 0$ the set $L_s(\alpha, P, k)$ is non-empty. To obtain a control of the size of these sets, Bowditch [2] uses the following observation (Lemma 4.1 in [2]) whose first part was earlier shown by Masur and Minsky (Lemma 5.1 of [19]).

Lemma 3.3 ([2]). *There is a number $k_1 \geq k_0$ with the following property. For all $\alpha, \beta \in \mathcal{MC}(S)$ which jointly full up S and every $a \in (0, \infty)$ there is some $\delta \in L_a(\alpha, \beta, k_1)$ such that for every $\gamma \in \mathcal{MC}(S)$ we have*

$$i(\delta, \gamma) \leq k_1 \max\{ai(\alpha, \gamma), i(\gamma, \beta)/ai(\alpha, \beta)\}.$$

In particular, for every $R > 0$, for all $\alpha, \beta \in \mathcal{MC}(S)$ and for every $a > 0$ the diameter of the set $L_a(\alpha, \beta, R)$ is not bigger than $2k_1R + 1$.

Proof. In [19, 2] it is shown that there is a number $\nu > 0$ only depending on the topological type of S and there is an embedded essential annulus in S whose width with respect to the piecewise euclidean metric defined by the quadratic differential $q(a\alpha, \beta/ai(\alpha, \beta))$ is at least ν. This means that the distance between the boundary circles of the annulus is at least ν. Assuming the existence of such an annulus, let δ be its core-curve. Then for every simple closed curve γ on S and for every essential intersection of γ with δ there is a subarc of γ which crosses through this annulus and hence whose length is at least ν; moreover, different subarcs of γ corresponding to different essential intersections between γ and δ are disjoint. Thus the length with respect to the singular euclidean metric on S of any simple closed curve γ on S is at least $\nu i(\gamma, \delta)$. On the other hand, by construction the minimal length with respect to this metric of a curve in the free homotopy class of γ is bounded from above by $2 \max\{ai(\alpha, \gamma), i(\beta, \gamma)/ai(\alpha, \beta)\}$ and therefore the core curve δ of the annulus has the properties stated in the first part of our lemma (see [2]).

The second part of the lemma is immediate from the first. Namely, let $\alpha, \beta \in \mathcal{MC}(S)$ and let $a > 0$. Choose $\delta \in L_a(\alpha, \beta, k_1)$ which satisfies the properties stated in the first part of the lemma. If $\gamma \in L_a(\alpha, \beta, R)$ for some $R > 0$ then we have $i(\gamma, \delta) \leq k_1R$ and hence $d(\gamma, \delta) \leq k_1R + 1$. $\qquad\square$

As an immediate consequence of Lemma 3.2 and Lemma 3.3 we observe that there is a universal number $D_1 > 0$ with the following property. Let P be a pants decomposition for S, let $\beta \in \mathcal{MC}(S)$ and let $(\tau_i)_{0 \leq i \leq m} \subset \mathcal{TT}$ be any $P \to \beta$-splitting sequence; then the Hausdorff distance in $\mathcal{C}(S)$ between the sets $\{\Phi(\tau_i) \mid 0 \leq i \leq m\}$ and $\cup_{a>0} L_a(\beta, P, k_1)$ is at most $D_1/16$. If $c > 0$ and if $j \leq m$ is such that there is a vertex cycle γ for $\tau(j)$ which is contained in $L_c(\beta, P, k_1)$ then the splitting sequence $(\tau_i)_{0 \leq i \leq j}$ is a $P \to \gamma$-splitting sequence and hence the Hausdorff distance between $\cup_{a>0} L_a(\gamma, P, k_1)$ and $\cup_{a \geq c} L_a(\beta, P, k_1)$ is at most $D_1/8$. Moreover, for every $\beta \in \mathcal{C}(S)$ and every simple multi-curve Q containing β as a component the Hausdorff-distance between the sets $\cup_{a>0} L_a(\beta, P, k_1)$ and $\cup_{a>0} L_a(Q, P, k_1)$ is at most $D_1/8$. Thus if Q, Q' are pants decompositions for S containing a common curve $\beta \in \mathcal{C}(S)$ then the Hausdorff distance between $\cup_a L_a(Q, P, k_1)$ and $\cup_a L_a(Q', P, k_1)$ is at most $D_1/4$.

On the other hand, for multi-curves $P, Q \in \mathcal{MC}(S)$ we have

$$\cup_{a>0} L_a(P, Q, k_1) = \cup_{a>0} L_a(Q, P, k_1).$$

Therefore from two applications of our above consideration we obtain the following. Let $\alpha, \beta \in \mathcal{C}(S)$ and let P, P', Q, Q' be any pants decompositions for S containing α, β; then the Hausdorff distance between $\cup_{a>0} L_a(P, Q, k_1)$ and $\cup_{a>0} L_a(P', Q', k_1)$ is not bigger than $D_1/2$. By our choice of D_1 this implies that the Hausdorff distance between the images under Φ of *any* $\alpha \to \beta$- or $\beta \to \alpha$-splitting sequences is bounded from above by D_1.

Now let $\alpha, \beta, \gamma \in \mathcal{C}(S)$ be such that their pairwise distance in $\mathcal{C}(S)$ is at least 3; then any two of these curves jointly fill up S. Choose pants decompositions $P_\alpha, P_\beta, P_\gamma$ containing α, β, γ. Then there are unique numbers $a, b, c > 0$ such that $abi(P_\alpha, P_\beta) = bci(P_\beta, P_\gamma) = aci(P_\gamma, P_\alpha) = 1$. By construction, we have

$$L_a(P_\alpha, P_\beta, k_1) = L_b(P_\beta, P_\alpha, k_1), \quad L_b(P_\beta, P_\gamma, k_1) = L_c(P_\gamma, P_\beta, k_1)$$

and $L_c(P_\gamma, P_\alpha, k_1) = L_a(P_\alpha, P_\gamma, k_1)$. Choose a point $\delta \in L_a(P_\alpha, P_\beta, k_1)$ such that for every ζ in $\mathcal{MC}(S)$ we have

$$i(\delta, \zeta) \leq k_1 \max\{ai(P_\alpha, \zeta), i(\zeta, P_\beta)/ai(P_\alpha, P_\beta)\};$$

such a point exists by Lemma 3.3. Applying this inequality to $\zeta = P_\gamma$ yields $ci(\delta, P_\gamma) \leq k_1$. For $\zeta = P_\beta$ we obtain

$$i(\delta, P_\beta)/ci(P_\gamma, P_\beta) \leq ai(P_\alpha, P_\beta)/ci(P_\gamma, P_\beta) = 1,$$

and for $\zeta = P_\alpha$ we obtain

$$i(\delta, P_\alpha)/ci(P_\gamma, P_\alpha) \leq 1/aci(P_\gamma, P_\alpha) = 1.$$

Therefore we have $\delta \in L_c(P_\gamma, P_\beta, k_1) \cap L_a(P_\alpha, P_\beta, k_1) \cap L_c(P_\gamma, P_\alpha, k_1)$. Together with Lemma 3.2 and our above remark we conclude that there is a universal constant $D_2 > 0$ such that the distance between $\phi(\alpha, \beta, \gamma) = \delta$ and the image under Φ of any $\alpha \to \beta$-splitting sequence, any $\alpha \to \gamma$-splitting sequence and any $\gamma \to \beta$-splitting sequence is bounded from above by D_2.

We use the map ϕ to derive the δ-thin triangle condition for triangles whose sides are images under the map Φ of splitting sequences in $\mathcal{TT}$.

Lemma 3.4. *There is a number $D_3 > 0$ with the following property. Let $\alpha, \beta, \gamma \in \mathcal{C}(S)$ and let a, b, c be the image under Φ of a $\beta \to \gamma$, $\gamma \to \alpha$, $\alpha \to \beta$-splitting sequence. Then the D_3-neighborhood of $a \cup b$ contains c.*

Proof. Let $\alpha, \beta, \gamma \in \mathcal{C}(S)$ and assume that $d(\beta, \gamma) \leq p$ for some $p > 0$. Let $(\tau_i)_{0 \leq i \leq m}$ be an $\alpha \to \beta$-splitting sequence and let $(\eta_j)_{0 \leq j \leq \ell}$ be an $\alpha \to \gamma$-splitting sequence; if $D_1 > 0$ is as above then the Hausdorff distance between $\{\Phi(\tau_i) \mid 0 \leq i \leq m\}$ and $\{\Phi(\eta_j) \mid 0 \leq j \leq \ell\}$ is at most $2pD_1$. Namely, we observed that the Hausdorff distance between the image under Φ of any two $\alpha \to \beta$-splitting sequences is bounded from above by D_1. Moreover, if $d(\beta, \gamma) = 1$ then $\beta \cup \gamma \in \mathcal{MC}(S)$ and hence there is an $\alpha \to \beta$-splitting sequence which also is an $\alpha \to \gamma$-splitting sequence. Thus the statement of the corollary holds for $p = 1$, and the general case follows from a successive application of this fact for the points on a geodesic in $\mathcal{C}(S)$ connecting β to γ.

Now let $\alpha, \beta, \gamma \in \mathcal{C}(S)$ be arbitrary points whose pairwise distance is at least 3. Let again $(\tau_i)_{0 \leq i \leq m}$ be an $\alpha \to \beta$-splitting sequence. By the definition of ϕ and the choice of the constant $D_2 > 0$ above there is some $i_0 \leq m$ such that the distance betweeen $\Phi(\tau_{i_0})$ and $\phi(\alpha, \beta, \gamma)$ is at most D_2. Let $(\eta_j)_{0 \leq j \leq \ell}$ be any $\alpha \to \phi(\alpha, \beta, \gamma)$-splitting sequence. By our above consideration, the Hausdorff distance between $\{\Phi(\tau_i) \mid 0 \leq i \leq i_0\}$ and $\{\Phi(\eta_j) \mid 0 \leq j \leq \ell\}$ is at most $2D_1 D_2$. Similarly, let $(\zeta_j)_{0 \leq j \leq n}$ be an $\alpha \to \gamma$-splitting sequence. Then there is some $j_0 > 0$ such that $d(\Phi(\zeta_{j_0}), \phi(\alpha, \beta, \gamma)) \leq D_2$. By our above argument, the Hausdorff distance between the sets $\{\Phi(\tau_i) \mid 0 \leq i \leq i_0\}$ and $\{\Phi(\zeta_j) \mid 0 \leq j \leq j_0\}$ is at most $4D_1 D_2$.

As a consequence, there are numbers $a(\alpha, \beta) > 0, a(\alpha, \gamma) > 0$ such that

$$\phi(\alpha, \beta, \gamma) \in L_{a(\alpha, \beta)}(\beta, \alpha, k_1) \cap L_{a(\alpha, \gamma)}(\gamma, \alpha, k_1)$$

and that the Hausdorff distance between $\cup_{a \geq a(\alpha, \beta)} L_a(\beta, \alpha, k_1)$ and $\cup_{a \geq a(\alpha, \gamma)} L_a(\gamma, \alpha, k_1)$ is at most $6D_1 D_2$. The same argument, applied to a $\beta \to \alpha$-splitting sequence and a $\beta \to \gamma$-splitting sequence, shows that $\cup_{a \leq a(\alpha, \beta)} L_a(\alpha, \beta, k_1)$ is contained in the $6D_1 D_2$-neighborhood of $\cup_a L_a(\beta, \gamma, k_1)$. Then $\{\Phi(\tau_i) \mid 0 \leq i \leq m\}$ is contained in the $12D_1 D_2$-neighborhood of the union of the image under Φ of a $\gamma \to \alpha$-splitting sequence and a $\beta \to \gamma$-splitting sequence. This shows the lemma. $\square$

Hyperbolicity of the curve graph now follows from Lemma 3.4 and the following criterion.

Proposition 3.5. *Let (X, d) be a geodesic metric space. Assume that there is a number $D > 0$ and for every pair of points $x, y \in X$ there is an arc $\eta(x, y) : [0, 1] \to X$ connecting $\eta(x, y)(0) = x$ to $\eta(x, y)(1) = y$ so that the following conditions are satisfied.*

(1) *If $d(x, y) \leq 1$ then the diameter of $\eta(x, y)[0, 1]$ is at most D.*
(2) *For $x, y \in X$ and $0 \leq s \leq t \leq 1$, the Hausdorff distance between $\eta(x, y)[s, t]$ and $\eta(\eta(x, y)(s), \eta(x, y)(t))[0, 1]$ is at most D.*
(3) *For any $x, y, z \in X$ the set $\eta(x, y)[0, 1]$ is contained in the D-neighborhood of $\eta(x, z)[0, 1] \cup \eta(z, y)[0, 1]$.*

Then (X, d) is δ-hyperbolic for a number $\delta > 0$ only depending on D.

Proof. Let (X, d) be a geodesic metric space. Assume that there is a number $D > 0$ and there is a family of paths $\eta(x, y) : [0, 1] \to X$, one for every pair of points $x, y \in X$, which satisfy the hypotheses in the statement of the proposition. To show hyperbolicity for X it is then enough to show the existence of a constant $\kappa > 0$ such that for all $x, y \in X$ and every geodesic $\nu : [0, \ell] \to X$ connecting x to y, the Hausdorff-distance between $\nu[0, \ell]$ and $\eta(x, y)[0, 1]$ is at most κ. Namely, if this

is the case then for every geodesic triangle with sides a, b, c the side a is contained in the $3\kappa + D$-neighborhood of $b \cup c$.

To show the existence of such a constant $\kappa > 0$, let $x, y \in X$ and let $c : [0, 2^k] \to X$ be *any* path of length $\ell(c) = 2^k$ parametrized by arc length connecting x to y. Write $\eta_1 = \eta(c(0), c(2^{k-1}))$ and write $\eta_2 = \eta(c(2^{k-1}), c(2^k))$. By our assumption, the D-neighborhood of $\eta_1 \cup \eta_2$ contains $\eta(c(0), c(2^k))$. Repeat this construction with the points $c(2^{k-2}), c(3 \cdot 2^{k-2})$ and the arcs η_1, η_2. Inductively we conclude that the path $\eta(c(0), c(2^k))$ is contained in the $(\log_2 \ell(c))D$-neighborhood of a path $\tilde{c} : [0, 2^k] \to \mathcal{C}(S)$ whose restriction to each interval $[m - 1, m]$ $(m \leq 2^k)$ equals up to parametrization the arc $\eta(c(m - 1), c(m))$. Since $d(c(m - 1), c(m)) \leq 1$, by assumption the diameter of each of the sets $\eta(c(m-1), c(m))[0, 1]$ is bounded from above by D and therefore the arc $\eta(c(0), c(2^k))$ is contained in the $(\log_2 \ell(c))D + D$-neighborhood of $c[0, 2^k]$.

Now let $c : [0, k] \to X$ be a geodesic connecting $c(0) = x$ to $c(k) = y$ which is parametrized by arc length. Let $t > 0$ be such that $\eta(x, y)(t)$ has maximal distance to $c[0, k]$, say that this distance equals χ. Choose some $s > 0$ such that $d(c(s), \eta(x, y)(t)) = \chi$ and let $t_1 < t < t_2$ be such that $d(\eta(x, y)(t), \eta(x, y)(t_u)) = 2\chi$ $(u = 1, 2)$. In the case that there is no $t_1 \in [0, t)$ (or $t_2 \in (t, 1]$) with $d(\eta(x, y)(t), \eta(x, y)(t_1)) \geq 2\chi$ (or $d(\eta(x, y)(t), \eta(x, y)(t_2)) \geq 2\chi$) we choose $t_1 = 0$ (or $t_2 = 1$). By our choice of χ, there are numbers $s_u \in [0, k]$ such that $d(c(s_u), \eta(x, y)(t_u)) \leq \chi$ $(u = 1, 2)$. Then the distance between $c(s_1)$ and $c(s_2)$ is at most 6χ. Compose the subarc $c[s_1, s_2]$ of c with a geodesic connecting $\eta(x, y)(t_1)$ to $c(s_1)$ and a geodesic connecting $c(s_2)$ to $\eta(x, y)(t_2)$. We obtain a curve ν of length at most 8χ. By our above observation, the $(\log_2(8\chi))D + D$-neighborhood of this curve contains the arc $\eta(\eta(x, y)(t_1), \eta(x, y)(t_2))$. However, the Hausdorff distance between $\eta(x, y)[t_1, t_2]$ and $\eta(\eta(x, y)(t_1), \eta(x, t)(t_2))$ is at most D and therefore the $(\log_2(8\chi))D + 2D$-neighborhood of the arc ν contains $\eta(x, y)[t_1, t_2]$. But the distance between $\eta(t)$ and our curve ν equals χ by construction and hence we have $\chi \leq (\log_2(8\chi))D + 2D$. In other words, χ is bounded from above by a universal constant $\kappa_1 > 0$, and $\eta(x, y)$ is contained in the κ_1-neighborhood of the geodesic c.

A similar argument also shows that the $3\kappa_1$-neighborhood of $\eta(x, y)$ contains $c[0, k]$. Namely, by the above consideration, for every $t \leq 1$ the set $A(t) = \{s \in [0, k] \mid d(c(s), \eta(x, y)(t)) \leq \kappa_1\}$ is a non-empty closed subset of $[0, k]$. The diameter of the sets $A(t)$ is bounded from above by $2\kappa_1$. Assume to the contrary that $c[0, k]$ is not contained in the $3\kappa_1$-neighborhood of $\eta(x, y)$. Then there is a subinterval $[a_1, a_2] \subset [0, k]$ of length $a_2 - a_1 \geq 4\kappa_1$ such that $d(c(s), \eta(x, y)[0, 1]) > \kappa_1$ for every $s \in (a_1, a_2)$. Since $d(c(s), c(t)) = |s - t|$ for all s, t we conclude that for every $t \in [0, 1]$ the set $A(t)$ either is entirely contained in $[0, a_1]$ or it is entirely contained in $[a_2, k]$. Define $C_1 = \{t \in [0, 1] \mid A(t) \subset [0, a_1]\}$ and $C_2 = \{t \in [0, 1] \mid A(t) \subset [a_2, 1]\}$. Then the sets C_1, C_2 are disjoint and their union equals $[0, 1]$; moreover, we have $0 \in C_1$ and $1 \in C_2$. On the other hand, the sets C_i are closed. Namely, let $(t_i) \subset C_1$ be a sequence converging to some $t \in [0, 1]$. Let $s_i \in A(t_i)$ and assume after passing to a subsequence that $s_i \to s \in [0, a_1]$. Now $\kappa_1 \geq d(c(s_i), \eta(x, y)(t_i)) \to d(c(s), \eta(x, y)(t))$ and therefore $s \in A(t)$ and hence $t \in C_1$. However, $[0, 1]$ is connected and hence we arrive at a contradiction. In other words, the geodesic c is contained in the $3\kappa_1$-neighborhood of $\eta(x, y)$. This completes the proof of the proposition. $\square$

As an immediate corollary, we obtain.

Theorem 3.6 ([19, 2]). *The curve graph is hyperbolic.*

Proof. Let $\Phi : \mathcal{T}T \to \mathcal{C}(S)$ be as before. For $\alpha, \beta \in \mathcal{C}(S)$ choose an $\alpha \to \beta$-splitting sequence $(\tau_i)_{0 \leq i \leq m}$. Define an arc $\eta(\alpha, \beta) : [0, 1] \to \mathcal{C}(S)$ by requiring that for $1 \leq i \leq m$ the restriction of $\eta(\alpha, \beta)$ to the interval $[\frac{i}{m+2}, \frac{i+1}{m+2}]$ is a geodesic connecting $\Phi(\tau_{i-1})$ to $\Phi(\tau_i)$ and that the restriction of $\eta(\alpha, \beta)$ to $[0, \frac{1}{m+2}]$ (or $[\frac{m-1}{m-2}, 1]$) is a geodesic connecting α to $\Phi(\tau_0)$ (or $\Phi(\tau_m)$ to β).

We claim that this family of arcs satisfy the assumptions in Proposition 3.5. Namely, we observed before that for all $\alpha, \beta \in \mathcal{C}(S)$, the Hausdorff distance between the image under Φ of *any* two $\alpha \to \beta$-splitting sequences is bounded from above by a universal constant. Now if $(\tau_i)_{0 \leq i \leq m}$ is any splitting sequence, then for all $0 \leq k \leq \ell \leq m$ the sequence $(\tau_i)_{k \leq i \leq \ell}$ is a $\Phi(\tau_k) \to \Phi(\tau_\ell)$-splitting sequence and hence our curve system satisfies the second condition in Proposition 3.5.

Moreover, curves $\alpha, \beta \in \mathcal{C}(S)$ with $d(\alpha, \beta) = 1$ can be realized disjointly, and $\alpha \cup \beta$ is a multi-curve. For such a pair of curves we can choose a *constant* $\alpha \to \beta$-splitting sequence; hence our curve system also satisfies the first condition stated in Proposition 3.5. Finally, the third condition was shown to hold in Lemma 3.4. Now hyperbolicity of the curve graph follows from Proposition 3.5. $\qquad\square$

A curve $c : [0, m] \to \mathcal{C}(S)$ is called an *unparametrized p-quasi-geodesic* for some $p > 1$ if there is a homeomorphism $\rho : [0, u] \to [0, m]$ for some $u > 0$ such that

$$d(c(\rho(s)), c(\rho(t)))/p - p \leq |s - t| \leq pd(c(\rho(s)), c(\rho(t))) + p$$

for all $s, t \in [0, u]$. We define a map $c : \{0, \ldots, m\} \to \mathcal{C}(S)$ to be an unparametrized q-quasi-geodesic if this is the case for the curve $\tilde{c}$ whose restriction to each interval $[i, i+1)$ coincides with $c(i)$. The following observation is immediate from Proposition 3.5 and its proof.

Corollary 3.7 ([20, 7]). *There is a number $p > 0$ such that the image under Φ of an arbitrary splitting sequence is an unparametrized p-quasi-geodesic.*

Proof. By Proposition 3.5, Theorem 3.6 and their proofs, there is a universal number $D > 0$ with the property that for every splitting sequence $(\tau_i)_{0 \leq i \leq m}$ and every geodesic $c : [0, s] \to \mathcal{C}(S)$ connecting $c(0) = \Phi(\tau_0)$ to $c(s) = \Phi(\tau_m)$, the Hausdorff distance between the sets $\{\Phi(\tau_i) \mid 0 \leq i \leq m\}$ and $c[0, s]$ is at most D. From this the corollary is immediate. $\qquad\square$

4. Geometry of Teichmüller space

In this section, we relate the geometry of the curve graph to the geometry of Teichmüller space equipped with the Teichmüller metric. For this we first define a map $\Psi : \mathcal{T}_{g,m} \to \mathcal{C}(S)$ as follows. By a well-known result of Bers (see [4]) there is a number $\chi > 0$ only depending on the topological type of S such that for every complete hyperbolic metric on S of finite volume there is a pants decomposition P for S which consists of simple closed geodesics of length at most χ. Since the distance between any two points $\alpha, \beta \in \mathcal{C}(S)$ is bounded from above by $i(\alpha, \beta) + 1$, the collar lemma for hyperbolic surfaces (see [4]) implies that the diameter in $\mathcal{C}(S)$ of the set of simple closed curves whose length with respect to the fixed metric is at most χ is bounded from above by a universal constant $D > 0$. Define $\Psi : \mathcal{T}_{g,m} \to \mathcal{C}(S)$ by assigning to a finite volume hyperbolic metric h on S a simple closed curve $\Psi(h)$ whose h-length is at most χ. Then for any two maps Ψ, Ψ' with

this property and every $h \in \mathcal{T}_{g,m}$ the distance in $\mathcal{C}(S)$ between $\Psi(h)$ and $\Psi'(h)$ is at most D. Moreover, the map Ψ is coarsely equivariant with respect to the action of the mapping class group $\mathcal{M}_{g,m}$ on $\mathcal{T}_{g,m}$ and $\mathcal{C}(S)$: For every $h \in \mathcal{T}_{g,m}$ and every $\phi \in \mathcal{M}_{g,m}$ we have $d(\Psi(\phi(h)), \phi(\Psi(h))) \leq D$.

The following result is due to Masur and Minsky (Theorem 2.6 and Theorem 2.3 of [19]). For its formulation, let d_T be the distance function on $\mathcal{T}_{g,m}$ induced by the Teichmüller metric.

Theorem 4.1 ([19]).   (1) *There is a number $a > 0$ such that $d(\Psi h, \Psi h') \leq a d_T(h, h') + a$ for all $h, h' \in \mathcal{T}_{g,m}$.*
  (2) *There is a number $\tilde{p} > 0$ with the following property. Let $\gamma : (-\infty, \infty) \to \mathcal{T}_{g,m}$ be any Teichmüller geodesic; then the assignment $t \to \Psi(\gamma(t))$ is an unparametrized $\tilde{p}$-quasi-geodesic in $\mathcal{C}(S)$.*

Proof. Let $\gamma : (-\infty, \infty) \to \mathcal{T}_{g,m}$ be any Teichmüller geodesic parametrized by arc length. Then the cotangent of γ at $t = 0$ is a quadratic differential q of area one defined by a pair $(\lambda, \mu) \in \mathcal{ML} \times \mathcal{ML}$ of measured geodesic laminations which jointly fill up S. The cotangent of γ at t is given by the quadratic differential $q(t)$ defined by the pair $(e^t \lambda, e^{-t} \mu)$. For $k_1 > 0$ as in Lemma 3.3 and $t \in \mathbb{R}$ let $\zeta(t) \in \mathcal{C}(S)$ be a curve whose $q(t)$-length is at most $2k_1$. For every $\beta \in [0, 1]$ the $q(t+\beta)$-length of $\zeta(t)$ is bounded from above by $2ek_1$ and therefore by Lemma 3.3, for every t the distance in $\mathcal{C}(S)$ between $\zeta(t)$ and $\zeta(t + \beta)$ is bounded from above by a universal constant $k_2 > 0$. In particular, the assignment $t \to \zeta(t)$ satisfies $d(\zeta(s), \zeta(t)) \leq k_2|s - t| + k_2$. Hence for the proof of our lemma, we only have to show that there is a constant $k_3 > 0$ such that for every $h \in \mathcal{T}_{g,m}$ and every holomorphic quadratic differential q of area one for h, the distance between $\Psi(h)$ and a curve on S whose q-length is bounded from above by $2k_1$ is uniformly bounded.

Thus let h be a complete hyperbolic metric of finite volume and let q be a holomorphic quadratic differential for h of area one. By the collar lemma of hyperbolic geometry, a simple closed geodesic c for h whose length is bounded from above by χ is the core curve of an embedded annulus A whose *modulus* is bounded from below by a universal constant $\epsilon > 0$; we refer to [27] for a definition of the modulus of an annulus and its properties. Then the *extremal length* of the core curve of A is bounded from above by a universal constant $m > 0$. Now the area of q equals one and therefore the q-length of the core curve c does not exceed $\sqrt{m}$ by the definition of extremal length (see e.g. [22]). In other words, the q-length of the curve $\Psi(h)$ is uniformly bounded which together with Lemma 3.3 implies our claim. The theorem follows. $\square$

There are also Teichmüller geodesics in Teichmüller space which are mapped by Ψ to *parametrized* quasi-geodesics in $\mathcal{C}(S)$. For their characterization, denote for $\epsilon > 0$ by $\mathcal{T}_{g,m}^\epsilon$ the subset of Teichmüller space consisting of all marked hyperbolic metrics for which the length of the shortest closed geodesic is at least ϵ. The set $\mathcal{T}_{g,m}^\epsilon$ is invariant under the action of the mapping class group and projects to a *compact* subset of moduli space. Moreover, every compact subset of moduli space is contained in the projection of $\mathcal{T}_{g,m}^\epsilon$ for some $\epsilon > 0$.

Cobounded Teichmüller geodesics. i.e. Teichmüller geodesic which project to a compact subset of moduli space, relate the geometry of Teichmüller space to the geometry of the curve graph. We have.

Proposition 4.2 ([8])**.** *The image under Ψ of a Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{T}_{g,m}$ is a parametrized quasi-geodesic in $\mathcal{C}(S)$ if and only if there is some $\epsilon > 0$ such that $\gamma(\mathbb{R}) \subset \mathcal{T}_{g,m}^{\epsilon}$.*

Minsky [21] discovered earlier that the Teichmüller metric near a cobounded geodesic line has properties similar to properties of a hyperbolic geodesic metric space. Namely, for a Teichmüller geodesic $\gamma : \mathbb{R} \to \mathcal{T}_{g,m}^{\epsilon}$ the map which associates to a point $h \in \mathcal{T}_{g,m}$ a point on $\gamma(\mathbb{R})$ which minimizes the Teichmüller distance is coarsely Lipschitz and contracts distances in a way which is similar to the contraction property of the closest point projection from a δ-hyperbolic geodesic metric space to any of its bi-infinite geodesics.

A hyperbolic geodesic metric space X admits a *Gromov boundary* which is defined as follows. Fix a point $p \in X$ and for two points $x, y \in X$ define the *Gromov product* $(x, y)_p = \frac{1}{2}(d(x, p) + d(y, p) - d(x, y))$. Call a sequence $(x_i) \subset X$ *admissible* if $(x_i, x_j)_p \to \infty$ $(i, j \to \infty)$. We define two admissible sequences $(x_i), (y_i) \subset X$ to be *equivalent* if $(x_i, y_i)_p \to \infty$. Since X is hyperbolic, this defines indeed an equivalence relation (see [3]). The Gromov boundary ∂X of X is the set of equivalence classes of admissible sequences $(x_i) \subset X$. It carries a natural Hausdorff topology. For the curve graph, the Gromov boundary was determined by Klarreich [15] (see also [7]). For the formulation of Klarreich's result, we say that a minimal geodesic lamination λ *fills up* S if every simple closed geodesic on S intersects λ transversely, i.e. if every complementary component of λ is an ideal polygon or a once punctured ideal polygon with geodesic boundary [5]. For any minimal geodesic lamination λ which fills up S, the number of geodesic laminations μ which contain λ as a sublamination is bounded by a universal constant only depending on the topological type of the surface S. Namely, each such lamination μ can be obtained from λ by successively subdividing complementary components P of λ which are different from an ideal triangle or a once punctured monogon by adding a simple geodesic line which either connects two non-adjacent cusps or goes around a puncture. Note that every leaf of μ which is not contained in λ is necessarily isolated in μ.

Recall that the space $\mathcal{L}$ of geodesic laminations on S equipped with the restriction of the Hausdorff topology for compact subsets of S is compact and metrizable. It contains the set $\mathcal{B}$ of minimal geodesic laminations which fill up S as a subset which is neither closed nor dense. We define on $\mathcal{B}$ a new topology which is coarser than the restriction of the Hausdorff topology as follows. Say that a sequence $(\lambda_i) \subset \mathcal{L}$ *converges in the coarse Hausdorff topology* to a minimal geodesic lamination μ which fills up S if every accumulation point of (λ_i) with respect to the Hausdorff topology contains μ as a sublamination. Define a subset A of $\mathcal{B}$ to be closed if and only if for every sequence $(\lambda_i) \subset A$ which converges in the coarse Hausdorff topology to a lamination $\lambda \in \mathcal{B}$ we have $\lambda \in A$. We call the resulting topology on $\mathcal{B}$ the *coarse Hausdorff topology*. The space $\mathcal{B}$ is not locally compact. Using this terminology, Klarreich's result [15] can be formulated as follows.

Theorem 4.3 ([15, 7])**.**   (1) *There is a natural homeomorphism Λ of $\mathcal{B}$ equipped with the coarse Hausdorff topology onto the Gromov boundary $\partial\mathcal{C}(S)$ of the complex of curves $\mathcal{C}(S)$ for S.*
  (2) *For $\mu \in \mathcal{B}$ a sequence $(c_i) \subset \mathcal{C}(S)$ is admissible and defines the point $\Lambda(\mu) \in \partial\mathcal{C}(S)$ if and only if (c_i) converges in the coarse Hausdorff topology to μ.*

Recall that every Teichmüller geodesic in $\mathcal{T}_{g,m}$ is uniquely determined by a pair of projective measured laminations which jointly fill up S. The following corollary is immediate from Theorem 4.1 and Theorem 4.3 with $\tilde{p} > 0$ as in Theorem 4.1.

Corollary 4.4. *Let $\lambda, \mu \in \mathcal{PML}$ be such that their supports are minimal and fill up S. Then the image under Ψ of the unique Teichmüller geodesic in $\mathcal{T}_{g,m}$ determined by λ and μ is a biinfinite unparametrized $\tilde{p}$-quasigeodesic in $\mathcal{C}(S)$, and every biinfinite unparametrized $\tilde{p}$-quasi-geodesic in $\mathcal{C}(S)$ is contained in a uniformly bounded neighborhood of a curve of this form.*

Our above discussion also gives information on images under Ψ of a convergent sequence of geodesic lines in Teichmüller space. Namely, if (γ_i) is such a sequence of Teichmüller geodesic lines converging to a Teichmüller geodesic which is determined by a pair of projective measured geodesic laminations (α, β) so that the support of α does not fill up S then there is a curve $\zeta \in \mathcal{C}(S)$, a number $m > 0$ and a sequence $j(i) \to \infty$ such that $d(\Psi(\gamma_i[0, j(i)]), \zeta) \leq m$.

On the other hand, the image under Ψ of "most" Teichmüller geodesics are unparametrized quasi-geodesics of infinite diameter which are not *parametrized* quasi-geodesics. For example, let $\lambda \in \mathcal{PML}$ be a projective measured geodesic lamination whose support λ_0 is minimal and fills up S but is not *uniquely ergodic*. This means that the dimension of the space of transverse measures supported in λ_0 is at least 2. Let $\gamma : [0, \infty) \to \mathcal{T}_{g,m}$ be a Teichmüller geodesic ray determined by a quadratic differential whose horizontal foliation corresponds to λ. By a result of Masur [17], the projection of γ to moduli space eventually leaves every compact set. On the other hand, since λ_0 is minimal and fills up S the points $\gamma(t)$ converge as $t \to \infty$ to λ viewed as a point in the *Thurston boundary* of the Thurston compactification of Teichmüller space [18] (compare also [6] for the construction of the Thurston compactification). By the definition of the map Ψ, the projective measured geodesic laminations defined by the curves $\Psi\gamma(t)$ converge as $t \to \infty$ to λ and therefore the curves $\Psi\gamma(t)$ converge in the coarse Hausdorff topology to λ_0. By Theorem 4.3, this implies that the diameter of $\Psi\gamma[0, \infty)$ is infinite.

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