Christmas Sheet

Remark: You are not required to solve all the problems. Some problems are standard, others amusing and some really hard!

8.1. Let $P_1, P_2$ be two copies of the real projective plane $\mathbb{R}P^2$. Calculate the fundamental group of the connected sum of $P_1, P_2$ which is the space $K$ obtained by removing from $P_1, P_2$ the interiors of two embedded discs and identifying the boundaries with a homeomorphism. Calculate the homology of $K$.

Calculate the fundamental group and the homology of the connected sum of $K$ with $\mathbb{R}P^2$.

8.2. Modify the construction of the Alexander horned sphere to produce an embedding of the two-sphere $S^2$ into $S^3$ so that both components of $S^3 - S^2$ are not simply connected.

8.3. Can you construct the three-sphere from two handlebodies of genus 2 by gluing them along their boundary? If yes, how many ways are there to do this up to homeomorphism of $S^3$?

8.4. Remove from the disc $D \subset \mathbb{C}$ the interior of two disjoint embedded subdiscs. Is there a homeomorphism of the resulting disc without fixed point? What about a homeomorphism of the complement of three open discs in $D$?

8.5. Let $U$ be an open cover of $T^2$. Assume that for any $U \in U$ the rank of the image of the map $H_1(U, \mathbb{Z}) \to H_1(T^2, \mathbb{Z})$ is at most one. Show that there are three distinct sets in $U$ with non-empty intersection.

8.6. Suppose you are given a picture attached to a rope. Can you hang up the picture with the rope on the wall with two nails so that if you remove any one of the nails, the picture falls down?

8.7. Adventskalender

Suppose you are given a set of squares of side length one each, made out of cardboard. Each square has a pair of opposite sides which are colored red, the other pair of sides is colored blue.

Can you construct from the square for each day in Advent (numbered 1 to 24) a surface whose genus is this number by gluing squares at their sides in such a way that red sides are glued to red sides and blue sides are glued to blue sides?

If yes, how many square you need at least in total?