ISOMETRY GROUPS OF PROPER HYPERBOLIC SPACES

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ABSTRACT. Let X be a proper hyperbolic geodesic metric space and let G be a closed subgroup of the isometry group $\mathrm{Iso}(X)$ of X. We show that if G is not elementary then for every $p \in (1, \infty)$ the second continuous bounded cohomology group $H^2_{cb}(G, L^p(G))$ does not vanish. As an application, we derive some structure results for closed subgroups of $\mathrm{Iso}(X)$.

1. INTRODUCTION

A geodesic metric space X is called δ-*hyperbolic* for some $\delta > 0$ if it satisfies the δ-thin triangle condition: For every geodesic triangle in X with sides a, b, c the side a is contained in the δ-neighborhood of $b \cup c$. If X is *proper* (i.e. if closed balls in X of finite radius are compact) then it can be compactified by adding the *Gromov boundary* ∂X. Moreover, the isometry group $\mathrm{Iso}(X)$ of X, equipped with the compact open topology, is a locally compact σ-compact topological group which acts as a group of homeomorphisms on ∂X. The *limit set* Λ of a subgroup G of $\mathrm{Iso}(X)$ is the set of accumulation points in ∂X of an orbit of the action of G on X. This limit set is a closed G-invariant subset of ∂X. The group G is called *elementary* if its limit set consists of at most two points or if G fixes globally a point in ∂X.

A *compact extension* of a topological group H is a topological group G which contains a compact normal subgroup K such that $H = G/K$ as topological groups. Extending earlier results of Monod-Shalom [13] and of Mineyev-Monod-Shalom [11], we show.

Theorem 1. *Let X be a proper hyperbolic geodesic metric space and let $G < \mathrm{Iso}(X)$ be a closed subgroup. Then one of the following three possibilities holds.*

(1) *G is elementary.*
(2) *Up to passing to an open subgroup of finite index, G is a compact extension of a simple Lie group of rank one.*
(3) *G is a compact extension of a totally disconnected group.*

The proof of the above result uses continuous second bounded cohomology for locally compact topological groups G with coefficients in a *Banach module* for G. Such a Banach module is a separable Banach space E together with a continuous homomorphism of G into the group of linear isometries of E. For every such

Banach module E for G and every $i \geq 1$, the group G naturally acts on the vector space $C_b(G^i, E)$ of continuous bounded maps $G^i \to E$. If we denote by $C_b(G^i, E)^G \subset C_b(G^i, E)$ the linear subspace of all G-invariant such maps, then the *second continuous bounded cohomology group* $H^2_{cb}(G, E)$ of G with coefficients E is defined as the second cohomology group of the complex

$$0 \to C_b(G, E)^G \xrightarrow{d} C_b(G^2, E)^G \xrightarrow{d} \dots$$

with the usual homogeneous coboundary operator d (see [12]). The trivial representation of G on $\mathbb{R}$ defines the real second bounded cohomology group $H^2_{cb}(G, \mathbb{R})$. Let $H^2_c(G, \mathbb{R})$ be the ordinary *continuous* real second cohomology group of G. Then there is a natural homomorphism $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ which in general is neither injective nor surjective.

A closed subgroup G of $\mathrm{Iso}(X)$ is a locally compact and σ-compact topological group and hence it admits a left invariant locally finite Haar measure μ. In particular, for every $p \in (1, \infty)$ the separable Banach space $L^p(G, \mu)$ of functions on G which are p-integrable with respect to μ is a Banach module for G with respect to the isometric action of G by left translation. Extending earlier results of Monod-Shalom [13], of Mineyev-Monod-Shalom [11] and of Fujiwara [7], we obtain the following non-vanishing result for second bounded cohomology.

Theorem 2. *Let G be a closed non-elementary subgroup of the isometry group of a proper hyperbolic geodesic metric space X with limit set $\Lambda \subset \partial X$. Then we have $H^2_{cb}(G, L^p(G, \mu)) \neq \{0\}$ for every $p \in (1, \infty)$. Moreover, one of the following two possibilities holds.*

(1) *G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ and the kernel of the natural homomorphism $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ is infinite dimensional.*
(2) *G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ and the kernel of the natural homomorphism $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ is trivial.*

Theorem 2 implies an extension of all the results of [13, 11] to arbitrary proper hyperbolic geodesic metric spaces without any additional assumptions.

We also investigate cocycles with values in $\mathrm{Iso}(X)$. Namely, let S be a standard Borel space and let ν be a Borel probability measure on S. Let Γ be a countable group which admits a measure preserving action on (S, ν). The action is called *mildly mixing* if there are no nontrivial recurrent sets, i.e. if for every Borel subset A of S with $\nu(A) \in (0, 1)$ and every sequence $g_i \to \infty$ in Γ we have $\liminf_{i \to \infty} \nu(A \triangle g_i A) \neq 0$. An $\mathrm{Iso}(X)$-valued *cocycle* for the action is a measurable map $\alpha : \Gamma \times S \to \mathrm{Iso}(X)$ such that

$$\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$$

for all $g, h \in \Gamma$ and for ν-almost every $x \in S$. The cocycle α is *cohomologous* to a cocycle $\beta : \Gamma \times S \to \mathrm{Iso}(X)$ if there is a measurable map $\varphi : S \to \mathrm{Iso}(X)$ such that

$$\varphi(gx)\alpha(g, x) = \beta(g, x)\varphi(x)$$

for all $g \in G$, ν-almost every $x \in S$.

A lattice Γ in a product $G_1 \times G_2$ of two locally compact σ-compact and non-compact topological groups is called *irreducible* if the projection of Γ into each factor is dense. Extending earlier results of Monod and Shalom [13] we show.

Theorem 3. *Let G be a semi-simple Lie group with finite center, no compact factors and of rank at least 2. Let $\Gamma < G$ be an irreducible lattice which admits a mildly mixing measure preserving action on a standard probability space (S, ν). Let X be a proper hyperbolic geodesic metric space and let $\alpha : \Gamma \times S \to \mathrm{Iso}(X)$ be a cocycle. Then one of the following two possibilities holds.*

(1) *α is cohomologous to a cocycle into an elementary subgroup of $\mathrm{Iso}(X)$.*
(2) *α is cohomologous to a cocycle into a subgroup H of $\mathrm{Iso}(X)$ which is a compact extension of a simple Lie group L of rank one, and there is a continuous surjective homomorphism $G \to L$.*

A proper hyperbolic geodesic metric space X is said to be of *bounded growth* if there is a constant $b > 0$ such that for every $R \geq 1$ an open metric ball in X of radius R contains at most be^{bR} disjoint open metric balls of radius 1. Every hyperbolic metric graph of bounded valence and every finite dimensional simply connected Riemannian manifold of bounded negative curvature has this property. We show.

Theorem 4. *Let Γ be a finitely generated group which admits a properly discontinuous isometric action on a hyperbolic geodesic metric space X of bounded growth. If $H_{cb}^2(\Gamma, \mathbb{R})$ or $H_{cb}^2(\Gamma, \ell^2(\Gamma))$ is finite dimensional then Γ is virtually nilpotent.*

The organization of this paper is as follows. In Section 2 we construct for every non-elementary closed subgroup G of the isometry group of a proper hyperbolic geodesic metric space X with limit set $\Lambda \subset \partial X$ and for every $p \in (1, \infty)$ a nontrivial continuous bounded $L^p(G \times G, \mu \times \mu)$-valued cocycle for the action of G on Λ where as before, μ is a Haar measure on G. We use this in Section 3 to show that for every $p \in (1, \infty)$ the second bounded cohomology group $H_{cb}^2(G, L^p(G, \mu))$ does not vanish. In Section 4 we deduce Theorem 1 from this non-vanishing result. The proof of Theorem 3 is contained in Section 5.

In Section 6 we construct for a closed non-elementary subgroup G of $\mathrm{Iso}(X)$ with limit set $\Lambda \subset \partial X$ which does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ infinitely many quasi-morphisms whose linear spans are pairwise inequivalent. This is then used to derive Theorem 2. The proof of Theorem 4 is contained in Section 7.

2. Continuous bounded cocycles

In this section let (X, d) always be a proper hyperbolic geodesic metric space. Recall that the *Gromov boundary* ∂X of X is defined as follows. Fix a point $x \in X$. For two points $y, z \in X$, define the *Gromov product* $(y, z)_x$ based at x by

$$(y, z)_x = \frac{1}{2}\big(d(y, x) + d(z, x) - d(y, z)\big).$$

Let $\mathcal{A} \subset \prod_{i \geq 0} X$ be the set of all sequences $(y_i) \subset X$ such that $(y_i, y_j)_x \to \infty$ $(i, j \to \infty)$. Call two sequences $(y_i), (z_j) \in \mathcal{A}$ *equivalent* if $(y_i, z_i)_x \to \infty$ $(i \to \infty)$. By hyperbolicity of X, this notion of equivalence defines an equivalence relation on $\mathcal{A}$ [4]. The boundary ∂X of X is the set of equivalence classes of this relation.

The Gromov product $(\ ,\ )_x$ for pairs of points in X can be extended to a product on ∂X by defining
$$(\xi, \eta)_x = \sup \liminf_{i,j \to \infty} (y_i, z_j)_x$$
where the supremum is taken over all sequences $(y_i), (z_j) \subset X$ whose equivalence classes define the points $\xi, \eta \in \partial X$.

For a number $\chi > 0$ only depending on the hyperbolicity constant of X, there is a distance δ_x on ∂X with the property that the distance $\delta_x(\xi, \eta)$ between two points $\xi, \eta \in \partial X$ is comparable to $e^{-\chi(\xi,\eta)_x}$ (see 7.3 of [8]). More precisely, there is a constant $\theta > 0$ not depending on x such that

(1) $e^{-\chi\theta} e^{-\chi(\xi,\eta)_x} \leq \delta_x(\xi, \eta) \leq e^{-\chi(\xi,\eta)_x}$

for all $\xi, \eta \in \partial X$. In particular, the diameter of δ_x is bounded from above independent of x. The distances δ_x $(x \in X)$ on ∂X are invariant under the natural action of the isometry group $\mathrm{Iso}(X)$ of X on $\partial X \times \partial X \times X$, and they satisfy

(2) $e^{-\chi d(x,y)} \delta_x \leq \delta_y \leq e^{\chi d(x,y)} \delta_x$ for all $x, y \in X$.

As a consequence, the topology on ∂X defined by the metric δ_x does not depend on x.

There is a natural topology on $X \cup \partial X$ which restricts to the given topology on X and to the topology on ∂X induced by any one of the metrics δ_x. With respect to this topology, a sequence $(y_i) \subset X$ converges to $\xi \in \partial X$ if and only if the sequence is contained in $\mathcal{A}$ (i.e. we have $(y_i, y_j)_x \to \infty$) and if moreover the equivalence class of (y_i) equals ξ. Since X is proper by assumption, the space $X \cup \partial X$ is compact and metrizable. Every isometry of X acts naturally on $X \cup \partial X$ as a homeomorphism (see [4] for all this).

We need the following simple observation.

Lemma 2.1. *$\partial X \times X$ admits a natural $\mathrm{Iso}(X)$-invariant distance function $\tilde{d}$ inducing the product topology.*

Proof. Let δ_x $(x \in X)$ be a family of metrics on ∂X which is invariant under the action of $\mathrm{Iso}(X)$ on $\partial X \times \partial X \times X$ and such that for some $\theta > 0, \chi \in (0, 1)$ the inequalities (1) and (2) above are valid.

For $(\xi, x), (\eta, y) \in \partial X \times X$ write
$$\tilde{d}_0((\xi, x), (\eta, y)) = d(x, y) + \frac{1}{2}(\delta_x(\xi, \eta) + \delta_y(\xi, \eta)).$$
Note that $\tilde{d}_0$ is a symmetric function on $(\partial X \times X) \times (\partial X \times X)$ which is invariant under the diagonal action of $\mathrm{Iso}(X)$.

Define a *chain* between $(\xi, x), (\eta, y)$ to be a finite sequence $\{(\xi_i, x_i) \mid 0 \leq i \leq m\} \subset \partial X \times X$ with $(\xi_0, x_0) = (\xi, x)$ and $(\xi_m, x_m) = (\eta, y)$. The *length* of the chain

is defined to be $\sum_i \tilde{d}_0((\xi_i, x_i), (\xi_{i+1}, x_{i+1}))$. Let $\tilde{d}((\xi, x), (\eta, y))$ be the infimum of the lengths of any chains connecting (ξ, x) to (η, y). Then $\tilde{d}$ is a symmetric $\text{Iso}(X)$-invariant function on $(\partial X \times X) \times (\partial X \times X)$ which is bounded from above by $\tilde{d}_0$. By construction, $\tilde{d}$ satisfies the triangle inequality. Moreover, by the triangle inequality for d and the definition, the function $\tilde{d}$ satisfies

$$\tilde{d}((\xi, x), (\eta, y)) \geq d(x, y) \, \forall \xi, \eta \in \partial X, \forall x, y \in X.$$

We claim that there is a constant $c > 0$ such that

(3) $$c\delta_x(\xi, \eta) \leq \tilde{d}((\xi, x), (\eta, x)) \leq \delta_x(\xi, \eta) \, \forall x \in X, \forall \xi, \eta \in \partial X.$$

Namely, the upper estimate for $\tilde{d}((\xi, x), (\eta, x))$ is immediate from the definitions. To show the lower estimate, note that the diameter of ∂X equipped with any one of the metrics δ_x is bounded from above by a universal constant $D > 0$. Let $x \in X$, let $\xi, \eta \in \partial X$ and let $\{(\xi_i, x_i) \mid 0 \leq i \leq m\}$ be a chain connecting $(\xi, x) = (\xi_0, x_0)$ to $(\eta, x) = (\xi_m, x_m)$. If there is some $i < m$ such that $d(x, x_i) \geq D$ then by the triangle inequality for d, the length of the chain is bigger than $2D$ and hence it is bigger than $\tilde{d}((\xi, x), (\eta, x)) + D$. On the other hand, if for every $i \leq m$ we have $d(x, x_i) \leq D$ then inequality (2) together with the triangle inequality for δ_x shows that the length of the chain is not smaller than $e^{-\chi D}\delta_x(\xi, \eta)$. Inequality (3) follows.

As a consequence, we have $\tilde{d}((\xi, x), (\eta, y)) = 0$ only if $\xi = \eta, x = y$. This shows that $\tilde{d}$ is a distance on $\partial X \times X$ which moreover induces the product topology. $\square$

For every proper metric space X, the isometry group $\text{Iso}(X)$ of X can be equipped with a natural locally compact σ-compact metrizable topology, the so-called *compact open topology*. With respect to this topology, a sequence $(g_i) \subset \text{Iso}(X)$ converges to some isometry g if and only if $g_i \to g$ uniformly on compact subsets of X. In this topology, a closed subset $A \subset \text{Iso}(X)$ is compact if and only if there is a compact subset K of X such that $gK \cap K \neq \emptyset$ for every $g \in A$. In particular, the action of $\text{Iso}(X)$ on X is proper. In the sequel we always equip subgroups of $\text{Iso}(X)$ with the compact open topology.

Let again X be a proper hyperbolic geodesic metric space and let $G < \text{Iso}(X)$ be a subgroup of the isometry group of X. The *limit set* Λ of G is the set of accumulation points in ∂X of one (and hence every) orbit of the action of G on X. If the closure of G is non-compact then its limit set is a compact non-empty G-invariant subset of ∂X. The group G is called *elementary* if its limit set consists of at most two points or if G fixes globally a point in ∂X. In particular, every compact subgroup of $\text{Iso}(X)$ is elementary. If G is non-elementary then its limit set Λ is uncountable without isolated points.

An element $g \in \text{Iso}(X)$ is called *hyperbolic* if it generates an infinite cyclic subgroup $G < \text{Iso}(X)$ whose limit set Λ consists of two points $a, b \in \partial X$ which are fixed points for g. Moreover, a is an *attracting fixed point*, b is a *repelling fixed point* and g acts with *north-south-dynamics* with respect to these fixed points. This means that for any two neighborhoods U of a, V of b there is some $k > 0$ such that $g^k(\partial X - V) \subset U$ and $g^{-k}(\partial X - U) \subset V$. If $G < \text{Iso}(X)$ is non-elementary then the fixed points of hyperbolic elements in G form a dense subset of Λ (8.26 of [8]).

Since X is proper, any two points $\xi \neq \eta \in \partial X$ can be connected by a geodesic in X. We need the following simple observation for which we did not find a precise reference in the literature.

Lemma 2.2. *Let X be a proper κ_0-hyperbolic geodesic metric space for some $\kappa_0 > 0$. Let g be a hyperbolic isometry of X and let x_0 be a point on a geodesic connecting the two fixed points $a \neq b \in \partial X$ for the action of g on ∂X. If $k > 0$ is such that $d(x_0, g^k x_0) \geq 4\kappa_0$ then there is no isometry h of X with $hx_0 = x_0, hg^k x_0 = g^k x_0$ and $h(a) = b, h(b) = a$.*

Proof. Let X be κ_0-hyperbolic for some $\kappa_0 > 0$, let $g \in \mathrm{Iso}(X)$ be a hyperbolic isometry and let $a \neq b \in \partial X$ be the attracting and the repelling fixed point for the action of g on ∂X, respectively. Let γ be a geodesic in X connecting b to a. Such a geodesic may not be unique, but the Hausdorff distance between γ and any other such geodesic is bounded from above by κ_0. Since g maps the geodesic γ connecting b to a to a geodesic $g\gamma$ connecting $gb = b$ to $ga = a$, for every $x \in \gamma$ the orbit of x under the infinite cyclic subgroup of $\mathrm{Iso}(X)$ generated by g is contained in the κ_0-neighborhood of γ.

Let $x_0 \in \gamma$ and let $k > 0$ be such that $d(x_0, g^k x_0) \geq 4\kappa_0$. Assume that γ is parametrized in such a way that $\gamma(0) = x_0$. Since $\gamma(t)$ converges as $t \to \infty$ to the attracting fixed point a for the action of g on ∂X, we have $d(g^k x_0, \gamma[3\kappa_0, \infty)) \leq \kappa_0$. Assume to the contrary that there is an isometry h of X which fixes both x_0 and $g^k x_0$ and exchanges a and b. Then h maps the geodesic ray $\gamma[0, \infty)$ connecting x_0 to a to a geodesic ray $h\gamma[0, \infty)$ connecting x_0 to b. Thus the Hausdorff distance between $h\gamma[0, \infty)$ and $\gamma(-\infty, 0]$ is at most κ_0. As a consequence, the κ_0-neighborhood of the ray $h\gamma[3\kappa_0, \infty)$ does not intersect the κ_0-neighborhood of $\gamma[3\kappa_0, \infty)$. However, the κ_0-neighborhood of $\gamma[3\kappa_0, \infty)$ contains $g^k x_0 = hg^k x_0$, and since h is an isometry, the point $hg^k x_0$ is contained in the κ_0-neighborhood of $h\gamma[3\kappa_0, \infty)$ as well. This is a contradiction and shows the lemma.   □

The following theorem is the main technical result of this paper. It gives a simplification and extension of some results of Monod and Shalom [13] and of Mineyev, Monod and Shalom [11] avoiding the difficult concept of homological bicombing. For its formulation, let again X be a proper hyperbolic geodesic metric space and let G be a *closed* subgroup of the isometry group of X with limit set Λ. Note that G is a locally compact σ-compact topological group. Assume that G is non-elementary and let $T \subset \Lambda^3$ be the space of triples of pairwise distinct points in Λ. Then T is an uncountable topological space without isolated points. The group G acts diagonally on T as a group of homeomorphisms.

For a Banach-module E for G define an *E-valued continuous bounded two-cocycle* for the action of G on Λ to be a continuous *bounded* G-equivariant map $\omega : T \to E$ which satisfies the following two properties.

(1) For every permutation σ of the three variables, the *anti-symmetry condition* $\omega \circ \sigma = \mathrm{sgn}(\sigma)\omega$ holds.

(2) For every quadruple (x, y, z, w) of distinct points in Λ, the *cocycle equality*

$$(4) \qquad \omega(y, z, w) - \omega(x, z, w) + \omega(x, y, w) - \omega(x, y, z) = 0$$

is satisfied.

Recall that every locally compact σ-compact topological group G admits a left invariant locally finite Haar measure μ. For $p \in (1, \infty)$ denote by $L^p(G \times G, \mu \times \mu)$ the Banach space of all functions on $G \times G$ which are p-integrable with respect to the product measure $\mu \times \mu$. The group G acts continuously and isometrically on $L^p(G \times G, \mu \times \mu)$ by left translation. We have.

Theorem 2.3. *Let X be a proper hyperbolic geodesic metric space and let $G <$ $\mathrm{Iso}(X)$ be a closed non-elementary subgroup. For every $p \in (1, \infty)$ and every triple $(a, b, \xi) \in T$ of pairwise distinct points in Λ such that (a, b) is the pair of fixed points of a hyperbolic element of G there is an $L^p(G \times G, \mu \times \mu)$-valued continuous bounded cocycle ω for the action of G on Λ with $\omega(a, b, \xi) \neq 0$.*

Proof. Let $G < \mathrm{Iso}(X)$ be a non-elementary closed subgroup of $\mathrm{Iso}(X)$. We divide the proof of the theorem into five steps.

Step 1:

Let $x_0 \in X$ be an arbitrary point and denote by G_{x_0} the stabilizer of x_0 in G. Then G_{x_0} is a compact subgroup of G, and the quotient space G/G_{x_0} is G-equivariantly homeomorphic to the orbit $Gx_0 \subset X$ of x_0. Note that Gx_0 is a closed subset of X and hence it is locally compact.

Since G is non-elementary by assumption, the limit set $\Lambda \subset \partial X$ of G is an uncountable closed subset of ∂X without isolated points [4]. In particular, Λ is compact. The group G acts on the locally compact space $\Lambda \times X$ as a group of homeomorphisms. Denote by Δ the diagonal in $\partial X \times \partial X$. The $\mathrm{Iso}(X)$-invariant metric $\tilde{d}$ on $\partial X \times X$ constructed in Lemma 2.1 induces a G-invariant metric on $\Lambda \times G/G_{x_0}$ and hence a G-invariant symmetrized product metric $\hat{d}$ on

$$V = (\Lambda \times \Lambda - \Delta) \times G/G_{x_0} \times G/G_{x_0}$$

by defining

$$(5) \qquad \hat{d}((\xi, \eta, x, y), (\xi', \eta', x', y'))$$
$$= \frac{1}{2} \big(\tilde{d}((\xi, x), (\xi', x')) + \tilde{d}((\eta, y), (\eta', y')) + \tilde{d}((\eta, x), (\eta', x')) + \tilde{d}((\xi, y), (\xi', y')) \big).$$

The distance $\hat{d}$ is also invariant under the involution $\iota : (\xi, \eta, x, y) \to (\eta, \xi, x, y)$ exchanging the first two factors. Moreover, $\hat{d}$ induces the product topology on V.

Since V is locally compact and $G < \mathrm{Iso}(X)$ is closed, the space $W = G \backslash V$ admits a natural metric d_0 as follows. Let

$$P : V \to W$$

be the canonical projection and define

$$(6) \qquad d_0(x, y) = \inf \{ \hat{d}(\tilde{x}, \tilde{y}) \mid P\tilde{x} = x, P\tilde{y} = y \}.$$

The topology induced by this metric is the quotient topology for the projection P. In particular, W is a locally compact metric space. Since the action of G commutes with the isometric involution ι, the map ι descends to an isometric involution of the metric space (W, d_0) which we denote again by ι.

Let $\kappa_0 > 0$ be a hyperbolicity constant for X. Let as before δ_z ($z \in X$) be a family of distance functions on ∂X which satisfy the properties (1) and (2) in the beginning of this section. There is a universal constant $c_0 > 0$ depending only on κ_0 with the following property. For any $\xi \neq \eta \in \partial X$, for every geodesic γ connecting η to ξ and for every point $z \in X$ which is contained in the κ_0-neighborhood of γ we have $\delta_z(a, b) \geq 2c_0$. Moreover, there is a constant $\kappa > 6\kappa_0$ only depending on κ_0 (and not on ξ, η) such that the set $\{z \in X \mid \delta_z(\xi, \eta) \geq c_0\}$ is contained in the κ-tubular neighborhood of any geodesic γ connecting η to ξ.

Choose a smooth function $\varphi_0 : [0, \infty) \to [0, 1]$ such that $\varphi_0(t) = 0$ for all $t \leq c_0$ and $\varphi_0(t) = 1$ for all $t \geq 2c_0$. Define a continuous function $\varphi : V \to [0, 1]$ by

$$(7) \qquad \varphi(\xi, \eta, gx_0, hx_0) = \varphi_0(\delta_{gx_0}(\xi, \eta))\varphi_0(\delta_{hx_0}(\xi, \eta)).$$

Note that φ is invariant under the diagonal action of G and under the involution ι. In particular, the support V_0 of φ is invariant under G, and it projects to a ι-invariant closed subset $W_0 = G \backslash V_0$ of W.

Let $\kappa_1 > 3\kappa$ and let $\psi : [0, \infty) \to [0, 1]$ be a smooth function which satisfies $\psi(t) = 1$ for $t \leq \kappa_1$ and $\psi(t) = 0$ for $t \geq 2\kappa_1$. Define a continuous function $\zeta : V \to [0, 1]$ by

$$(8) \qquad \zeta(\xi, \eta, ux_0, hx_0) = \varphi(\xi, \eta, ux_0, hx_0)\psi(d(ux_0, hx_0)).$$

Note that ζ is invariant under the diagonal action of G and under the involution ι. In particular, ζ projects to a continuous function ζ_0 on $W = G \backslash V$.

Step 2:

In equation (6) in Step 1 above, we defined a distance d_0 on the space $W = G \backslash V$. With respect to this distance, the involution ι acts non-trivially and isometrically. Choose a small closed metric ball B in W which is disjoint from its image under ι. In Step 5 below we will construct explicitly such balls B, however for the moment, we simply assume that such a ball exists.

Let $\mathcal{H}$ be the vector space of all Hölder continuous functions $f : W \to \mathbb{R}$ supported in B. An example of such a function can be obtained as follows.

Let z be an interior point of B and let $r > 0$ be sufficiently small that the closed metric ball $B(z, r)$ of radius r about z is contained in B. Choose a smooth function $\alpha : \mathbb{R} \to [0, 1]$ such that $\alpha(t) = 1$ for $t \in [r/2, \infty)$ and $\alpha(t) = 0$ for $t \in (-\infty, 0]$ and define $f(y) = \alpha(r - d_0(y, z))$. Since the function $y \to r - d_0(y, z)$ is one-Lipschitz on W and α is smooth, the function f on W is Lipschitz, does not vanish at z and is supported in B.

Since B is disjoint from $\iota(B)$ by assumption and since ι is an isometry, every function $f \in \mathcal{H}$ admits a natural extension to a Hölder continuous function f_0 on W supported in $B \cup \iota(B)$ whose restriction to B coincides with the restriction of f

and which satisfies $f_0(\iota z) = -f_0(z)$ for all $z \in W$. The function $\hat{f} = f_0 \circ P : V \to \mathbb{R}$ is invariant under the action of G, and it is *anti-invariant* under the involution ι of V, i.e. it satisfies $\hat{f}(\iota(v)) = -\hat{f}(v)$ for all $v \in V$ (here as before, $P : V \to W$ denotes the canonical projection).

Equip $\tilde{V} = (\Lambda \times \Lambda - \Delta) \times G \times G$ with the product topology. The group G acts on $G \times G$ by left translation, and it acts diagonally on $\tilde{V}$. Denote again by ι the involution of $\tilde{V}$ exchanging the first two factors. There is a natural continuous projection $\Pi : \tilde{V} \to V$ which is equivariant with respect to the action of G and with respect to the action of the involution ι on $\tilde{V}$ and V. The function $\hat{f}$ on V lifts to a G-invariant ι-anti-invariant continuous function $\tilde{f} = \hat{f} \circ \Pi$ on $\tilde{V}$. The lift $\tilde{\zeta} = \zeta \circ \Pi$ of the function ζ defined in equation (8) in Step 1 is G-invariant and ι-invariant.

For $\xi \neq \eta \in \Lambda$ write $F(\xi, \eta) = \{(\xi, \eta, z) \mid z \in G \times G\}$. The sets $F(\xi, \eta)$ define a G-invariant foliation $\mathcal{F}$ of $\tilde{V}$. The leaf $F(\xi, \eta)$ of $\mathcal{F}$ can naturally be identified with $G \times G$. For all $\xi \neq \eta \in \Lambda$ and every function $f \in \mathcal{H}$ we denote by $f_{\xi,\eta}$ the restriction of the function $\tilde{f}$ to $F(\xi, \eta)$, viewed as a continuous function on $G \times G$. Similarly, define $\zeta_{\xi,\eta}$ to be the restriction of the function $\tilde{\zeta}$ to $F(\xi, \eta)$. For every $f \in \mathcal{H}$, all $\xi \neq \eta \in \Lambda$ and all $g \in G$ we then have $f_{g\xi,g\eta} \circ g = f_{\xi,\eta} = -f_{\eta,\xi}$, moreover $\zeta_{g\xi,g\eta} \circ g = \zeta_{\xi,\eta} = \zeta_{\eta,\xi}$.

Recall that the limit set Λ of G is uncountable without isolated points. For $f \in \mathcal{H}$ and for an ordered triple (ξ, η, β) of pairwise distinct points in Λ define

$$(9) \qquad \omega(\xi, \eta, \beta) = f_{\xi,\eta}\zeta_{\xi,\eta} + f_{\eta,\beta}\zeta_{\eta,\beta} + f_{\beta,\xi}\zeta_{\beta,\xi}.$$

Then $\omega(\xi, \eta, \beta)$ is a continuous bounded function on $G \times G$. Since $f_{\xi,\eta} = -f_{\eta,\xi}$ and $\zeta_{\xi,\eta} = \zeta_{\eta,\xi}$ for all $\xi \neq \eta \in \Lambda$, we have $\omega \circ \sigma = (\mathrm{sgn}(\sigma))\omega$ for every permutation σ of the three variables. Since the functions $f_{\xi,\eta}, \zeta_{\xi,\eta}$ are restrictions to the leaves of the foliation $\mathcal{F}$ of globally continuous bounded functions on $\tilde{V}$, the assignment $(\xi, \eta, \beta) \in T \to \omega(\xi, \eta, \beta) \in C^0(G \times G)$ is continuous with respect to the compact open topology on $C^0(G \times G)$. Moreover, it, is equivariant with respect to the natural action of G on T and on $C^0(G \times G)$. This means that ω is a continuous bounded cocycle for the action of G on Λ with values in $C^0(G \times G)$.

Step 3:

Recall the choices of the constants $0 < \kappa_0 < \kappa < \kappa_1$ from Step 1 above. We may assume that $\kappa > 1$. In this technical step we obtain some control on the functions $f_{\xi,\eta}\zeta_{\xi,\eta}$ for a pair of distinct points $\xi \neq \eta \in \Lambda$.

By hyperbolicity of X and the choice of κ_0, for every triple (ξ, η, β) of pairwise distinct points in ∂X there is a point $y_0 \in X$ which is contained in the κ_0-neighborhood of every side of a geodesic triangle with vertices ξ, η, β. Let $\gamma : \mathbb{R} \to X$ be a geodesic connecting ξ to η with $d(y_0, \gamma(0)) \leq \kappa_0$. Then the Gromov product $(\xi, \beta)_{\gamma(t)}$ based at $\gamma(t)$ between ξ, β satisfies

$$(\xi, \beta)_{\gamma(t)} \geq t - c_0 \text{ for all } t \geq 0$$

with a constant $c_0 > 0$ only depending on X (see [4]). Thus by the properties (1) and (2) of the distance functions δ_x, there is a number $r_0 > 0$ (depending on κ_1 and the hyperbolicity constant κ_0 for X) such that if $t \geq 0$ and if $y \in X$ satisfies

$d(\gamma(t), y) < 3\kappa_1$ then $\delta_y(\xi, \beta) \leq r_0 e^{-\chi t}$ where $\chi > 0$ is as in inequality (1). The triangle inequality for δ_y then yields

$$(10) \qquad |\delta_y(\xi, \eta) - \delta_y(\eta, \beta)| \leq r_0 e^{-\chi t}.$$

This implies the following. First, recall that the auxiliary function φ_0 which we used in the definition (7) of the function φ is smooth and constant outside a compact set and hence it is uniformly Lipschitz continuous. Therefore by the estimate (10) there is a constant $r_1 > r_0$ such that

$$(11) \qquad |\varphi_0(\delta_y(\xi, \eta)) - \varphi_0(\delta_y(\eta, \beta))| \leq r_1 e^{-\chi t}$$

whenever $d(y, \gamma(t)) \leq 3\kappa_1$ for some $t \geq 0$.

Now let $0 \leq t$ and let $u, h \in G$ be such that

$$d(ux_0, \gamma(t)) < 3\kappa_1, d(hx_0, \gamma(t)) < 3\kappa_1.$$

Since the function φ_0 assumes values in $[0, 1]$, we obtain from the definition (7) of the function φ and the estimate (11) that

$$(12) \qquad |\varphi(\xi, \eta, ux_0, hx_0) - \varphi(\beta, \eta, ux_0, hx_0)|$$
$$\leq |(\varphi_0(\delta_{ux_0}(\xi, \eta)) - \varphi_0(\delta_{ux_0}(\beta, \eta)))\varphi_0(\delta_{hx_0}(\xi, \eta))|$$
$$+ |\varphi_0(\delta_{ux_0}(\beta, \eta))(\varphi_0(\delta_{hx_0}(\xi, \eta)) - \varphi_0(\delta_{hx_0}(\beta, \eta)))| \leq 2r_1 e^{-\chi t}.$$

Similarly, the function ψ used in the definition (8) of the function ζ assumes values in $[0, 1]$ and hence we conclude from (12) that also

$$(13) \qquad |\zeta(\xi, \eta, ux_0, hx_0) - \zeta(\beta, \eta, ux_0, hx_0)| \leq 2r_1 e^{-\chi t}.$$

Moreover, by the definition (5) of the distance function $\hat{d}$ on V and by the inequality (3) for the distance function $\tilde{d}$ on $\partial X \times X$, we have

$$(14) \qquad \hat{d}\big((\xi, \eta, hx_0, ux_0), (\beta, \eta, hx_0, ux_0)\big) \leq \frac{1}{2}(\delta_{hx_0}(\xi, \beta) + \delta_{ux_0}(\xi, \beta)) \leq r_0 e^{-\chi t}.$$

The function $\hat{f} : V \to \mathbb{R}$ constructed in Step 2 from a function $f \in \mathcal{H}$ is Hölder continuous and ι-anti-invariant. Therefore by the estimate (14) there are numbers $\alpha > 0, r_2 > r_1$ only depending on the Hölder norm for f with the following property. Let $0 \leq t$ and let $u, h \in G$ be such that $d(ux_0, \gamma(t)) < 3\kappa_1, d(hx_0, \gamma(t)) < 3\kappa_1$; then

$$(15) \qquad |\hat{f}(\xi, \eta, ux_0, hx_0) + \hat{f}(\eta, \beta, ux_0, hx_0)| \leq r_2 e^{-\chi \alpha t}.$$

The functions f and ζ are bounded in absolute value by a universal constant. Hence using a calculation as in (12) above, from the definition of the functions $f_{\xi, \eta}$ and $f_{\eta, \beta}$ and from the estimates (13) and (15) we obtain the existence of a constant $r > r_2$ (depending on the Hölder norm of f) such that

$$(16) \qquad |(f_{\xi, \eta}\zeta_{\xi, \eta} + f_{\eta, \beta}\zeta_{\eta, \beta})(u, h)| \leq r e^{-\chi \alpha t}.$$

Step 4:

Let $\nu = \mu \times \mu$ be the left invariant product measure on $G \times G$. Our goal is to show that for every $p \in (0, \infty)$, the cocycle ω defined in equation (9) above is in fact a bounded cocycle with values in $L^p(G \times G, \nu)$. For this we show that for every $(\xi, \eta, \beta) \in T$ the function $\omega(\xi, \eta, \beta)$ on $G \times G$ is contained in a fixed bounded

subset of $L^p(G \times G, \nu)$ and that moreover the assignment $(\xi, \eta, \beta) \to \omega(\xi, \eta, \beta) \in L^p(G \times G, \nu)$ is continuous.

For a subset C of X write

$$C_{G,2\kappa_1} = \{(u,h) \in G \times G \mid ux_0 \in C, d(ux_0, hx_0) \leq 2\kappa_1\}$$

and for $r > 0$ let $N(C, r)$ be the r-neighborhood of C in X. We claim that there is a number $m > 0$ such that for every subset C of X of diameter at most one the ν-mass of the set $N(C, 2\kappa)_{G,2\kappa_1}$ is at most m. Namely, the subset

$$D = \{(u,h) \in G \times G \mid d(ux_0, x_0) \leq 6\kappa, d(ux_0, hx_0) \leq 2\kappa_1\}$$

of $G \times G$ is compact and hence its ν-mass is finite, say this mass equals $m > 0$. On the other hand, if $C \subset X$ is a set of diameter at most one and if there is some $g \in G$ such that $gx_0 \in N(C, 2\kappa)$ then any pair $(u, h) \in N(C, 2\kappa)_{G,2\kappa_1}$ is contained in gD. Our claim now follows from the fact that ν is invariant under left translation.

As in Step 3 above, let (ξ, η, β) a triple of pairwise distinct points in Λ and let $y_0 \in X$ be a point which is contained in the κ_0-neighborhood of every side of a geodesic triangle in X with vertices ξ, η, β. Let $\gamma : \mathbb{R} \to X$ be a geodesic connecting ξ to η with $d(y_0, \gamma(0)) \leq \kappa_0$. Also, let $\rho : \mathbb{R} \to X$ be a geodesic connecting β to η which is parametrized in such a way that $d(y_0, \rho(0)) \leq \kappa_0$. Then $\gamma[0, \infty), \rho[0, \infty)$ are two sides of a geodesic triangle in X with vertices $\gamma(0), \rho(0), \eta$. Since κ_0 is a hyperbolicity constant for X and $d(\gamma(0), \rho(0)) \leq 2\kappa_0$, the ray $\gamma[3\kappa_0, \infty)$ is contained in the κ_0-tubular neighborhood of $\rho[0, \infty)$, and the ray $\rho[3\kappa_0, \infty)$ is contained in the κ_0-tubular neighborhood of $\gamma[0, \infty)$. Thus the Hausdorff distance between the geodesic rays $\gamma[0, \infty)$ and $\rho[0, \infty)$ is at most $6\kappa_0 \leq \kappa$. In particular, the κ-neighborhood of $\rho[0, \infty)$ is contained in the 2κ-neighborhood of $\gamma[0, \infty)$.

Recall that $\varphi_0(\delta_{ux_0}(\xi, \eta)) \neq 0$ only if ux_0 is contained in the κ-neighborhood of γ. Moreover, we have $\zeta(\xi, \eta, ux_0, hx_0) \neq 0$ only if $d(ux_0, hx_0) \leq 2\kappa_1$. This implies that the support of the function $f_{\xi,\eta}\zeta_{\xi,\eta}$ is contained in $N(\gamma(\mathbb{R}), \kappa)_{G,2\kappa_1}$ and similarly for the functions $f_{\eta,\beta}\zeta_{\eta,\beta}, f_{\beta,\xi}\zeta_{\beta,\xi}$.

As a consequence, there is an open subset U of $G \times G$ with compact closure and there is a number $T > 0$ only depending on κ_1 and the hyperbolicity constant for X with the following property. The support of the function ω defined in (9) above is the disjoint union of the three sets

$$N(\gamma[T, \infty), 2\kappa)_{G,2\kappa_1}, N(\gamma(-\infty, -T], 2\kappa)_{G,2\kappa_1}, N(\rho(-\infty, -T], 2\kappa)_{G,2\kappa_1}$$

with U. Moreover, the restriction of ω to $N(\gamma[T, \infty), 2\kappa)_{G,2\kappa_1}$ coincides with the restriction of the function $f_{\xi,\eta}\zeta_{\xi,\eta} + f_{\eta,\beta}\zeta_{\eta,\beta}$ and similarly for the other two sets different from U in the above decomposition of the support of ω. Thus to show that ω is contained in $L^p(G \times G, \nu)$ it is enough to show that there is constant $c_p > 0$ only depending on p and the Hölder norm of f such that

$$\int_{N(\gamma[T,\infty),2\kappa)_{G,2\kappa_1}} |f_{\xi,\eta}\zeta_{\xi,\eta} + f_{\eta,\beta}\zeta_{\eta,\beta}|^p d\nu < c_p.$$

However, this is immediate from the estimate (16) together with the control on the ν-mass of subsets of $N(\gamma[T, \infty), 2\kappa)_{G,2\kappa_1}$ $(k > 0)$. Namely, we showed that for every integer $k \geq 0$ the ν-mass of the set $N(\gamma[T+k, T+k+1], 2\kappa)_{G,2\kappa_1}$ is bounded

from above by a universal constant $m > 0$. Moreover, for every $p \geq 1$ the value of the function $|f_{\xi,\eta}\zeta_{\xi,\eta} + f_{\beta,\eta}\zeta_{\beta,\eta}|^p$ on this set does not exceed $r^p e^{-p\chi\alpha(T+k)}$. Thus the inequality holds true with $c_p = mr^p \sum_{k=0}^{\infty} e^{-p\chi\alpha(T+k)}$.

Since the function $\tilde{f}$ on $\tilde{V}$ is globally continuous, the same consideration also shows that $\omega(\xi, \eta, \beta) \in L^p(G \times G, \nu)$ depends continuously on (ξ, η, β). Namely, let $((\xi_i, \eta_i, \beta_i)) \subset T$ be a sequence of triples of pairwise distinct points converging to a triple $(\xi, \eta, \beta) \in T$. By the above consideration, for every $\epsilon > 0$ there is a compact subset A of $G \times G$ such that $\int_{G \times G - A} |\omega(\xi_i, \eta_i, \beta_i)|^p d\nu \leq \epsilon$ for all sufficiently large $i > 0$ and that the same holds true for $\omega(\xi, \eta, \beta)$. Let χ_A be the characteristic function of A. By continuity of the function $\tilde{f}\tilde{\zeta}$ on $\tilde{V}$ and compactness, the functions $\chi_A\omega(\xi_i, \eta_i, \beta_i)$ converge as $i \to \infty$ in $L^p(G \times G, \nu)$ to $\chi_A\omega(\xi, \eta, \beta)$. Since $\epsilon > 0$ was arbitrary, the required continuity follows.

Moreover, the assignment $(\xi, \eta, \beta) \to \omega(\xi, \eta, \beta)$ is equivariant under the action of G on the space T of triples of pairwise distinct points in Λ and on $L^p(G \times G, \nu)$ and satisfies the cocycle equality (4). In other words, ω defines a continuous $L^p(G \times G, \nu)$-valued bounded cocycle for the action of G on Λ as required.

Step 5:

Let $g \in G$ be a hyperbolic isometry, let $a \neq b \in \partial X$ be the attracting and repelling fixed point for the action of g on ∂X, respectively, and let $\xi \in \Lambda - \{a, b\}$. We have to show that we can find a cocycle ω as in (9) above with $\omega(a, b, \xi) \neq 0$.

For this let γ be a geodesic in X connecting b to a and choose the basepoint x_0 for the above construction on γ. Let $\kappa > \kappa_0$ be as above. Then the orbit of x_0 under the infinite cyclic subgroup of G generated by g is contained in the κ_0-neighborhood of γ (compare the discussion in the proof of Lemma 2.2). Since $g^j x_0 \to a, g^{-j}x_0 \to b$ $(j \to \infty)$, there are numbers $k < \ell$ such that the κ-neighborhood of a geodesic connecting a to ξ and of a geodesic connecting b to ξ contains at most one of the points $g^k x_0, g^\ell x_0$ and that moreover the distance between $g^k x_0, g^\ell x_0$ is at least $4\kappa_0$. Choose $\kappa_1 > 2d(g^k x_0, g^\ell x_0)$.

Using this constant κ_1 for the definition of the function ζ in equation (8), by the definition of φ_0 we have $\zeta(a, b, g^k x_0, g^\ell x_0) > 0$ and $\zeta(a, \xi, g^k x_0, g^\ell x_0) = \zeta(b, \xi, g^k x_0, g^\ell x_0) = 0$. Thus the pair of points $(g^k, g^\ell) \in G \times G$ is contained in the support of the function $\zeta_{a,b}$ but not in the support of any of the two functions $\zeta_{a,\xi}, \zeta_{b,\xi}$. Moreover, by Lemma 2.2, there is no $h \in G$ with $hg^k x_0 = g^k x_0, hg^\ell x_0 = g^\ell x_0$ and $h(a) = b, h(b) = a$. Therefore the G-orbit of $(a, b, g^k x_0, g^\ell x_0) \in V$ does not contain the point $(b, a, g^k x_0, g^\ell x_0)$. This means that the projection of $(a, b, g^k x_0, g^\ell x_0)$ into W is not fixed by the involution ι.

As a consequence, we can find a function $f \in \mathcal{H}$ (for a suitable choice of a support ball B) whose lift $\tilde{f}$ to $\tilde{V}$ does not vanish at (a, b, g^k, g^ℓ). By the choice of κ_1, this means that $f_{a,b}\zeta_{a,b}(g^k, g^\ell) \neq 0$ and $f_{b,\xi}\zeta_{b,\xi}(g^k, g^\ell) = f_{\xi,a}\zeta_{\xi,a}(g^k, g^\ell) = 0$. In other words, the cocycle ω constructed as above from f does not vanish at (a, b, ξ). This shows the theorem. $\square$

Remark: The construction in the proof of Theorem 2.3 yields in fact for every triple $(a, b, \xi) \in \Lambda^3$ such that (a, b) is the pair of fixed points of a hyperbolic element of G an infinite dimensional space of continuous bounded $L^p(G \times G, \mu \times \mu)$-valued cocycles which do not vanish at (a, b, ξ).

3. Bounded cohomology

In this section we use Theorem 2.3 to construct nontrivial second bounded cohomology classes for closed non-elementary subgroups of the isometry group $\mathrm{Iso}(X)$ of a proper hyperbolic geodesic metric space X.

Every locally compact σ-compact topological group G admits a *strong boundary* [10] which is a standard Borel G-space B with a quasi-invariant ergodic probability measure λ such that the action of G on (B, λ) is amenable and doubly ergodic with respect to any separable Banach module for G ([12], Definition 11.1.1). The following lemma is basically contained in [13] and follows from the arguments of Zimmer (see [17]). For later use we formulate it more generally for *cocycles*.

Namely, let G be a locally compact σ-compact topological group which admits a measure preserving ergodic action on a standard probability space (S, ν). An $\mathrm{Iso}(X)$-valued *cocycle* for this action is a measurable map $\alpha : G \times S \to \mathrm{Iso}(X)$ such that
$$\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$$
for all $g, h \in G$ and almost all $x \in S$. The cocycle α is *cohomologous* to a cocycle β if there is a measurable map $\psi : S \to \mathrm{Iso}(X)$ such that $\psi(gx)\alpha(g, x) = \beta(g, x)\psi(x)$ for almost every $x \in S$, all $g \in G$. We have.

Lemma 3.1. *Let X be a proper hyperbolic geodesic metric space with isometry group $\mathrm{Iso}(X)$. Let G be a locally compact σ-compact group which admits a measure preserving ergodic action on a standard probability space (S, ν) and let $\alpha : G \times S \to \mathrm{Iso}(X)$ be a cocycle. Assume that $\alpha(G \times S)$ is contained in a closed non-elementary subgroup H of $\mathrm{Iso}(X)$ and that α is not cohomologous to a cocycle into an elementary subgroup of H. Let Λ be the limit set of H and let (B, λ) be a strong boundary for G; then there is an α-equivariant measurable map $\Psi : B \times S \to \Lambda$.*

Proof. Let X be a proper hyperbolic geodesic metric space and let G be a locally compact σ-compact group with a measure preserving ergodic action on a standard probability space (S, ν). Let $\alpha : G \times S \to \mathrm{Iso}(X)$ be a cocycle. Assume that the closure H of $\alpha(G \times S)$ in $\mathrm{Iso}(X)$ is non-elementary, with limit set Λ, and that α is not cohomologous to a cocycle into an elementary subgroup of H. Let (B, λ) be a strong boundary for G. Since the action of G on B is amenable, there is an α-equivariant Furstenberg map $f : B \times S \to \mathcal{P}(\Lambda)$ where $\mathcal{P}(\Lambda)$ denotes the space of Borel probability measures on Λ [17].

Let $\mathcal{P}_{\geq 3}(\Lambda)$ be the space of probability measures on Λ whose support contains at least 3 points. By Corollary 5.3 of [2], the H-action on $\mathcal{P}_{\geq 3}(\Lambda)$ is tame with compact point stabilizers. Moreover, since B is a strong boundary for G, the action of G on $B \times S$ is ergodic. Thus if the image under f of a set of positive measure in $B \times S$ is contained in $\mathcal{P}_{\geq 3}(\Lambda)$ then by ergodicity we have $f(B \times S) \subset \mathcal{P}_{\geq 3}(\Lambda)$

(almost everywhere). Moreover by the results of Zimmer [17], the cocycle α is cohomologous to a cocycle into a compact subgroup of H. This is a contradiction to our assumption on α.

As a consequence, we have $f(B \times S) \subset \mathcal{P}_{\leq 2}(\Lambda)$ where $\mathcal{P}_{\leq 2}(\Lambda)$ is the set of measures whose support contains at most 2 points. To show that the measures in $f(B \times S)$ are in fact supported in a single point of Λ we argue as in the proof of Lemma 3.4 of [13]. Namely, the group H is locally compact and σ-compact and its action on the space of triples of pairwise distinct points in Λ is proper (see [2]). Thus the assumptions in Lemma 23 of [11] are satisfied. We can then use Lemma 23 of [11] as in the proof of Lemma 3.4 of [13] to conclude that the image of f is in fact contained in the set of Dirac masses on Λ, i.e. there is an α-equivariant map $B \times S \to \Lambda$ as claimed. $\square$

We use Lemma 3.1 and Theorem 2.3 to deduce.

Corollary 3.2. *Let G be a locally compact σ-compact group and let ρ be a homomorphism of G into the isometry group $\mathrm{Iso}(X)$ of a proper hyperbolic geodesic metric space X. Let H be the closure of $\rho(G)$ in $\mathrm{Iso}(X)$ and let μ be Haar measure on H. If H is non-elementary, then $H_{cb}^2(G, L^p(H, \mu)) \neq \{0\}$ for every $p \in (1, \infty)$.*

Proof. Let $\rho : G \to \mathrm{Iso}(X)$ be a homomorphism and let H be the closure of $\rho(G)$ in $\mathrm{Iso}(X)$. Assume that H is non-elementary and let $\Lambda \subset \partial X$ be the limit set of H. By Lemma 3.1, applied to the homomorphism ρ viewed as a cocycle for the trivial action of G on a point, there is a measurable ρ-equivariant map φ from a strong boundary (B, λ) of G into Λ. Since $\rho(G)$ is dense in H, the set Λ is also the limit set of $\rho(G)$ and therefore the $\rho(G)$-orbit of every point in Λ is dense in Λ (8.27 in [8]). By equivariance, the image under φ of λ-almost every G-orbit on B is dense in Λ. In particular, the support of the measure class $\varphi_*\lambda$ is all of Λ.

Let μ be a Haar measure of H. By Theorem 2.3, for every $p \in (1, \infty)$ there is a nontrivial bounded continuous $L^p(H \times H, \mu \times \mu)$-valued cocycle ω on the space of triples of pairwise distinct points in Λ. Then the $L^p(H \times H, \mu \times \mu)$-valued λ-measurable bounded cocycle $\omega \circ \varphi^3$ on $B \times B \times B$ is non-trivial on a set of positive measure. Since B is a strong boundary for G, this cocycle then defines a *non-trivial* class in $H_{cb}^2(G, L^p(H \times H, \mu \times \mu))$ (see [12]). On the other hand, the isometric G-representation space $L^p(H \times H, \mu \times \mu)$ is a direct integral of copies of the isometric G-representation space $L^p(H, \mu)$ and therefore by Corollary 2.7 of [13] and Corollary 3.4 of [14], if $H_{cb}^2(G, L^p(H, \mu)) = \{0\}$ then also $H_{cb}^2(G, L^p(H \times H, \mu \times \mu)) = \{0\}$. This implies the corollary. $\square$

Corollary 3.2 applied to a closed non-elementary subgroup $G < \mathrm{Iso}(X)$ shows that $H_{cb}^2(G, L^p(G, \mu)) \neq \{0\}$ for every $p \in (1, \infty)$.

4. Structure of the isometry group

In this section we analyze the structure of the isometry group of a proper hyperbolic geodesic metric space X using the results of Section 2 and Section 3. For this recall that an *extension* of a locally compact group H by a topological group K is a locally compact group G which contains K as a normal subgroup and such that $H = G/K$ as topological groups. The following easy observation of Monod and Shalom [13] describes non-elementary closed subgroups of $\mathrm{Iso}(X)$ which are extensions by amenable groups.

Lemma 4.1. *Let X be a proper hyperbolic geodesic metric space and let G be a closed non-elementary subgroup of $\mathrm{Iso}(X)$. Then a maximal normal amenable subgroup of G is compact.*

Proof. Let G be a closed non-elementary subgroup of $\mathrm{Iso}(X)$ and let $H \lhd G$ be a maximal normal amenable subgroup. Since H is amenable and the Gromov boundary ∂X of X is compact, there is an H-invariant probability measure μ on ∂X. Thus H is compact if the support A of μ contains at least three points (see the proof of Theorem 21 in [11]).

Now assume that A contains at most two points. Since G is non-elementary by assumption, the limit set of G is the smallest G-invariant closed subset of ∂X (8.27 of [8]). Thus the set A is not invariant under the action of G. Let $g \in G$ be such that $A \cup gA \cup g^2 A$ contains at least 3 points. Since H is a normal subgroup of G and A is H-invariant, the set $A \cup gA \cup g^2 A$ is preserved by H and hence as before, the group H is necessarily compact. $\square$

Let G be a locally compact σ-compact topological group and let $N \lhd G$ be a compact normal subgroup. For an isometric representation ρ of G into a separable Banach space E let E^N be the closed subspace of E of all N-invariant vectors. Then ρ induces a representation of G/N into E^N. Since compact groups are amenable the following observation is a special case of Corollary 8.5.2 of [12].

Lemma 4.2. *Let G be an extension of a topological group H by a compact group N and let ρ be an isometric representation of G into a separable Banach space E. Then the projection $P : G \to H$ induces an isomorphism of $H^2_{cb}(H, E^N)$ onto $H^2_{cb}(G, E)$.*

We use Corollary 3.2, Lemma 4.1 and Lemma 4.2 to complete the proof of Theorem 1 from the introduction.

Proposition 4.3. *Let X be a proper hyperbolic geodesic metric space and let $G <$ $\mathrm{Iso}(X)$ be a closed subgroup. Then one of the following three possibilities holds.*

(1) *G is elementary.*
(2) *G contains an open subgroup G' of finite index which is a compact extension of a simple Lie group of rank 1.*
(3) *G is a compact extension of a totally disconnected group.*

Proof. Let G be a closed subgroup of the isometry group $\mathrm{Iso}(X)$ of a proper hyperbolic geodesic metric space X. Then G is locally compact. Assume that G is non-elementary; then by Lemma 4.1, the maximal normal amenable subgroup H of G is compact, and the quotient $V = G/H$ is a locally compact σ-compact topological group. Denote by $\pi : G \to G/H = V$ the canonical projection.

By the solution to Hilbert's fifth problem (see Theorem 11.3.4 in [12]), after possibly replacing G by an open subgroup of finite index (which we denote again by G for simplicity), the quotient $V = G/H$ splits as a direct product $V = V_0 \times Q$ where V_0 is a semi-simple connected Lie group with finite center and without compact factors and Q is totally disconnected. If V_0 is trivial then G is a compact extension of a totally disconnected group.

Now assume that V_0 is nontrivial. Then V_0 is not compact and the limit set Λ of $\pi^{-1}(V_0) < G$ is nontrivial. Since Q commutes with V_0, the group $\pi^{-1}(Q) < G$ commutes with $\pi^{-1}(V_0) < G$ up to the compact normal subgroup H and hence $\pi^{-1}(Q)$ acts trivially on Λ. Namely, let $x \in X$ and let $(g_i) \subset \pi^{-1}(V_0)$ be a sequence such that $(g_i x)$ converges to $\xi \in \Lambda$. If $h \in Q$ is arbitrary then for every $i > 0$ there is some $h_i' \in H$ with $hg_i = g_i h h_i'$ for all i. Now H is compact and therefore the sequence $(g_i h h_i' x)$ converges as $i \to \infty$ to ξ. This implies that $(hg_i x)$ converges to $h\xi = \xi$ $(i \to \infty)$. Thus if Λ consists of one or two points then the action of G on ∂X fixes one or two points. Since G is non-elementary by assumption, this is impossible.

As a consequence, Λ contains at least three points. Then $\pi^{-1}(Q)$ fixes at least three points in ∂X and hence $\pi^{-1}(Q) < G$ is necessarily a *compact* normal subgroup of G containing H. Since by assumption H is a maximal compact normal subgroup of G we conclude that $\pi^{-1}(Q) = H$ and that Q is trivial. In other words, if G is non-elementary then up to passing to an open subgroup of finite index, either G is a compact extension of a totally disconnected group or G is a compact extension of a semi-simple Lie group V_0 with finite center and without compact factors.

We are left with showing that if G is a compact extension of a semi-simple Lie group V_0 with finite center and without compact factors then V_0 is simple and of rank 1. For this note from Corollary 3.2 and Lemma 4.2 that $H^2_{cb}(V_0, L^2(V_0, \mu)) \neq \{0\}$. On the other hand, if V_0 is a *simple* Lie group of non-compact type and rank at least 2 then Theorem 1.4 of [13] shows that $H^2_{cb}(V_0, L^2(V_0, \mu)) = \{0\}$. The case that $V_0 = V_1 \times V_2$ is a non-trivial product of semi-simple Lie groups V_1, V_2 of non-compact type can be ruled out in the same way. Namely, in this case the representation space $L^2(V_0, \mu)$ does not admit any nontrivial V_i-invariant vectors $(i = 1, 2)$ and therefore $H^2_{cb}(V_0, L^2(V_0, \mu)) = \{0\}$ by the Burger-Monod super-rigidity result for cohomology [6] (see Theorem 14.2.2 in [12]). Together we conclude that necessarily V_0 is a simple Lie group of rank one. This finishes the proof of the proposition. $\square$

5. SUPER-RIGIDITY OF COCYCLES

Let again X be a proper hyperbolic geodesic metric space. The goal of this section is to study $\mathrm{Iso}(X)$-valued cocycles and derive Theorem 3 from the introduction.

Let S be a standard Borel space and let ν be a Borel probability measure on S. Let Γ be any countable group which admits a measure preserving ergodic action on (S, ν). This action then defines a natural continuous unitary representation of Γ into the Hilbert space $L^2(S, \nu)$ of square integrable functions on S. Let $\alpha : \Gamma \times S \to \mathrm{Iso}(X)$ be a cocycle, i.e. $\alpha : G \times S \to \mathrm{Iso}(X)$ is a Borel map which satisfies $\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$ for all $g, h \in \Gamma$ and ν-almost every $x \in S$.

If Γ is a lattice in a locally compact σ-compact topological group G then Γ admits a measure preserving action on the product space $G \times S$. Since $\Gamma < G$ is a lattice, the quotient space $(G \times S)/\Gamma$ can be viewed as a bundle over G/Γ with fibre S. If $\Omega \subset G$ is a finite measure Borel fundamental domain for the action of Γ on G then $\Omega \times S \subset G \times S$ is a finite measure Borel fundamental domain for the action of Γ on $G \times S$. Thus if we denote by μ a Haar measure on G then up to normalization, the product measure $\mu \times \nu$ projects to a probability measure λ on $(G \times S)/\Gamma$. The action of G on $G \times S$ by left translation commutes with the action of Γ and hence it projects to an action on $(G \times S)/\Gamma$ preserving the measure λ (see p.75 of [17]).

Recall that the action of Γ on (S, ν) is mildly mixing if $\liminf_{g \to \infty} \nu(A \Delta gA) \neq 0$ for every Borel set $A \subset S$ with $0 < \nu(A) < 1$. The following lemma is due to Schmidt and Walters [15].

Lemma 5.1. *Let Γ be an irreducible lattice in a product $G = G_1 \times G_2$ of two semi-simple non-compact Lie groups G_1, G_2 with finite center. If the action of Γ on (S, ν) is mildly mixing then the induced action of G_1 on $((G \times S)/\Gamma, \lambda)$ is ergodic.*

The following proposition completes the proof of Theorem 3 from the introduction and follows as in [13] from Lemma 5.1, the rigidity results for bounded cohomology of Burger and Monod [5, 6] and the work of Zimmer [17].

Proposition 5.2. *Let G be a semi-simple Lie group with finite center, no compact factors and of rank at least 2. Let $\Gamma < G$ be an irreducible lattice and let (S, ν) be a mildly mixing Γ-space. Let X be a proper hyperbolic geodesic metric space and let $\alpha : \Gamma \times S \to \mathrm{Iso}(X)$ be a cocycle. Then either α is cohomologous to a cocycle into an elementary subgroup of $\mathrm{Iso}(X)$ or there is a closed subgroup H of $\mathrm{Iso}(X)$ which is a compact extension of a simple Lie group L of rank one and there is a surjective homomorphism $G \to L$.*

Proof. Let G be a semi-simple Lie group of non-compact type with finite center and of rank at least 2 and let Γ be an irreducible lattice in G. Let (S, ν) be a mildly mixing Γ-space with invariant Borel probability measure ν. Let $\alpha : \Gamma \times S \to \mathrm{Iso}(X)$ be a cocycle into the isometry group of a proper hyperbolic geodesic metric space X. Assume that α is not cohomologous to a cocycle into an elementary subgroup of $\mathrm{Iso}(X)$. Let $H < \mathrm{Iso}(X)$ be a closed subgroup with limit set Λ which contains the image of α.

Let $\Omega \subset G$ be a Borel fundamental domain for the action of Γ on G. Let λ be the G-invariant Borel probability measure on $(G \times S)/\Gamma = \Omega \times S$. We obtain a λ-measurable function $\beta : G \times (G \times S)/\Gamma \to H$ as follows. For $z \in \Omega$ and $g \in G$ let $\eta(g, z) \in \Gamma$ be the unique element such that $gz \in \eta(g, z)\Omega$ and define

$\beta(g,(z,\sigma)) = \alpha(\eta(g,z),\sigma)$. By construction, β satisfies the cocycle equation for the action of G on $(G \times S)/\Gamma$.

Let (B, ρ) be a strong boundary for G; we may assume that B is also a strong boundary for Γ. By Lemma 5.1, the action of G on $(G \times S)/\Gamma$ is ergodic and hence by Lemma 3.1, there is a measurable β-equivariant map $\psi_0 : B \times (G \times S)/\Gamma \to \Lambda$. Hence we can define a β-equivariant map $\psi : B^3 \times (G \times S)/\Gamma \to \Lambda^3$ by

$$\psi(a,b,c,x) = (\psi_0(a,x), \psi_0(b,x), \psi_0(c,x)) \ (x \in (G \times S)/\Gamma).$$

Let $\Lambda' \subset \Lambda$ be the support of the measure $(\psi_0)_*(\rho \times \lambda)$. Then Λ' is a closed subset of Λ. For every $g \in \Gamma$ and almost every $u \in S$ the element $\alpha(g,u) \in H$ stabilizes Λ', and α is cohomologous to a cocycle with values in the intersection of H with the stabilizer of Λ'. Since the stabilizer of Λ' is a closed subgroup of $\mathrm{Iso}(X)$ we may assume without loss of generality that $\Lambda' = \Lambda$, i.e. that the support of the measure $(\psi_0)_*(\rho \times \lambda)$ equals the limit set of H.

For simplicity, write $L^2(H \times H)$ for the space of functions on $H \times H$ which are square integrable with respect to a Haar measure on $H \times H$, Let $L^{[2]}((G \times S)/\Gamma, L^2(H \times H))$ be the space of all measurable maps $(G \times S)/\Gamma \to L^2(H \times H)$ with the additional property that for each such map ζ the function $x \to \|\zeta(x)\|$ is square integrable on $(G \times S)/\Gamma$ (where $\|\ \|$ is the L^2-norm on $L^2(H \times H)$). Then $L^{[2]}((G \times S)/\Gamma, L^2(H \times H))$ has a natural structure of a separable Hilbert space, and the group G acts on $L^{[2]}((G \times S)/\Gamma, L^2(H \times H))$ as a group of isometries. In other words, $L^{[2]}((G \times S)/\Gamma, L^2(H \times H))$ is a Hilbert module for G.

By Theorem 2.3, there is a continuous $L^2(H \times H)$-valued bounded nontrivial cocycle $\varphi : \Lambda^3 \to L^2(H \times H)$. The composition of the map $\psi : B^3 \times (G \times S)/\Gamma \to \Lambda^3$ with the cocycle φ can be viewed as a nontrivial β-invariant measurable bounded map $B^3 \to L^{[2]}((G \times S)/\Gamma, L^2(H \times H))$. Since B is a strong boundary for G, this map defines a nontrivial cohomology class in $H^2_{cb}(G, L^{[2]}((G \times S)/\Gamma, L^2(H \times H)))$.

Now if G is simple then by the results of Monod and Shalom [13] there is a β-equivariant map $(G \times S)/\Gamma \to L^2(H \times H)$. Since the action of G on $(G \times S)/\Gamma$ is ergodic, by the cocycle reduction lemma of Zimmer [17] the cocycle β and hence α is cohomologous to a cocycle into a compact subgroup of H. On the other hand, if $G = G_1 \times G_2$ for semi-simple Lie groups G_1, G_2 with finite center and without compact factors, then the results of Burger and Monod [5, 6] show that via possibly exchanging G_1 and G_2 we may assume that there is an equivariant map $(G \times S)/\Gamma \to L^2(H \times H)$ for the restriction of β to $G_1 \times (G \times S)/\Gamma$, viewed as a cocycle for G_1. By Lemma 5.1 the action of G_1 on $(G \times S)/\Gamma$ is ergodic and therefore the cocycle reduction lemma of Zimmer [17] shows that the restriction of β to G_1 is equivalent to a cocycle into a compact subgroup of H. We now follow the proof of Theorem 1.2 of [13] and find a minimal such compact subgroup K of H. The cocycle β and hence α is cohomologous to a cocycle into the normalizer N of K in H. Moreover, there is a continuous homomorphism of G onto N/K. Since G is connected, the image of G under this homomorphism is connected as well and hence by Proposition 4.3, N/K is a simple Lie group of rank one. This completes the proof of the proposition. $\qquad \square$

We formulate Proposition 5.2 once more in the particular case that the measure space S consists of a single point.

Corollary 5.3. *Let G be a connected semi-simple Lie group with finite center, no compact factors and of rank at least 2. Let Γ be an irreducible lattice in G, let X be a proper hyperbolic geodesic metric space and let $\rho : \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. Let $H < \mathrm{Iso}(X)$ be the closure of $\rho(\Gamma)$. If H is non-elementary, then H is a compact extension of a simple Lie group L of rank one, and if $\pi : H \to L$ is the canonical projection then $\pi \circ \rho$ extends to a continuous surjective homomorphism $G \to L$.*

A proper hyperbolic geodesic metric space X is *of bounded growth* if there is a number $b > 0$ such that for every $x \in X$ and every $R > 0$ the closed ball of radius R about x contains at most be^{bR} disjoint balls of radius 1. This implies that every elementary subgroup of $\mathrm{Iso}(X)$ is amenable (Proposition 8 of [11]). Thus we can use Corollary 5.3 to show.

Corollary 5.4. *Let G be a connected semi-simple Lie group with finite center, no compact factors, and no factors of rank one. Let Γ be an irreducible lattice in G, let X be a proper hyperbolic geodesic metric space of bounded growth and let $\rho : \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. Then the closure of $\rho(\Gamma)$ in $\mathrm{Iso}(X)$ is compact.*

Proof. Let G, Γ, X be as in the corollary and let $\rho : \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. Since G has no factor of rank one by assumption, G does not admit any surjective homomorphism onto a simple Lie group of rank one. Thus by Corollary 5.3, the closure H of $\rho(\Gamma)$ in $\mathrm{Iso}(X)$ is elementary and hence amenable since X is of bounded growth by assumption. By Margulis' normal subgroup theorem, either the kernel K of ρ has finite index in Γ and $\rho(\Gamma)$ is finite, or K is finite and the group $\Gamma/K = \rho(\Gamma)$ has Kazhdan's property T. But $\rho(\Gamma)$ is a dense subgroup of H and therefore H has property T if this is the case for $\rho(\Gamma)$. In other words, H is an amenable group with property T and hence H is compact (see [17] for details). This shows the corollary. $\qquad\square$

6. Totally disconnected groups of isometries

In this section we investigate closed subgroups of the isometry group of a proper hyperbolic geodesic metric space X which are compact extensions of totally disconnected groups. We continue to use the assumptions and notations from Sections 2-5. We begin with a simple observation on topological properties of a closed subgroup G of $\mathrm{Iso}(X)$ on its limit set.

Lemma 6.1. *Let X be a proper hyperbolic geodesic metric space and let $G < \mathrm{Iso}(X)$ be a closed subgroup of $\mathrm{Iso}(X)$ with limit set Λ. Let Δ be the diagonal in $\Lambda \times \Lambda$. For every hyperbolic element $g \in G$ with fixed points $a \neq b \in \Lambda$, the G-orbit of (a, b) is a closed subset of $\Lambda \times \Lambda - \Delta$.*

Proof. Let X be a proper hyperbolic geodesic metric space and let $G < \mathrm{Iso}(X)$ be a closed subgroup with limit set Λ. Assume without loss of generality that G is non-elementary. Let $g \in G$ be a hyperbolic element with fixed points $a \neq b \in \Lambda$. We have to show that the G-orbit of (a, b) is a closed subset of $\Lambda \times \Lambda - \Delta$.

Thus let $(\xi_i, \eta_i) \subset \Lambda \times \Lambda - \Delta$ be a sequence of pairs of points in this orbit which converges to some $(a', b') \in \Lambda \times \Lambda - \Delta$. Then there are $u_i \in G$ such that $(u_i a, u_i b) = (\xi_i, \eta_i)$. Let $\Gamma = \{g^k \mid k \in \mathbb{Z}\}$ be the cyclic subgroup of G generated by g. Let γ be a geodesic connecting a to b and let $x_0 = \gamma(0)$. The Hausdorff distance between γ and the orbit Γx_0 of x_0 under the action of Γ is finite, say this Hausdorff distance is not bigger than some number $k_0 > 0$. As a consequence, every point on the geodesic $u_i \gamma$ is contained in the k_0-neighborhood of the orbit of $u_i x_0$ under the action of the infinite cyclic group $u_i \Gamma u_i^{-1}$.

Since $(\xi_i, \eta_i) \to (a', b') \in \Lambda \times \Lambda - \Delta$, the geodesics $u_i \gamma$ connecting the points ξ_i, η_i intersect a fixed compact subset K of X which is independent of i. Thus via composing u_i with g^{ℓ_i} for a suitable number $\ell_i \in \mathbb{Z}$ we may assume that the image of the point x_0 under the isometry u_i is contained in the closed k_0-neighborhood K' of the compact set K. Since the subset of $\mathrm{Iso}(X)$ of all isometries which map x_0 to a fixed compact subset of X is compact and since $G < \mathrm{Iso}(X)$ is closed by assumption, after passing to a subsequence we may assume that the sequence $(u_i) \subset G$ converges in G to an element $u \in G$ which maps (a, b) to (a', b'). Therefore (a', b') is contained in the G-orbit of (a, b). This shows the lemma. $\square$

Following [5], for any locally compact σ-compact topological group G, every element in the kernel of the homomorphism $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ can be represented by a *continuous quasi-morphism*. Such a continuous quasi-morphism is a continuous map $\rho : G \to \mathbb{R}$ such that

$$\sup_{g,h \in G} |\rho(g) + \rho(h) - \rho(gh)| < \infty.$$

The set of all continuous quasi-morphisms for G is naturally a vector space. A continuous quasi-morphism is bounded on every *compact* subset of G and is bounded on each fixed conjugacy class in G. The quasi-morphism ρ defines a non-trivial element in the kernel of the natural homomorphism $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ if and only if there is no continuous homomorphism $\chi : G \to \mathbb{R}$ such that $\rho - \chi$ is bounded.

We say that the quasi-morphism ρ *separates* an element $g \in G$ from a subset $A \subset G$ if $\liminf_{k \to \infty} \frac{1}{k}|\rho(g^k)| > 0$ and $\limsup_{k \to \infty} \frac{1}{k}|\rho(h^k)| = 0$ for every $h \in A$. The following proposition extends earlier results of Fujiwara [7] (see also the papers [3, 9] for generalizations of [7] in a different direction).

Proposition 6.2. *Let X be a proper hyperbolic geodesic metric space and let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ. Let $g_1, \dots, g_k \in G$ be any hyperbolic elements and let a_i, b_i be the attracting and repelling fixed point for the action of g_i on Λ, respectively. If the ordered pairs $(a_i, b_i), (b_j, a_j) \in \Lambda \times \Lambda$ $(i, j \leq k)$ are contained in pairwise distinct orbits for the action of G on $\Lambda \times \Lambda$ then for every i there is a continuous quasi-morphism $\rho_i : G \to \mathbb{R}$ which separates g_i from $\{g_j \mid j \neq i\}$.*

Proof. We follow the strategy from the proof of Theorem 2.3. Let $\kappa_0 > 0$ be a hyperbolicity constant for X. Let $\Lambda \subset \partial X$ be the limit set of a closed subgroup $G < \mathrm{Iso}(X)$. Then for every hyperbolic element $g \in G$ with fixed points $a, b \in \Lambda$, for every geodesic γ connecting a to b and for every $x_0 \in \gamma$, the orbit $\{g^k x_0 \mid k \in \mathbb{Z}\}$ of

x_0 under the action of the infinite cyclic subgroup of G generated by g is contained in the κ_0-neighborhood of γ.

Let δ_z $(z \in X)$ be a family of metrics on ∂X which satisfies the properties (1),(2) from Section 2. As in the proof of Theorem 2.3, choose $c_0 > 0$ such that $\delta_z(\xi, \eta) \geq 2c_0$ for all $\xi \neq \eta \in \Lambda$ and every $z \in X$ whose distance to a geodesic connecting ξ to η is at most κ_0. Let $\varphi_0 : [0, \infty) \to [0, 1]$ be a smooth function such that $\varphi_0(t) = 0$ for all $t \leq c_0$ and $\varphi_0(t) = 1$ for all $t \geq 2c_0$ and define a function φ on $V = (\Lambda \times \Lambda - \Delta) \times X \times X$ by $\varphi(\xi, \eta, x, y) = \varphi_0(\delta_x(\xi, \eta))\varphi_0(\delta_y(\xi, \eta))$.

Let $\kappa > 0$ be sufficiently large that the set $\{x \in X \mid \delta_x(\xi, \eta) \geq c_0\}$ is contained in the κ-tubular neighborhood of any geodesic connecting ξ to η. Let $\psi : [0, \infty) \to [0, 1]$ be a smooth function which satisfies $\psi(t) = 1$ for $t \leq \kappa$ and $\psi(t) = 0$ for $t \geq 2\kappa$ and define a function ζ on V by $\zeta(\xi, \eta, x, y) = \varphi(\xi, \eta, x, y)\psi(d(x, y))$. The function ζ is invariant under the diagonal action of G and hence it projects to a function on $W = G \backslash V$ whose support we denote by W_0. Let $\hat{d}$ be the G-invariant metric on V defined as in equation (5) in the proof of Theorem 2.3 and let d_0 be its projection to $W = G \backslash V$.

Define a G-invariant foliation $\mathcal{F}$ of V by requiring that the leaf of $\mathcal{F}$ through (ξ, η, x, y) equals $F(\xi, \eta) = \{(\xi, \eta, x, y) \mid x, y \in X\}$. The foliation $\mathcal{F}$ then projects to a foliation $\hat{\mathcal{F}}$ on W. Let $\iota : V \to V$ be the G-equivariant involution which maps a point $(\xi, \eta, x, y) \in V$ to (η, ξ, x, y). This involution projects to an involution of W which we denote again by ι. Let $P : V \to W$ be the canonical projection.

Let $g_1, \ldots, g_k \in G$ be hyperbolic elements with attracting and repelling fixed points $a_i, b_i \in \Lambda$, respectively. Assume that the G-orbits of the ordered pairs $(a_i, b_i), (b_j, a_j) \in \Lambda \times \Lambda$ $(i, j \in \{1, \ldots, k\})$ are pairwise disjoint. Then we have $\iota(PF(a_i, b_i)) \cap PF(a_j, b_j) = \emptyset$ for all i, j. In particular, the action of ι on $W_0 \cap \bigcup_i PF(a_i, b_i)$ is non-trivial. Since by Lemma 6.1 for every $i \in \{1, \ldots, k\}$ the G-orbit of (a_i, b_i) is a closed subset of $\Lambda \times \Lambda - \Delta$, we can find a small ball $B \subset W$ which is disjoint from its image under ι and such that the interior of the preimage $\tilde{B}$ of B in V meets the intersection of $F(a_i, b_i)$ with the support of ζ and that $\tilde{B} \cap F(a_j, b_j) = \emptyset$ for $j \neq i$.

As in the proof of Theorem 2.3, let $\mathcal{H}$ be the vector space of Hölder continuous functions on W supported in B. For $f \in \mathcal{H}$ denote by $\tilde{f}$ the lift of f to a G-invariant ι-anti-invariant function on V. For $\xi \neq \eta \in \Lambda$ define $f_{\xi, \eta}$ to be the restriction of $\tilde{f}\zeta$ to the leaf $F(\xi, \eta)$, viewed as a function on $X \times X$ (note that this is slightly inconsistent with the notations in the proof of Theorem 2.3). For a subset C of X and a number $r > 0$ define $N(C, r) = \{(x, y) \in X \times X \mid d(x, C) \leq r, d(y, C) \leq r, d(x, y) \leq 2\kappa\}$. Then for all $\xi \neq \eta \in \Lambda$ and every geodesic γ connecting η to ξ the support of $f_{\xi, \eta}$ is contained in $N(\gamma, \kappa)$. By Proposition 5.2 of [13], there is a G-invariant Radon measure μ on X with full support such that the μ-mass of a ball in X of radius 4κ is at most one. Let $\nu = \mu \times \mu$; it follows from the discussion in the proof of Theorem 2.3 that for every subset C of X of diameter at most R the ν-mass of the intersection $N(C, \kappa) \times N(\gamma, \kappa)$ is bounded from above by mR for a universal constant $m > 0$.

Choose an arbitrary point $\xi \in \Lambda$ and a small open neighborhood A of ξ in $X \cup \partial X$. Identifying each leaf of the foliation $\mathcal{F}$ with $X \times X$, for $f \in \mathcal{H}$ and an element $g \in G$ with $g\xi \neq \xi$ define

$$\Phi(f)(g) = \int_{F(\xi,g\xi)-A\times A-gA\times gA} f_{\xi,g\xi}\,d\nu$$

and if $g\xi = \xi$ define $\Phi(f)(g) = 0$. By construction, the intersection of the support of $f_{\xi,g\xi}\zeta_{\xi,g\xi}$ with $X \times X - A \times A - gA \times gA$ is compact and hence this integral exists.

We claim that $\Phi(f)(g)$ depends continuously on g. Namely, if $(g_i) \subset G$ is any sequence which converges to some $g \in G$ $(i \to \infty)$ then $g_i(X\cup\partial X - A)$ converges to $g(X\cup\partial X - A)$ in the Hausdorff topology for compact subsets of $X \cup \partial X$, moreover $g_i\xi \to g\xi$. Since $\tilde{f}\zeta$ is continuous and bounded, the functions $f_{\xi,g_i\xi}$ converge locally in $L^1(X \times X, \nu)$ to $f_{\xi,g\xi}$ and consequently $\Phi(f)(g_i) \to \Phi(f)(g)$ $(i \to \infty)$. In other words, $g \to \Phi(f)(g)$ is continuous.

Next we claim that $\Phi(f)$ is a quasi-morphism for G. More precisely, there is some number $c > 0$ only depending on the Hölder norm of f such that

$$|\Phi(f)(g) + \Phi(f)(h) - \Phi(f)(gh)| \leq c$$

for all $g, h \in G$. To see this simply observe that by invariance of $\tilde{f}\zeta$ under the action of G and by anti-invariance of $\tilde{f}\zeta$ under the involution ι we have

$$\Phi(f)(g) + \Phi(f)(h) - \Phi(f)(gh) = \int_{F(\xi,g\xi)-A\times A-gA\times gA} f_{\xi,g\xi}\,d\nu$$

$$+ \int_{F(g\xi,gh\xi)-gA\times gA-ghA\times ghA} f_{g\xi,gh\xi}\,d\nu + \int_{F(gh\xi,\xi)-ghA\times ghA-A\times A} f_{gh\xi,\xi}\,d\nu$$

and therefore our claim is an immediate consequence of the estimates in Step 3 and Step 4 of the proof of Theorem 2.3.

For every $j \in \{1,\ldots,k\}$ the infinite cyclic subgroup Γ generated by g_j acts properly on the intersection of $F(a_j, b_j)$ with the support of the function ζ, with compact fundamental domain D_j. By our choice of B, for every $q \in \mathbb{R}$ there is a function $f \in \mathcal{H}$ with the property that $\int_{D_i} f_{a_i,b_i}\,d\nu = q$ and $\int_{D_\ell} f_{a_\ell,b_\ell}\,d\nu = 0$ for $\ell \neq i$. Now the arguments in the proof of Theorem 3.1 of [9] show that $\lim_{k\to\infty} \Phi(f)(g_i^k)/k = q$ and $\lim_{k\to\infty} \Phi(f)(g_\ell^k)/k = 0$ for all $\ell \neq i$. In other words, $\Phi(f)$ is a continuous quasi-morphism for G which separates g_i from $\{g_j \mid j \neq i\}$. $\square$

The following corollary shows the first part of Theorem 2 from the introduction and extends earlier results of Fujiwara [7] (see also [3, 9]).

Corollary 6.3. *Let X be a proper hyperbolic geodesic metric space and let G be a closed non-elementary subgroup of $\mathrm{Iso}(X)$ with limit set $\Lambda \subset \partial X$. If the action of G on the complement of the diagonal in $\Lambda \times \Lambda$ is not transitive then the kernel of the natural homomorphism $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ is infinite dimensional.*

Proof. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ. Denote by Δ the diagonal in $\Lambda \times \Lambda$ and assume that G does not act transitively on $\Lambda \times \Lambda - \Delta$. The set of pairs of fixed points of hyperbolic elements of G is dense in $\Lambda \times \Lambda - \Delta$, and the action of G on $\Lambda \times \Lambda - \Delta$ has a dense orbit [8]. Let $g \in G$ be a hyperbolic element and let $(a, b) \in \Lambda \times \Lambda$ be the ordered pair of fixed points for the action of G on ∂X. By Lemma 6.1, the G-orbit $G(a, b)$ of (a, b) is a closed subset of $\Lambda \times \Lambda - \Delta$. Since G does not act transitively on $\Lambda \times \Lambda - \Delta$ by assumption, the complement of $G(a, b)$ in $\Lambda \times \Lambda - \Delta$ contains a pair of fixed points (a', b') for a hyperbolic element $h \in G$. The orbit $G(a', b')$ of (a', b') is distinct from the orbit $G(a, b)$ of (a, b).

Let γ, γ' be geodesics connecting b, b' to a, a'. By Lemma 6.1 and its proof, for every $m > 0$ there is a number $R > 0$ such that for every subsegment η of γ of length R, there is no $u \in G$ which maps η into the m-neighborhood of γ'. In other words, the group G satisfies the assumption in Theorem 1 of [3]. As a consequence of Proposition 2 of [3] (whose proof is valid without the assumption that the space X is a graph or that the group of isometries is countable), there is a free subgroup $\Gamma < G$ with two generators consisting of hyperbolic elements and with the following properties.

   i) For a fixed point $x_0 \in X$, the orbit map $u \in \Gamma \to ux_0 \in X$ is a quasi-isometric embedding.
   ii) There are infinitely many elements $u_i \in \Gamma$ $(i > 0)$ with fixed points $a_i, b_i \in \Lambda$ such that for all $i > 0$ the G-orbit of $(a_i, b_i) \in \Lambda \times \Lambda - \Delta$ is distinct from the orbit of (b_j, a_j) $(j > 0)$ or (a_j, b_j) $(j \neq i)$.

Choose $\{h_1, \ldots, h_n\} \subset \{u_i \mid i > 0\} \subset \Gamma$ as in ii) above. By Proposition 6.2, for every i there is a continuous quasi-morphism ρ_i for G which separates h_i from $\{h_j \mid j \neq i\}$. This implies that the dimension of the kernel of the natural homomorphism $H^2_{cb}(G, \mathbb{R}) \to H_c(G, \mathbb{R})$ is at least n. Since $n > 0$ was arbitrary we conclude that the kernel of the natural map $H^2_{cb}(G, \mathbb{R}) \to H_c(G, \mathbb{R})$ is indeed infinite dimensional. $\square$

Remark: The proof of Corollary 6.3 also shows the following. If $G < \mathrm{Iso}(X)$ is a closed subgroup with limit set Λ whose action on $\Lambda \times \Lambda - \Delta$ is not transitive then there is an infinite dimensional vector space of continuous bounded G-invariant functions $\omega : \Lambda^3 \to \mathbb{R}$ which are anti-symmetric under permutations of the three variables and satisfy the cocycle equation (4).

The following proposition completes the proof of Theorem 2 from the introduction.

Proposition 6.4. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ. If G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then the kernel of the natural homomorphism $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ is trivial.*

Proof. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup which acts transitively on the space A of pairs of distinct points in Λ. Since every element in the kernel of the natural map $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ can be represented by a continuous

unbounded quasi-morphism it suffices to show that such a continuous unbounded quasi-morphism for G does not exist.

Let $a \in \Lambda$ be any point. For $x, y \in X$ and a geodesic ray $\gamma : [0, \infty) \to X$ connecting x to a write $\beta(y, \gamma) = \limsup_{t \to \infty}(d(y, \gamma(t)) - t)$ and define the *Busemann function*

$$(17) \qquad \beta_a(y, x) = \sup\{\beta(y, \gamma) \mid \gamma \text{ is a geodesic ray connecting } x \text{ to } a\}.$$

By Lemma 8.1 of [8], there is a constant $c > 0$ with the following properties. Let $\gamma : \mathbb{R} \to X$ be any geodesic with $\gamma(t) \to a$ $(t \to \infty)$. Then for every fixed $s \in \mathbb{R}$ and all sufficiently large $T > 0$ we have

$$(18) \qquad |\beta_a(\cdot, \gamma(s)) - (d(\cdot, \gamma(T)) - T + s)| \leq c.$$

This implies that

$$(19) \qquad |\beta_a(\cdot, \gamma(s)) - \beta_a(\cdot, \gamma(t)) + s - t| \leq 2c$$

for all $s, t \in \mathbb{R}$. Morever, by Proposition 8.2 of [8], for all $x, y \in X$ we have

$$(20) \qquad |\beta_a(\cdot, y) - \beta_a(\cdot, x) + \beta_a(y, x)| \leq c$$

and consequently

$$(21) \qquad |\beta_a(x, y) + \beta_a(y, x)| \leq 2c.$$

Define the *horosphere* at a through x to be the set

$$(22) \qquad H_a(x) = \beta_a(\cdot, x)^{-1}[-4c, 4c].$$

By inequality (21), if $z \in X$ is any point with $|\beta_a(x, z)| \leq 2c$ then $|\beta_a(z, x)| \leq 4c$ and hence $z \in H_a(x)$.

We claim that there is a universal constant $c_0 > 0$ with the following property. Let $\gamma : \mathbb{R} \to X$ be a biinfinite geodesic with $\gamma(t) \to a$ $(t \to \infty)$. Then for every point $y \in X$ and every $t \in \mathbb{R}$ we have

$$(23) \qquad d(\gamma(t), H_a(y)) \leq |\beta_a(y, \gamma(t))|.$$

To see this let $p = \beta_a(y, \gamma(t))$. Then the estimate (19) shows that $|\beta_a(y, \gamma(t + p))| \leq 2c$. Thus by the inequality (21) we have $|\beta_a(\gamma(t + p), y)| \leq 4c$ and hence $\gamma(t + p) \in H_a(y)$. This shows the claim

Since G is non-elementary by assumption, G contains a hyperbolic element $g \in G$. Let $a \in \Lambda$ be the attracting fixed point of g and let $b \in \Lambda - \{a\}$ be the repelling fixed point. Then g preserves the set of geodesics connecting b to a. Let $W(a, b) \subset X$ be the closed non-empty subset of all points in X which lie on a geodesic connecting b to a. The set $W(a, b)$ is contained in a tubular neighborhood of fixed radius $\kappa_0 > 0$ about a fixed geodesic connecting b to a. The isometry $g \in G$ is hyperbolic with fixed points $a, b \in \Lambda$ and therefore it preserves $W(a, b)$. If we denote by Γ the infinite cyclic subgroup of G generated by g then $W(a, b)/\Gamma$ is compact. As a consequence of the estimate (18), there is a number $\nu > 0$ and for every $x \in W(a, b)$ and every $t \in \mathbb{R}$ there is a number $k(t) \in \mathbb{Z}$ with $|\beta_a(g^{k(t)}(x), x) - t| < \nu$. It follows from this and (20) that for every $y \in X$ there is some $k = k(y) \in \mathbb{Z}$ such that $|\beta_a(y, g^k(x))| \leq \nu + c$. Since $g^k(x)$ is contained in

a biinfinite geodesic converging to a, we obtain from the estimate (23) above that the distance between $g^k(x)$ and the horosphere $H_a(y)$ is at most $\delta_0 = \nu + c$.

Since G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$, the stabilizer $G_a < G$ of the point $a \in \Lambda$ acts transitively on $\Lambda - \{a\}$. Thus there is for every $\zeta \in \Lambda - \{a\}$ an element $h_\zeta \in G_a$ with $h_\zeta(b) = \zeta$. Let again $x \in W(a,b)$. Since $h_\zeta \in G_a$ we have $h_\zeta^{-1}(H_a(x)) = H_a(h_\zeta^{-1}(x))$. By our above consideration, there is a number $\ell \in \mathbb{Z}$ such that the distance between $g^\ell(x)$ and $H_a(h_\zeta^{-1}(x))$ is at most δ_0 and therefore the distance between $h_\zeta \circ g^\ell(x)$ and $H_a(x)$ is at most δ_0. Hence via replacing h_ζ by $h_\zeta \circ g^\ell \in G_a$ we may assume that the distance between $h_\zeta(x)$ and $H_a(x)$ is at most δ_0.

For $x \in X$ denote by $N_{a,x} \subset G_a$ the set of all elements $h \in G_a$ with the property that the distance between $H_a(x)$ and hx is at most δ_0. The above consideration shows that for every $x \in W(a,b)$ and every $\zeta \in \Lambda - \{a\}$ there is some $h_\zeta \in N_{a,x}$ which maps b to ζ.

For every $h \in N_{a,x}$ the sequence $(g^{-\ell} \circ h \circ g^\ell)_{\ell > 0}$ is contained in G_a. We claim that there is a number $\delta_1 > 0$ not depending on h such that $(g^{-\ell} \circ h \circ g^\ell)(x)$ is contained in the δ_1-neighborhood of $H_a(x)$ for every $\ell > 0$. Namely, inequality (18) together with the triangle inequaltity shows that

$$(24) \qquad |\beta_a(y,x) - \beta_a(z,x)| \leq d(y,z) + 2c$$

for all $z, y \in X$ and hence we conclude that $|\beta_a(h(x),x)| \leq \delta_0 + 6c$ for every $h \in N_{a,x}$. As a consequence of this and inequality (20), we obtain that

$$|\beta_a(\cdot, x) - \beta_a(\cdot, h(x))| \leq \delta_0 + 7c.$$

Let again $\gamma : \mathbb{R} \to X$ be a geodesic connecting b to a with $\gamma(0) = x$. Then h maps the geodesic γ to a geodesic $h(\gamma)$ connecting $h(b)$ to a. By inequality (19), we have $|\beta_a(\cdot, h(\gamma(t))) - \beta_a(\cdot, h(x)) + t| \leq 2c$ for all $t \in \mathbb{R}$ and the same estimate for the geodesic γ then implies that

$$|\beta_a(\cdot, \gamma(t)) - \beta_a(\cdot, h(\gamma(t)))| \leq \delta_0 + 12c \text{ for all } t \in \mathbb{R}.$$

Together with the estimate (23) we deduce that for every $t > 0$ the distance between $h(\gamma(t))$ and $H_a(\gamma(t))$ is bounded from above by $\delta_0 + 12c$. Since the infinite cyclic group Γ preserves the set of geodesics connecting b to a we conclude that there is indeed a universal constant $\delta_1 > 0$ such that for every $\ell \geq 0$ the distance between $h(g^\ell x)$ and $H_a(g^\ell(x))$ is at most δ_1. This shows the above claim.

Now for every $h \in N_{a,x}$ the sequence $(g^{-\ell} \circ h \circ g^\ell(b))_{\ell > 0} \subset \Lambda$ converges as $\ell \to \infty$ to b. This means that there is a number $\delta_2 > \delta_1$ such that for sufficiently large ℓ the element $g^{-\ell} \circ h \circ g^\ell \in G_a$ maps the point x into the closed δ_2-neighborhood B_x of x. The group Γ acts on $W(a,b)$ cocompactly and hence if $C \subset W(a,b)$ is a compact fundamental domain for this action, then $B = \cup_{x \in C} B_x$ is compact. Moreover, for every $x \in W(a,b)$, every element $h \in N_{a,x}$ is conjugate in G to an element in the compact subset $K = \{u \in G \mid uB \cap B \neq \emptyset\}$ of G. As a consequence, the restriction to $N_{a,b} = \cup_{x \in W(a,b)} N_{a,x}$ of any continuous quasi-morphism q on G is uniformly bounded. By our assumption on G the sets $N_{a,b}$ $((a,b) \in A)$ are pairwise conjugate in G and hence q is uniformly bounded on $\cup_{(a,b) \in A} N_{a,b} = N$.

Next we show that the restriction of a quasi-morphism q to the subgroup $G_{a,b}$ of G_a which stabilizes both points $a, b \in \Lambda$ is bounded. For this consider an arbitrary element $u \in G_{a,b}$. We may assume that $u \notin N_{a,b}$. Let $x \in C \subset W(a, b)$ where as before, C is a compact fundamental domain for the action of the infinite cyclic group Γ on $W(a, b)$.

By assumption, G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ and hence there is an element $h \in G$ such that $h(a) = b$ and $h(b) = a$. Then $hW(a, b) = W(a, b)$ and hence via composition of h with an element of Γ we may assume that $hx \in C$. Thus if $\delta_3 > 0$ is the diameter of C then

$$|\beta_b(hux, x) - \beta_a(ux, x)| \le |\beta_b(hux, hx) - \beta_a(ux, x)| + c + d(hx, x) \le \delta_3 + c$$

by the estimate (20).

On the other hand, from another application of the estimate (18) we obtain the existence of a constant $\delta_4 > 0$ such that $|\beta_b(z, y) - \beta_a(y, z)| \le \delta_4$ for all $y, z \in W(a, b)$. But this just means that the distance between hux and $u^{-1}x$ is uniformly bounded and hence the distance between $uhux$ and x is uniformly bounded as well. As a consequence, the element uhu is contained in a fixed compact subset of G. As before, we conclude from this that $|q(u)|$ is bounded from above by a universal constant not depending on u and hence the restriction of q to $G_{a,b}$ is uniformly bounded.

Now let $h \in G$ be arbitrary and assume that $ha = x, hb = y$. We showed above that there are $h_y \in N \cap G_x, h_x \in N \cap G_b$ with $h_y(y) = b$ and $h_x(x) = a$; then $h' = h_x h_y h \in G_{a,b}$ and $|q(h') - q(h)|$ is uniformly bounded. As a consequence, q is bounded and hence it defines the trivial bounded cohomology class. This completes the proof of the proposition. $\qquad\square$

Examples:

1) Let T be a regular k-valent tree for some $k \ge 3$. Then T is a proper hyperbolic geodesic metric space, and its isometry group G is totally disconnected. If ∂T denotes the Gromov boundary of T then the group G acts transitively on the space Y of *triples* of pairwise distinct points in ∂T. Moreover, there is a G-invariant measure class λ on ∂T with the property that $(\partial T, \lambda)$ is a strong boundary for G (see [1, 10]). As a consequence, every bounded cohomology class $\omega \in H^2_{cb}(G, \mathbb{R})$ can be represented by a G-invariant λ^3-measurable bounded function $\omega : Y \to \mathbb{R}$ [6, 12] which is anti-symmetric under permutations of the three variables and which satisfies the cocycle condition (4). Since the action of G on Y is transitive, such a function has to vanish. In other words, $H^2_{cb}(G, \mathbb{R}) = \{0\}$.

2) Let G be a simple rank-one Lie group of non-compact type. Then G is the isometry group of a negatively curved symmetric space X. The limit set of G is the full Gromov boundary ∂X of X, moreover the action of G on the complement of the diagonal in $\partial X \times \partial X$ is transitive. By Proposition 6.4 and well known results on the usual continuous cohomology of G, the second bounded cohomology group $H^2_{cb}(G, \mathbb{R})$ is trivial if $G \ne SU(n, 1)$ and equals $\mathbb{R}$ for $G = SU(n, 1)$ for some $n \ge 2$.

7. Proper hyperbolic spaces of bounded growth

In this section we investigate more restrictively proper hyperbolic geodesic metric spaces of bounded growth. This means that there is a number $b > 1$ such that for every $R > 1$, every metric ball of radius R contains at most be^{bR} disjoint metric balls of radius 1. The following proposition is Theorem 4 from the introduction.

Proposition 7.1. *Let Γ be a finitely generated group which admits a proper isometric action on a proper hyperbolic geodesic metric space of bounded growth. If $H^2_{cb}(\Gamma, \mathbb{R})$ or $H^2_{cb}(\Gamma, \ell^2(\Gamma))$ is finite dimensional then Γ is virtually nilpotent.*

Proof. Let Γ be a finitely generated group which admits a proper isometric action on a proper hyperbolic geodesic metric space X of bounded growth. Assume that Γ is infinite and that $H^2_{cb}(\Gamma, \mathbb{R})$ or $H^2_{cb}(\Gamma, \ell^2(\Gamma))$ is finite dimensional. By the results of Fujiwara [7] (for real coefficients) and by [9] (for coefficients $\ell^2(\Gamma)$), the subgroup Γ of $\mathrm{Iso}(X)$ is elementary. We have to show that Γ is virtually nilpotent. Since Γ is infinite and acts properly on X by assumption, the limit set of Γ is nontrivial and hence it either consists of exactly two points or there is a global fixed point $a \in \partial X$ for the action of Γ on ∂X. Assume first that $a \in \partial X$ is fixed by Γ.

We recall some notations from the proof of Proposition 6.4. Namely, since X is locally compact, for every $x \in X$ there is a geodesic ray $\gamma : [0, \infty) \to X$ connecting x to a. For $x \in X$ and $a \in \partial X$ let $y \to \beta_a(y, x)$ be the Busemann function determined by a and x as defined in (17) in the proof of Proposition 6.4. Recall also from (20) that there is a number $c > 0$ such that $|\beta_a(\cdot, y) - \beta_a(\cdot, x) + \beta_a(y, x)| \le c$ for all $x, y \in X$. For $x \in X$ let $H_a(x)$ be the horosphere through x and a defined in (22). By the estimate (24), for all $x, y \in X$ and all $r \in \mathbb{R}$ the distance between $\beta_a(\cdot, x)^{-1}(r)$ and y is at least $|\beta_a(y, x) - r| - 2c$. Together with the estimate (20), this implies that the distance in X between any two horospheres $H_a(x), H_a(y)$ for $x, y \in X$ is at least $|\beta_a(y, x)| - 10c$.

We claim that there is a number $\kappa > 0$ only depending on the hyperbolicity constant of X such that for every $x \in X$ the group Γ maps the horosphere $H_a(x)$ into $\beta_a(\cdot, x)^{-1}[-\kappa, \kappa]$. To see this let $g \in \Gamma, x \in X$ be arbitrary. Then $gH_a(x) = H_a(gx)$ and therefore if for some $r > 4c$ the image of $H_a(x)$ under g is not contained in $\beta_a(\cdot, x)^{-1}[-r, r]$ then the distance between $H_a(x)$ and $gH_a(x)$ is at least $r - 10c$.

Thus if for *every* $x \in X$ the image of $H_a(x)$ under g is *not* contained in $\beta_a(\cdot, x)^{-1}[-r, r]$ then the minimal displacement $\inf\{d(y, gy) \mid y \in X\}$ of g is at least $r - 10c$. However, by Proposition 8.24 in [8] there is a universal constant $r_0 > 0$ such that every isometry of X whose minimal displacement is at least $r_0 - 10c$ is hyperbolic and hence it generates an infinite cyclic group of isometries of X whose limit set consists of two points. Since Γ fixes a, this violates the assumption that the action of Γ on X is proper. As a consequence, for every $g \in \Gamma$ there is a point $x(g) \in X$ such that

$$gH_a(x(g)) \subset \beta_a(\cdot, x(g))^{-1}[-r_0, r_0].$$

In particular, we have $|\beta_a(g(x(g)), x(g))| \le r_0$.

From inequality (20) above with $x = x(g), y = g(x(g))$ we conclude that

$$|\beta_a(\cdot, x(g)) - \beta_a(\cdot, g(x(g)))| \leq r_0 + c.$$

On the other hand, inequality (20) applied to an arbitrary point $y \in X$ and to $x = x(g)$ shows that $|\beta_a(\cdot, y) - \beta_a(\cdot, x(g)) + \beta_a(y, x(g))| \leq c$ and similarly $|\beta_a(\cdot, gy) - \beta_a(\cdot, g(x(g))) + \beta_a(gy, g(x(g)))| \leq c$. Since g fixes the point a we have $\beta_a(y, x(g)) = \beta_a(gy, g(x(g)))$ and therefore $|\beta_a(\cdot, y) - \beta_a(\cdot, gy)| \leq r_0 + 3c$ for every $y \in X$. Since $\beta_a(gz, gy) = \beta_a(z, y)$ for all $g \in \Gamma$, all $y, z \in X$ this shows the above claim.

Let $x \in X$ be an arbitrary point and let $\kappa > 0$ be such that for every $g \in \Gamma$ the horosphere $gH_a(x)$ is contained in $\beta_a(\cdot, x)^{-1}[-\kappa, \kappa]$. Choose a finite symmetric generating set $g_1, \ldots, g_{2k}$ for Γ; such a set exists since Γ is finitely generated by assumption. Let $q = \max\{d(x, g_i x) \mid i = 1, \ldots, 2k\}$. Since for every $t \in \mathbb{R}$ the distance between $\beta_a(\cdot, x)^{-1}(t)$ and $H_a(x)$ is at least $|t| - 6c$, each of the points $g_i x$ can be connected to x by a curve of length at most q which is contained in $\beta_a(\cdot, x)^{-1}[-q - 6c, q + 6c]$. Similarly, by the choice of κ, for every $g \in \Gamma$ the points $gx, gg_i x$ can be connected by a curve of length at most q contained in

$$V = \beta_a(\cdot, x)^{-1}[-\kappa - q - 6c, \kappa + q + 6c].$$

For $g \in \Gamma$ let $|g|$ be the *word norm* of g with respect to the generating set $g_1, \ldots, g_{2k}$, i.e. $|g|$ is the minimal length of a word in $g_1, \ldots, g_{2k}$ representing g. Then for every $g \in \Gamma$ the point gx can be connected to x by a curve which is contained in V and whose length is at most $q|g|$.

For $y, z \in V$ define $\delta(y, z) \in [0, \infty]$ to be the infimum of the lengths of any curve in X which connects y to z and is contained in V. Then the restriction of δ to a path-connected component W of V containing the Γ-orbit of x is a distance which is not smaller than the restriction of d. For $y \in W$ let $B_W(y, R) \subset W$ be the δ-ball of radius R about y. Let $\gamma : [0, \infty) \to X$ be a geodesic ray connecting x to a. We claim that there is a number $\chi \geq 1$ such that for every $R > 0$ the δ-ball $B_W(x, e^R)$ of radius e^R about x is contained in the ball in X of radius $\chi R + \chi$ about $\gamma(\chi R)$.

To see this we argue as in [1]. Recall from hyperbolicity that there is a number $\kappa_0 \geq 1$ such that for every geodesic triangle $\triangle \subset X$ with sides a, b, c, the side c is contained in the κ_0-neighborhood of $a \cup b$. Let $y \in W$ be such that $\delta(x, y) \leq 2^m$ for some $m \geq 0$. Then there are $2^m + 1$ points $x_0 = x, \ldots, x_{2^m} = y \in W \subset V$ such that $d(x_i, x_{i+1}) \leq 1$ for all i. Write $\tilde{q} = \kappa + q + 1 + 12c$. Then for $i \leq 2^m$ a geodesic $\gamma_{i,1}$ in X connecting x_{i-1} to x_i is contained in $\beta_a(\cdot, x)^{-1}[-\tilde{q}, \infty)$.

For each $i \leq 2^{m-1}$ connect the points x_{2i-2} and x_{2i} by a geodesic $\gamma_{i,2}$. Then $\gamma_{i,2}$ is contained in the κ_0-neighborhood of $\gamma_{2i-1,1} \cup \gamma_{2i,1}$ and hence

$$\gamma_{i,2} \subset \beta_a(\cdot, x)^{-1}[-\tilde{q} - \kappa_0 - 12c, \infty).$$

Write $\alpha = \kappa_0 + 12c$. By induction, for $j \leq m$ choose a geodesic $\gamma_{i,2^j}$ connecting the points $x_{(i-1)2^j}$ and x_{i2^j} $(i = 1, \ldots, 2^{m-j})$. A successive application of our argument implies that $\gamma_{i,2^j}$ is contained in $\beta_a(\cdot, x)^{-1}[-j\alpha - \tilde{q}, \infty)$. In particular, a geodesic ζ connecting x to y is contained in $\beta_a(\cdot, x)^{-1}[-m\alpha - \tilde{q}, \infty)$. On the other hand, ζ is a side of an (ideal) geodesic triangle with vertices x, y, a and the given geodesic ray γ as a second side. Since for $t \geq 0$ we have $|\beta_a(\gamma(t), x) - t| \leq c$ by inequality (18), we conclude that for any geodesic ray η connecting y to a the distance between $\eta(\alpha m)$

and $\gamma(\alpha m)$ is bounded by a constant $\tilde{c} > 0$ only depending on the hyperbolicity constant of X. As a consequence, every point $y \in W$ with $\delta(x, y) \leq 2^m$ is contained in the ball of radius $\alpha m + \tilde{c}$ about $\gamma(\alpha m)$. This is just the statement of our claim with $\chi = \max\{\alpha \log 2, \tilde{c}\}$.

From this observation we conclude that the group Γ has *polynomial growth*. By definition, this means that there is a number $p > 0$ such that the number of elements $g \in \Gamma$ of word norm at most ℓ is not bigger than $p\ell^p$. Namely, we observed above that the image of x under an element $g \in \Gamma$ of word norm at most ℓ is contained in the ball $B_W(x, q\ell)$ where $q > 0$ is as above. Since Γ acts properly and isometrically on X by assumption, there is a number $j > 0$ such that there are at most j elements $g \in \Gamma$ with $gx \in B(x, 4)$. Then for every $z \in X$ the ball $B(z, 2)$ contains at most j points from the orbit Γx of x counted with multiplicity. Thus the number of elements of Γ of word norm at most ℓ is not bigger than j times the maximal number of disjoint balls of radius 1 contained in $B_W(x, q\ell + 1)$ where $q > 0$ is as above. Namely, if this number equals $k > 0$ then there are k balls of radius 2 which cover $B_W(x, q\ell + 1)$.

On the other hand, by our above observation the ball $B_W(x, q\ell + 1)$ is contained in a ball of radius $\chi(\log(q\ell)) + \chi$ in X for a universal number $\chi > 0$. By assumption, a ball in X of radius $R > 1$ contains at most be^{bR} disjoint balls of radius 1, where $b > 0$ is a universal constant. Together this shows that Γ is indeed of polynomial growth. By a well known result of Gromov, groups of polynomial growth are virtually nilpotent. This shows the proposition in the case that Γ fixes a point in ∂X.

If the limit set Λ of Γ consists of two distinct points $a \neq b \in \partial X$ then by invariance of Λ under the action of Γ, every element of Γ maps a geodesic in X connecting these two points to a geodesic with the same properties. Now the Hausdorff distance between any two such geodesics is bounded from above by a universal constant. Since the action of Γ on X is moreover proper, this implies immediately that the group Γ is of polynomial growth and hence virtually nilpotent. $\square$

Example: Proposition 7.1 is easily seen to be false for proper hyperbolic geodesic metric spaces which are not of bounded growth. Namely, let (M, g) be a symmetric space of non-compact type and higher rank and consider the space $N = M \times \mathbb{R}$ with the warped product metric $e^{2t}g \times dt$. Then N is a complete simply connected Riemannian manifold whose curvature is bounded from above by a negative constant and hence N is a proper hyperbolic geodesic metric space. However, the isometry group of N contains an elementary subgroup which is a semi-simple Lie group of non-compact type and higher rank. Note that the curvature of N is not bounded from below and N is not of bounded growth.

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