ON THE COHOMOLOGY OF STRATA OF ABELIAN DIFFERENTIALS

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Abstract. For $g \geq 3$, we study the cohomology classes in the closure of a stratum of abelian differentials defined by the boundary strata of codimension one. As an application, we find an explicit stratification of the spin moduli space for an odd spin structure consisting of $g-1$ strata $\mathcal{D}_j$ of codimension $j-1$ such that $\mathcal{D}_j$ does not contain a complete variety for all j. We also recover some results of Korotkin and Zograf and of Chen using a unified topological argument.

1. Introduction

For $g \geq 3$ the *moduli space* $\mathcal{M}_g$ of complex curves of genus g is a complex orbifold. More precisely, it is the quotient of a bounded domain in $\mathbb{C}^{3g-3}$, the so-called *Teichmüller space* $\mathcal{T}_g$ of genus g, under the action of a discrete group of biholomorphic automorphisms, the *mapping class group* $\mathrm{Mod}(S_g)$. The following question can be found in [FL08], see also [HL98] for a motivation.

Question. *Does $\mathcal{M}_g$ admit a stratification with all strata affine subvarieties of codimension $\leq g-1$?*

The moduli space admits a compactification $\overline{\mathcal{M}}_g$, the so-called *Deligne Mumford compactification*, which equips $\mathcal{M}_g$ with the structure of a quasi-projective variety. The complement of an irreducible effective ample divisor in $\overline{\mathcal{M}}_g$ is affine. This was used by Fontanari and Looijenga to show that the complement of the *Thetanull* divisor in $\mathcal{M}_g$ parameterizing curves with an effective even theta characteristic is affine for every $g \geq 4$ (Proposition 2.1 of [FL08]). They also show that the answer to the question is yes for all $g \leq 5$. Another approach towards an answer to this question which is closer to our viewpoint is due to Chen [Ch19].

The main goal of this article is to give some additional evidence that the answer to the above question is affirmative. To this end consider the *Hodge bundle* over $\mathcal{M}_g$ whose fiber over a complex curve X is just the g-dimensional vector space of holomorphic one-forms on X. The projectivization $P : \mathcal{P} \to \mathcal{M}_g$ of the Hodge bundle if a holomorphic fiber bundle over $\mathcal{M}_g$ in the orbifold sense. It admits a natural stratification whose strata consist of projective differentials with the same number and multiplicities of zeros. These strata need not be connected, but the number of connected components is at most 3 [KtZ03].

The *tautological ring* of $\mathcal{M}_g$ is the subring of the rational cohomology ring of $\mathcal{M}_g$ generated by the *Mumford Morita Miller* classes $\kappa_k \in H^{2k}(\mathcal{M}_g, \mathbb{Q})$ (see [M87]

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and [Lo95] for a comprehensive discussion of these classes). Denote by η the Chern class of the tautological line bundle over the fibers of $\mathcal{P}$. We use the zeros of the differentials in a component $\mathcal{Q}$ of a stratum to analyze the cohomology classes on the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ defined by the boundary components of $\mathcal{Q}$ of codimension one. We find that these classes are all contained in the subspace of $H^*(\overline{\mathcal{Q}},\mathbb{Q})$ spanned by the restrictions to $\overline{\mathcal{Q}}$ of the pull-back $P^*\kappa_1$ and the class η.

Denote by $\mathbb{P}\mathcal{H}(k_1,\dots,k_m)$ the stratum of projective abelian differentials with m zeros of order k_j (here $k_j \geq 1$ and $\sum_j k_j = 2g-2$). As an application, we obtain a topological proof of the following result of Chen [Ch17].

Theorem 1. *Let $\mathcal{Q}\subset \mathbb{P}\mathcal{H}(k_1,\dots,k_m)$ be a component of a stratum of projective abelian differentials with $m\geq 1$ zeros of order k_i $(i\leq m)$; then for all $\ell\geq 1$ we have*

$$P^*\kappa_\ell|\mathcal{Q} = (-1)^{\ell+1}\sum_i\Big(\sum_{j=0}^{\ell}\frac{k_i}{(k_i+1)^{\ell-j}}\Big)\eta^\ell.$$

In the case $\ell=1$ this reads

$$P^*\kappa_1|\mathcal{Q} = \sum_i\Big(k_i+1-\frac{1}{k_i+1}\Big)\eta|\mathcal{Q}.$$

In particular, the restriction to $\mathcal{Q}$ of the pull-back of the tautological ring of $\mathcal{M}_g$ coincides with the subring of $H^*(\mathcal{Q},\mathbb{Q})$ generated by the restriction of η.

Since all but the first Mumford Morita Miller classes vanish on $\mathcal{M}_3$ [Lo95], Theorem 1 for two strata in $g=3$ is also due to Looijenga and Mondello [LM14].

The *moduli space of curves with odd theta characteristic* $\mathcal{M}_{g,\mathrm{odd}}$ is the moduli space of pairs (X,L) where X is a complex curve of genus g and where L is a square root of the canonical bundle so that $h^0(X,L)$ is odd. This is a finite orbifold cover of $\mathcal{M}_g$. We use our cohomological computation to give some evidence towards the question in [FL08]. We show

Theorem 2. *The spin moduli space $\mathcal{M}_{g,\mathrm{odd}}$ admits an explicit stratification into complex strata $\mathcal{D}_j$ of codimension $j-1$ $(1\leq j\leq g-1)$ such that for all $j\leq g-1$, the restriction of the class κ_1 to the stratum $\mathcal{D}_j$ vanishes. In particular, $\mathcal{D}_j$ does not contain a complete subvariety.*

The stratum $\mathcal{D}_j$ is defined as follows. Let $\mathcal{Q}=\mathbb{P}\mathcal{H}(2,\dots,2)^{\mathrm{odd}}$ be the component of the stratum of abelian differentials with all zeros of order 2 and odd parity [KtZ03]. The closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathcal{P}$ projects onto $\mathcal{M}_{g,\mathrm{odd}}$. For $j\leq g-1$ let $\overline{\mathcal{Q}}_j\subset\overline{\mathcal{Q}}$ be the closure of the union of all boundary components of $\mathcal{Q}$ of codimension $j-1$ and define $\mathcal{D}_j = P\overline{\mathcal{Q}}_j - P\overline{\mathcal{Q}}_{j+1}$. Note that $\mathcal{D}_{g-1}$ is the projection of the union of those components of $\mathbb{P}\mathcal{H}(2g-2)$ which have an odd spin structure. The number of such components is one for $g\equiv 0,3 \mod 4$, and it equals two otherwise. We conjecture that the strata $\mathcal{D}_j$ are in fact affine for all j.

That components of strata do not contain complete subvarieties is due to Gendron [G20].

The organization of this article is as follows. In Section 2 we study the pull-back $P^*\mathcal{C}$ of the *universal curve* $\mathcal{C}\to\mathcal{M}_g$ to the moduli space of projective abelian differentials. We obtain some information on the cohomology class $P^*\kappa_1$ by analyzing the subvariety of $P^*\mathcal{C}$ which intersects the fiber over q in the zeros of q. This locus can be used to gain some information on $P^*\kappa_1$ via Poincare duality in surface bundles as in [H20].

In Section 3 we begin the study of the second cohomology group of closures of components of strata, and we establish Theorem 1. This is used in Section 4 to give a purely topological proof of the following result of Korotkin and Zograf. For its formulation, recall that the rational cohomology ring of the projectivized Hodge bundle $P:\mathcal{P}\to\mathcal{M}_g$ is the ring

$$H^*\mathcal{P},\mathbb{Q})=P^*H^*(\mathcal{M}_g,\mathbb{Q})[\eta]/\big(\eta^g+c_1(\mathcal{H})\eta^{g-1}+\cdots+c_g(\mathcal{H})\big)$$

where η is the tautological class of the fiber and $c_i(\mathcal{H})$ is the i-th Chern class of $\mathcal{H}$. These Chern classes are polynomials in the odd Mumford Morita Miller classes (see Section 2 of [M87]). Let $\xi\in H^2(\mathcal{P},\mathbb{Q})$ be the class dual to the stratum $\mathcal{P}(1)$ of codimension one.

Theorem 3 (Korotkin and Zograf [KZ11])**.**

$$\xi=2P^*\kappa_1-(6g-6)\eta.$$

The computation in [KZ11] extends to the boundary of the Deligne Mumford compactification, however we do not pursue such a computation in this work. An algebraic geometric proof is due to Chen [Ch13].

In Section 5 we obtain some information on the second cohomology classes of the closure of a stratum defined by its codimension one boundary strata. This is then used in Section 6 to show Theorem 2.

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2. The zero sets of strata of abelian differentials

The goal of this section is to establish some geometric properties of strata of abelian differentials and use this to obtain some first information on the first Mumford Morita Miller class. Throughout we assume that $g\geq 3$.

Let $\Upsilon:\mathcal{C}\to\mathcal{M}_g$ be the *universal curve*, that is, the fiber bundle (in the orbifold sense) whose fiber over a point $X\in\mathcal{M}_g$ is just the Riemann surface X. Consider the pull-back

$$\Pi:P^*\mathcal{C}\to\mathcal{P}$$

of the universal curve to the projectivized Hodge bundle. For each $q\in\mathcal{P}$, the zeros of q define a subset of the fiber of $P^*\mathcal{C}$ of cardinality at most $2g-2$. Denote by $\Delta\subset P^*\mathcal{C}$ the locus of all these zeros.

The following is Proposition 2.2 of [H20]. For its formulation, a closed subvariety Y of codimension one of a smooth variety X is a *local complete intersection* if the ideal sheaf $\mathcal{F}_Y$ of Y in X can be locally generated by a single element at every point.

Proposition 2.1. *The subset $\Delta\subset P^*\mathcal{C}$ is a subvariety of $P^*\mathcal{C}$ of codimension one, and it is a local complete intersection.*

Write $\mathcal{P}=\cup_k\mathcal{P}(k)$ where for $0\leq k\leq 2g-3$ the set $\mathcal{P}(k)$ is the locus of projective differentials with precisely $2g-2-k$ zeros. Then $\mathcal{P}(k)$ is a smooth suborbifold of $\mathcal{P}$ of codimenision k, and it is a disjoint union of strata. Furthermore, we have

$\overline{\mathcal{P}}(k) = \cup_{j\geq k}\mathcal{P}(j)$ and hence this decomposition gives $\mathcal{P}$ the structure of a complex stratified space. The set $\mathcal{P}(1)$ consists of the single stratum $\mathbb{P}\mathcal{H}(1,\dots,1,2)$.

Now let us consider a component $\mathcal{D} \subset \mathcal{P}$ of a stratum of projective abelian differentials. By Proposition 2.1, if $q \in \mathcal{D}$ and if $z \in P^*\mathcal{C}$ is a point in the fiber of $P^*\mathcal{C}$ over q which is a zero of q of order $k \geq 1$, then there is a neighborhood U of q in $\mathcal{D}$ and a holomorphic section $\zeta : U \to P^*\mathcal{C}$ with image in Δ and such that $\zeta(q) = z$. If z is the only zero of q of order k, then the map which associates to a differential in $\mathcal{D}$ its unique zero of order k is a global holomorphic section of $P^*\mathcal{C}$ over $\mathcal{D}$.

The following proposition gives some further information on the variety Δ. It is a more explicit version of a result in Section 8 of [EMZ03].

Proposition 2.2. *Let $\mathcal{Q} \subset \mathcal{P}$ be a component of a stratum of projective abelian differentials and let $\mathcal{D}$ be an irreducible component of codimension one of the boundary of $\mathcal{Q}$, obtained by colliding two zeros of differentials in $\mathcal{Q}$ of order $m_1, m_2 \geq 1$ to a single zero of order $m = m_1 + m_2 \geq 2$.*

1. *Assume that the differentials in $\mathcal{D}$ have a single zero of order m. Then $\mathcal{Q}\cup \mathcal{D} \subset \mathcal{P}$ is a smooth complex orbifold. Let $\zeta : \mathcal{D} \to P^*\mathcal{C}$ be the holomorphic section defined by the zero of order m.*
   - *If $m_1 = m_2$ then the normal bundle of $\mathcal{D} \subset \mathcal{Q} \cup \mathcal{D}$ is isomorphic to the square $\zeta^*(\nu)^2$ of the pull-back $\zeta^*(\nu)$ of the vertical tangent bundle of $P^*\mathcal{C}$ along ζ.*
   - *If $m_1 \neq m_2$ then the normal bundle of $\mathcal{D} \subset \mathcal{Q} \cup \mathcal{D}$ is isomorphic to $\zeta^*(\nu)$.*
2. *If $\mathcal{D}$ consists of differentials with $k \geq 2$ zeros of order m, then $\mathcal{Q}\cup\mathcal{D}$ has a normal crossing singularity along $\mathcal{D}$ consisting of k smooth local branches which intersect transversely along $\mathcal{D}$.*

Proof. Consider first the case that differentials in $\mathcal{D}$ have a unique zero of order m. Equivalently, a zero of a differential $q \in \mathcal{D}$ arising from a collision of two zeros of a differential in $\mathcal{Q}$ is distinguished by its multiplicity.

Consider the preimages $\mathcal{D}_0$ and $\mathcal{Q}_0$ of $\mathcal{D}$ and $\mathcal{Q}$, respectively, in the moduli space of abelian differentials, that is, in the complement $\mathcal{H}^*$ of the zero section of the Hodge bundle $\mathcal{H}$. Then $\mathcal{D}_0, \mathcal{Q}_0$ admit a natural holomorphic action of the group $\mathbb{C}^*$ by complex multiplication, with quotients $\mathcal{D}$ and $\mathcal{Q}$.

A neighborhood of $q \in \mathcal{D}_0$ in $\mathcal{Q}_0\cup\mathcal{D}_0$ is obtained from a neighborhood of q in $\mathcal{D}_0$ by opening the distinguished zero of order $m \geq 2$ to two zeros of prescribed order $m_1 \leq m_2$ with $m_1+m_2 = m$ as explained in Section 8 of [EMZ03]. We have to show that this operation is equivariant with respect to the $\mathbb{C}^*$-action and compatible with the complex structure on the quotients $\mathcal{D}, \mathcal{Q}$, and we have to compute the normal bundle.

We proceed as on p.86 of [EMZ03]. An abelian differential q on a Riemann surface X of genus g determines a flat metric on X in the conformal class of X, with singularities at the zeros of q. Assume that $q \in \mathcal{D}_0$, let $x \in X$ be the distinguished zero of q and let $\epsilon > 0$ be sufficiently small that the closed disk $D(\epsilon)$ of radius ϵ about x for the flat metric defined by q is a topological disk embedded in X. Let $\delta < \epsilon/2$ and let γ be a straight line segment of length δ with one endpoint at x, parameterized proportional to arc length on the interval $[0, 1]$. We claim that γ determines uniquely a point in $\mathcal{Q}_0$ with a saddle connection of length 2δ connecting a zero of order m_1 to a zero of order m_2.

Namely, the closed disk $D(\epsilon)$ can be represented as a union of $2m+2$ flat half-disks of radius ϵ. Their oriented straight line boundary segments are segments whose direction for the flat metric is up to sign the direction of γ. These half-disks are glued in circular order along the half-segments of length ϵ. Let $\hat{\gamma}$ be the straight line segment of the same length δ as γ, with one endpoint at x, which makes an angle of $(2m_1+1)\pi$ with γ, measured for the flat metric in counter clockwise direction. The direction of $\hat{\gamma}$ for the flat metric is opposite to the direction of γ. Both oriented segments $\gamma, \hat{\gamma}$ are contained in the oriented boundary of one of the embedded flat half-disk of radius ϵ determined by the direction of γ, with center at x.

Cut $D(\epsilon)$ open along the line segments in the boundary of these two half-disks containing $\gamma, \hat{\gamma}$ and glue these two half-disks along the segment of length 2δ centered at x, leaving a pair of free line segments of length $\epsilon - \delta$ on the boundary of each of the half-disks. The remaining half-disks determined by the direction of γ can be glued to these two half-disks isometrically along the boundary in a circular fashion as illustrated on p.87 of [EMZ03]. The result of this construction is a new flat metric on the surface S_g, of the same area. This flat metric is defined by an abelian differential $q(\gamma) \in \mathcal{Q}_0$ which is uniquely determined by q, the choice of γ and the decomposition $m = m_1 + m_2$. It has a distinguished saddle connection of length 2δ connecting the two newborn zeros of order m_1, m_2. If we denote by $x(\gamma)$ the midpoint of this saddle connection, then the complements of the disks of radius ϵ about x and $x(\gamma)$ for the flat metrics defined by $q, q(\gamma)$ are isometric.

For fixed $q \in \mathcal{D}_0$ and as the endpoint of the geodesic segment γ different from x varies in the punctured disk of radius $\epsilon/2$ about x, the above construction defines a family of abelian differentials in $\mathcal{Q}_0$ depending on a complex parameter varying in a punctured disk in $\mathbb{C}$, that is, in a punctured coordinate disk about x in X. As explained on p.87 of [EMZ03], for a suitable choice of a basis of relative homology of the closed surface S_g of genus g, marked at the zeros of q, all but perhaps one coefficient of the corresponding period coordinates are constant, and the remaining period coordinate is changed by $-\gamma$ (the missing factor $1/2$ in our description stems from a slight variation in the setup).

As a consequence, this construction gives rise to a holomorphic map from of a disk in $\mathbb{C}$ into $\mathcal{H}^*$ which intersects $\mathcal{D}_0$ in the single point q. As it commutes with multiplication of a differential with a nonzero complex number, it descends to a holomorphic map from of a disk in $\mathbb{C}$ into $\mathcal{P}$ which intersects $\mathcal{D}$ in a single point, and this point is the projection of q. Furthermore, by naturality with respect to suitable period coordinates, chosen as in the previous paragraph, it depends in a holomorphic fashion on $q \in \mathcal{D}_0$.

If $m_1 = m_2 = m/2$, then the two flat surfaces obtained from this construction from segments γ_1, γ_2 of the same length δ which make an angle of $(m+1)\pi$ at x for the flat cone metric, are isometric. As a consequence, the holomorphic involution $z \to -z$ in the tangent space of X at the distinguished zero of order m extends to an involution of the local parameter space for opening a zero of order m, and the map which associates to a point in this local parameter space the resulting area one abelian differential factors through the quotient of this involution.

Lemma 8.1 of [EMZ03] shows that in the open and dense subset of $\mathcal{D}_0$ consisting of flat metrics which do not admit any isometry, this is the only identification. More precisely, the lemma states that each direction for the flat metric of q gives rise to precisely $m+1$ distinct flat surfaces with *labeled* zeros. The isometry between two of these flat metrics arising from the involution $z \to -z$ as discussed in the previous paragraph exchanges the two newborn zeros of order $m_1 = m_2$ of the differentials in $\mathcal{Q}_0$ and hence changes the labels. As we do not fix labels here, by Lemma 8.1 of [EMZ03] the above construction defines a holomorphic parameterization of a neighborhood of $\mathcal{D}_0$ in $\mathcal{Q}_0 \cup \mathcal{D}_0$. By equivariance under the action of $\mathbb{C}^*$, this parameterization descends to a parameterization of a neighborhood of $\mathcal{D}$ in $\mathcal{Q}$. Moreover, the normal bundle of $\mathcal{D}$ in $\mathcal{Q}$ is the square of the pull-back of the vertical tangent bundle of $P^*\mathcal{C}$ at the distinguished zero as described in the proposition as its fiber over a fixed zero x is doubly covered by the fiber of the vertical tangent bundle at x.

If $m_1 \neq m_2$, then locally near $\mathcal{D}_0$ the two newborn zeros of the differentials arising from the above construction can not be exchanged. Thus in this case Lemma 8.1 of [EMZ03] shows that the above construction defines a holomorphic parameterization of a neighborhood of $\mathcal{D}_0$ in $\mathcal{Q}_0$ and hence it parameterizes a neighborhood of $\mathcal{D}$ in $\mathcal{Q}$. Furthermore, the normal bundle of $\mathcal{D}$ equals the pull-back of the vertical tangent bundle of $P^*\mathcal{C}$ at the distinguished zero of order m. This shows the first part of the proposition.

Now let us assume that differentials in $\mathcal{D}$ have $k \geq 2$ zeros of order m. Let $q \in \mathcal{D}$ and let $q_0 \in \mathcal{D}_0$ be a preimage of q. If $x_1 \neq x_2$ are two zeros of the same order m for q_0, then for differentials in a neighborhood V_0 of q_0 in $\mathcal{D}_0$ we can open up the zero x_1 locally in a neighborhood of x_1, preserving the flat metric on the complement of a small disk about x_1, in particular near x_2, and we obtain a differential in $\mathcal{Q}_0$. This construction determines the structure of a smooth complex orbifold on the union of the projection V of V_0 to $\mathcal{D}$ with some open subset $U(x_1, V)$ of $\mathcal{Q}$ which is compatible with the complex structure and the topology of $\mathcal{P}$. Similarly, preserving x_1 and opening x_2 gives rise to the structure of a smooth complex orbifold on the union of V with an open subset $U(x_2, V)$ of $\mathcal{Q}$.

If the flat metric defined by $q \in \mathcal{D}_0$ does not admit an isometry which exchanges x_1 and x_2, that is, if q belongs to the open and dense set of smooth points of the orbifold $\mathcal{P}$, then for a suitable choice of the neighborhood V_0 of q, the sets $U(x_1, V)$ and $U(x_2, V)$ are disjoint from each other, and the unions $U(x_1, V) \cup V$ and $U(x_2, V) \cup V$ intersect transversely along V. As this construction is local, it can be extended to more than two zeros of the same order m and yields the second part of the proposition. $\square$

By Proposition 2.1, the zeros of a projective abelian differential q on a Riemann surface X define a codimension one complex subvariety Δ in the holomorphic fiber bundle $P^*\mathcal{C} \to \mathcal{P}$. Over a fixed stratum $\mathcal{Q}$, this subvariety is just a holomorphic *multisection*, that is, it can locally be described as consisting of ℓ holomorphic sections of $P^*\mathcal{C} \to \mathcal{P}$ where $\ell \geq 1$ is the number of zeros of differentials in $\mathcal{Q}$. Note that this a purely local statement.

Our next goal is to describe the behavior of these multisections as the differentials approach a boundary component of $\mathcal{Q}$ of codimension one, given by a collision of two of these zeros.

For the formulation of this description, recall that a two-sheeted holomorphic branched covering $\zeta : S \to B$ of two complex curves S, B, branched at a point $x \in S$, is given in suitable holomorphic coordinates z, w on S, B near x and $\zeta(x)$, respectively, with $x = \{z = 0\}, \zeta(x) = \{w = 0\}$, as $w = z^2$. The following definition is a special case of well known constructions and is included here for clarity of the exposition.

Definition 2.3. For a number $m \geq 2$, an *m-sheeted holomorphic branched covering* of two complex manifolds M, N of the same complex dimension, *doubly branched* along a complex hypersurface $H \subset M$, is a surjective holomorphic map $\zeta : M \to N$ with the following properties.

(1) The restriction of ζ to H is a biholomorphism onto its image.
(2) The restriction of ζ to $M - \zeta^{-1}(\zeta(H))$ is an m-sheeted unbranched holomorphic covering.
(3) There exists a neighborhood V of $\zeta^{-1}(\zeta(H)) - H$ such that the restriction of ζ to V is an $(m-2)$-sheeted unbranched holomorphic covering.
(4) For every $x \in H$ there are holomorphic coordinates $(z_1, \dots, z_n)$ on an open neighborhood U of x in M and holomorphic coordinates $(w_1, \dots, w_n)$ on a neighborhood of $\zeta(x)$ in N, with $H \cap U = \{z_1 = 0\}$ and the property that in these coordinates, the map ζ is defined by $(z_1, z_2, \dots, z_n) \to (z_1^2, z_2, \dots, z_n)$.

We also have to look at a two-sheeted holomorphic branched covering from a singular variety M onto a smooth complex manifold N which is branched along a normal crossing divisor of M in the following sense.

Definition 2.4. Let Z be a codimension one complex subvariety of a smooth complex variety M, smooth away from a codimension one subvariety $H \subset Z$, and assume that Z has a normal crossing singularity along H. A *holomorphic branched covering* of Z onto a smooth complex variety N, doubly branched along the singular hypersurface H, is a surjective holomorphic map $\zeta : Z \to N$ with the following properties.

(1) The restriction of ζ to H is a biholomorphic map onto its image.
(2) The restriction of ζ to $Z - \zeta^{-1}(\zeta(H))$ is an m-sheeted unbranched holomorphic covering.
(3) There exists a neighborhood V of $\zeta^{-1}(\zeta(H)) - H$ such that the restriction of ζ to V is an $(m-2)$-sheeted unbranched holomorphic covering.
(4) For every $x \in H$ there are holomorphic coordinates $(z_0, z_1, \dots, z_n)$ on a neighborhood U of x in M, and holomorphic coordinates $(w_1, \dots, w_n)$ on N near $\zeta(x)$, with $Z \cap U = \{z_0^2 - z_1^2 = 0\}$ and $H \cap U = \{z_0 = z_1 = 0\}$ and the property that in these coordinates, the map ζ is the restriction to $Z \cap U$ of the map defined by $(z_0, z_1, \dots, z_n) \to (z_1, z_2, \dots, z_n)$.

The following statement uses Definition 2.3 and Definition 2.4 in the orbifold sense. That is, the definitions (which are mainly local) apply after perhaps passing to a finite manifold cover.

Proposition 2.5. *Let $\mathcal{D}$ be a codimension one irreducible boundary component of a stratum $\mathcal{Q} \subset \mathcal{P}$, obtained by colliding two zeros of differentials in $\mathcal{Q}$ of order $m_1, m_2 \geq 1$ to a single zero of order $m = m_1 + m_2 \geq 2$, and let $Z = \Delta \cap \Pi^{-1}(\mathcal{Q} \cup \mathcal{D})$.*

(1) *Assume that $\mathcal{D}$ consists of differentials with a single zero of order m.*
   - *If $m_1 = m_2$ then Z is a smooth complex suborbifold of $\Pi^{-1}(\mathcal{Q} \cup \mathcal{D})$ of codimension one. The projection $\Pi | Z : Z \to \mathcal{Q} \cup \mathcal{D}$ is a holomorphic branched covering, doubly branched along the zeros of order m of the differentials in $\mathcal{D}$.*

- *If $m_1 \neq m_2$ then Z has a normal crossing singularity at the zeros of order m of the differentials in $\mathcal{D}$. The projection $\Pi|Z : Z \to \mathcal{Q} \cap \mathcal{D}$ is a holomorphic branched covering, doubly branched along the zeros of order m of the differentials in $\mathcal{D}$.*

(2) *If $\mathcal{D}$ consists of differentials with $k \geq 2$ zeros of order m, then property (1) holds true for each of the k local branches of $\mathcal{Q}$ which intersect transversely along $\mathcal{D}$.*

Proof. As in the proof of Proposition 2.2, denote by $\mathcal{D}_0, \mathcal{Q}_0$ the preimage of $\mathcal{D}, \mathcal{Q}$ in the moduli space of abelian differentials. Let x be a zero of order m for an abelian differential $q \in \mathcal{D}_0$ and let $\epsilon > 0$ be such that the closed disk of radius ϵ about x for the flat metric defined by q is isometrically embedded in the Riemann surface X underlying q. Then there is a canonical complex coordinate z for X near x, so that in this coordinate, the differential q equals the differential $z^m dz$.

Let γ be a straight line segment of length $\delta < \epsilon/2$ for the flat metric defined by q issuing from x. Opening up the zero of q into two zeros of order m_1, m_2 along γ as described in Section 8 of [EMZ03] and recorded in the proof of Proposition 2.2 defines a differential $q(\gamma)$ with a saddle connection of length 2δ. The direction of the saddle connection equals the direction of γ, and the midpoint of the saddle connection is the natural image of the zero x of q.

Let us consider a model for this situation. It is given by a complex coordinate z on the Riemann surface underlying $q(\gamma)$, containing 0 in its range, a straight line segment through 0 for the flat metric defined by $q(\gamma)$ of length 2δ, and a zero of order m_1, m_2 at the endpoints. For a suitable choice of such a complex coordinate z, the differential can be represented as

$$(z-a)^{m_1}(z+a)^{m_2}dz$$

where $a = a(\gamma) \in \mathbb{C}^*$ can be computed from the length and direction of the saddle connection. Solving $(z-a)^{m_1}(z+a)^{m_2}dz = dw$ near $z = 0$ expresses the local coordinate w describing the flat metric of the differential $q(\gamma)$, normalized to vanish at the point $z = 0$, as a polynomial of degree $m+1$ in the coordinate z, with coefficients depending holomorphically on the complex variable $a \in \mathbb{C}^*$.

As in the proof of Proposition 2.2, for a fixed basis of relative homology of the surface X marked at the zeros of q, period coordinates for the differentials in a neighborhood of q in $\mathcal{D}_0$ extend to period coordinates on a neighborhood of q in $\mathcal{Q}_0 \cup \mathcal{D}_0$ using as an extra parameter the distinguished saddle connection between the newborn zeros of length less than ϵ (here $\epsilon > 0$ is a constant which depends on the flat metric defined by q), and these coordinates also define holomorphic coordinates on a neighborhood V of q in $\mathcal{Q} \cup \mathcal{D}$ by equivariance under the action of $\mathbb{C}^*$. In other words, the differentials resulting from this construction depend in a holomorphic fashion on period coordinates for q and the endpoint of the straight line segment γ.

Let us assume that the differentials in $\mathcal{D}$ have a single zero of order m. By Proposition 2.2, in this case $\mathcal{D}_0$ is a smooth suborbifold of $\mathcal{Q}_0 \cup \mathcal{D}_0$. Let $\Pi_0 : P_0^*\mathcal{C} \to \mathcal{H}^*$ be the pull-back of the universal curve to the complement $\mathcal{H}^*$ of the zero section in the Hodge bundle. Considering again a differential $q \in \mathcal{D}_0$ with a zero x of order m, the above discussion shows that there are holomorphic local functions $(z, v_1, \ldots, v_k)$ on a neighborhood U of x in $\Pi_0^{-1}(\mathcal{Q}_0 \cup \mathcal{D}_0)$, with $\{v_i = \text{const}\}$ defining the foliation into the fibers of the bundle $P_0^*\mathcal{C} \to \mathcal{H}^*$, and the following additional properties.

(1) The functions v_i are pull-backs by Π_0 of holomorphic local functions $\hat{v}_i$ on $\mathcal{Q}_0 \cup \mathcal{D}_0$.
(2) $\{\hat{v}_1 = 0\} = \Pi_0(U) \cap \mathcal{D}_0 \subset \mathcal{D}_0 \cup \mathcal{Q}_0$, and $(\hat{v}_2, \dots, \hat{v}_k)$ are holomorphic coordinates for $\mathcal{D}_0$ (in the orbifold sense).
(3) The restriction of the function z to a fiber of Π_0 over $\Pi_0(U)$ is a holomorphic coordinate on the fiber.
(4) The zeros of order m for the projective differentials in $\Pi_0(U) \cap \mathcal{D}_0$, viewed as points in the fiber of $P_0^*\mathcal{C}$ over the points in $\mathcal{D}_0$, are contained in the domain of the fiber coordinate z. The abelian differentials $u \in \Pi_0(U)$ are given in the fiber coordinate z by $u = (z - \hat{v}_1(u))^{m_1}(z + \hat{v}_1(u))^{m_2}dz$.

By Proposition 2.2 and its proof, in the case $m_1 = m_2$, the differentials parameterized by $(\hat{v}_1, \hat{v}_2, \dots, \hat{v}_m)$ and $(-\hat{v}_1, \hat{v}_2, \dots, \hat{v}_m)$ coincide and hence $(\hat{v}_1^2, \hat{v}_2, \dots, \hat{v}_m)$ are complex coordinates for $\mathcal{Q}_0 \cup \mathcal{D}_0$ (in the usual sense which equips the quotient of the unit disk $\{|z| < 1\}$ by the involution $z \to -z$ with the structure of a Riemann surface, biholomorphic to the disk). Putting $w = v_1^2$, the equation for the subvariety $Z = \Delta \cap \Pi_0^{-1}(\mathcal{D}_0 \cup \mathcal{Q}_0)$ of $P_0^*\mathcal{C}$ near the zero x of order m equals $z^2 - w = 0$. For fixed $\hat{v}_2, \dots, \hat{v}_k$ this locus is parameterized by $z \to (z, z^2)$ in the coordinate functions (z, w). Thus Z is a smooth suborbifold of $\Pi_0^{-1}(\mathcal{Q}_0 \cup \mathcal{D}_0)$ whose tangent space at the zero x of order m of a differential in $\mathcal{D}_0$ contains the tangent space of the fiber of Π_0 at x. Furthermore, in these coordinates, near the zero of order m the projection map $\Pi_0|Z : Z \to \mathcal{Q}_0 \cup \mathcal{D}_0$ is of the form required in Definition 2.3. The first item in part (1) of the proposition follows from invariance under the action of $\mathbb{C}^*$.

In the case $m_1 \neq m_2$ the tuple of functions $(\hat{v}_1, \hat{v}_2, \dots, \hat{v}_k)$ defines coordinates on $\mathcal{Q}_0 \cup \mathcal{D}_0$. The equation $(z - v_1)^{m_1}(z + v_1)^{m_2} = 0$ is the equation of a union of two complex lines in $\mathbb{C}^2$ which intersect transversely in a single point 0. Thus the union of these two planes has a normal crossing singularity at 0. As this applies to a neighborhood of the zero of order m in the fiber over any point in $\mathcal{D}_0$, it follows that Z is a complex variety with a normal crossing singularity at the zero of order m. Furthermore, in these coordinates, near the zero of order m the projection $\Pi_0|Z : Z \to \mathcal{Q}_0 \cup \mathcal{D}_0$ is of the form required in Definition 2.4. The second item in part (1) of the proposition follows again from invariance under the action of $\mathbb{C}^*$.

If the differentials in $\mathcal{D}$ have $k \geq 2$ zeros of order m, then there are k sheets for the intersection of $\mathcal{Q} \cup \mathcal{D}$ with $\mathcal{D}$, with a normal crossing intersection, and the above discussion applies separately to each of these sheets. This shows part (2) of the proposition. □

3. On the cohomology of the closure of a stratum

In this section we begin the investigation of the second cohomology of the closure $\overline{\mathcal{Q}}$ of a projective stratum $\mathcal{Q}$ of abelian differentials, and we establish Theorem 1 (see [Ch17]).

As in Section 2, let $P^*\mathcal{C}$ be the pull-back of the universal curve $\mathcal{C} \to \mathcal{M}_g$ to $\mathcal{P}$ and let Δ be the codimension one subvariety defined by the zeros of the projective differentials in $\mathcal{P}$. Let ν be the vertical tangent bundle of $P^*\mathcal{C}$ and let $\tau \to P^*\mathcal{C}$ be the pull-back of the tautological line bundle on $\mathcal{P}$.

In the sequel we always denote by ζ^* the dual of a complex line bundle ζ, or, equivalently, the inverse of ζ in the group of all complex line bundles on $P^*\mathcal{C}$. We are interested in topological properties of holomorphic line bundles, that is, in their Chern class. Some of the statements below also hold in the holomorphic setting. An example is the following

Lemma 3.1. *The line bundle $\nu \otimes \tau$ is trivial on $P^*\mathcal{C} - \Delta$.*

Proof. Let $\alpha \in \nu^*$ be any vector in the vertical cotangent bundle of $P^*\mathcal{C}$ at a point $y \in P^*\mathcal{C} - \Delta$. We may view α as a $\mathbb{C}$-linear functional on the holomorphic tangent space ν_y of the fiber of $P^*\mathcal{C}$ through y.

The fiber $\tau_{\Pi(y)}$ of τ at the point $\Pi(y) \in \mathcal{P}$ consists of the line of holomorphic one-forms on the Riemann surface $P\Pi(y)$ in the projective class defined by $\Pi(y)$. As y is not a zero of a differential in this projective class and as the dimension of the complex vector space of $\mathbb{C}$-linear functionals $\nu_y \to \mathbb{C}$ equals one, there is precisely one holomorphic one-form $\Lambda(\alpha) \in \tau_{\Pi(y)}$ whose restriction to ν_y coincides with α. Then $\alpha \to \Lambda(\alpha)$ defines an isomorphism between ν^* and τ on $P^*\mathcal{C} - \Delta$, and hence it defines a nowhere vanishing section of the bundle $(\nu^*)^* \otimes \tau = \nu \otimes \tau$ on $P^*\mathcal{C} - \Delta$ which is what we wanted to show. $\square$

The codimension one complex subvariety $\Delta \subset P^*\mathcal{C}$ is a Weil divisor and hence a Cartier divisor in the smooth complex orbifold $P^*\mathcal{C}$. Thus it defines a holomorphic line bundle $L \to P^*\mathcal{C}$ whose first Chern class $c_1(L)$ is dual to Δ in the sense of intersection (see [Fu84] for more and for references). This line bundle is trivial on the complement of Δ and restricts to the normal bundle on the regular part of Δ. Lemma 3.1 indicates that this line bundle may be a power of the bundle $\nu \otimes \tau$ in the Picard group of $P^*\mathcal{C}$.

Instead of pursuing this line of idea, we identify the cohomology class defined by the bundle L which is a weaker statement, but sufficient for our purpose. Thus the following statement is meant in the topological sense, and it can be viewed as a version of Theorem 3.13 of [H20]. By the usual exact sequence in sheaf cohomology defined by the exponential function, it is equivalent to stating that the Chern classes of these line bundles coincide.

Proposition 3.2. *$\nu^* \otimes \tau^* = L$ on $P^*\mathcal{C}$.*

Proof. Let Σ be a closed oriented surface and let $\varphi : \Sigma \to P^*\mathcal{C}$ be a smooth map. By transversality, after changing φ with a homotopy we may assume that $\Pi \circ \varphi$ intersects $\mathcal{P}(1)$ in only isolated points and that furthermore, if $x \in \Sigma$ is such that $\Pi \circ \varphi(x) \in \mathcal{P}(1)$ then $\varphi(x) \notin \Delta$ (see [H20] for a detailed discussion). Moreover, as $\Delta \cap \Pi^{-1}(\mathcal{P}(0))$ is the image of a holomorphic multisection of the restriction of $\Pi^{-1}(\mathcal{P}(0))$ to $\mathcal{P}(0)$, we may assume that $\varphi(\Sigma) \cap \Delta$ consists of finitely many transverse intersection points, say the points $x_1, \dots, x_s$. Each of these points x_i is a simple zero of the differential $\Pi(x_i)$. It now suffices to show that $\varphi^*(c_1(\nu^* \otimes \tau^*))[\Sigma]$ equals the number of intersection points of $\varphi(\Sigma)$ with Δ, counted with sign and multiplicity. Here $[\Sigma]$ denotes the fundamental cycle of Σ.

Let us without loss of generality assume in addition that Σ is equipped with a complex structure and that the restriction of φ to a disk neighborhood D_i of x_i in Σ is a holomorphic or antiholomorphic embedding of D_i into a fiber of Π. The latter can be achived by modifying φ with a small homotopy.

By Lemma 3.1, the restriction of the line bundle $\nu^* \otimes \tau^*$ to the complement of Δ admits a natural trivialization whose restriction to the boundary $\varphi(\partial D_i)$ of the disk $\varphi(D_i)$ can be described as follows.

Choose a trivialization Y of the tangent bundle of $D_i \sim \varphi(D_i)$. Choose furthermore a trivialization ξ of $\nu^* \otimes \tau^*$ over $\varphi(D_i)$. We may assume that the contraction of ξ with Y is constant, that is, it is the pull-back of a fixed nontrivial vector q in the fiber of τ^* over $\Pi\varphi(D_i)$.

By Lemma 3.1 and its proof, the contraction of the vector field Y with the restriction of the trivialization of $\nu^* \otimes \tau^*$ on $P^*\mathcal{C} - \Delta$ to $\varphi(\partial D_i)$ equals the section of τ^* which associates to a point $p \in \varphi(\partial D_i)$ the element of τ_p^* which is defined by the following linear functional ζ_p. Recall that the fixed vector q is a holomorphic differential on the fiber of $P^*\mathcal{C}$ containing $\varphi(D_i)$ which does not vanish at p; we then have $\zeta_p(aq) = aq(Y_p)$.

Since x_i is a zero of the differential q and the only zero of q in D_i, if we equip $\varphi(\partial D_i)$ with the orientation defined by the Riemann surface structure of the fiber of $P^*\mathcal{C}$ containing $\varphi(D_i)$, then for this orientation, the map $S^1 = \varphi(\partial D_i) \to \mathbb{C}^*$ defined by $\zeta_p(q) = q(Y_p)$ has rotation number one. This shows that if the restriction of φ to D_i is holomorphic, then the rotation number of the restriction to ∂D_i of the trivialization of the bundle $\varphi^*(\nu^* \otimes \tau^*)$ on $P^*\mathcal{C} - \Delta$ to ∂D_i with respect to a trivialization of $\varphi^*(\nu^* \otimes \tau^*)$ on the disk D_i equals one, and it equals -1 otherwise.

As a consequence, the value $c_1(\nu^* \otimes \tau^*)[\Sigma]$ indeed equals the number of intersections of $\varphi(\Sigma)$ with Δ, counted with sign and multiplicities. □

Let now $\mathcal{Q} \subset \mathcal{P}$ be a component of a stratum of projective abelian differentials, with m zeros of order k_i $(i = 1, \dots, s)$. The closure in $P^*\mathcal{C}$ of the locus of the zeros of order k_i in $\Pi^{-1}\mathcal{Q}$ is a complex subvariety Δ_{k_i} of $\Delta \cap \Pi^{-1}\overline{\mathcal{Q}}$. Its intersection with $\Pi^{-1}\mathcal{Q}$ is a holomorphic multisection of $\Pi^{-1}\mathcal{Q}$ and hence a smooth complex orbifold. We have

Lemma 3.3. $(\nu^*)^{\otimes(k_j+1)}|\Delta_{k_j} \cap \Pi^{-1}\mathcal{Q} = \tau|\Delta_{k_j} \cap \Pi^{-1}\mathcal{Q}$.

Proof. The lemma is well known, and a proof is contained in [Ch17], see also [EKZ14]. We give a topological proof.

A point $y \in \Delta_{k_j}$ is a zero of order k_j of a projective holomorphic one-form on the fiber of $P^*\mathcal{C}$ containing y. The fiber τ_y of τ at y can be identified with the complex line of holomorphic one-forms in this projective class.

As y is a zero of this holomorphic one-form of order k_j, there is a holomorphic local coordinate z on $\Pi^{-1}(\Pi(y))$ near y, with y corresponding to $z = 0$, such that a nonzero differential in the line τ_y can locally near y be written in the form $az^{k_j}dz$ for some $a \in \mathbb{C}^*$. This differential then defines a singular euclidean metric near y, which has a cone point of cone angle $2\pi(k_j + 1)$ at y.

A geodesic arc γ in the fiber $\Pi^{-1}(\Pi(y))$ with one endpoint at y and no singular point in its interior defines a $\mathbb{C}$-valued functional β_γ on τ_y by associating to a differential $\omega \in \tau_y$ the *complex* length of γ with respect to the singular euclidean metric defined by ω, that is, we distinguish real and imaginary part of this length, and we distinguish the orientation.

If γ is not trivial then we have $\beta_{\gamma'} = \beta_\gamma$ if and only if γ' is obtained from γ by a rotation at y by the angle $e^{2\pi i\ell/(k_j+1)}$ for some $\ell \in \mathbb{Z}$ in the complex coordinate z. As a consequence, for a nontrivial arc γ the map which associates to $\theta \in S^1$ the functional defined by the image of γ by rotation with angle θ defines a k_j+1-sheeted covering of the fiber of τ^* at y. As $y \in \Delta_{k_j}$ was arbitrary, this shows that $\nu|\Delta_{k_j}$ is a k_j+1-th root of $\tau^*|\Delta_{k_j}$. Equivalently, we have $\tau|\Delta_{k_j} = (\nu^*)^{\otimes(k_j+1)}|\Delta_{k_j}$. As this identification is natural and hence depends in a holomorphic fashion on $y \in \Delta_{k_j}$ this shows the lemma. $\square$

Remark 3.4. Let us consider the restriction of the bundle $\nu^* \otimes \tau^*$ to the preimage $\Pi^{-1}(\mathcal{Q})$ of a stratum $\mathcal{Q}$ with m zeros of order k_i. Proposition 3.2 shows that the restriction of $\nu^* \otimes \tau^*$ to $\Pi^{-1}\mathcal{Q} - \Delta$ is trivial, and taking the tensor product of the equation in Lemma 3.3 with ν and dualizing yields that its restriction to Δ_{k_j} equals the restriction of ν^{k_j}. This is consistent as the restriction of a holomorphic line bundle to a defining divisor equals the normal bundle of the divisor, and since $\Delta \cap \Pi^{-1}\mathcal{Q}$ is a multisection of $\Pi^{-1}\mathcal{Q}$ over $\mathcal{Q}$, this normal bundle equals the vertical tangent bundle ν. Furthermore, the multiplicity of Δ_{k_j} in $\Delta \cap \Pi^{-1}\mathcal{Q}$ equals k_j.

We use the results obtained so far to show Theorem 1 from the introduction (see [Ch17] and also [EKZ14]). To this end recall that the ℓ-th Mumford Morita Miller class $\kappa_\ell \in H^{2\ell}(\mathcal{M}_g, \mathbb{Q})$ is defined as follows [M87]. Let as before $\Upsilon : \mathcal{C} \to \mathcal{M}_g$ be the universal curve, let B be closed oriented manifold and let $\varphi : B \to \mathcal{M}_g$ be a smooth map; then $E = \varphi^*\mathcal{C}$ is a surface bundle over B with vertical tangent bundle ν. We have

$$\kappa_\ell(\varphi(B)) = \Upsilon_*(c_1(\nu)^\ell)(\varphi(B))$$

where Υ_* is the Gysin push-forward map obtained by integration over the fiber.

The following proposition treats the case $\ell = 1$ and is included here to make the argument more transparent.

Proposition 3.5. *Let $\mathcal{Q}$ be a component of a stratum $\mathbb{P}\mathcal{H}(\ell_1, \dots, \ell_m)$ of projective abelian differentials (here the ℓ_j are counted with multiplicity); then $P^*\kappa_1|\mathcal{Q} = \sum_j (\ell_j + 1 - \frac{1}{\ell_j+1})\eta|\mathcal{Q}$.*

Proof. It suffices to evaluate $P^*\kappa_1$ on the image of a smooth map $\varphi : B \to Q$ where B is a closed oriented surface.

To this end let $1 \leq k_1 < \cdots < k_m$ be the distinct orders of the zeros of the differentials in $\mathcal{Q}$, and let $d_i \geq 1$ be the multiplicity of the zero of order k_i. Let $\Pi^E : E \to B$ be the surface bundle $(P \circ \varphi)^*\mathcal{C}$. The hypersurface Δ_{k_j} in $P^*\mathcal{C}$ pulls back to a smooth multisection of $E \to B$ which defines a homology class $\delta_{k_j} \in H_2(E, \mathbb{Q})$. Write $\delta = \sum_j k_j \delta_{k_j} \in H_2(E, \mathbb{Q})$.

By Proposition 3.2 and naturality of Chern classes under pull-back by inclusions, the first Chern class $\varphi^*(c_1(\nu^*) - c_1(\tau))$ of the pull-back bundle $\varphi^*(\nu^* \otimes \tau^*)$ is Poincaré dual to the homology class δ. Denoting again by ν^* the vertical cotangent bundle of $E \to B$ and omitting the pull-back by φ in our notation, we have

$$(c_1(\nu^*) - c_1(\tau)) \cup \xi[E] = \xi(\delta)$$

for every $\xi \in H^2(E, \mathbb{Q})$, where $[E], [B]$ is the fundamental cycle of E, B.

As a consequence, we compute (see [H20] for details)

$$\begin{aligned} (1) \qquad \varphi^*P^*\kappa_1[B] &= c_1(\nu^*) \cup c_1(\nu^*)[E] \\ &= (c_1(\nu^*) - c_1(\tau)) \cup c_1(\nu^*)[E] + c_1(\tau) \cup c_1(\nu^*)[E] \\ &= c_1(\nu^*)(\delta) + c_1(\tau) \cup c_1(\nu^*)[E]. \end{aligned}$$

By Lemma 3.3, the restriction of $(\nu^*)^{\otimes(k_j+1)}$ to Δ_{k_j} is equivalent to the restriction of τ and therefore

$$c_1(\nu^*)(\delta_{k_j}) = \frac{1}{k_j+1}c_1(\tau)(\delta_{k_j}).$$

The restriction $\Pi|\Delta_{k_j}\cap\Pi^{-1}\mathcal{Q} : \Delta_{k_j}\cap\Pi^{-1}\mathcal{Q}\to\mathcal{Q}$ is an (unbranched) covering of degree d_j. Since $c_1(\tau)$ is the pull-back to E of the class $\varphi^*(\eta)\in H^2(\mathcal{Q},\mathbb{Q})$, we have

$$c_1(\tau)(\delta_{k_j}) = d_j\eta(\varphi_*[B]).$$

Thus by the definition of δ, we have

$$c_1(\nu^*)(\delta) = \sum_j \frac{k_jd_j}{k_j+1}\eta(\varphi_*[B]). \tag{2}$$

Using once more that $c_1(\tau)$ is the pull-back of the class $\varphi^*(\eta)$ on B and that the evaluation of $c_1(\nu^*)$ on a fiber of $P^*\mathcal{C}$ equals $2g-2$, we also have

$$c_1(\nu^*)\cup c_1(\tau)[E] = (2g-2)\eta(\varphi_*[B]). \tag{3}$$

Now $\sum_j k_jd_j = 2g-2$ and hence we conclude from equations (1), (2) and (3) that

$$\begin{aligned}\kappa_1(\varphi_*[B]) &= \sum_j(1-\frac{1}{k_j+1})d_j\eta(\varphi_*[B]) + \sum_j k_jd_j\eta(\varphi_*[B])\\ &= \sum_j(k_j+1-\frac{1}{k_j+1})d_j\eta(\varphi_*[B]).\end{aligned}$$

Since the degree d_j equals the number of zeros of order k_j which are contained in $\Delta_{k_j}\cap\Pi^{-1}(y)$ for any $y\in B$, this concludes the proof of the proposition. ☐

Example 3.6. Let us consider the principal stratum $\mathcal{Q}=\mathcal{P}-\mathcal{P}_1$. Then $k_j=1$ for all j and hence Proposition 3.5 shows that

$$P^*\kappa_1|\mathcal{Q} = \sum_j(2-\frac{1}{2})\eta|\mathcal{Q} = (3g-3)\eta|\mathcal{Q},$$

which is consistent with Theorem 3.

Proof of Theorem 1. Let $\mathcal{Q}\subset\mathcal{P}$ be a component of a stratum of projective abelian differentials with m zeros of order k_i. It suffices to evaluate $P^*\kappa_\ell$ for $\ell\geq 1$ on the image of a finite 2ℓ-dimensional simplicial Poincaré duality complex B of homogeneous dimension 2ℓ under a continuous map $\varphi: B\to\mathcal{Q}$.

To this end consider the surface bundle $\Pi: E\to B$ defined by $P\circ\varphi$. Let $[E]$ be the fundamental class of E. The zeros of the differentials in $\varphi(B)$ of order k_i define define a multisection $\Delta^E_{k_i}$ of E, and each of these multisections defines a homology class $\delta_{k_i}\in H_{2\ell}(E,\mathbb{Q})$. By Lemma 5.2, the homology class $\delta=\sum_j k_j\delta_{k_j}$ is Poincare dual to $c_1(\nu^*)-c_1(\tau)$ (compare the discussion in the proof of Proposition 3.5 which carries over without change; here we omit in our notation that all classes on E are pull-backs of classes on $P^*\mathcal{C}$). In particular, by the definition of the ℓ-th Mumford Morita Miller class κ_ℓ [M87] and the fact that $c_1(\tau)^{\ell+1}[E]=0$ since $c_1(\tau)$ is the pull-back of a cohomology class on $\mathcal{Q}$, using the Ansatz in the proof of Proposition 3.5 we have

$$\begin{aligned}(4)\qquad (-1)^{\ell+1}\kappa_\ell(P\varphi_*[B]) &= c_1(\nu^*)^{\ell+1}[E] \\ &= (c_1(\nu^*)-c_1(\tau))\cup c_1(\nu^*)^\ell[E] + c_1(\tau)\cup c_1(\nu^*)^\ell[E].\end{aligned}$$

Taking into account the fact that all contributions are of degree two and hence their cup products commute, we can expand the second summand in this equation as

$$\begin{aligned}(5)\qquad & c_1(\tau)\cup c_1(\nu^*)^\ell[E] \\ &= (c_1(\tau)\cup(c_1(\nu^*)-c_1(\tau))\cup c_1(\nu^*)^{\ell-1}[E] + c_1(\tau)^2\cup c_1(\nu^*)^{\ell-1}[E] \\ &= (c_1(\nu^*)-c_1(\tau))\cup c_1(\tau)\cup c_1(\nu^*)^{\ell-1}[E] + c_1(\tau)^2\cup c_1(\nu^*)^{\ell-1}[E].\end{aligned}$$

Proceeding inductively, we obtain the equation

$$\begin{aligned}(6)\qquad & (-1)^{\ell+1}\kappa_\ell(P\varphi_*[B]) \\ &= \sum_{j=0}^{\ell}(c_1(\nu^*)-c_1(\tau))\cup\big(c_1(\tau)^j\cup c_1(\nu^*)^{\ell-j}\big)[E] + c_1(\tau)^{\ell+1}[E] \\ &= \sum_{j=0}^{\ell} c_1(\tau)^j\cup c_1(\nu^*)^{\ell-j}(\delta).\end{aligned}$$

The restriction of Π to each of the sets $\Delta^E_{k_i}\subset\Delta^E$ is a covering. Moreover, by Lemma 3.3, we have $(\nu^*)^{\otimes(k_i+1)}|\Delta^E_{k_i} = \tau|\Delta^E_{k_i}$. This yields

$$c_1(\tau)^j\cup c_1(\nu^*)^{\ell-j}(\delta_{k_i}) = \frac{1}{(k_i+1)^{\ell-j}}c_1(\tau)^\ell(\delta_{k_i})$$

and therefore

$$\sum_{j=0}^{\ell} c_1(\tau)^j\cup c_1(\nu^*)^{\ell-j}(\delta) = \sum_i\Big(\sum_{j=0}^{\ell}\frac{k_i}{(k_i+1)^{\ell-j}}\Big)c_1(\tau)^\ell(\delta_{k_i})$$

and hence the theorem follows from the fact that $c_1(\tau)$ is the pull-back of the class $\varphi^*(\eta)\in H^2(\mathcal{Q},\mathbb{Q})$ and that furthermore the restriction of the projection Π to $\Delta^E_{k_i}$ is an unbranched covering of degree d_i. $\square$

4. The Poincaré dual of $\mathcal{P}_1$

In this section we apply the results of Section 2 to represent the signature of a surface bundle as an intersection number. This yields a purely topological proof of Theorem 3 [KZ11].

The projectivized Hodge bundle $P:\mathcal{P}\to\mathcal{M}_g$ extends to a bundle over the Deligne Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ which we denote by $P:\overline{\mathcal{P}}\to\overline{\mathcal{M}}_g$. A standard spectral sequence argument shows that the second rational cohomology group of $\overline{\mathcal{P}}$ is generated by the pull-back of the second rational cohomology group of $\overline{\mathcal{M}}_g$ together with the cohomology class η of the tautological line bundle of the fibre (see Lemma 1 of [KZ11] for details). Since $\overline{\mathcal{P}}$ is a Poincaré duality space, there is a cohomology class $\xi\in H^2(\overline{\mathcal{P}},\mathbb{Q})$ which is Poincaré dual to the closure $\overline{\mathcal{P}(1)}$ of $\mathcal{P}(1)$. The set $\overline{\mathcal{P}(1)}$ is in fact a (singular) complex hypersurface in $\overline{\mathcal{P}}$.

The class ξ can be expressed as a rational linear combination of the class η and the pull-back of a set of generators of $H^2(\overline{\mathcal{M}}_g,\mathbb{Q})$. Such a set of generators consists of the first Chern class λ of the Hodge bundle as well as the Poincaré duals δ_j $(0\leq j\leq \lfloor g/2\rfloor)$ of the irreducible components of the boundary divisor $\overline{\mathcal{M}}_g-\mathcal{M}_g$. We refer to [HM98] for more information.

The first Chern class λ of the Hodge bundle is non-zero precisely when $g\geq 3$ [H83, HM98]. The class δ_0 is dual to the divisor of stable curves with a single non-separating node, and for $1\leq j\leq g/2$, the class δ_j is dual to the divisor of stable curves with a node separating the stable curve into a curve of genus j and a curve of genus $g-j$.

Korotkin and Zograf calculated this linear combination (the formula before Remark 2 on p.456 of [KZ11]) using ideas from mathematical physics. An algebraic geometric proof of Theorem 4.1 is due to Chen [Ch13].

Theorem 4.1 (Korotkin and Zograf [KZ11])**.**

$$\xi = 24P^*\lambda-(6g-6)\eta - P^*(2\delta_0-3\sum_{j=1}^{\lfloor g/2\rfloor}\delta_j).$$

Let $\Pi^E:E\to B$ be a surface bundle over a surface, defined by a smooth map $f:B\to\mathcal{M}_g$. Let $\mathcal{S}\to\mathcal{M}_g$ be the sphere subbundle of the Hodge bundle over $\mathcal{M}_g$; it admits a fibration $\Xi:\mathcal{S}\to\mathcal{P}$ with fiber a circle. Let $F:B\to\mathcal{S}$ be a lift of f to $\mathcal{S}$. Such a lift exists since the dimension of the fiber of $\mathcal{S}$ equals $2g-1>2$ (see for example Lemma 3.5 of [H20]). Denote by $\Delta_E\subset F^*\Xi^*P^*\mathcal{C}$ the pull-back of the variety Δ in the pull-back of the universal curve to B. Then Δ_E defines a homology class $[\Delta_E]\in H_2(E,\mathbb{Q})$ which is Poincaré dual to the Chern class $c_1(\nu^*)$ of the vertical cotangent of E by Proposition 3.2 and the fact that the pull-back to $\mathcal{S}$ of the tautological bundle on $\mathcal{P}$ is trivial. We refer to Section 3 for a more detailed discussion. Denoting as before by $[E]$ the fundamental class of E we have

Corollary 4.2. $c_1(\nu^*)\cup c_1(\nu^*)[E]=[\Delta_E]\cdot[\Delta_E]=c_1(\nu^*)[\Delta_E]$.

Proof. Both equations follows from Poincaré duality for the surface bundle E. □

By the definition of the first Mumford Morita Miller class κ_1 as explained before Proposition 3.5, we obtain

Corollary 4.3. $f^*\kappa_1[B]=c_1(\nu^*)[\Delta_E]$.

In view of the identity $\kappa_1=12\lambda$ on $\mathcal{M}_g$ [HM98], the formula in Theorem 3 is a special case of Theorem 4.1. The following lemma is the first step towards a purely topological proof. For its formulation, recall that we can look at the intersection number between $\varphi(B)$ and the hypersurface $\mathcal{P}(1)\subset\mathcal{P}$.

Lemma 4.4. $\Xi F(B)\cdot\mathcal{P}(1)=2[\Delta_E]\cdot[\Delta_E]=2c_1(\nu^*)[\Delta_E]$*; in particular, the restriction of the class* ξ *to* $\mathcal{P}|\mathcal{M}_g$ *satisfies*

$$\xi=2P^*\kappa_1+a\eta=24P^*\lambda+a\eta$$

for some $a\in\mathbb{Q}$*.*

Proof. Since the complex codimension of $\mathcal{P}(2)\subset\mathcal{P}$ equals two and since by Proposition 2.2 the union $\mathcal{P}(0)\cup\mathcal{P}(1)$ is a smooth orbifold (recall to this end that $\mathcal{P}(1)$ is just the stratum of abelian differentials with a single zero of order two and all other zeros of order one), by transversality we may assume that $\Xi F(B) \subset \mathcal{P}(0) \cup \mathcal{P}(1)$ and that furthermore $\Xi F(B)$ intersects $\mathcal{P}(1)$ transversely in finitely many points.

Let as before $\Delta_E \subset E = F^*\Xi^*P^*\mathcal{C}$ be the pull-back of Δ in the surface bundle $E \to B$. By the first part of Proposition 2.5, Δ_E is a smoothly embedded surface in E. Furthermore, the restriction of the projection $\Pi^E : E \to B$ to Δ_E is a branched covering, doubly branched at each double zero of a differential in the finitely many intersection points of $\Xi F(B)$ with $\mathcal{P}(1)$.

At each branch point $x \in \Delta_E$, the surface Δ_E is tangent to the fiber of E at x, and the orientation of Δ_E coincides with the orientation of the fiber if and only if the intersection point $\Xi F(x)$ of $\Xi F(B)$ with $\mathcal{P}(1)$ is positive.

Assigning to each branch point in Δ_E this sign defines a divisor A on the surface Δ_E. The tangent bundle of Δ_E can be represented in the form $(\Pi_E|\Delta_E)^*(TB) \otimes (-H)$ where H is the line bundle on Δ_E with divisor A. Thus the normal bundle N of Δ_E can be written as $N = \nu \otimes H^+(\otimes H^-)^{-1}$ where H^+ is the line bundle defined by the divisor on Δ_E which corresponds to the positive intersection points of $\Xi F(B)$ with $\mathcal{P}(1)$, and H^- is the line bundle defined by the divisor on Δ_E which corresponds to the negative intersection points.

This implies that the self-intersection number in E of the surface $\Delta_E \subset E$ equals

$$[\Delta_E] \cdot [\Delta_E] = c_1(\nu)[\Delta_E] + b$$

where $b = \Xi F(B) \cdot \mathcal{P}(1)$ is the number of branch points of $\Pi_E|\Delta_E$, counted with sign.

By Poincaré duality (see Corollary 4.2), we have

$$c_1(\nu^*)[\Delta_E] = [\Delta_E] \cdot [\Delta_E] = c_1(\nu)[\Delta_E] + b = -c_1(\nu^*)[\Delta_E] + b$$

and hence $b = 2c_1(\nu^*)[\Delta_E]$. Together with Corollary 4.3 and the fact that $\kappa_1 = 12\lambda$ as classes in $H^2(\mathcal{M}_g, \mathbb{Q})$ [HM98], this completes the proof of the lemma. □

For the proof of Theorem 3 we are left with computing the constant $a \in \mathbb{Q}$.

Proof of Theorem 3. To calculate the coefficient $a \in \mathbb{Q}$ in the expression in Lemma 4.4 note first that in the case $g = 2$, we have $\lambda = 0$ [HM98] and

$$\xi = 2P^*\kappa_1 - 6\eta = -6\eta.$$

Namely, for $g = 2$ the complex rank of the Hodge bundle equals 2 and hence the fibre of the bundle $\mathcal{P} \to \mathcal{M}_2$ over the moduli space of genus 2 complex curves is just $\mathbb{C}P^1$. A *Weierstrass point* on a genus 2 complex curve X is a double zero of a holomorphic one-form on X. Now X has precisely $6 = -3\chi(S_2)$ Weierstrass points and hence the intersection number of the fibre of the bundle $\mathcal{P} \to \mathcal{M}_2$ with the divisor $\mathcal{P}(1)$ equals 6. As the evaluation on $\mathbb{C}P^1$ of the Chern class of the tautological line bundle on $\mathbb{C}P^1$ equals -1, the formula in the theorem follows from Poincaré duality.

For arbitrary $g \geq 3$ choose a complex curve $X \in \mathcal{M}_g$ which admits an unbranched cover of degree $g-1$ onto a curve $Y \in \mathcal{M}_2$. The projective line of projective holomorphic one-forms on Y pulls back to a projective line of projective holomorphic one-forms on X. The pull-back of a projective differential with two simple zeros is a differential with only simple zeros, but the pull-back q of a differential with a double zero is a differential with $g-1$ double zeros. By Proposition 2.5, such a differential is contained in a component $\mathcal{D}$ of a stratum of differentials with $g-1$ double zeros, and it is the locus of a $g-1$-fold normal crossing of its union with the (connecting) stratum $\mathcal{Q} = \mathbb{P}\mathcal{H}(1,1,2,\dots,2)$.

The projective line $\mathbb{C}P^1$ of holomorphic one-forms pulled back from Y is transverse at q to each of the d local branches through q of the closure of $\mathcal{Q}$. As a consequence, if we fix a small disk D about q in the fiber $\mathbb{C}P^1$ of the bundle $\mathcal{P}$, then the homological intersection number of $(D, \partial D)$ with each of these d local branches, that is, the intersection number of a deformation of D with fixed boundary, counted with sign and multiplicities, equals one. As a consequence, the intersection number with $\mathcal{P}(1)$ of this pulled-back $\mathbb{C}P^1$ equals $6d = 6g-6$. This completes the proof of Theorem 3. □

Remark 4.5. Theorem 3 also shows the following. Let X be a Riemann surface of genus g. Then the intersection of $\overline{\mathcal{P}(1)}$ with the complex projective space $\mathbb{C}P^{g-1}$ defined as the projectivization of the vector space of holomorphic one-forms on X is a complex hypersurface in $\mathbb{C}P^{g-1}$. The degree of this hypersurface equals $6g-6$.

5. Boundary divisor computation

Let $\mathcal{Q}$ be a component of a stratum of projective abelian differentials with at least two zeros, with closure $\overline{\mathcal{Q}}$. Assume that the differential in $\mathcal{Q}$ have d_i zeros of order k_i. These zeros define a multisection of the restriction of $P^*\mathcal{C}|\mathcal{Q} = \Pi^{-1}\mathcal{Q}$ to $\mathcal{Q}$ whose closure in $P^*\mathcal{C}$ will be denoted by Δ_{k_i}. The goal of this section is to use the hypersurfaces $\Delta_{k_i} \subset \Pi^{-1}\overline{\mathcal{Q}}$ to obtain some information on the second cohomology group of $\overline{\mathcal{Q}}$.

The following is immediate from Proposition 2.1 and Proposition 2.5.

Lemma 5.1. *Δ_{k_i} is a codimension one subvariety of $P^*\mathcal{C}|\overline{\mathcal{Q}}$ which is a local complete intersection. Thus Δ_{k_i} defines a class in $H^2(P^*\mathcal{C}|\overline{\mathcal{Q}}, \mathbb{Q})$.*

Proof. By Proposition 2.1 and Proposition 2.5, Δ_{k_i} is a codimension one subvariety of the variety $P^*\mathcal{C}|\overline{\mathcal{Q}}$ which is a local complete intersection. Thus Δ_{k_i} is a Weil divisor in $P^*\mathcal{C}|\overline{\mathcal{Q}}$ and hence a Cartier divisor since $P^*\mathcal{C}|\overline{\mathcal{Q}}$ is a complex variety. Via intersection, such a divisor defines a class in $H^2(P^*\mathcal{C}|\overline{\mathcal{Q}}, \mathbb{Q})$ [Fu84]. □

Denote by L_{k_i} the holomorphic line bundle which defines Δ_{k_i}, that is, so that Δ_{k_i} is the zero set of a rational section of L_{k_i}. On the regular subset of Δ_{k_i}, the restriction of L_{k_i} coincides with the normal bundle of Δ_{k_i}. Since $\Delta_{k_i} \cap \Pi^{-1}(\mathcal{Q})$ is a holomorphic multi-section of $\Pi^{-1}\mathcal{Q} \to \mathcal{Q}$, the line bundle L_{k_i} has the following properties.

1. (1) $L_{k_i}|\Pi^{-1}\overline{\mathcal{Q}} - \Delta_{k_i}$ is trivial.
2. (2) The restriction of L_{k_i} to $\Delta_{k_i} \cap \Pi^{-1}(\mathcal{Q})$ coincides with the restriction of vertical tangent bundle ν.
3. (3) The degree of the restriction of L_{k_i} to a fiber of the bundle $\Pi^{-1}\overline{\mathcal{Q}}$ equals the multiplicity d_i of the zero k_i.

The last property follows from the fact that the restriction of the bundle L_{k_i} to a fiber X of $\Pi^{-1}\overline{\mathcal{Q}}$ is the line bundle on X defined by the effective divisor $\Delta_{k_i} \cap X$, and the degree of this divisor equals d_i.

The tautological line bundle over the fibers of $\mathcal{P}$ pulls back via the projection P to a line bundle τ on $P^*\mathcal{C}$. The following is an easy consequence of Proposition 3.2.

Lemma 5.2. $c_1(\nu^* \otimes \tau^*)|\Pi^{-1}\overline{\mathcal{Q}} = \sum_i k_i c_1(L_{k_i})$.

Proof. The Weil divisor $\sum_i k_i\Delta_{k_i}$ is the intersection of the hypersurface $\Delta \subset P^*\mathcal{C}$ with $\Pi^{-1}(\overline{\mathcal{Q}})$. As the cohomology class dual to a divisor is defined by intersection [Fu84], the intersection $\Delta\cap\Pi^{-1}\overline{\mathcal{Q}}$, counted with multiplicites, defines the restriction to $\Pi^{-1}\overline{\mathcal{Q}}$ of the cohomology class dual to Δ. By Proposition 3.2, this cohomology class is the class $c_1(\nu^*\otimes\tau^*)$.

On the other hand, since all constructions are natural, $\sum_i k_i\Delta_{k_i}$ also defines the comology class $\sum_i k_i c_1(L_{k_i})$. This is what we wanted to show. □

Let for the moment X, Y be arbitrary connected locally path connected Hausdorff spaces and let $\varphi : X \to Y$ be an open and closed continuous map. The *degree* of φ is defined as

$$\deg(\varphi) = \sup\{\sharp\varphi^{-1}(y) \mid y \in Y\},$$

and the *local degree* of φ at $x \in X$ is defined as

$$\deg(\varphi, x) = \inf_U \sup\{\sharp\varphi^{-1}\varphi(z) \cap U \mid z \in U\},$$

where U ranges over the neighborhoods of x. The following is taken from [Ed76].

Definition 5.3. An open and closed continuous map $\varphi : X \to Y$ is a *finite branched covering* if $\deg(\varphi) < \infty$ and for each $y \in Y$,

$$\deg(\varphi) = \sum_{x\in\varphi^{-1}(y)} \deg(\varphi, x).$$

The relevance for our purpose is Theorem 2.1 of [Ed76].

Theorem 5.4 (Edmonds [Ed76])**.** *Let $f : X \to Y$ be a finite branched covering. Then there is a transfer homomorphism*

$$\tau : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$$

such that $\tau \circ f^* = \deg(f)\cdot 1$.

As in [H20], we have

Lemma 5.5. *For each i the restriction of the projection Π to Δ_{k_i} is a finite branched covering.*

Proof. The restriction of the projection Π to $\Delta_{k_i} \subset \Pi^{-1}\overline{\mathcal{Q}}$ is clearly open and closed, and its restriction to $\Delta_{k_i} \cap \Pi^{-1}\mathcal{Q}$ is an unbranched covering of degree d_i. Thus it suffices to observe the following. Let $z \in \Delta_{k_i}$; then the local degree of $\varphi = \Pi|\Delta_{k_i}$ at z equals the multiplicity of z in the divisor supported in $\Pi^{-1}(\Pi(z)) \cap \Delta_{k_i}$ which defines the restriction of the line bundle L_{k_i} to $\Pi^{-1}(\Pi(z))$.

To this end choose a sequence of points $z_j \in \Delta_{k_i} \cap \Pi^{-1}\mathcal{Q}$ such that $z_j \to z$ and hence $q_j = \Pi(z_j) \to q = \Pi(z)$.

For each j the intersection $\Delta_{k_i} \cap \Pi^{-1}(q_j)$ is an effective divisor D_j of degree d_i where d_i is the multiplicity of the zero of order k_i for q_j. By the definition of the topology on $\mathcal{P}$, as $i \to \infty$ these divisors converge to an effective divisor D of degree d_i containing z. If the multiplicity of z in D equals $m \geq 1$ then it follows as in Lemma 3.2 of [H20] that the local degree of $\Pi|\Delta_{k_i}$ at z equals m. This implies the lemma. □

By Lemma 5.5 and Theorem 5.4, there is a transfer map

$$H^*(\Delta_{k_i},\mathbb{Q}) \to H^*(\mathcal{P},\mathbb{Q}).$$

Thus we can define a cohomology class

$$\kappa^{k_i} \in H^2(\overline{\mathcal{Q}}, \mathbb{Q})$$

as the image of $c_1(L_{k_i})|\Delta_{k_i}$ under the transfer map $H^*(\Delta_{k_i}, \mathbb{Q}) \to H^*(\overline{\mathcal{Q}}, \mathbb{Q})$. In other words, if A is cycle in $\overline{\mathcal{Q}}$ which defines a homology class $[A] \in H_2(\overline{\mathcal{Q}}, \mathbb{Q})$, then $\kappa^{k_i}[A]$ is the evaluation of $c_1(L_{k_i})$ on $\Pi^{-1}(A) \cap \Delta_{k_i}$ as specified in Theorem 5.4.

Denote by $\mathcal{D}^c_{i,j}$ the closure of a connected component of the boundary divisor $\mathcal{D}_{i,j}$ of $\mathcal{Q}$ which is obtained by colliding two zeros of not necessarily distinct order k_i, k_j. Recall to this end from [KtZ03] that a stratum may have several connected components, and the boundary of a component of a stratum may contain more than one of these components of codimension one. By Proposition 2.2, if differentials in $\mathcal{Q}$ do not have a zero of order $k_i + k_j$, then differential in $\mathcal{D}^c_{i,j}$ contain a single zero of order $k_i + k_j$, and the boundary component $\mathcal{D}^c_{i,j}$ of $\mathcal{Q}$ is a smooth suborbifold of $\mathcal{Q} \cup \mathcal{D}$. Otherwise it is the locus of a normal crossing singularity. Thus the closure of such a boundary component is a divisor in $\overline{\mathcal{Q}}$ which defines a dual line bundle on $\overline{\mathcal{Q}}$. Our goal is to compute the Chern class of this line bundle using the classes κ^{k_ℓ}.

We begin with computing the normal bundle of Δ_{k_ℓ} in $\mathcal{Q} \cup \mathcal{D}^c_{i,j}$. The following is similar to Lemma 4.4.

Lemma 5.6. *Assume that the multiplicity of the zero of order $k_i + k_j$ in $\mathcal{D}^c_{i,j}$ equals one. Then Δ_{k_ℓ} is a smooth suborbifold of $\Pi^{-1}(\mathcal{Q} \cup \mathcal{D}^c_{i,j})$.*

1. *If $i = j$ and $\ell = i$ then the normal bundle of Δ_{k_i} equals $(\nu|\Delta_{k_i}) \otimes N$ where N is the line bundle on Δ_{k_i} defined by the divisor which equals the locus of the zeros of order $2k_i$ of the points in $\mathcal{D}^c_{i,i}$.*
2. *If $i = j$ and $\ell \neq i$ or if $i \neq j$ then the normal bundle of Δ_{k_ℓ} equals the restriction of ν.*

Proof. Recall that for all ℓ, the normal bundle of the intersection $\Delta_{k_\ell} \cap \Pi^{-1}(\mathcal{Q})$ equals the bundle ν.

Now if $i = j$ then by Proposition 2.5, the intersection $\Delta_{k_i} \cap \Pi^{-1}(\mathcal{Q} \cup \mathcal{D}^c_{i,j})$ is tangent to the fibers of the bundle $P^*\mathcal{C}$ at the zeros of order $2k_i$ in Δ_{k_i}. The locus of these zeros equals the branch locus of the projection $\Delta_{k_i} \to \mathcal{Q} \cup \mathcal{D}^c_{i,j}$, and it is a subvariety of Δ_{k_i} of codimension one. Furthermore, the projection $\Pi|\Delta_{k_i}$ is doubly branched along its branch locus. Therefore the normal bundle of Δ_{k_i} equals the tensor product of the restriction of the vertical tangent bundle ν with the line bundle which is dual to this locus of tangency. We refer to the proof of Lemma 4.4 for more details on this well known fact, with a slightly different but equivalent viewpoint.

If $i = j$ and $\ell \neq i$ then the restriction of Π to $\Delta_\ell \cap \Pi^{-1}(\mathcal{Q} \cup \mathcal{D}^c_{i,j})$ is an unbranched covering and hence the normal bundle of $\Delta_\ell \cap \Pi^{-1}(\mathcal{Q} \cup \mathcal{D}^c_{i,j})$ coincides with the vertical tangent bundle ν.

If $i \neq j$ then for each ℓ the intersection $\Delta_{k_\ell} \cap \Pi^{-1}(\mathcal{Q} \cup \mathcal{D}^c_{i,j})$ is smooth, with normal bundle ν. Note however that the hypersurfaces Δ_{k_i} and Δ_{k_j} intersect along the zeros of order $k_i + k_j$. ☐

To keep notations transparent, from now on we denote by $[A]$ the second cohomology class defined by a Cartier divisor A. In particular, for a component $\mathcal{Q}$ of a stratum, with zeros of order k_i, k_j, we obtain a class $[\mathcal{D}^c_{i,j}] \in H^2(\mathcal{Q} \cup \mathcal{D}^c_{i,i}, \mathbb{Q})$ defined by a boundary component $\mathcal{D}^c_{i,j}$ of $\mathcal{Q}$ which is obtained by merging a zero of order k_i with a zero of order k_j. Let as before τ be the pull-back of the tautological bundle on $\mathcal{P}$ to $P^*\mathcal{C}$. The following statement is the main technical result required for the proof of Theorem 2, and it is of independent interest.

Proposition 5.7. *For all i,j the cohomology class $[\mathcal{D}_{i,j}^c]$ is contained in the subgroup of $H^2(\mathcal{Q}\cup\mathcal{D}_{i,j}^c,\mathbb{Q})$ spanned by the restrictions of $P^*\kappa_1$ and η.*

Proof. By Proposition 2.2, if differentials in $\mathcal{D}_{i,j}^c$ have a single zero of order k_i+k_j, then $\mathcal{Q}\cup\mathcal{D}_{i,j}^c$ is a smooth complex orbifold, and otherwise $\mathcal{Q}\cup\mathcal{D}_{i,j}^c$ has a normal crossing singularity along $\mathcal{D}_{i,j}^c$. As a consequence, there is a well defined desingularization of $\mathcal{Q}\cup\mathcal{D}_{i,j}^c$ which is a smooth complex orbifold. This desingularization contains $d\geq 1$ copies of $\mathcal{D}_{i,j}^c$ where d is the multiplicity of the zero of order k_i+k_j in $\mathcal{D}_{i,j}^c$, and $\mathcal{Q}\cup\mathcal{D}_{i,j}^c$ is obtained from this desingularization by identifying these d copies of $\mathcal{D}_{i,j}^c$.

As a consequence, for a smooth closed surface B we can talk about a smooth map $\varphi: B\to\mathcal{Q}\cup\mathcal{D}_{i,j}^c$ which intersects $\mathcal{D}_{i,j}^c$ transversely. Such a smooth map is the projection of a smooth map of B into the desingularization of $\mathcal{Q}\cup\mathcal{D}_{i,j}^c$ which intersects the preimage of the hypersurface $\mathcal{D}_{i,j}^c$ transversely in finitely many points. Each of these intersection points then descends to an intersection point of $\varphi(B)$ with $\mathcal{D}_{i,j}^c$.

Consider the pull-back $\Pi^E: E=\varphi^*P^*\mathcal{C}\to B$ of the universal curve to B. This is a smooth fiber bundle over B. Denote by $\Delta_{k_\ell}^E$ the pull-back of Δ_{k_ℓ} to E. Since $\varphi(B)$ intersects $\mathcal{D}_{i,j}^c$ transversely in finitely many points, the restriction of the projection $\Pi_E: E\to B$ to $\Delta_{k_\ell}^E$ is a branched multi-section, and $\Delta_{k_\ell}^E\subset E$ is a cycle which defines a homology class $\delta_\ell=\delta_{k_\ell}\in H_2(E,\mathbb{Q})$.

Write $\delta=\sum_i k_i\delta_{k_i}$. In view of the fact that $c_1(\nu^*)\cup c_1(\nu^*)[E]=P^*\kappa_1(\varphi_*[B])$, Proposition 3.2 and naturality with respect to pull-back implies that

$$\begin{aligned}\delta\cdot\delta &= c_1(\nu^*\otimes\tau^*)(\delta)=(c_1(\nu^*)-c_1(\tau))\cup(c_1(\nu^*)-c_1(\tau))[E]\\ &= P^*\kappa_1(\varphi_*[B])-2(2g-2)\eta(\varphi_*[B]).\end{aligned}\tag{7}$$

Recall that $c_1(\nu^*)\cup c_1(\tau)[E]=(2g-2)\eta(\varphi_*[B])$.

For the proof of the proposition, we analyze the evaluation of the the cohomology classes in $H^2(E,\mathbb{Q})$ which are Poincare dual to the classes δ_ℓ. To this end we distinguish three cases.

Case 1: $\ell\neq i,j$.

By Lemma 3.3, the restriction of the bundle $\nu^{\otimes(k_\ell+1)}$ to $\Delta_{k_\ell}\cap\Pi^{-1}\mathcal{Q}$ coincides with the restriction of the bundle τ^*. We claim that this holds true on $\Delta_{k_\ell}\cap\Pi^{-1}(\mathcal{Q}\cup\mathcal{D}_{i,j}^c)$. To this end note that since $\ell\neq i,j$ by assumption, the local computation carried out in Lemma 3.3 is valid as well for preimages of points in $\mathcal{D}_{i,j}^c$.

By statement (2) of Lemma 5.6, in this case $\Delta_{k_\ell}^E$ is a smooth multisection of E, and its normal bundle can be identified with the restriction of the vertical tangent bundle ν of E. Since the line bundle L_{k_ℓ} on $P^*\overline{\mathcal{Q}}$ is dual to Δ_{k_ℓ} in the sense of intersections, by naturality of Chern classes under pull-back we conclude that the class δ_ℓ is Poincaré dual to the Chern class $\varphi^*c_1(L_{k_\ell})$ of $\varphi^*(L_{k_\ell})$.

As the restriction of the projection Π^E to $\Delta_{k_\ell}^E$ is an unbranched covering of degree d_ℓ where d_ℓ is the multiplicity of the zero of order k_ℓ in $\mathcal{Q}$, using the transfer map in cohomology for this covering map $\Delta^E_{k_\ell}\to B$ we obtain

$$\begin{aligned}\kappa^{k_\ell}(\varphi_*[B]) &= \varphi^*c_1(L_{k_\ell})(\delta_\ell)=\delta_\ell\cdot\delta_\ell\\ &= c_1(\nu)(\delta_\ell)=\frac{-1}{k_\ell+1}\varphi^*c_1(\tau)(\delta_\ell)=\frac{-d_\ell}{k_\ell+1}\eta(\varphi_*[B]).\end{aligned}\tag{8}$$

Case 2: $i=j=\ell$.

By construction, in this case we have $\Delta^E_{k_q}\cap\Delta^E_{k_u}=\emptyset$ for $q\neq u$. Thus we obtain from equation (7), from Lemma 5.2 and from equation (8) the equation

$$(P^*\kappa_1-2(2g-2)\eta)(\varphi_*[B])=\sum_{\ell\neq i}\frac{-k_\ell d_\ell}{k_\ell+1}\eta(\varphi_*[B])+k_i\kappa^{k_i}(\varphi_*[B]).\tag{9}$$

Solving for $\kappa^{k_i}(\varphi_*[B])$ shows that

$$\kappa^{k_i}(\varphi_*[B])=\frac{1}{k_i}\big(P^*\kappa_1(\varphi_*[B])+(\sum_{\ell\neq i}\frac{k_\ell d_\ell}{k_\ell+1}-2(2g-2))\eta(\varphi_*[B])\big)\tag{10}$$

and hence the restriction of the class κ^{k_i} to $\mathcal{Q}\cup\mathcal{D}^c_{i,j}$ is contained in the subgroup generated by $P^*\kappa_1$ and η.

To show that $[\mathcal{D}^c_{i,i}]$ also is contained in the subgroup generated by $P^*\kappa_1$ and η, recall from Proposition 2.5 that the pull-back of Δ_{k_i} to E is a branched multisection of E, with a single branch point in each fiber of E over the points $x_u\in B$ with $\varphi(x_u)\in\mathcal{D}^c_{i,i}$, and each of these branch points is a zero of order $2k_i$ of the differential $\varphi(x_u)$. Thus by Lemma 5.6 and Lemma 3.3, we deduce as in the proof of Lemma 4.4 that

$$\kappa^{k_i}(\varphi[B])=c_1(L_{k_i})(\delta_i)=\delta_i\cdot\delta_i=c_1(\nu)(\delta_i)+b\tag{11}$$

where b is the number of intersection points between $\varphi(B)$ and $\mathcal{D}^c_{i,i}$, counted with sign and multiplicities.

On the other hand, we have

$$P^*\kappa_1(\varphi_*[B])=\sum_j k_jc_1(\nu^*)(\delta_j)+2(2g-2)\eta(\varphi_*[B])\tag{12}$$

and therefore as in Section 4, we conclude that

$$-c_1(\nu)(\delta_i)=c_1(\nu^*)(\delta_i)=\frac{1}{k_i}\big(P^*\kappa_1\varphi_*[B]-\sum_{u\neq i}c_1(\nu^*)(\delta_u)-2(2g-2)\eta(\varphi_*[B])\big).\tag{13}$$

Since by equation (8) in Case 1 above, for all $u\neq i$ the value $c_1(\nu)(\delta_u)$ is a multiple of $\eta(\varphi_*[B])$, we conclude from equations (11,13) and the fact that the class κ^{k_i} is a linear combination of the restriction to $\overline{\mathcal{Q}}$ of the classes $P^*\kappa_1$ and η that the same holds true for the class $[\mathcal{D}^c_{i,i}]$. This completes the proof of the proposition in the case $k_i=k_j$.

Case 3: $\ell=i\neq j$.

Let $x\in B$ be such that $\varphi(x)\in\mathcal{D}^c_{i,j}$. By modifying φ with an isotopy, we may assume that for some complex structure on B (whose orientation may be opposite to the orientation of B in the case that the intersection index of φ is negative), the map φ is a holomorphic embedding near x, and that the intersection of $\varphi(B)$ with $\mathcal{D}^c_{i,j}$ is transverse at $\varphi(x)$. The pull-back $\Delta^E_{k_i}$ of Δ_{k_i} to E contains a point p_0 in the fiber of E over x which is a zero of the differential $\varphi(x)\in\mathcal{D}^c_{i,j}$ of degree k_i+k_j. Furthermore, it follows from the discussion in the proof of Proposition 2.5 that there are holomorphic local coordinates (z,y) for E near $p_0=(0,0)$ whose range is a polydisk $\{(z,y)\in\mathbb{C}^2\mid |z|<\epsilon,|y|<\epsilon\}$, and with the following properties.

(1) In these coordinates, the projection $\Pi^E:E\to B$ is the second factor projection $(z,y)\to y$.
(2) On the disk $\{(z,y)\mid |z|<\epsilon\}$, the differential $\varphi(y)=\varphi(\Pi^E(z,y))$ is the projectivization of the differential $\Phi(y)=(z-y)^{k_i}(z+y)^{k_j}dz$.

Note that $(z,y)\to\Phi(y)(z)$ defines a section of the pull-back of the tautological bundle τ on $\mathcal{P}$ over the domain of the coordinates (z,y). Moreover, in these coordinates, the locus $\Delta_{k_i}^E$ is the diagonal $D=\{(y,y)\mid |y|<\epsilon\}$. The normal bundle of this diagonal is spanned by the restriction to D of the holomorphic vector field $\frac{\partial}{\partial z}-\frac{\partial}{\partial y}$, and the section $\Psi(z,y)=\frac{1}{(z-y)^{k_i}}(\frac{\partial}{\partial z}-\frac{\partial}{\partial y})$ of the holomorphic tangent bundle of the polydisk is meromorphic, with a pole of order k_i along D.

Now $(z,y)\to\Phi(y)(z)$ also can be viewed as a local holomorphic section of the vertical cotangent bundle. Pairing this section Φ with the meromorphic vector field Ψ defines an isomorphism between the restriction of the line bundle τ to the punctured disk $D-p_0\subset\Delta_{k_i}$ and the k_i+1-th power of the conormal bundle of $D-p_0$. This isomorphism is the restriction of the isomorphism constructed in the proof of Lemma 3.3. Since for $(y,y)\in D$ we have $\Phi(\Psi(y,y))=(2y)^{k_j}$, the rotation number of the image of this isomorphism with respect to a section of the vertical cotangent bundle which extends across p_0 equals k_j.

Using this analysis for all intersection points of $\varphi(B)$ with $\mathcal{D}_{i,j}^c$, we conclude that the restriction of the bundle τ to $\Delta_{k_i}^E$ can be identified with the bundle $(\nu^*)^{\otimes(k_i+1)}\otimes\xi^{k_j}$ where ξ is the bundle with divisor the zeros of order k_i+k_j for the differentials in $\varphi(B)$.

As a consequence, we have

$$\kappa^{k_i}(\varphi_*[B])=\delta_{k_i}\cdot\delta_{k_i}=c_1(\nu)(\delta_{k_i})=\frac{1}{k_i+1}(-d_i\eta(\varphi_*[B])+k_jb) \tag{14}$$

where b is the number of intersections of $\varphi_*(B)$ and $\mathcal{D}_{i,j}^c$, counted with sign and multiplicity, and where d_i is the multiplicity of the zero of order k_i. Similarly, the same equation also holds true if we replace k_i by k_j.

As $\sum_\ell k_\ell\kappa^{k_\ell}(\varphi_*[B])=(P^*\kappa_1-(2g-2)\eta)(\varphi_*[B])$, from Case 1 above we infer that

$$(k_i\kappa^{k_i}+k_j\kappa^{k_j})(\varphi_*[B])=(P^*\kappa_1+(\sum_{\ell\neq i,j}\frac{k_\ell d_\ell}{k_\ell+1}-(2g-2))\eta)(\varphi_*[B]) \tag{15}$$

and hence $k_i\kappa^{k_i}+k_j\kappa^{k_j}$ is contained in the subgroup of $H^2(\overline{\mathcal{Q}},\mathbb{Q})$ generated by $P^*\kappa_1$ and η.

On the other hand, by equation (14) we know that

$$\begin{aligned}&k_ik_j(\frac{1}{k_i+1}+\frac{1}{k_j+1})b\\&=(k_i\kappa^{k_j}+k_j\kappa^{k_j})(\varphi_*[B])+(\frac{k_id_i}{k_i+1}+\frac{k_jd_j}{k_j+1})\eta(\varphi_*[B]).\end{aligned}$$

This yields that indeed, the intersection number b with $\mathcal{D}_{i,j}^c$ is a rational linear combination of $P^*\kappa_1$ and η. This completes the proof of the proposition. □

Remark 5.8. The proof of Proposition 5.7 also shows the following. Let $\mathcal{D}^1,\mathcal{D}^2$ be two boundary components of codimension one of a stratum $\mathcal{Q}$. Let us assume that $\mathcal{D}^1, \mathcal{D}^2$ are distinct components of the same stratum of projective abelian differentials. The cohomology class defined by $\mathcal{D}^i$ can be represented in the form $aP^*\kappa_1 + b\eta$ for both $i = 1, 2$, that is, it is the same linear combination of the classes $P^*\kappa_1$ and η.

Remark 5.9. The computations in the proof of Proposition 5.7 can be used to establish an explicit formula for the classes of the boundary divisors $\mathcal{D}_{i,j}^c$ in $H^2(\mathcal{Q} \cup \mathcal{D}_{i,j}^c, \mathbb{Q})$. As these formulas are rather involved and we do not know any interesting application, we omit this discussion.

6. A stratification of the spin moduli space

The goal of this section is to prove Theorem 2. We begin with the following well known

Lemma 6.1. *Let $\mathcal{V} \subset \mathcal{M}_g$ be any subvariety. If $\kappa_1 = 0$ on $\mathcal{V}$ then $\mathcal{V}$ does not contain any complete complex subvariety.*

Proof. We evoke the following result of Wolpert [W86]: There exists a holomorphic line bundle L on $\mathcal{M}_g$ with Chern class κ_1, and there is a Hermitian metric on L with curvature form $\omega = \frac{1}{2\pi^2}\omega_{WP}$ where ω_{WP} is the Weil Petersson Kähler form on $\mathcal{M}_g$. In particular, ω is positive. As a consequence, if V is a compact complex variety of dimension $k \geq 1$ and if $\zeta : V \to \mathcal{M}_g$ is a holomorphic map which does not factor through a map from a variety of smaller dimension, then

$$\kappa_1^k(\zeta(V)) = \int_V (\zeta^*\omega)^k > 0. \tag{16}$$

This shows that if $\mathcal{V} \subset \mathcal{M}_g$ is a complex subvariety which contains a complete complex subvariety, then $\kappa_1 \neq 0$ on $\mathcal{V}$. □

The following is the main result of [G20]. Its proof is completely elementary.

Theorem 6.2 (Gendron [G20])**.** *A stratum of abelian differentials does not contains a nontrivial complete complex subvariety.*

Let $\mathcal{M}_{g,\mathrm{odd}}$ be the finite orbifold cover of $\mathcal{M}_g$ which is the moduli space of curves with odd theta characteristic. By definition, this is the quotient of Teichmüller space by the finite index subgroup of the mapping class group $\mathrm{Mod}(S_g)$ which preserves an *odd spin structure* on the surface S_g of genus g. Such an odd spin structure is defined as a quadratic form on $H_2(S_g, \mathbb{Z}/2\mathbb{Z})$ with odd Arf invariant (see [KtZ03] for more information). Each of the curves $X \in \mathcal{M}_{g,\mathrm{odd}}$ admits an odd *theta characteristic*, which by definition is a holomorphic line bundle L whose square equals the canonical bundle of X and such that $h^0(X, L)$ is odd. The square of a holomorphic section of L is a holomorphic one-form on X with all zeros of even multiplicity.

All bundles over $\mathcal{M}_g$ will be pulled back to $\mathcal{M}_{g,\mathrm{odd}}$ and will be denoted by the same symbols. Let $\overline{\mathcal{Q}}$ be the closure in $\mathcal{P}$ of the stratum $\mathcal{Q} = \mathbb{P}\mathcal{H}(2, \dots, 2)^{\mathrm{odd}}$ of projective abelian differentials with all zeros of order two and odd spin structure. Then the restriction of the projection $P : \mathcal{P} \to \mathcal{M}_{g,\mathrm{odd}}$ to $\overline{\mathcal{Q}}$ is surjective.

Recall that $\overline{\mathcal{Q}}$ admits a stratification of depth $g-1$ into subspaces $\mathcal{Q}_j$ of codimension $j-1$. Here $\mathcal{Q}_j$ is the union of all components of strata in $\overline{\mathcal{Q}}$ of codimension $j-1$. In particular, we have $\mathcal{Q}_1 = \mathcal{Q}$ and $\mathcal{Q}_{g-1}$ is the union of those components of $\mathbb{P}\mathcal{H}(2g-2)$ with an odd spin structure [KtZ03].

For $r \geq 2$ let

$$\mathcal{M}^r_{g,\mathrm{odd}} = \{(X,L) \in \mathcal{M}_{g,\mathrm{odd}} \mid h^0(X,L) \geq r+1\}.$$

The first part of the following statement is Clifford's theorem (for $g \leq 4$), the second and third parts are due to Teixidor i Bigas [TiB87], in particular Theorem 2.13 in that article.

Theorem 6.3 (Clifford, Teixidor i Bigas [TiB87])**.** (1) *For $g \leq 4$, the locus $\mathcal{M}^r_{g,\mathrm{odd}}$ is empty for all $r \geq 2$.*
(2) *For $g \geq 5$ the locus $\mathcal{M}^2_{g,\mathrm{odd}}$ has pure codimension 3 in $\mathcal{M}_{g,\mathrm{odd}}$.*
(3) *Any component of $\mathcal{M}^r_{g,\mathrm{odd}}$ has dimension at most $3g-2r-2$.*

To avoid technical difficulties we occasionally pass to a finite orbifold cover $\hat{\mathcal{M}}$ of $\mathcal{M}_{g,\mathrm{odd}}$ which is a complex manifold. Then strata of abelian differentials over $\hat{\mathcal{M}}$ are complex manifolds as well. The properties we are interested in do not change by this modification. By abuse of notion, we still work with $\mathcal{M}_{g,\mathrm{odd}}$, adopting the convention that whenever we talk about smooth complex orbifolds, by which we mean the quotient of a smooth complex manifold by a finite group of biholomorphic automorphisms.

Recall from [KtZ03] that a *hyperelliptic component* of (projective) abelian differentials consists of differentials on hyperelliptic curves which are invariant under the hyperelliptic involution. There are two such components in each genus $g \geq 3$, the components $\mathbb{P}\mathcal{H}(g-1,g-1)^{\mathrm{hyp}}$ and $\mathbb{P}\mathcal{H}(2g-2)^{\mathrm{hyp}}$. The projection P maps each of these components onto the locus Hyp of hyperelliptic curves in $\mathcal{M}_g$.

By [KtZ03], for $g \geq 4$ the stratum $\mathbb{P}\mathcal{H}(2g-2)$ has three connected components. There is an odd non-hyperelliptic component $\mathbb{P}\mathcal{H}(2g-2)^{\mathrm{odd}}$, the hyperelliptic component $\mathbb{P}\mathcal{H}(2g-2)^{\mathrm{hyp}}$ and an even component $\mathbb{P}\mathcal{H}(2g-2)^{\mathrm{even}}$. The parity of the hyperelliptic component $\mathbb{P}\mathcal{H}(2g-2)^{\mathrm{hyp}}$ is odd if and only if $g \equiv 1,2 \bmod 4$. In the case $g=3$, the even component coincides with the hyperellipitic component. To keep notations uniform, we put $\mathbb{P}\mathcal{H}(4)^{\mathrm{even}} = \emptyset$. We have

Lemma 6.4. *The image of the projection $P : \mathbb{P}\mathcal{H}(2g-2)^{\mathrm{odd}} \cup \mathbb{P}\mathcal{H}(2g-2)^{\mathrm{even}} \to \mathcal{M}_g$ is disjoint from the hyperelliptic locus.*

Proof. Let $q \in \mathbb{P}\mathcal{H}(2g-2) - \mathbb{P}\mathcal{H}(2g-2)^{\mathrm{hyp}}$ be a projective abelian differential with a single zero on a Riemann surface X which is not contained in the hyperelliptic component of $\mathbb{P}\mathcal{H}(2g-2)$. The zero of the projective differential q is a Weierstrass point on X, and q is uniquely determined by this Weierstrass point.

If X is a hyperelliptic surface, then as Weierstrass points are fixed by the hyperelliptic involution, the projective differential q is invariant under the hyperelliptic involution. But this implies that q is contained in the hyperelliptic component of $\mathbb{P}\mathcal{H}(2g-2)$, a contradiction. □

Example 6.5. If $g=3$ then the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q} = \mathbb{P}\mathcal{H}(2,2)^{\mathrm{odd}}$ in $\mathcal{P}$ consists precisely of squares of projective sections of an odd theta characteristic. By the first part of Theorem 6.3, this implies that the restriction of the projection $P : \mathcal{P} \to \mathcal{M}_{g,\mathrm{odd}}$ to $\overline{\mathcal{Q}}$ is a biholomorphism. Since the spin structure of the hyperelliptic component of $\mathcal{H}(4)$ is even [KtZ03], we have $\overline{\mathcal{Q}} = \mathbb{P}\mathcal{H}(2,2)^{\mathrm{odd}} \cup \mathbb{P}\mathcal{H}(4)^{\mathrm{odd}}$.

As the restriction of the projection P to $\overline{\mathcal{Q}}$ is a biholomorphism, it induces an isomorphism in cohomology. Now $H^2(\mathcal{M}_{3,\mathrm{odd}},\mathbb{Q}) = \mathbb{Q}$ is generated by κ_1

[H83, RW14]. Since $P\mathbb{P}\mathcal{H}(4)^{\mathrm{odd}}$ is a divisor in $\mathcal{M}_{3,\mathrm{odd}}$ and hence defines a second cohomology class $\xi \in H^2(\mathcal{M}_{3,\mathrm{odd}}, \mathbb{Q})$, this class then is a multiple of κ_1. Now a line bundle defined by a divisor is trivial on the complement of the divisor, the restriction of κ_1 to $P\mathcal{H}(2,2)^{\mathrm{odd}}$ vanishes.

On the other hand, by Lemma 6.4, the divisor $P\mathbb{P}\mathcal{H}(4)^{\mathrm{odd}}$ is disjoint from the hyperelliptic locus $\mathrm{Hyp} = P\mathbb{P}\mathcal{H}(4)^{\mathrm{hyp}}$ which is a divisor in $\mathcal{M}_{3,\mathrm{odd}}$. This divisor also defines a multiple of κ_1. In other words, the Chern class of the line bundle defined by the divisor Hyp is a multiple of κ_1. As the line bundle dual to a divisor is trivial on the complement of the divisor, we conclude that the restriction of κ_1 to $P\mathbb{P}\mathcal{H}(4)^{\mathrm{odd}}$ vanishes.

As a consequence, $\mathcal{M}_{3,\mathrm{odd}}$ is stratified into two strata, namely the stratum $\mathcal{M}_{3,\mathrm{odd}} - P\mathbb{P}\mathcal{H}(4)^{\mathrm{odd}}$ and the stratum $P\mathbb{P}\mathcal{H}(4)^{\mathrm{odd}}$, and the restriction of κ_1 to each of these strata vanishes. In particular, these strata do not contain a complete subvariety. Together we obtain Theorem 2 in the case $g = 3$. The article [FL08] contains a stronger result.

Example 6.6. For $g = 4$, Clifford's theorem shows that the restriction of the projection $P : \mathcal{P} \to \mathcal{M}_{4,\mathrm{odd}}$ to the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q} = \mathbb{P}\mathcal{H}(2,2,2)^{\mathrm{odd}}$ is a biholomorphism. Since the spin structure of the hyperelliptic component of $\mathbb{P}\mathcal{H}(6)^{\mathrm{hyp}}$ is even, we have $\overline{\mathcal{Q}} = \mathbb{P}\mathcal{H}(2,2,2)^{\mathrm{odd}} \cup \mathbb{P}\mathcal{H}(2,4)^{\mathrm{odd}} \cup \mathbb{P}\mathcal{H}(6)^{\mathrm{odd}}$.

Since $H^2(\mathcal{M}_{4,\mathrm{odd}}, \mathbb{Q}) = \mathbb{Q}$, we know that $P\mathbb{P}\mathcal{H}(2,4)^{\mathrm{odd}}$ is dual to a multiple of κ_1. This also follows from Proposition 5.7. Namely, as $P|\overline{\mathcal{Q}}$ is a biholomorphism and the class of the boundary divisor $\overline{\mathbb{P}\mathcal{H}(2,4)^{\mathrm{odd}}}$ in $\overline{\mathcal{Q}}$ is a rational linear combination of the class η and $P^*\kappa_1$, the class of the divisor $P\mathbb{P}\mathcal{H}(2,4)^{\mathrm{odd}}$ is a multiple of κ_1. In particular, the class κ_1 vanishes on $\mathcal{M}_{g,\mathrm{odd}} - P\mathbb{P}\mathcal{H}(2,4)^{\mathrm{odd}}$.

Similarly, by Proposition 5.7, the class of the boundary divisor $\mathbb{P}\mathcal{H}(6)^{\mathrm{odd}}$ in $\overline{\mathbb{P}\mathcal{H}(2,4)^{\mathrm{odd}}}$ is a rational linear combination of the class $P^*\kappa_1$ and η. Thus as before, the class of the divisor $P\mathbb{P}\mathcal{H}(6)^{\mathrm{odd}} \subset \overline{P\mathbb{P}\mathcal{H}(2,4)^{\mathrm{odd}}}$ is a rational multiple of κ_1, and the restriction of κ_1 to $P\mathbb{P}\mathcal{H}(2,4)^{\mathrm{odd}}$ vanishes.

This discussion can not be used to show that the restriction of κ_1 to $P\mathbb{P}\mathcal{H}(6)^{\mathrm{odd}}$ vanishes as well. To this end we need a different argument as explained below. Assuming this result, we obtain a stratification of $\mathcal{M}_{4,\mathrm{odd}}$ into 3 strata such that the restriction of κ_1 to each of these strata vanishes. A stronger result is contained in [FL08].

Our next goal is to show that the restriction of κ_1 to $P\mathbb{P}\mathcal{H}(2g-2) \subset \mathcal{M}_g$ vanishes. This then implies that the restriction of κ_1 to $P\mathbb{P}\mathcal{H}(2g-2)^{\mathrm{odd}} \subset \mathcal{M}_{g,\mathrm{odd}}$ vanishes.

This vanishing statement is certainly well known. As we were not able to locate a precise statement in the literature, we provide a proof which also illustrates the use of the results in Section 5 for applications beyond the strict context of that section.

A zero of order $2g-2$ for an abelian differential on a Riemann surface of genus g is a Weierstrass point. As a complex curve of genus g has $(g-1)g(g+1)$ Weierstrass points counted with multiplicity, this implies that the restriction of the projection P to $\mathbb{P}\mathcal{H}(2g-2)$ is a finite morphism onto its image. Although by Theorem 1 the restrictions of $P^*\kappa_1$ and η to $\mathbb{P}\mathcal{H}(2g-2)$ are positive multiples of each other, this does not immediately imply that $\kappa_1 = 0$ on $P\mathbb{P}\mathcal{H}(2g-2)$ as $\mathbb{P}\mathcal{H}(2g-2)$ may be a twisted multisection over its projection, similar to a section of a trivial surface bundle over a surface obtained from a map of non-zero degree from the base onto the fiber (see [H12]).

Instead we directly apply the results of Section 5 towards our goal. Namely, define $\mathcal{O} = \mathbb{P}\mathcal{H}(1, 2g-3)$. By [KtZ03], the stratum is connected. Furthermore, it contains the entire stratum $\mathbb{P}\mathcal{H}(2g-2)$ in its boundary. The dimension of $\mathcal{O}$ equals $2g$.

The following is due to Gendron [G18].

Lemma 6.7. *$P\overline{\mathcal{O}} \subset \mathcal{M}_g$ is a complex variety of dimension $2g$.*

Corollary 6.8. *The restriction of κ_1 to $P\mathbb{P}\mathcal{H}(2g-2)$ vanishes.*

Proof. By Lemma 6.7, $P\overline{\mathcal{O}}$ is a complex variety of dimension $2g$. By Lemma 6.4, it contains two disjoint subvarieties $\mathcal{V}_1, \mathcal{V}_2$ of codimension one. Here $\mathcal{V}_1$ is the projection of the boundary components $\mathbb{P}\mathcal{H}(2g-2)^{\rm odd}$, and $\mathcal{V}_2$ is the projection of $\mathbb{P}\mathcal{H}(2g-2)^{\rm hyp}$. Each of these varieties defines a dual cohomology class. By Proposition 5.7 and Remark 5.8, each of the distinct boundary components of $\mathcal{O}$ defines the same linear combination of the class κ_1 and η. As a consequence, each of the two components $\mathcal{V}_1, \mathcal{V}_2$ define the same multiple of κ_1 in $P\overline{\mathcal{O}}$. Since $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint and being divisors, they define a nontrivial cohomology class on $P\overline{\mathcal{Q}}$, and this class is a multiple of the restriction of κ_1. As $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint, this implies as in the proof of Corollary 6.8 that the restriction of κ_1 to $\mathcal{V}_1$ and $\mathcal{V}_2$ vanishes.

Since in this argument, we may replace $\mathbb{P}\mathcal{H}(2g-2)^{\rm odd}$ by $\mathbb{P}\mathcal{H}(2g-2)^{\rm even}$, this implies the corollary. ☐

Example 6.6 and Corollary 6.8 prove Theorem 2 for $g = 4$. Thus for the remainder of this article, we restrict to the case $g \geq 5$.

For the formulation of the following lemma, note that as strata are smooth complex suborbifolds of $\mathcal{P}$, the intersection of the closure in $\mathcal{P}$ of a stratum with a fiber of $P : \mathcal{P} \to \mathcal{M}_{g,\rm odd}$ is a compact complex variety. We use Theorem 1 to show the following well known fact.

Lemma 6.9. *Let $X \in \mathcal{M}^2_{g,\rm odd}$ and $j \geq 0$ be such that $\dim(\mathcal{Q}_j \cap P^{-1}(X)) > 0$. Then $X \in P(\cup_{\ell \geq j+1} \mathcal{Q}_\ell)$.*

Proof. As both the fiber of $\mathcal{P} \to \mathcal{M}_{g,\rm odd}$ and the union of strata $\mathcal{Q}_j$ are smooth complex suborbifolds of $\mathcal{P}$, for each $X \in \mathcal{M}_{g,\rm odd}$ the intersection $\mathcal{Q}_j \cap P^{-1}(X)$ is a complex (possibly singular) variety.

Let us assume that this variety has a component Y of positive dimension $k \geq 1$. As $P^{-1}(X)$ is compact and $\cup_{\ell \geq j} \mathcal{Q}_\ell \subset \mathcal{P}$ is closed, either $X \in P(\cup_{\ell \geq j+1} \mathcal{Q}_\ell)$ or $Y \subset \mathcal{Q}_j$ is compact.

In the second case, Y is a complex subvariety of $P^{-1}(X) \cap \mathcal{Q}_j$, and as $P^{-1}(X)$ is a complex projective space, we conclude as in the proof of Lemma 6.1 that $\eta^k(Y) > 0$. Namely, η is the Chern class of the tautological bundle over $\mathcal{P}$ whose restriction to a fiber is positive. On the other hand, $Y \subset P^{-1}(X)$ implies that $P^*\kappa_1^k(Y) = 0$. But this contradicts the fact that by Proposition 3.5, the restriction of the class $P^*\kappa_1$ to $\mathcal{Q}_j$ is a positive multiple of η. The lemma is proven. ☐

The following statement illustrates the main remaining step towards the proof of Theorem 2. Recall from Theorem 6.3 the definition of the locus $\mathcal{M}^2_{g,\rm odd}$. Define

$$\mathcal{Z} = \mathcal{M}^2_{g,\rm odd} - \mathcal{M}^4_{g,\rm odd}$$

to be the locus of pairs (X,L) with $h^0(X,L)=3$.

Proposition 6.10. *For $g\geq 5$ the preimage $P^{-1}(\mathcal{M}^2_{g,\mathrm{odd}})\cap\overline{\mathcal{Q}}=\mathcal{E}$ is a divisor in $\overline{\mathcal{Q}}$ which is dual to the restriction of the class η to $H^2(\overline{\mathcal{Q}},\mathbb{Q})$.*

Proof. By Theorem 6.3, for $g\geq 5$ the locus $\mathcal{M}^2_{g,\mathrm{odd}}$ is of pure codimension 3, and for $r\geq 4$ the dimension of the locus $\mathcal{M}^r_{g,\mathrm{odd}}\subset\mathcal{M}_{g,\mathrm{odd}}$ is at most $3g-2r-2$. Thus $\mathcal{M}^2_{g,\mathrm{odd}}$ is the closure of $\mathcal{Z}=\mathcal{M}^2_{g,\mathrm{odd}}-\mathcal{M}^4_{g,\mathrm{odd}}$.

The dimension of the preimage of $\mathcal{M}^2_{g,\mathrm{odd}}$ in $\overline{\mathcal{Q}}$ equals $3g-4$ and hence this preimage is a divisor $\mathcal{E}$ in the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}=\mathbb{P}\mathcal{H}(2,\dots,2)^{\mathrm{odd}}$. We have to show that $\mathcal{E}$ is dual to η.

By counting dimensions, the codimension of the locus $P^{-1}\mathcal{M}^4_{g,\mathrm{odd}}\cap\overline{\mathcal{Q}}$ is at least three. As a consequence, it suffices to show that $\mathcal{E}-P^{-1}(\mathcal{M}^4_{g,\mathrm{odd}})$ is dual to η in the sense of intersections in the complex orbifold $\overline{\mathcal{Q}}-P^{-1}(\mathcal{M}^4_{g,\mathrm{odd}})$.

If $(X,L)\in\mathcal{Z}=\mathcal{M}^2_{g,\mathrm{odd}}-\mathcal{M}^4_{g,\mathrm{odd}}$ then $h^0(X,L)=3$. Thus as $\mathcal{Z}$ is a (non-closed) complex subvariety of $\mathcal{M}_{g,\mathrm{odd}}$ of complex codimension 3, the restriction of the projection P to $P^{-1}(\mathcal{Z})\cap\overline{\mathcal{Q}}$ defines on $P^{-1}(\mathcal{Z})\cap\overline{\mathcal{Q}}$ the structure of a $\mathbb{C}P^2$-bundle over $\mathcal{Z}$. For each point $x\in\mathcal{Z}$, the fiber of this bundle is a projective subplane of the fiber of $\mathcal{P}$. In particular, the restriction of the fiberwise tautological line bundle for $\mathcal{P}$ to this projective plane coincides with the tautological line bundle of this plane.

Since the restriction of P to $\overline{\mathcal{Q}}-P^{-1}\mathcal{M}^2_{g,\mathrm{odd}}$ is a biholomorphism, $P^{-1}(\mathcal{M}_{g,\mathrm{odd}}-\mathcal{M}^4_{g,\mathrm{odd}})\cap\overline{\mathcal{Q}}$ is biholomorphic to the blow-up of the codimension three subvariety $\mathcal{Z}$ in $\mathcal{M}_{g,\mathrm{odd}}-\mathcal{M}^4_{g,\mathrm{odd}}$ by uniqueness of blow-ups as explained on p.604 of [GH78]. The normal bundle of the blow-up of $\mathcal{Z}$ is equivalent to the fiberwise tautological bundle over the blow-up fibers. By the discussion in the previous paragraph, this bundle is just the restriction of the line bundle τ over $\mathcal{P}$. By naturality of Chern classes under pull-back by inclusions, the restriction to $P^{-1}(\mathcal{Z})$ of the Chern class of this normal bundle equals the restriction of η.

Let π be the restriction of the projection P to $\overline{\mathcal{Q}}$. The rational cohomology of the blow-up of $\mathcal{M}_{g,\mathrm{odd}}-\mathcal{M}^4_{g,\mathrm{odd}}$ along $\mathcal{Z}$ equals

$$\pi^*H^*(\mathcal{M}_{g,\mathrm{odd}}-\mathcal{M}^4_{g,\mathrm{odd}},\mathbb{Q})\oplus H^*(\pi^{-1}(\mathcal{Z}),\mathbb{Q})/\pi^*H^*(\mathcal{Z},\mathbb{Q})$$

(see p.605 of [GH78]), and the cohomology $H^*(\pi^{-1}(\mathcal{Z}),\mathbb{Q})$ is the cohomology of a $\mathbb{C}P^2$-bundle over $\mathcal{Z}$ whose cohomology ring is a quotient of $H^*(\mathcal{Z},\mathbb{Q})[\hat{\eta}]$ where $\hat{\eta}$ is the restriction of η (p.406 of [GH78]). Since $H^2(\mathcal{M}_{g,\mathrm{odd}},\mathbb{Q})$ is spanned by κ_1, this yields that the class defined by the divisor $\pi^{-1}(\mathcal{Z})$ is of the form $\eta|\overline{\mathcal{Q}}$. $\square$

Define

$$\mathcal{Y}=P(\overline{\mathcal{Q}}-\mathcal{Q})\subset\mathcal{M}_{g,\mathrm{odd}}.$$

For reasons of dimension, $\mathcal{Y}$ is a divisor in $\mathcal{M}_{g,\mathrm{odd}}$ which contains $\mathcal{Z}$ as a subvariety of codimension two by Lemma 6.9. Thus by Proposition 6.10, $\mathcal{W}=P^{-1}(\mathcal{Y})\cap\overline{\mathcal{Q}}$ is a divisor in $\overline{\mathcal{Q}}$ containing the closure $\mathcal{E}$ of $P^{-1}(\mathcal{M}^2_{g,\mathrm{odd}})\cap\overline{\mathcal{Q}}$ as an irreducible component.

As $H^2(\mathcal{M}_{g,\mathrm{odd}},\mathbb{Q})$ is generated by the class κ_1, the divisor $\mathcal{Y}$ is dual to a line bundle ξ whose Chern class $c_1(\xi)$ is a multiple of κ_1. The line bundle ξ is trivial on $\mathcal{M}_{g,\mathrm{odd}}-\mathcal{Y}$, and its pull-back to $\overline{\mathcal{Q}}$ is trivial on $\overline{\mathcal{Q}}-P^{-1}\mathcal{Y}\subset\mathcal{Q}$. By naturality of the duality between divisors and line bundles under birational maps, we conclude that $P^{-1}\mathcal{Y}$ is the defining divisor for the pull-back of ξ to $\overline{\mathcal{Q}}$. In other words, $P^*\kappa_1$ vanishes on $\overline{\mathcal{Q}} - P^{-1}\mathcal{Y}$.

Recall that $P^{-1}\mathcal{Y} = \mathcal{E} \cup (\overline{\mathcal{Q}} - \mathcal{Q})$ is reducible, and the irreducible component $\mathcal{E}$ dual to η may intersect $\mathcal{Q}$ non-trivially. As a consequence, the cohomology class in $H^2(\overline{\mathcal{Q}}, \mathbb{Q})$ defined by the irreducible component $\overline{\mathcal{Q}} - \mathcal{Q}$ (which is the closure of the connected stratum $\mathbb{P}\mathcal{H}(2,\dots,2,4)^{\mathrm{odd}}$) is a rational linear combination of the classes η and $P^*\kappa_1$ as was shown in Proposition 5.7.

Recall that we denoted by $\mathcal{Q}_j$ the union of the components of strata in $\overline{\mathcal{Q}}$ of codimension $j-1$. We are now ready to complete the main step in the proof of Theorem 2 from the introduction. Note that by a result of Diaz [D84], the maximal dimension of a complete subvariety of $\mathcal{M}_g$ and hence of $\mathcal{M}_{g,\mathrm{odd}}$ is not bigger than $g-2$. We do not have information on a sharp bound.

Proposition 6.11. *For $k \leq g-1$ define $\mathcal{D}_k = P(\mathcal{Q}_k) - \overline{P(\mathcal{Q}_{k+1})}$; then for all k, the restriction of κ_1 to the locus $\mathcal{D}_k$ vanishes. As a consequence, $\mathcal{D}_k$ does not contain a complete variety of positive dimension, and $\mathcal{M}_{g,\mathrm{odd}} - \cup_{j\geq k+1}\mathcal{D}_j$ does not contain a complete variety of dimension at least k.*

Proof. By Corollary 6.8, it suffices to show the proposition in the case $k \leq g-2$. Thus let $k \leq g-2$ and let $\mathcal{A}$ be a component of $\mathcal{Q}_k$. This is a component of a stratum of abelian differentials with all zeros a multiple of 2 and odd spin structure. We have to show that the restriction of κ_1 to $P\mathcal{A} - \overline{P\mathcal{Q}_{k+1}}$ is trivial. By Lemma 6.1, this then implies that $P\mathcal{Q}_k - \overline{P\mathcal{Q}_{k+1}}$ does not contain a complete variety of positive dimension.

As the projection P is closed, $P\overline{\mathcal{A}}$ is a closed subvariety of $\mathcal{M}_{g,\mathrm{odd}}$. Define $\mathcal{R}$ to be the closure of the set $\{z \in P\overline{\mathcal{A}} \mid \dim(P^{-1}(z) \cap \overline{\mathcal{A}}) > 0\}$. Then $\mathcal{R}$ is a closed subvariety of $P\overline{\mathcal{A}}$ which is contained in $\overline{P\mathcal{Q}_{k+1}}$ by Lemma 6.9. Since the restriction of P to each component of a stratum in $\overline{\mathcal{Q}}$ is generically finite-to-one [G18], its codimension in $P\overline{\mathcal{A}}$ is at least one (in fact, $\mathcal{R}$ may be empty). Furthermore, the preimage $\hat{\mathcal{R}}$ of $\mathcal{R}$ in $\overline{\mathcal{A}}$ is of codimension at least one as well.

By naturality of pull-backs under birational maps, the pull-back by $P|\overline{\mathcal{A}}$ of the cohomology class dual to the divisor $P(\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1})$ is the cohomology class dual to $\hat{\mathcal{R}} \cup (\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1})$.

Consider first the case that the codimension of $\hat{\mathcal{R}}$ is at least two. Then this class coincides with the class defined by the divisor $\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1}$ in $\overline{\mathcal{A}}$. By Proposition 5.7, this class is a linear combination of $P^*\kappa_1$ and η. By naturality under pull-back, we conclude that $P(\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1})$ defines a multiple of the restriction of κ_1. As a consequence, the restriction of κ_1 to $P\overline{\mathcal{A}} - P\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1}$ vanishes as claimed in the proposition.

If the codimension of $\hat{\mathcal{R}}$ equals one, then $P^{-1}(P(\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1})) = \hat{\mathcal{R}} \cup (\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1})$ is reducible, and it defines the pull-back of the class of $P(\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1}) \subset P\overline{\mathcal{A}}$. By Proposition 5.7, the class of $\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1}$ is a rational linear combination of $P^*\kappa_1$ and η. On the other hand, we know that the class of the divisor $\hat{\mathcal{R}}$ is not the pull-back of a second cohomology class on $P\overline{\mathcal{A}}$. As in the proof of Proposition 6.10, in this case we deduce that it defines a multiple of the fiber class η, and the divisor $P(\overline{\mathcal{A}} \cap \overline{\mathcal{Q}}_{k+1})$ in $P\overline{\mathcal{A}}$ defines a multiple of κ_1. Together with Corollary 6.8, this completes the first part of the proposition.

To show the second part of the proposition, let $V \subset \mathcal{M}_{g,\mathrm{odd}}$ be a complete variety of dimension $k \geq 1$ and assume that $V \subset \mathcal{M}_{g,\mathrm{odd}} - \mathcal{D}_k$. As $\mathcal{M}_{g,\mathrm{odd}} - \mathcal{D}_2$

does not contain a complete variety, the variety V has to intersect $\mathcal{D}_2$ nontrivially. Since $\mathcal{D}_2 \subset \mathcal{M}_{g,\mathrm{odd}}$ is a closed subvariety of codimension one, this intersection is a complete variety V_2 whose dimension is at least $k-1$.

Repeat this reasoning with $V_2 \subset \mathcal{D}_2$ and the subvariety $\mathcal{D}_3$. In finitely many such steps we conclude that if $V \subset \mathcal{M}_{g,\mathrm{odd}} - \mathcal{D}_{k+1}$ has dimension k, then $V \cap \mathcal{D}_k$ is a complete variety of dimension at least one which is disjoint from $\mathcal{D}_{k+1}$. By the above, this is impossible. This completes the proof of the proposition. □

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