ACTIONS OF FINITELY GENERATED GROUPS ON COMPACT METRIC SPACES

ABSTRACT. Let Γ be a finitely generated group which admits an action by homeomorphisms on a metrizable space X. We show that there is a metric on X defining the original topology such that for this metric, the action is by bi-Lipschitz transformations.

1. INTRODUCTION

The homeomorphism group $\mathrm{Homeo}(S^1)$ of the circle S^1 contains many large and interesting subgroups, but there are also many rigidity results stating that large classes of groups do not embed into $\mathrm{Homeo}(S^1)$. Most of the known rigidity results require however that the action is by diffeomorphisms of class at least $C^{1,\alpha}$ for some $\alpha > 0$, but it is generally believed that many of these results are true in larger generality. We refer to the recent book [KK21] for an account of what is known to date.

On the other hand, for *countable* groups acting on the circle, some additional regularity can always be assumed. The following is Theorem D of [DKN07].

Theorem 1 (Deroin, Kleptsyn and Navas)**.** *An action of a countable group Γ on S^1 is conjugate to an action by bi-Lipschitz transformations.*

This result is sharp (see however [Na14] for more refined information on circle actions): There are homeomorphisms of manifolds of dimensions different from 1 and 4 which are not conjugate to Lipschitz maps. We refer to [H79] for examples of such maps (and to an account of non-improvability of the regularity of C^r-diffeomorphisms by conjugation with a homeomorphism). On the other hand, a countable group of homeomorphisms of a compact manifold is conjugate to a group of homeomorphisms preserving the Lebesgue measure class (2.3.17 of [Na11]).

The goal of this note is to point out that from the point of view of rigidity of actions on metric spaces, Theorem 1 is not specific to groups acting on S^1.

Theorem 2. *Let Γ be a finitely generated group acting as a group of homeomorphisms on a metrizable space X. Then there exists a metric d on X defining the original topology such that the action is by bi-Lipschitz transformations for d.*

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For finitely generated groups, Theorem 1 follows from Theorem 2 as will be discussed at the end of this note. It is in this conclusion where specific properties of the circle enter.

After this work was completed, Sang-hyun Kim [Ki23] informed me that Theorem 2 extends in fact to all countable groups, and there are versions for locally compact topological groups as well.

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2. Proof of the theorem

Consider a finitely generated group Γ. Denote by $\mathcal{C}$ the Cayley graph of Γ with respect to some symmetric finite generating set. Giving all edges length one gives $\mathcal{C}$ the structure of a locally finite geodesic metric graph. The group Γ acts freely and cocompactly from the left on $\mathcal{C}$ as a group of isometries. Let dist be the Γ-invariant distance function on $\mathcal{C}$.

The *critical exponent* $\delta(\Gamma)$ of Γ with respect to the chosen finite generating set is the infimum of all numbers $s > 0$ such that the *Poincaré series*

$$\sum_{\psi\in\Gamma} e^{-s\,\mathrm{dist}(y,\psi x)}$$

converges for some and hence all $x, y \in \mathcal{C}$. Since vertices of $\mathcal{C}$ of distance ℓ to the identity correspond to reduced words in the generating set of length ℓ, the critical exponent of Γ is finite.

Note that the critical exponent depends in a sensitive way on the generating set and hence on the resulting word metric. However, all what matters for our purpose is the existence of some left invariant metric on Γ of finite critical exponent.

Lemma 2.1. *For any $s > \delta(\Gamma)$ the value of the convergent series*

$$\sum_{\psi\in\Gamma} e^{-s\,\mathrm{dist}(y,\psi x)}$$

is bounded independently of $x, y \in \mathcal{C}$.

Proof. For $\zeta \in \Gamma$ we have

$$\sum_{\psi\in\Gamma} e^{-s\,\mathrm{dist}(\zeta y,\psi x)} = \sum_{\psi\in\Gamma} e^{-s\,\mathrm{dist}(\zeta y,\zeta\psi x)} = \sum_{\psi\in\Gamma} e^{-s\,\mathrm{dist}(y,\psi x)}$$

and hence the claim is immediate from continuity of the distance function and cocompactness of the action of Γ on $\mathcal{C}$. $\square$

Let us now assume that the group Γ acts as a group of homeomorphisms on the metrizable space X. Choose a metric $\hat{\delta}$ on X defining its topology. By possibly replacing $\hat{\delta}$ by the metric $\hat{\delta}_0 = \min\{\hat{\delta}, 1\}$ (which defines the same topology) we may assume that the diameter of $\hat{\delta}$ is finite.

View the identity e of Γ as a basepoint in $\Gamma \subset \mathcal{C}$. For $\psi \in \Gamma$ write

$$\hat{\delta}_\psi = \hat{\delta} \circ \psi^{-1};$$

this defines a Γ-equivariant family of distance functions on X indexed by the elements of Γ, with $\hat{\delta}_e = \hat{\delta}$. Let $s > \delta(\Gamma)$ and for $p \in \mathcal{C}$ define

$$\delta_p = \sum_{\psi \in \Gamma} e^{-s \operatorname{dist}(p,\psi)} \hat{\delta}_\psi. \tag{1}$$

Lemma 2.2. *δ_p is a distance function on X for each $p \in \mathcal{C}$.*

Proof. As the sum of two distance functions is a distance function, all we have to show is that for each $p \in \mathcal{C}$ and any two points $x, y \in X$ the increasing sequence

$$\sum_{\operatorname{dist}(p,\psi) \le n} e^{-s \operatorname{dist}(p,\psi)} \hat{\delta}_\psi(x, y)$$

converges as $n \to \infty$.

However, since the diameter of $\hat{\delta}_\psi$ is a finite number $D > 0$ not depending on ψ, for $m > n$ we have

$$\begin{aligned} |\sum_{\operatorname{dist}(p,\psi) \le m} e^{-s \operatorname{dist}(p,\psi)} \hat{\delta}_\psi(x, y) - \sum_{\operatorname{dist}(p,\psi) \le n} e^{-s \operatorname{dist}(p,\psi)} \hat{\delta}_\psi(x, y)| \\ \le \sum_{\operatorname{dist}(p,\psi) > n} e^{-s \operatorname{dist}(p,\psi)} D, \end{aligned}$$

and the last term in this inequality converges to zero as $n \to \infty$ by Lemma 2.1. □

The following proposition summarizes the properties of these distance functions we are interested in.

Proposition 2.3.
(1) *The distances δ_p $(p \in \mathcal{C})$ are mutually bi-Lipschitz equivalent, with bi-Lipschitz constant bounded by a Γ-invariant function on $\mathcal{C} \times \mathcal{C}$, and they define the original topology on X.*
(2) *For all $p \in \mathcal{C}, \psi \in \Gamma$ we have $\delta_{\psi p} = \delta_p \circ \psi^{-1}$. Furthermore, each $\psi \in \Gamma$ acts on (X, δ_e) as a bi-Lipschitz transformation.*
(3) *Up to adjusting the parameter $s > 0$, if the action of Γ on $(X, \hat{\delta})$ is by bi-Lipschitz transformations, then the metric δ_e is bi-Lipschitz equivalent to $(X, \hat{\delta})$.*

Proof. By the definition of the distances δ_p, for all $\eta \in \Gamma$ we have

$$\delta_p \circ \eta^{-1} = (\sum_\psi e^{-s \operatorname{dist}(p,\psi)} \hat{\delta}_e \circ \psi^{-1}) \circ \eta^{-1} = \sum_{\eta\psi} e^{-s \operatorname{dist}(\eta p, \eta\psi)} \hat{\delta}_{\eta\psi} = \delta_{\eta p} \tag{2}$$

which shows equivariance.

To show that the distances δ_p are mutually bi-Lipschitz equivalent, it suffices to observe that for all $p \in \mathcal{C}$ we have

$$e^{-s\,\mathrm{dist}(p,e)}\delta_e \le \delta_p \le e^{s\,\mathrm{dist}(p,e)}\delta_e. \tag{3}$$

To this end note that by definition and the triangle inequality, for $p \in \mathcal{C}$ we have

$$\delta_p = \sum_{\psi\in\Gamma} e^{-s\,\mathrm{dist}(p,\psi)}\hat{\delta}_\psi \ge e^{-s\,\mathrm{dist}(p,e)} \sum_{\psi\in\Gamma} e^{-s\,\mathrm{dist}(e,\psi)}\hat{\delta}_\psi,$$

which yields

$$\delta_p \ge e^{-s\,\mathrm{dist}(p,e)}\delta_e.$$

The reverse estimate $\delta_e \ge e^{-s\,d(p,e)}\delta_p$ follows from exactly the same argument.

By formula (2), for all $\eta \in \Gamma$ the map $\eta : (X, \delta_e) \to (X, \delta_\eta)$ is an isometry. It now follows from bi-Lipschitz equivalence of the metrics δ_ψ ($\psi \in \Gamma$) that $\psi : (X, \delta_e) \to (X, \delta_e)$ is bi-Lipschitz, with controlled bi-Lipschitz constant. The second part of the proposition follows.

We claim that the distances δ_p define the original topology on X. To this end note that by definition, we have $\delta_e \ge \hat{\delta}_e = \hat{\delta}$ and hence the identity map $(X, \delta_e) \to (X, \hat{\delta})$ is one-Lipschitz. In particular, the identity map $(X, \delta_e) \to (X, \hat{\delta})$ is continuous.

To show that the identity map $(X, \hat{\delta}) \to (X, \delta_e)$ is continuous as well it suffices to show that for every $x \in X$ and for every $\epsilon > 0$ the open ball $B_e(x, \epsilon)$ of radius ϵ about x for the metric δ_e contains a neighborhood of x for the topology defined by the metric $\hat{\delta} = \hat{\delta}_e$.

To this end let $D > 0$ be the diameter of $\hat{\delta}_e$. For $\epsilon > 0$ there is a finite subset $A \subset \Gamma$ so that

$$\sum_{\psi\notin A} e^{-s\,\mathrm{dist}(e,\psi)} D < \epsilon/2. \tag{4}$$

Let $b > 0$ be such that

$$\sum_{\psi\in A} e^{-s\,\mathrm{dist}(e,\psi)} b < \epsilon/2$$

and let $C = \cap_{\psi\in A}\hat{B}_\psi(x, b)$ where $\hat{B}_\psi(x, b)$ denotes the open ball of radius b about x for the metric $\hat{\delta}_\psi$. Note that C is an open neighborhood of x for the topology induced by the metric $\hat{\delta}_e$ because the group Γ acts on $(X, \hat{\delta}_e)$ as a group of homeomorphisms and A is finite.

If $y \in C$ then $\hat{\delta}_\psi(x, y) < b$ for all $\psi \in A$ and hence

$$\sum_{\psi\in A} e^{-s\,\mathrm{dist}(e,\psi)}\hat{\delta}_\psi(x, y) < \epsilon/2.$$

Together with (4), this yields $\delta_e(x, y) < \epsilon$. As $C \subset X$ is open for the topology defined by $\hat{\delta}_e$, this implies that the ball of radius ϵ about x for the metric δ_e contains an open neighborhood of x for $\hat{\delta}_e$. Since $\epsilon > 0$ was arbitrary, we conclude that a subset of X which is open for the topology induced by δ_e also is open for the topology induced by $\hat{\delta}_e$. In other words, the identity $(X, \hat{\delta}_e) \to (X, \delta_e)$ is indeed continuous. This completes the proof of part (1) of the proposition.

To show the third part of the proposition, we have to show that if Γ acts by bi-Lipschitz transformations then up to adjusting s, the identity $(X,\hat{\delta}) \to (X,\delta_e)$ is Lipschitz. To this end let $\psi_1,\dots,\psi_k$ be the symmetric generating set defining the Cayley graph $\mathcal{C}$. Assume that the action of Γ on $(X,\hat{\delta})$ is by bi-Lipschitz transformations and let $L_i \geq 1$ be the bi-Lipschitz constant of the element ψ_i. Write $L = \max\{L_i \mid i\}$. Then the bi-Lipschitz constant of any $\psi \in \Gamma$ does not exceed $L^{\mathrm{dist}(e,\psi)}$.

Now assume that $s > 0$ is large enough that $e^s > e^\delta L$ where as before, $\delta > 0$ is the critical exponent of Γ, say $e^{-s}L \leq e^{-u}$ for some $u > \delta$. Then for all $x, y \in X$ and all $\psi \in \Gamma$ we have

$$e^{-s\,\mathrm{dist}(e,\psi)} L^{\mathrm{dist}(e,\psi)} < e^{-u\,\mathrm{dist}(e,\psi)}.$$

As a consequence, the identity

$$(X,\hat{\delta}) \to (X, e^{-s\,\mathrm{dist}(e,\psi)}\hat{\delta}_\psi)$$

is $e^{-u\,\mathrm{dist}(e,\psi)}$-Lipschitz. Summing over $\psi \in \Gamma$ then yields that the identity $(X,\hat{\delta}) \to (X,\delta_e)$ is $\sum_\psi e^{-u\,\mathrm{dist}(e,\psi)}$-Lipschitz. Since this sum converges, Lipschitz equivalence of $(X,\hat{\delta})$ and (X,δ_e) follows. □

Remark 2.4. The proof of Theorem 2 rests on the existence of a left invariant metric on the group Γ whose growth is bounded from above by an exponential function. Although the Birkhoff-Kakutani theorem gives a left invariant metric on any countable group, this metric may not fulfill a growth condition. However, it was pointed out by Kim [Ki23] that this difficulty can be overcome.

As an easy consequence we obtain the proof of Theorem 1 for finitely generated groups.

Corollary 2.5. *A finitely generated group of homeomorphisms of S^1 is conjugate to a group of Lipschitz homeomorphisms.*

Proof. For a fixed basepoint on S^1 and the choice of an orientation, a volume normalized length metric on S^1 is just a Borel probability measure μ on S^1 of full support and with no atoms. The standard normalized Lebesgue measure λ on S^1 corresponds to the standard distance function.

A probability measure μ on S^1 of full support without atoms defines a homeomorphism $\Psi_\mu : S^1 \to S^1$ by $\Psi_\mu(t) = s$ if $\mu[0,s] = t$. This homeomorphism satisfies $(\Psi_\mu)_*\lambda = \mu$.

Starting with the standard distance d and the Lebesgue measure λ on S^1, the distance function δ constructed in the proof of Proposition 2.3 corresponds to the measure

$$\nu = \sum_\psi e^{-s\,\mathrm{dist}(\psi,e)}\psi_*\lambda.$$

Putting $\mu = \nu/\nu(S^1)$, the conjugation of the action of Γ on (S^1,d) with a Lipschitz action is given by the homeomorphism Ψ_μ considered in the previous paragraph. □

References

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