

DYNAMICAL PROPERTIES OF THE WEIL-PETERSSON METRIC

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ABSTRACT. Let $\mathcal{T}(S)$ be the Teichmüller space of an oriented surface S of finite type. We discuss the action of subgroups of the mapping class group of S on the CAT(0)-boundary of the completion of $\mathcal{T}(S)$ with respect to the Weil-Petersson metric. We show that the set of invariant Borel probability measures for the Weil-Petersson flow on moduli space which are supported on a closed orbit is dense in the space of all ergodic invariant probability measures.

1. Introduction

For an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and *complexity* $3g - 3 + m \geq 2$ let $\mathcal{T}(S)$ be the *Teichmüller space* of all isotopy classes of complete hyperbolic metrics on S of finite volume. Then $\mathcal{T}(S)$ is a contractible manifold which can be equipped with the *Weil-Petersson metric*, an incomplete Kähler metric of negative sectional curvature.

In spite of the lack of completeness, any two points in $\mathcal{T}(S)$ can be connected by a unique Weil-Petersson geodesic which depends smoothly on its endpoints [9]. As a consequence, $\mathcal{T}(S)$ can be completed to a *Hadamard space* $\overline{\mathcal{T}(S)}$, i.e. a complete simply connected CAT(0)-space which however is not locally compact.

A Hadamard space X admits a *visual boundary* ∂X , and the action of the isometry group of X extends to an action on ∂X . For surfaces S of complexity at most three, the visual boundary $\partial \overline{\mathcal{T}(S)}$ of $\overline{\mathcal{T}(S)}$ was identified by Brock and Masur [7], but for higher complexity it is not known. However, it follows from the work of Brock [6] that the boundary is not locally compact.

The *mapping class group* $\text{Mod}(S)$ of all isotopy classes of orientation preserving diffeomorphisms of S acts on $\mathcal{T}(S)$ as a group of isometries. Since every isometry of $\mathcal{T}(S)$ extends to an isometry of the completion $\overline{\mathcal{T}(S)}$, the mapping class group also acts isometrically on $\overline{\mathcal{T}(S)}$.

An isometry g of $\overline{\mathcal{T}(S)}$ is called *axial* if g admits an axis, i.e. if there is a geodesic $\gamma : \mathbb{R} \rightarrow \overline{\mathcal{T}(S)}$ and a number $\tau > 0$ such that $g\gamma(t) = \gamma(t + \tau)$ for all t . The endpoints $\gamma(\infty), \gamma(-\infty)$ of γ are then fixed points for the action of g on $\partial \overline{\mathcal{T}(S)}$. Every *pseudo-Anosov* mapping class $g \in \text{Mod}(S)$ is axial [9]. The *limit set* Λ of a

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subgroup G of $\text{Mod}(S)$ is the set of accumulation points in $\overline{\partial\mathcal{T}(S)}$ of an orbit of the action of G on $\overline{\mathcal{T}(S)}$. The group G is called *non-elementary* if its limit set contains at least three points. We show

THEOREM 1.1. *Let $G < \text{Mod}(S)$ be a non-elementary subgroup with limit set Λ which contains a pseudo-Anosov element.*

- (1) Λ does not have isolated points, and the G -action on Λ is minimal.
- (2) Pairs of fixed points of pseudo-Anosov elements are dense in $\Lambda \times \Lambda$.
- (3) There is a dense orbit for the action of G on $\Lambda \times \Lambda$.

There is a compactification of $\mathcal{T}(S)$ whose boundary consists of the sphere \mathcal{PML} of projective measured geodesic laminations on S . The mapping class group acts on $\mathcal{T}(S) \cup \mathcal{PML}$ as a group of homeomorphisms. Limit sets for subgroups of $\text{Mod}(S)$ in \mathcal{PML} were investigated by McCarthy and Papadopoulos [16].

We also look at properties of the *Weil-Petersson geodesic flow* Φ^t on the quotient of the unit tangent bundle $T1\mathcal{T}(S)$ of $\mathcal{T}(S)$ under the action of the mapping class group. Even though this quotient space $T1\mathcal{M}(S)$ is non-compact and this flow is not everywhere defined, it admits many invariant Borel probability measures. Particular such measures are measures supported on periodic orbits. Each of these measures is ergodic.

The space of all Φ^t -invariant Borel probability measures on $T1\mathcal{M}(S)$ can be equipped with the weak*-topology. Our second result is a version of Theorem 1 for the Weil-Petersson geodesic flow.

THEOREM 1.2. *A Φ^t -invariant Borel probability measure on $T1\mathcal{M}(S)$ can be approximated in the weak*-topology by measures supported on periodic orbits.*

The organization of this note is as follows. In Section 2 we review some geometric properties of Hadamard spaces. Section 3 explains some geometric properties of pseudo-Anosov mapping classes. In Section 4 we look at groups of isometries and establish the first and the second part of Theorem 1.1. In Section 5, we complete the proof of Theorem 1.1 and show Theorem 1.2.

2. Basic CAT(0)-geometry

The purpose of this section is to collect some general geometric properties of CAT(0)-spaces which are needed for the investigation of Weil-Petersson space.

A CAT(0)-space is defined as follows. A *triangle* Δ in a geodesic metric space consists of three vertices connected by three (minimal) geodesic arcs a, b, c . A *comparison triangle* $\bar{\Delta}$ for Δ in the euclidean plane is a triangle in \mathbb{R}^2 with the same side-lengths as Δ . By the triangle inequality, such a comparison triangle exists always, and it is unique up to isometry. For a point $x \in a \subset \Delta$ the comparison point of x in the comparison triangle $\bar{\Delta}$ is the point on the side \bar{a} of $\bar{\Delta}$ corresponding to a whose distance to the endpoints of \bar{a} coincides with the distance of x to the corresponding endpoints of a .

A geodesic metric space (X, d) is called a CAT(0)-*space* if for every geodesic triangle Δ in X with sides a, b, c and every comparison triangle $\bar{\Delta}$ in the euclidean plane with sides $\bar{a}, \bar{b}, \bar{c}$ and for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$ we have

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

A complete CAT(0)-space is called a *Hadamard space*.

In a Hadamard space X , the distance function is convex: If γ, ζ are two geodesics in X parametrized on the same interval then the function $t \rightarrow d(\gamma(t), \zeta(t))$ is convex. For two geodesics γ, ζ issuing from the same point $\gamma(0) = \zeta(0)$, the *Alexandrov angle* between γ, ζ is defined. If X is a Riemannian manifold of non-positive curvature, then this angle coincides with the angle between the tangents of γ, ζ at $\gamma(0)$ (see [5]). In the sequel, by angle we always mean the Alexandrov angle.

For a fixed point $x \in X$, the *visual boundary* ∂X of X is defined to be the space of all geodesic rays issuing from x equipped with the topology of uniform convergence on bounded sets. This definition is independent of the choice of x . We denote the point in ∂X defined by a geodesic ray $\gamma : [0, \infty) \rightarrow X$ by $\gamma(\infty)$. We also say that γ *connects* x to $\gamma(\infty)$. The union $X \cup \partial X$ has a natural topology which restricts to the usual topology on X and such that X is dense in $X \cup \partial X$. The isometry group of X acts as a group of homeomorphisms on $X \cup \partial X$.

A subset $C \subset X$ is *convex* if for $x, y \in C$ the geodesic connecting x to y is contained in C as well. For every complete convex set $C \subset X$ and every $x \in X$ there is a unique point $\pi_C(x) \in C$ of smallest distance to x (Proposition II.2.4 of [5]). Now let $J \subset \mathbb{R}$ be a closed connected set and let $\gamma : J \rightarrow X$ be a geodesic arc. Then $\gamma(J) \subset X$ is complete and convex and hence there is a shortest distance projection $\pi_{\gamma(J)} : X \rightarrow \gamma(J)$. The projection $\pi_{\gamma(J)} : X \rightarrow \gamma(J)$ is distance non-increasing.

The following definition is due to Bestvina and Fujiwara (Definition 3.1 of [3]).

DEFINITION 2.1. A geodesic arc $\gamma : J \rightarrow X$ is *B-contracting* for some $B > 0$ if for every closed metric ball K in X which is disjoint from $\gamma(J)$ the diameter of the projection $\pi_{\gamma(J)}(K)$ does not exceed B .

We call a geodesic *contracting* if it is *B-contracting* for some $B > 0$. As an example, every geodesic in a $\text{CAT}(\kappa)$ -space for some $\kappa < 0$ is *B-contracting* for a number $B = B(\kappa) > 0$ only depending on κ .

The next lemma (Lemma 3.2 and 3.5 of [3]) shows that a triangle containing a *B-contracting* geodesic as one of its sides is uniformly thin.

LEMMA 2.2. *Let $\gamma : [a, b] \rightarrow X$ be a B-contracting geodesic. If $x \in X$ and if $a = \pi_{\gamma[a, b]}(x)$ then for every $t \in [a, b]$ the geodesic connecting x to $\gamma(t)$ passes through the $3B + 1$ -neighborhood of $\gamma(a)$.*

On the other hand, thinness of triangles with a fixed geodesic γ as one of the three sides guarantees that γ is contracting. This is formulated in the following useful criterion to detect contracting geodesics.

LEMMA 2.3. *Let $\gamma : J \rightarrow X$ be a geodesic such that there is a number $B > 0$ with the following property. Assume that for all $[a, b] \subset J$ with $|b - a| \geq B/4$ and every geodesic quadrangle Q in X with one side $\gamma[a, b]$ and an angle at least $\pi/2$ at $\gamma(a), \gamma(b)$ the geodesic arc connecting the two vertices of Q which are distinct from $\gamma(a), \gamma(b)$ passes through the $B/4$ -neighborhood of $\gamma[a, b]$. Then $\gamma(J)$ is B-contracting.*

PROOF. Let $\gamma : J \rightarrow X$ be a geodesic which satisfies the assumption in the lemma. We have to show that $d(\pi_{\gamma(J)}(x), \pi_{\gamma(J)}(y)) \leq B$ for all $x \in X$ with $d(x, \gamma(J)) = R > 0$ and every $y \in X$ with $d(x, y) < R$.

Thus let $x \in X$ with $d(x, \gamma(J)) = R > 0$. Assume that $\pi_{\gamma(J)}(x) = \gamma(a)$. Let $y \in X$ and write $\pi_{\gamma(J)}(y) = \gamma(c)$ where $c \geq a$ without loss of generality (otherwise reverse the orientation of γ). The angle at $\gamma(c)$ between the geodesic connecting $\gamma(c)$ to y and the subarc of γ (with reversed orientation) connecting $\gamma(c)$ to $\gamma(a)$ is not smaller than $\pi/2$. Since the angle sum of a triangle in a CAT(0)-space does not exceed π , this implies that if $c > a + B/4$ then the angle at $\gamma(a + B/4)$ of the quadrangle with vertices $x, \gamma(a), \gamma(a + B/4), y$ is not smaller than $\pi/2$. Thus by the assumption in the lemma, the geodesic connecting x to y passes through a point z in the $B/4$ -neighborhood of $\gamma[a, a + B/4]$. Then $\pi_{\gamma(J)}(z) \in \gamma[a - B/4, a + B/2]$, moreover also $d(x, z) \geq R - B/4$.

Now assume that $c \geq a + B$. Then we have $d(\pi_{\gamma(J)}(z), \pi_{\gamma(J)}(y)) \geq B/2$. Since the projection $\pi_{\gamma(J)}$ is distance non-increasing we conclude that

$$d(x, y) = d(x, z) + d(z, y) \geq R - B/4 + B/2 > R.$$

In other words, $\gamma(J)$ is B -contracting. \square

3. CAT(0)-geometry of Weil-Petersson space

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$. The metric completion $\overline{\mathcal{T}(S)}$ of the Teichmüller space $\mathcal{T}(S)$ of S with respect to the Weil-Petersson metric d_{WP} is a Hadamard space. The completion locus $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$ of $\mathcal{T}(S)$ can be described as follows [13].

A *surface with nodes* is defined by a degenerate hyperbolic metric on S where at least one essential simple closed curve on S (i.e. a curve which is homotopically nontrivial and not freely homotopic into a puncture) has been pinched to a pair of punctures. For the free homotopy class of an essential simple closed curve c on S , the degenerate surfaces with a single node at c define a *stratum* $\mathcal{T}(S)_c$ in the completion locus $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$ of Teichmüller space. This stratum equipped with the induced metric is isometric to the Teichmüller space equipped with the Weil-Petersson metric of the (possibly disconnected) surface obtained from $S - c$ by replacing each of the two ends corresponding to c by a cusp. If S_c is disconnected then a point in the Teichmüller space of $S - c$ is given by a pair of points, one for each of the two components of $S - c$. The stratum $\mathcal{T}(S)_c$ is a convex subset of $\overline{\mathcal{T}(S)}$. The completion locus $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$ is the union of the completions of the strata $\mathcal{T}(S)_c$ where c runs through all free homotopy classes of simple closed curves and with the obvious identifications.

The *extended mapping class group* of all isotopy classes of diffeomorphisms of S acts on $(\mathcal{T}(S), d_{WP})$ properly discontinuously as a group of isometries. Every isometry of $\mathcal{T}(S)$ extends to an isometry of the completion $\overline{\mathcal{T}(S)}$. This fact was used by Masur and Wolf [14] to show

PROPOSITION 3.1. *The isometry group of $(\mathcal{T}(S), d_{WP})$ coincides with the extended mapping class group.*

For $\epsilon > 0$ let $\mathcal{T}(S)_\epsilon$ be the subset of $\mathcal{T}(S)$ of all hyperbolic metrics whose *systole*, i.e. the length of a shortest closed geodesic, is at least ϵ . The mapping class group preserves $\mathcal{T}(S)_\epsilon$ and acts on it properly discontinuously and cocompactly. In particular, the Weil-Petersson distance between $\mathcal{T}(S)_\epsilon$ and the completion locus $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$ of Teichmüller space is positive. Moreover, the sectional curvature of

the restriction of the Weil-Petersson metric to $\mathcal{T}(S)_\epsilon$ is bounded from above by a negative constant.

This fact together with Lemma 2.3 is used to show that Weil-Petersson geodesic segments which are entirely contained in the thick part of Teichmüller space are contracting. An analogous result for the *Teichmüller metric* on Teichmüller space (which is however much more difficult) was established by Minsky [17]. As a convention, in the sequel a Weil-Petersson geodesic is always parametrized on a closed connected subset of \mathbb{R} .

LEMMA 3.2. *For every $\epsilon > 0$ there is a number $B = B(\epsilon) > 0$ such that every geodesic $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ is B -contracting.*

PROOF. It was shown in Lemma 3.1 of [10] (see also [3, 8] for an earlier argument along the same line) that for every $\epsilon > 0$ there is a constant $B = B(\epsilon) > 0$ only depending on ϵ such that every geodesic $\gamma : J \rightarrow \mathcal{T}(S)_\epsilon$ satisfies the hypothesis in Lemma 2.3 for B . \square

For an isometry g of $\mathcal{T}(S)$ define the *displacement function* d_g of g to be the function $x \rightarrow d_g(x) = d_{WP}(x, gx)$.

DEFINITION 3.3. An isometry g of $\overline{\mathcal{T}(S)}$ is called *semisimple* if d_g achieves its minimum in $\overline{\mathcal{T}(S)}$. If g is semisimple and $\min d_g = 0$ then g is called *elliptic*. A semisimple isometry g with $\min d_g > 0$ is called *axial*.

By the above definition, an isometry is elliptic if and only if it fixes at least one point in $\overline{\mathcal{T}(S)}$. By Proposition 3.3 of [1], an isometry g of $\overline{\mathcal{T}(S)}$ is axial if and only if there is a geodesic $\gamma : \mathbb{R} \rightarrow \overline{\mathcal{T}(S)}$ such that $g\gamma(t) = \gamma(t + \tau)$ for every $t \in \mathbb{R}$ where $\tau = \min d_g > 0$. Such a geodesic is called an *oriented axis* for g . Note that the geodesic $t \rightarrow \gamma(-t)$ is an oriented axis for g^{-1} . The endpoint $\gamma(\infty)$ of γ is a fixed point for the action of g on $\partial\overline{\mathcal{T}(S)}$ which is called the *attracting fixed point*. The closed convex set $A \subset \overline{\mathcal{T}(S)}$ of all points for which the displacement function of g is minimal is isometric to $C \times \mathbb{R}$ where $C \subset A$ is closed and convex (Theorem II.2.14 of [5]). For each $x \in C$ the set $\{x\} \times \mathbb{R}$ is an axis of g .

By the Nielsen-Thurston classification, a mapping class $g \in \text{Mod}(S)$ either is *pseudo-Anosov* or it is of finite order or it is *reducible*. An example of a reducible mapping class is a *multi-twist* which can be represented in the form $\phi_1^{k_1} \circ \dots \circ \phi_\ell^{k_\ell}$ where each ϕ_i is a Dehn-twist about a simple closed curve c_i in S and where the curves c_i are pairwise disjoint. We allow the multi-twist to be trivial. We have (see also [4])

LEMMA 3.4. *Every isometry ϕ of $\overline{\mathcal{T}(S)}$ is semi-simple, and ϕ is elliptic if and only if there is some $k \geq 1$ such that ϕ^k is a multi-twist.*

PROOF. In a Hadamard space X , an isometry g with a finite orbit on X has a fixed point which is the center of the orbit. This means the following. For a fixed orbit $\{x_1, \dots, x_k\} \subset X$ for g there is a unique point $y \in X$ such that the radius of the smallest closed metric ball centered at y which contains the set $\{x_1, \dots, x_k\}$ is minimal (Proposition II.2.7 of [5]). Since this point is defined by purely metric properties, it is a fixed point for g .

As a consequence, an element $g \in \text{Mod}(S)$ is elliptic if and only if this holds true for g^k for every $k > 0$, and every element of finite order is elliptic. Similarly, an element $g \in \text{Mod}(S)$ is axial if and only if this holds true for g^k for every $k > 0$.

Namely, if g^k is axial for some $k > 1$ then the set of all points for which the displacement function d_{g^k} of g^k assumes a minimum is a closed convex subset of $\overline{\mathcal{T}(S)}$. This set is isometric to a product $A \times \mathbb{R}$ where for each $x \in A$, $\{x\} \times \mathbb{R}$ is an axis of g^k . The infinite cyclic group generated by g acts on A with a finite orbit and hence there is a fixed point $x \in A$ for this action. The set $\{x\} \times \mathbb{R}$ is an axis of g .

Now assume that $g \in \text{Mod}(S)$ is a multi-twist about a multi-curve $c = c_1 \cup \dots \cup c_\ell$. Let $\overline{\mathcal{T}(S)}_c$ be the completion of the stratum in $\overline{\mathcal{T}(S)}$ of all surfaces with nodes at the curves c_1, \dots, c_ℓ . Then g fixes each point in $\overline{\mathcal{T}(S)}_c$ and hence g is elliptic.

If g is pseudo-Anosov then it was shown in [9] that g has an axis in $\mathcal{T}(S)$ and hence it is axial. Now assume that g is reducible. Then up to replacing g by g^k for some $k > 0$, g preserves a non-trivial multi-curve c component-wise, and it preserves each connected component of $S - c$. Moreover, the multi-curve c can be chosen in such a way that for every component S_0 of $S - c$, either S_0 is a three-holed sphere or the restriction of g to S_0 is pseudo-Anosov. If g is not a multi-twist then there is at least one component S_0 of $S - c$ such that the restriction of g to S_0 is pseudo-Anosov. Then the restriction of g to S_0 , viewed as an element of the mapping class group of S_0 , has an axis in $\mathcal{T}(S_0)$. The Weil-Petersson metric on the stratum $\mathcal{T}(S)_c$ induced from the Weil-Petersson metric on S is the product of the Weil-Petersson metrics on the Teichmüller spaces of the connected components of $S - c$. This implies that the restriction of g to $\mathcal{T}(S)_c$ has an axis. Now the completion $\overline{\mathcal{T}(S)}_c \subset \overline{\mathcal{T}(S)}$ of $\mathcal{T}(S)_c$ is a closed convex subset of $\overline{\mathcal{T}(S)}$. The shortest distance projection $\overline{\mathcal{T}(S)} \rightarrow \overline{\mathcal{T}(S)}_c$ is distance non-increasing and equivariant with respect to the action of g . Therefore the infimum of the displacement function d_g of g equals the infimum of d_g on $\overline{\mathcal{T}(S)}_c$. Thus this infimum is a minimum and once again, g is axial. \square

Remark: 1) The flat strip theorem (Theorem II.2.14 of [5]) states that two geodesic lines in a Hadamard space X whose endpoints in the visual boundary ∂X coincide bound a flat strip. Since the sectional curvature of the Weil-Petersson metric is negative, the proof of Lemma 3.4 shows that an axial isometry of $\overline{\mathcal{T}(S)}$ which admits an axis intersecting $\mathcal{T}(S)$ is pseudo-Anosov.

2) The celebrated solution of the Nielsen realization problem states that each finite subgroup of $\text{Mod}(S)$ has a fixed point in $\mathcal{T}(S)$ [12]. The discussion in the proof of Lemma 3.4 immediately implies that such a group has a fixed point in $\overline{\mathcal{T}(S)}$. It is not difficult to establish that there is also a fixed point in $\mathcal{T}(S)$, however we omit this discussion here.

The following definition is due to Bestvina and Fujiwara (Definition 5.1 of [3]).

DEFINITION 3.5. An isometry g of a CAT(0)-space X is called *B-rank-one* for some $B > 0$ if g is axial and admits a B -contracting axis.

We call an isometry g *rank-one* if g is B -rank-one for some $B > 0$. Since a pseudo-Anosov element has an axis γ in $\mathcal{T}(S)$, by invariance and cocompactness of the action of g on γ , the geodesic γ entirely remains in $\mathcal{T}(S)_\epsilon$ for some $\epsilon > 0$. Thus the following result (Proposition 8.1 of [3]) is an immediate consequence of Lemma 3.2.

PROPOSITION 3.6. *A pseudo-Anosov element in $\text{Mod}(S)$ is rank-one.*

Example: An axial isometry g of $\overline{\mathcal{T}(S)}$ which admits an axis γ bounding a flat half-plane is not rank-one. An example of such an axial isometry of $\overline{\mathcal{T}(S)}$ can be obtained as follows. Let c be a simple closed separating curve on S such that none of the two components S_1, S_2 of $S - c$ is a three-holed sphere. Let $g \in \text{Mod}(S_1)$ be a pseudo-Anosov mapping class. Then g defines a reducible element in $\text{Mod}(S)$. If γ_1 is the axis for the action of g on $\mathcal{T}(S_1)$ then for each point $z \in \mathcal{T}(S_2)$ the curve (γ_1, z) is an axis for the action of g on the stratum $\mathcal{T}(S)_c \subset \overline{\mathcal{T}(S)}$ and hence (γ_1, z) is an axis for the action of g on $\overline{\mathcal{T}(S)}$. In particular, for every infinite geodesic $\zeta : \mathbb{R} \rightarrow \mathcal{T}(S_2)$ the set $\{(\gamma(t), \zeta(s)) \mid s, t \in \mathbb{R}\} \subset \mathcal{T}(S_1) \times \mathcal{T}(S_2) \subset \overline{\mathcal{T}(S)}$ is an isometrically embedded euclidean plane in $\overline{\mathcal{T}(S)}$ containing an axis for g . Thus g is axial but not rank-one.

A homeomorphism g of a topological space K is said to act with *north-south dynamics* if there are two fixed points $a \neq b \in K$ for the action of g such that for every neighborhood U of a , V of b there is some $k > 0$ such that $g^k(K - V) \subset U$ and $g^{-k}(K - U) \subset V$. The point a is called the *attracting fixed point* for g , and b is the *repelling fixed point*.

Teichmüller space equipped with the *Teichmüller metric* can be compactified by adding the *Thurston boundary* \mathcal{PML} of projective measured geodesic laminations which is a topological sphere. This compactification however is different from the CAT(0)-boundary $\partial\overline{\mathcal{T}(S)}$ of $\overline{\mathcal{T}(S)}$. The action of the extended mapping class group on $\mathcal{T}(S)$ naturally extends to an action on \mathcal{PML} . An element $g \in \text{Mod}(S)$ acts on \mathcal{PML} with north-south-dynamics if and only if g is pseudo-Anosov. Lemma 3.3.3 of [1] shows that a rank-one isometry of a proper Hadamard space X acts on the boundary ∂X with north-south dynamics. The proof of this fact given in [11] (proof of Lemma 4.4) does not use the assumption of properness of X . Thus we obtain.

LEMMA 3.7. *A rank-one isometry g of $\overline{\mathcal{T}(S)}$ acts with north-south dynamics on $\partial\overline{\mathcal{T}(S)}$.*

4. Non-elementary groups of isometries

In this section we investigate the action on $\partial\overline{\mathcal{T}(S)}$ of non-elementary subgroups of $\text{Mod}(S)$ which contain a pseudo-Anosov element. We begin with recalling some standard terminology used for groups of isometries on Hadamard spaces.

Let $G < \text{Mod}(S)$ be any subgroup. The *limit set* Λ of G is the set of accumulation points in $\partial\overline{\mathcal{T}(S)}$ of one (and hence every) orbit of the action of G on $\overline{\mathcal{T}(S)}$. If $g \in G$ is axial with axis γ , then $\gamma(\infty), \gamma(-\infty) \in \Lambda$. In other words, the two fixed points for the action of a pseudo-Anosov element on $\partial\overline{\mathcal{T}(S)}$ are contained in Λ .

LEMMA 4.1. *The limit set of $\text{Mod}(S)$ is the entire boundary $\partial\overline{\mathcal{T}(S)}$ of $\overline{\mathcal{T}(S)}$.*

PROOF. For sufficiently small $\epsilon > 0$, the set $\mathcal{T}(S)_\epsilon \subset \mathcal{T}(S)$ of all hyperbolic metrics whose systole is at least ϵ is connected, and the mapping class group $\text{Mod}(S)$ acts cocompactly on $\mathcal{T}(S)_\epsilon$. There is a number $R_0 > 0$ such that the Weil-Petersson distance between any point in $\overline{\mathcal{T}(S)}$ and $\mathcal{T}(S)_\epsilon$ is at most R_0 [18]. Thus there is a number $R_1 > R_0$ such that for all $x \in \mathcal{T}(S)_\epsilon$ and all $y \in \overline{\mathcal{T}(S)}$ there is some $g \in \text{Mod}(S)$ with $d(gx, y) \leq R_1$. This just means that $\partial\overline{\mathcal{T}(S)}$ is the limit set of $\text{Mod}(S)$. \square

LEMMA 4.2. *Let $G < \text{Mod}(S)$ be a subgroup which contains a pseudo-Anosov element g . Then the limit set Λ of G is the closure in $\overline{\partial\mathcal{T}(S)}$ of the set of fixed points of conjugates of g in G . If G is non-elementary then Λ does not have isolated points.*

PROOF. Let $G < \text{Mod}(S)$ be a subgroup which contains a pseudo-Anosov element $g \in G$. Let Λ be the limit set of G . We claim that Λ is contained in the closure of the G -orbit of the two fixed points of g . For this let $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$ be an axis of g and let $x_0 = \gamma(0)$. By Proposition 3.6, γ is B -contracting for some $B > 0$. Let $\xi \in \Lambda$ and let $(g_i) \subset G$ be a sequence such that $(g_i x_0)$ converges to ξ . There are two cases possible.

In the first case, up to passing to a subsequence, the geodesics $g_i \gamma$ eventually leave every bounded set. For $i \geq 1$ let $x_i = \pi_{g_i \gamma(\mathbb{R})}(x_0)$. Then $d_{WP}(x_0, x_i) \rightarrow \infty$ ($i \rightarrow \infty$). On the other hand, $g_i \gamma$ is B -contracting and hence by Lemma 2.2, a geodesic ζ_i connecting x_0 to $g_i x_0$ passes through the $3B + 1$ -neighborhood of x_i , and the same is true for a geodesic η_i connecting x_0 to $g_i \gamma(\infty)$. By CAT(0)-comparison, the angles at x_0 between the geodesics ζ_i, η_i converge to zero as $i \rightarrow \infty$. Since $g_i x_0 \rightarrow \xi$, the sequence $(g_i \gamma(\infty))$ converges to ξ as well. But $g_i \gamma(\infty)$ is a fixed point of the conjugate $g_i g g_i^{-1}$ of g . Thus ξ is contained in the closure of the fixed points of all conjugates of g .

In the second case there is a bounded neighborhood K of x_0 in $\overline{\mathcal{T}(S)}$ such that $g_i \gamma \cap K \neq \emptyset$ for all i . For $i > 0$ let ζ_i be the geodesic connecting x_0 to $g_i x_0$. Since $g_i x_0 \rightarrow \xi$, the geodesics ζ_i converge as $i \rightarrow \infty$ locally uniformly to the geodesic ray connecting x_0 to ξ .

After passing to a subsequence and perhaps a change of orientation of γ we may assume that for large i the point $g_i x_0$ lies between a point $z_i \in g_i \gamma \cap K$ and $g_i \gamma(\infty)$ along $g_i \gamma$. This means that $g_i x_0$ is contained in the geodesic connecting z_i to $g_i \gamma(\infty)$. Since the distance between z_i and x_0 is uniformly bounded, by CAT(0)-comparison the Alexandrov angle at $g_i x_0$ between the inverse of the geodesic ζ_i (which connects $g_i x_0$ to x_0) and the inverse of the geodesic $g_i \gamma$ (which connects $g_i x_0$ to z_i) tends to zero as $i \rightarrow \infty$. This implies that the angle at $g_i x_0$ of the ideal triangle in $\overline{\mathcal{T}(S)}$ with vertices $x_0, g_i x_0, g_i \gamma(\infty)$ tends to π as $i \rightarrow \infty$.

Since in a CAT(0)-space the sum of the Alexandrov angles of a geodesic triangle (with possibly one vertex at infinity) does not exceed π , the angle at x_0 between the geodesic ζ_i and the geodesic ρ_i connecting x_0 to $g_i \gamma(\infty)$ tends to zero as $i \rightarrow \infty$. But $g_i x_0 \rightarrow \xi$ and therefore the points $g_i \gamma(\infty)$ converge to ξ ($i \rightarrow \infty$) in $\overline{\partial\mathcal{T}(S)}$. Thus ξ is indeed contained in the closure of the fixed points of conjugates of g .

Now assume that the limit set Λ of G contains at least 3 points. Let g be any pseudo-Anosov element of G . Since by Lemma 3.7 g acts with north-south dynamics on $\overline{\partial\mathcal{T}(S)}$, the set Λ contains at least one point ξ which is not a fixed point of g . The sequence $(g^k \xi)$ consists of pairwise distinct points which converge as $k \rightarrow \infty$ to the attracting fixed point of g . Similarly, the sequence $(g^{-k} \xi)$ consists of pairwise distinct points which converge as $k \rightarrow \infty$ to the repelling fixed point of g . Moreover, by the above, a point $\xi \in \Lambda$ which is not a fixed point of a pseudo-Anosov element of G is a limit of fixed points of pseudo-Anosov elements. This shows that Λ does not have isolated points and completes the proof of the lemma. \square

We need the following simple (and well known to the experts) observation which parallels the properties of the action of $\text{Mod}(S)$ on the space of projective measured

geodesic laminations. This observation follows immediately from the work of Brock, Masur and Minsky [8].

LEMMA 4.3. *Let $g, h \in \text{Mod}(S)$ be pseudo-Anosov elements. If there is a common fixed point for the action of g, h on $\overline{\partial\mathcal{T}(S)}$ then the fixed point sets of g, h coincide.*

PROOF. Let $g, h \in \text{Mod}(S)$ be pseudo-Anosov elements and assume that there is a common fixed point for the action of g, h on $\overline{\partial\mathcal{T}(S)}$. Since g, h act with north-south dynamics on $\overline{\partial\mathcal{T}(S)}$, this implies that the axis γ for g and the axis η for h have a common endpoint, say $\gamma(\infty) = \eta(\infty)$. We may also assume that $\gamma(\infty)$ is the attracting fixed point for both g, h .

By Theorem 1.5 of [8], up to a reparametrization we have $d_{WP}(\gamma(t), \eta(t)) \rightarrow 0$ ($t \rightarrow \infty$). After another reparametrization, there is a number $r > 0$ such that the semi-group $\{g^k \mid k \geq 0\}$ acts cocompactly on the closed $2r$ -neighborhood $N \subset \mathcal{T}(S)$ of $\gamma[0, \infty)$, and that the r -neighborhood of $\gamma[0, \infty)$ contains $\eta[0, \infty)$.

Let τ_0 be the translation length of g on γ . If there are no integers $k, \ell > 0$ such that $g^k = h^\ell$ then there are infinitely many distinct elements of $\text{Mod}(S)$ of the form $g^{-m}h^n$ which map $\eta(0)$ into the r -neighborhood of $\gamma[0, \tau_0]$. Namely, let $n > 0$ be arbitrary. Then there is a unique number $m \in \mathbb{Z}$ such that $\pi_{\gamma(\mathbb{R})}h^n(\eta(0)) \in \gamma[m\tau_0, (m+1)\tau_0)$. Since the point $h^n(\eta(0))$ is contained in the r -neighborhood of $\gamma(\mathbb{R})$, the point $g^{-m}h^n(\eta(0))$ is contained in the r -neighborhood of $\gamma[0, \tau_0]$. However, this violates the fact that $\text{Mod}(S)$ acts properly discontinuously on $\mathcal{T}(S)$. Thus there are number $k, \ell > 0$ with $g^k = h^\ell$ and hence the fixed point sets for the action of g, h on $\overline{\partial\mathcal{T}(S)}$ coincide. \square

The action of a group G on a topological space Y is called *minimal* if every G -orbit is dense.

LEMMA 4.4. *Let $G < \text{Mod}(S)$ be a non-elementary group with limit set Λ which contains a pseudo-Anosov element $g \in G$ with fixed points $a \neq b \in \Lambda$. Then for every non-empty open set $V \subset \Lambda$ there is some $u \in G$ with $u\{a, b\} \subset V$. Moreover, the action of G on Λ is minimal.*

PROOF. Let $G < \text{Mod}(S)$ be a non-elementary subgroup with limit set Λ which contains a pseudo-Anosov element $g \in G$. Let $a, b \in \Lambda$ be the attracting and repelling fixed points of g , respectively, and let $V \subset \Lambda$ be a non-empty open set. By Lemma 4.2, the limit set Λ does not have isolated points and up to replacing g by g^{-1} (and exchanging a and b) there is an element $v \in G$ which maps a to $v(a) \in V - \{a, b\}$. Then $h = vgv^{-1}$ is a pseudo-Anosov element with fixed points $v(a) \in V - \{a, b\}, v(b) \in \Lambda$. By Lemma 4.3, we have $v(b) \notin \{a, b\}$. By Lemma 3.7, h acts with north-south dynamics on $\overline{\partial\mathcal{T}(S)}$ and hence $h^k\{a, b\} \subset V$ for all sufficiently large k .

Every closed G -invariant subset A of $\overline{\partial\mathcal{T}(S)}$ contains every fixed point of every pseudo-Anosov element. Namely, if $a \neq b$ are the two fixed points of a pseudo-Anosov element $g \in G$ and if there is some $\xi \in A - \{a, b\}$ then also $\{a, b\} \subset A$ since A is closed and g acts with north-south dynamics on $\overline{\partial\mathcal{T}(S)}$. On the other hand, if $a \in A$ then there is some $h \in G$ with $h(a) \in \Lambda - \{a, b\}$ and once again, we conclude by invariance that $b \in A$ as well. Now the set of all fixed points of pseudo-Anosov elements of G is G -invariant and hence the smallest non-empty closed G -invariant subset of $\overline{\partial\mathcal{T}(S)}$ is the closure of the set of fixed points of pseudo-Anosov elements.

This set contains the limit set Λ of G by Lemma 4.2 and hence it coincides with Λ . In other words, the action of G on Λ is minimal. The lemma is proven. \square

Note the following immediate corollary of Lemma 4.3.

COROLLARY 4.5. *Let $G < \text{Mod}(S)$ be a non-elementary subgroup which contains a pseudo-Anosov element. Then G does not fix a point in $\overline{\partial\mathcal{T}(S)}$.*

Example: There are non-elementary groups $G < \text{Mod}(S)$ which fix a point in $\overline{\partial\mathcal{T}(S)}$. Namely, let c be a simple closed separating curve on S so that $S - c = S_1 \cup S_2$ where neither S_1 nor S_2 is a three-holed sphere. Let $\gamma_i \in \text{Mod}(S_i)$ be a pseudo-Anosov element ($i = 1, 2$). Then γ_1, γ_2 generate a free abelian subgroup G of $\text{Mod}(S)$ whose limit set is a circle which is fixed pointwise by G .

We are now ready to show.

PROPOSITION 4.6. *Let $G < \text{Mod}(S)$ be a non-elementary subgroup with limit set Λ which contains a pseudo-Anosov element.*

- (1) *The pairs of fixed points of pseudo-Anosov elements of G are dense in $\Lambda \times \Lambda$.*
- (2) *For any two non-empty open subsets W_1, W_2 of $\Lambda \times \Lambda$ there is some $g \in G$ with $gW_1 \cap W_2 \neq \emptyset$.*

PROOF. Let $G < \text{Mod}(S)$ be a non-elementary subgroup with limit set Λ . Assume that G contains a pseudo-Anosov element g with attracting fixed point $a \in \Lambda$ and repelling fixed point $b \in \Lambda$.

Let $U \subset \Lambda \times \Lambda$ be a non-empty open set. Our goal is to show that U contains a pair of fixed points of a pseudo-Anosov element. Since Λ does not have isolated points, for this we may assume that there are small open sets $V_i \subset \overline{\partial\mathcal{T}(S)} - \{a, b\}$ with disjoint closure $\overline{V_i}$ ($i = 1, 2$) and such that $U = V_1 \times V_2 \cap \Lambda \times \Lambda$.

Choose some $u \in G$ which maps $\{a, b\}$ into V_1 . Such an element exists by Lemma 4.4. Then $v = ugu^{-1}$ is a pseudo-Anosov element with fixed points $ua, vb \in V_1$. Similarly, there is a pseudo-Anosov element $w \in G$ with both fixed points in V_2 . Via replacing v, w by sufficiently high powers we may assume that $v(\overline{\partial\mathcal{T}(S)} - V_1) \subset V_1, v^{-1}(\overline{\partial\mathcal{T}(S)} - V_1) \subset V_1$ and that $w(\overline{\partial\mathcal{T}(S)} - V_2) \subset V_2, w^{-1}(\overline{\partial\mathcal{T}(S)} - V_2) \subset V_2$. Then we have $wv(\overline{\partial\mathcal{T}(S)} - V_1) \subset V_2$ and $v^{-1}w^{-1}(\overline{\partial\mathcal{T}(S)} - V_2) \subset V_1$. By a result of McCarthy [15], up to possibly replacing v and w by even higher powers we may assume that wv is pseudo-Anosov. Then wv acts on $\overline{\partial\mathcal{T}(S)}$ with north-south dynamics. Since $wv(\overline{V_2}) \subset V_2$ and $v^{-1}w^{-1}(\overline{V_1}) \subset V_1$, the pair of fixed points of wv is necessarily contained in $V_1 \times V_2$ and hence in U . The first part of the proposition is proven.

To show the second part of the proposition, let $W_1, W_2 \subset \Lambda \times \Lambda$ be non-empty open sets. We have to show that there is some $g \in G$ such that $gW_1 \cap W_2 \neq \emptyset$. For this we may assume without loss of generality that $W_1 = U_1 \times U_2, W_2 = U_3 \times U_4$ where U_1, U_2 and U_3, U_4 are non-empty open subsets of Λ with disjoint closure. Since Λ does not have isolated points, by possibly replacing U_i by proper non-empty open subsets we may assume that the sets U_i are pairwise disjoint.

By the first part of the proposition, there is a pseudo-Anosov element $u \in G$ with attracting fixed point in U_1 and repelling fixed point in U_4 . Since u acts on $\overline{\partial\mathcal{T}(S)}$ with north-south dynamics, there is some $k > 0$ and a small open neighborhood $U_5 \subset U_1$ of the attracting fixed point of u such that $u^{-k}(U_5 \times U_2) \subset U_1 \times U_4$.

The same argument produces an element $w \in G$ with attracting fixed point in U_3 and repelling fixed point in $u^{-k}U_2 \subset U_4$, a number $\ell > 0$ and an open subset U_6 of U_2 such that $w^\ell(u^{-k}(U_5 \times U_6)) \subset U_3 \times U_4$. This completes the proof of the proposition. \square

As noted in the example after Corollary 4.5, in general the second part of Proposition 4.6 does not hold true for non-elementary subgroups of $\text{Mod}(S)$ which do not contain a pseudo-Anosov element.

5. The Weil-Petersson geodesic flow

In this section we discuss some implications of the results in the previous section to the dynamics of the Weil-Petersson geodesic flow on moduli space.

Let $T^1\mathcal{T}(S)$ be the unit tangent bundle of $\mathcal{T}(S)$ for the Weil-Petersson metric. The *Weil-Petersson geodesic flow* Φ^t acts on $T^1\mathcal{T}(S)$ by associating to a direction v and a number $t > 0$ the unit tangent $\Phi^t v$ at t of the geodesic with initial velocity v . Note that this flow is not everywhere defined due to the existence of finite length geodesics which end in a point in $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$. Define $\mathcal{G} \subset T^1\mathcal{T}(S)$ to be the space of all directions of biinfinite geodesics, i.e. such that the flow line of Φ^t through a point $v \in \mathcal{G}$ is defined for all times. Note that the set \mathcal{G} is invariant under the action of the extended mapping class group.

The following result is due to Wolpert [18, 19].

LEMMA 5.1. *\mathcal{G} is a dense G_δ -subset of $T^1\mathcal{T}(S)$ of full Lebesgue measure.*

PROOF. A direction at a point $x \in \mathcal{T}(S)$ either defines a geodesic ray (i.e. a geodesic defined on the half-line $[0, \infty)$) or a geodesic which ends at a point in $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$. The set $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$ is a countable union of closed convex strata, each of real codimension two. Since by the Cat(0)-property, any two points $x, y \in \overline{\mathcal{T}(S)}$ can be connected by a unique geodesic depending continuously on x, y , the set of directions of geodesics issuing from a point in $\mathcal{T}(S)$ and which terminate in the closure of a fixed stratum is a closed subset of $T^1\mathcal{T}(S)$ of real codimension one. Thus \mathcal{G} is the complement in $T^1\mathcal{T}(S)$ of a countable union of closed subsets of codimension one, i.e. it is a dense G_δ -set (we refer to [18, 19] for details).

Wolpert [18, 19] also observed that for every $x \in \mathcal{T}(S)$ the set of directions of geodesic rays issuing from x has full Lebesgue measure in the unit sphere at x . Then \mathcal{G} has full Lebesgue measure. \square

To each $v \in \mathcal{G}$ we can associate the ordered pair $\pi(v) \in \overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$ of endpoints of the biinfinite geodesic γ with initial velocity v (here ordered means that $\pi(v) = (\gamma(\infty), \gamma(-\infty))$). The map

$$\pi : \mathcal{G} \rightarrow \overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$$

clearly is invariant under the action of the geodesic flow on $T^1\mathcal{T}(S)$ and hence it factors through a map of the quotient space \mathcal{G}/Φ^t . Since the Weil-Petersson metric is negatively curved, by the flat strip theorem (Theorem II.2.13 of [5]) the induced map $\mathcal{G}/\Phi^t \rightarrow \overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$ is injective. This means that the set $\pi(\mathcal{G})$ can be equipped with two natural topologies: the topology as a quotient of \mathcal{G} , and the induced topology as a subset of $\overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$. We next observe that these two topologies coincide.

LEMMA 5.2. *The map π factors through a $\text{Mod}(S)$ -equivariant homeomorphism of \mathcal{G}/Φ^t equipped with the quotient topology onto $\pi(\mathcal{G})$ equipped with the topology as a subset of $\overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$.*

PROOF. By the definition of the topology of $\overline{\partial\mathcal{T}(S)}$, the map π is continuous. Moreover, it is clearly equivariant under the action of $\text{Mod}(S)$.

By the definition of the quotient topology on \mathcal{G}/Φ^t , it suffices to show that the map π is open for the topology of \mathcal{G} as a subset of $T1\mathcal{T}(S)$ and for the topology of $\pi\mathcal{G}$ as a subset of $\overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$. For this let $v \in \mathcal{G}$ and let U be a neighborhood of v in \mathcal{G} . We have to find a neighborhood V of $\pi(v)$ in $\overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$ such that the unit tangent line of every biinfinite geodesic whose pair of endpoints is contained in V passes through U .

For this let $x \in \mathcal{T}(S)$ be the footpoint of v . Since a geodesic depends smoothly on its initial velocity, if no such neighborhood V of $\pi(v)$ exists then there is a sequence of points $(a_i, b_i) \subset \overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$ with the following properties.

- (1) (a_i, b_i) can be connected by a geodesic line γ_i in $\overline{\mathcal{T}(S)}$.
- (2) $(a_i, b_i) \rightarrow \pi(v)$ ($i \rightarrow \infty$).
- (3) There is a number $\epsilon > 0$ such that $d_{WP}(x, \gamma_i(\mathbb{R})) \geq \epsilon$ for all i .

Let Δ_i be the ideal triangle in $\mathcal{T}(S)$ with vertices x, a_i, b_i . Since $(a_i, b_i) \rightarrow \pi(v)$ ($i \rightarrow \infty$), the angles at x of the triangles Δ_i converge to π as $i \rightarrow \infty$. Connect each point on the geodesic ray from x to b_i to the point $a_i \in \overline{\partial\mathcal{T}(S)}$ by a geodesic ray. This defines a *ruled surface* in $\mathcal{T}(S)$ with smooth interior which we denote again by Δ_i . The intrinsic Gauß curvature of this surface at a point y is bounded from above by an upper bound for the curvature of the Weil-Petersson metric at y . Since the Weil-Petersson metric is negatively curved, there is a number $r < \epsilon$ such that the Gauß curvature of the intersection of Δ_i with the ball of radius r about x is bounded from above by $-r$. (Such an argument has been used in the literature many times. We refer to [8] for a more detailed explanation and for additional references).

Let $\zeta_i : [0, \infty) \rightarrow \mathcal{T}(S)$ be the side of Δ_i connecting x to a_i . The intrinsic angle at x of the triangle Δ_i coincides with the Weil-Petersson angle at x . By assumption, the distance between x and the side γ_i of Δ_i is at least $\epsilon > r$. If $i > 0$ is sufficiently large that the angle of Δ_i at x exceeds $\pi/2$ then by convexity of the distance function, a geodesic in Δ_i for the intrinsic metric which issues from a point in $\zeta_i[0, r/2]$ and is perpendicular to ζ_i does not intersect the side of Δ_i connecting x to b_i . Therefore the maximal length of such a geodesic is not smaller than $r/2$. The union of these geodesic segments is an embedded strip in Δ_i which is contained in the ball of radius r about x in $\mathcal{T}(S)$. Hence the Gauß curvature of Δ_i at each point in the strip is at most $-r$. Moreover, comparison with the euclidean plane shows that the area of the strip is at least $r^2/4$. Since the Gauß curvature of Δ_i is negative, this implies that the integral of the Gauß curvature over Δ_i does not exceed $-r^3/4$.

On the other hand, the angle of Δ_i at x tends to π as $i \rightarrow \infty$. Since the Gauß curvature of Δ_i is negative, the Gauß-Bonnet theorem shows that the integral of the Gauß curvature of Δ_i tends to zero as $i \rightarrow \infty$. This is a contradiction to the estimate in the previous paragraph. As a consequence, the image of the open set U under the projection π contains indeed an open subset of $\pi(\mathcal{G})$ equipped with the topology as a subspace of $\overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$ which shows the lemma. \square

Let $T^1\mathcal{M}(S)$ be the quotient of the unit tangent bundle $T^1\mathcal{T}(S)$ of $\mathcal{T}(S)$ under the action of the (extended) mapping class group. By equivariance, the Weil-Petersson geodesic flow projects to a flow on $T^1\mathcal{M}(S)$.

Every pseudo-Anosov element $g \in \text{Mod}(S)$ defines a periodic orbit for the Weil-Petersson flow. This periodic orbit is the projection of the unit tangent line of an axis of g . These are the only periodic orbits. Namely, if $\{\Phi^t v \mid t\}$ is a periodic orbit in $T^1\mathcal{M}(S)$ then there is a biinfinite Weil-Petersson geodesic γ in $\mathcal{T}(S)$ whose unit tangent line projects to the orbit. This geodesic is invariant under an element $g \in \text{Mod}(S)$. Then g is axial, with axis $\gamma \subset \mathcal{T}(S)$, and hence g is pseudo-Anosov.

Proposition 4.6, applied to the full mapping class group, shows together with Lemma 5.2 immediately the following result of Brock, Masur and Minsky [8].

PROPOSITION 5.3. *Periodic orbits are dense in $T^1\mathcal{M}(S)$.*

PROOF. Since the subset \mathcal{G} of $T^1\mathcal{T}(S)$ is dense and $\text{Mod}(S)$ -invariant, it suffices to show that the unit tangents of all axes of all pseudo-Anosov elements are dense in \mathcal{G} . Now by Lemma 4.1 and Proposition 4.6, the set of pairs of endpoints of all axes of pseudo-Anosov elements in $\text{Mod}(S)$ is dense in $\overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$ and hence in $\pi\mathcal{G}$ and therefore the proposition follows from Lemma 5.2. \square

We now use local compactness of Teichmüller space to complete the proof of Theorem 1 from the introduction.

PROPOSITION 5.4. *Let $G < \text{Mod}(S)$ be a non-elementary subgroup with limit set Λ which contains a pseudo-Anosov element. Then there is a dense orbit for the action of G on $\Lambda \times \Lambda$.*

PROOF. Let $G < \text{Mod}(S)$ be a non-elementary subgroup which contains a pseudo-Anosov element. Let as before $\mathcal{G} \subset T^1\mathcal{T}(S)$ be the space of all directions of biinfinite Weil-Petersson geodesics in $\mathcal{T}(S)$ and let $\mathcal{G}_0 \subset \mathcal{G}$ be the space of all directions of geodesics with both endpoints in Λ . Then \mathcal{G}_0 is a closed G -invariant subset of the (non-locally compact) space \mathcal{G} . By Lemma 5.2, the restriction to \mathcal{G}_0 of the map π factors through a homeomorphism $\mathcal{G}_0/\Phi^t \rightarrow \pi(\mathcal{G}_0) \subset \Lambda \times \Lambda$. Since a pair of fixed points of a pseudo-Anosov element $g \in G$ is contained in $\pi(\mathcal{G})$, by Proposition 4.6 the set $\pi\mathcal{G}_0$ is dense in $\Lambda \times \Lambda$.

Let $P : T^1\mathcal{T}(S) \rightarrow T^1\mathcal{T}(S)/G = N$ be the canonical projection. We claim that for all nonempty open sets $U, V \subset N$ with $U \cap P\mathcal{G}_0 \neq \emptyset$ and $V \cap P\mathcal{G}_0 \neq \emptyset$ and every $t > 0$ there is some $u \in U \cap P\mathcal{G}_0$ and some $T > t$ such that $\Phi^T u \in V$. For this let \tilde{U}, \tilde{V} be the preimages of U, V in $T^1\mathcal{T}(S)$. Then \tilde{U}, \tilde{V} are open G -invariant subsets of $T^1\mathcal{T}(S)$. By Lemma 5.2, the projection $\pi : \mathcal{G} \rightarrow \pi\mathcal{G} \subset \overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$ is open. Therefore there are open subsets W_1, W_2 of $\overline{\partial\mathcal{T}(S)} \times \overline{\partial\mathcal{T}(S)}$ with

$$W_1 \cap \pi\mathcal{G} \subset \pi(\tilde{U} \cap \mathcal{G}), \quad W_2 \cap \pi\mathcal{G} \subset \pi(\tilde{V} \cap \mathcal{G})$$

and such that the intersections of W_1, W_2 with $\Lambda \times \Lambda$ are non-empty.

Since $W_i \cap \Lambda \times \Lambda \neq \emptyset$, by the second part of Proposition 4.6 there is some $h \in G$ such that $W = W_1 \cap h^{-1}W_2 \neq \emptyset$ and that $W \cap \Lambda \times \Lambda \neq \emptyset$. Let $(a, b) \in W \cap \pi\mathcal{G}_0$ be the pair of endpoints of an axis of a pseudo-Anosov element $g \in G$. Such an element exists by the first part of Proposition 4.6. Then the axis of the conjugate hgh^{-1} of g in G has a pair of endpoints $(ha, hb) \in W_2 \cap \pi\mathcal{G}_0$. Since the unit tangent lines of axes of pseudo-Anosov elements which are conjugate in G project to the same periodic orbit in N for the Weil-Petersson flow, this implies that the projection of

the unit tangent line of the axis of g passes through both U and V . In particular, for $t > 0$ and for a point $x \in U \cap P\mathcal{G}_0$ contained in this periodic orbit, there is some $T > t$ such that $\Phi^T x \in V$. This shows our claim.

We use this observation to complete the proof of the corollary. Namely, the closure $\overline{P\mathcal{G}_0}$ of $P\mathcal{G}_0$ in N is locally compact and separable since this is the case for N . Moreover, $\overline{P\mathcal{G}_0}$ is invariant under the Weil-Petersson flow Φ^t . Hence we can choose a countable basis U_i of open sets for $\overline{P\mathcal{G}_0}$ which are moreover relatively compact. Let $V_1 = U_1$ and for each $i \geq 2$ define inductively a nonempty open set V_i in $\overline{P\mathcal{G}_0}$ with $\overline{V_i} \subset V_{i-1} \subset U_1$ and a number $t_i > t_{i-1}$ such that $\Phi^{t_i} \overline{V_i} \subset U_i$. This is possible by the above consideration and by continuity of the Weil-Petersson flow. Then $\cap_i \overline{V_i} \neq \emptyset$, and the forward Φ^t -orbit of any point in $\cap_i \overline{V_i}$ is infinite and dense in $\overline{P\mathcal{G}_0}$. With the same argument we can also guarantee that the backward Φ^t -orbit of a point $v \in \cap_i \overline{V_i}$ is infinite and dense in $\overline{P\mathcal{G}_0}$. But this just means that for a lift \tilde{v} of v the G -orbit of $\pi(v) \in \Lambda \times \Lambda$ is dense. This completes the proof of the corollary. \square

A flow Φ^t is called *topologically transitive* if it admits a dense orbit. As an immediate consequence of Lemma 5.2 and Proposition 5.4 we obtain the following result of Brock, Masur and Minsky [8].

COROLLARY 5.5. *The Weil-Petersson geodesic flow on $T^1\mathcal{M}(S)$ is topologically transitive.*

Remark: For two points $\xi \neq \eta \in \overline{\partial\mathcal{T}(S)}$ it is in general difficult to decide whether $(\xi, \eta) \in \pi\mathcal{G}$, i.e. whether there is a geodesic line connecting ξ to η . However, Brock, Masur and Minsky [8] showed the following. Let $\gamma : [0, \infty) \rightarrow \mathcal{T}(S)$ be a geodesic ray such that there is a number $\epsilon > 0$ and a sequence $t_i \rightarrow \infty$ with $\gamma(t_i) \in \mathcal{T}(S)_\epsilon$. Then $\gamma(\infty)$ can be connected to every $\xi \in \overline{\partial\mathcal{T}(S)} - \{\gamma(\infty)\}$ by a geodesic.

Even though the space $T^1\mathcal{M}(S)$ is non-compact and the Weil-Petersson geodesic flow Φ^t on $T^1\mathcal{M}(S)$ is not everywhere defined, it admits uncountably many invariant Borel-probability measures. Namely, the space of Φ^t -invariant Borel probability measures for the Weil-Petersson flow can be equipped with the weak*-topology. With respect to this topology, it is a closed convex set in the topological vector space of all finite signed Borel measures on $T^1\mathcal{M}(S)$. The extreme points of this convex set are the *ergodic* measures. Specific such ergodic measures are measures which are supported on a single periodic orbit. It was shown in [10] that there is a continuous injection from the space of invariant probability measures for the Teichmüller flow into the space of invariant probability measures for the Weil-Petersson geodesic flow.

Theorem 2 from the introduction is an immediate consequence of the following

PROPOSITION 5.6. *Every Φ^t -invariant ergodic Borel probability measure on $T^1\mathcal{M}(S)$ can be approximated in the weak*-topology by measures supported on periodic orbits.*

PROOF. Let ν be a Borel probability measure on $T^1\mathcal{M}(S)$ which is invariant and ergodic for the Weil-Petersson geodesic flow Φ^t . We have to find a sequence of periodic orbits for Φ^t so that the normalized Lebesgue measures ν_i supported on

these orbits converge weakly as $i \rightarrow \infty$ to ν . This means that for every continuous function $f : T^1\mathcal{M}(S) \rightarrow \mathbb{R}$ with compact support we have

$$\int f d\nu_i \rightarrow \int f d\nu.$$

For this let $v \in T^1\mathcal{M}(S)$ be a (typical) density point for the measure ν . By the Birkhoff ergodic theorem, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\Phi^s v) ds = \int f d\nu$$

for every continuous function f on $T^1\mathcal{M}(S)$ with compact support. Thus it suffices to find a sequence of numbers $t_i \rightarrow \infty$ and a sequence of periodic orbits for Φ^t which are the supports of normalized Φ^t -invariant measures ν_i such that

$$\left| \frac{1}{t_i} \int_0^{t_i} f(\Phi^s v) ds - \int f d\nu_i \right| \rightarrow 0 \quad (i \rightarrow \infty)$$

for every continuous function $f : T^1\mathcal{M}(S) \rightarrow \mathbb{R}$ with compact support.

The Weil-Petersson metric induces a Riemannian metric and hence a distance function d_S on (the orbifold) $T^1\mathcal{M}(S)$ (the so-called Sasaki metric). Since a continuous function $f : T^1\mathcal{M}(S) \rightarrow \mathbb{R}$ with compact support is bounded and uniformly continuous, for every $\epsilon > 0$ there is a number $\delta > 0$ depending on f such that

$$\left| \frac{1}{T} \int_0^T f(\Phi^t w) dt - \frac{1}{T} \int_0^T f(\Phi^t u) dt \right| < \epsilon$$

whenever $w, u \in T^1\mathcal{M}(S)$ are such that $d_S(\Phi^t u, \Phi^t w) < \delta$ for all $t \in [\delta T, (1 - \delta)T]$.

Since the sectional curvature of the Weil-Petersson metric is negative, comparison with the euclidean plane shows that the Sasaki distance in the covering space $T^1\mathcal{T}(S)$ can geometrically be estimated as follows. Let $P : T^1\mathcal{T}(S) \rightarrow \mathcal{T}(S)$ be the canonical projection. Then for every $\delta > 0$ there is a number $R = R(\delta) > 0$ with the following property. Let $w, u \in T^1\mathcal{T}(S)$ be two unit tangent vectors such that the flow-lines $\Phi^t w, \Phi^t u$ of w, u are defined on the interval $[-R, R]$. If $d_{WP}(P\Phi^t u, P\Phi^t w) \leq 1/R$ for all $t \in [-R, R]$ then $d_S(u, w) < \delta$.

Let $\gamma : \mathbb{R} \rightarrow \mathcal{T}(S)$ be a geodesic whose initial tangent $\gamma'(0)$ is a preimage of the density point v . By the above discussion and convexity of the distance function, it suffices to find a sequence of numbers $t_i \rightarrow \infty$ and a sequence $(g_i) \subset \text{Mod}(S)$ of pseudo-Anosov elements with the following properties.

- (1) There is a number $p > 0$ such that the translation length of g_i is contained in the interval $[t_i - p, t_i + p]$ for all i .
- (2) For every number $\delta > 0$ there is a number $T = T(\delta) > 0$ not depending on i such that the distance between the points $\gamma(T), \gamma(t_i - T)$ and the axis γ_i of g_i is at most δ for all sufficiently large i .

For the construction of such a sequence of pseudo-Anosov elements, note that by the Poincaré recurrence theorem we may assume that there is a sequence $t_i \rightarrow \infty$ such that $\Phi^{t_i} v \rightarrow v$ ($i \rightarrow \infty$). This means that there is a sequence $(g_i) \subset \text{Mod}(S)$ such that

$$g_i^{-1} \gamma'(t_i) \rightarrow \gamma'(0) \quad (i \rightarrow \infty).$$

In particular, $d_{WP}(\gamma(t_i), g_i \gamma(0)) \rightarrow 0$ ($i \rightarrow \infty$). We show that the sequence (g_i) has the properties 1), 2) above.

Let $\zeta_i : [0, r_i] \rightarrow \mathcal{T}(S)$ be the geodesic connecting x_0 to $g_i(x_0)$. By the above discussion, it suffices to show that for sufficiently large i the element $g_i \in \text{Mod}(S)$ is pseudo-Anosov and its axis has property 2) above where the geodesic arc $\gamma[0, t_i]$ in the statement is replaced by the geodesic arc $\zeta_i[0, r_i]$.

For this recall from Lemma 3.4 that for each i , the element $g_i \in \text{Mod}(S)$ either is axial or it is elliptic. If g_i is axial then let $\gamma_i : \mathbb{R} \rightarrow \overline{\mathcal{T}(S)}$ be an oriented axis of g_i . Let $x_0 = \gamma(0)$ and assume that γ_i is parametrized in such a way that $\gamma_i(0) = x_i = \pi_{\gamma_i(\mathbb{R})}(x_0)$. By invariance under g_i we have

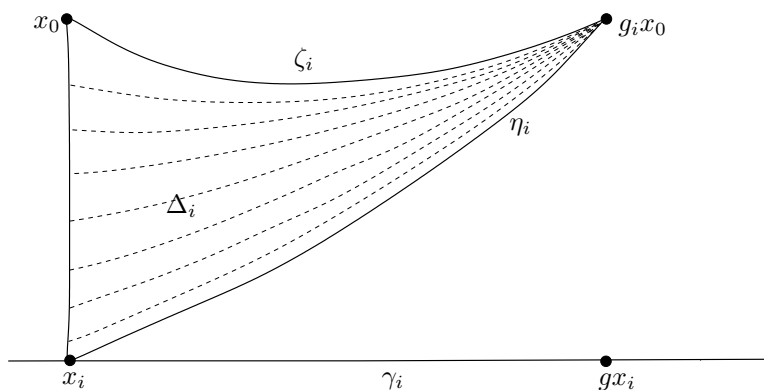
$$\pi_{\gamma_i(\mathbb{R})}(g_i x_0) = g_i x_i = \gamma_i(\tau_i)$$

where $\tau_i > 0$ is the minimum of the displacement function of g_i . Note that

$$\tau_i \leq d_{WP}(x_0, g_i x_0) \leq t_i + 1 \text{ for all large } i.$$

If g_i is not axial then let x_i be a fixed point of g_i .

Consider the (possibly degenerate) geodesic quadrangle Q_i in $\overline{\mathcal{T}(S)}$ with vertices $x_0, x_i, g_i x_i, g_i x_0$. Since $d_{WP}(x_0, x_i) = d_{WP}(g_i x_0, g_i x_i)$, by convexity of the distance function the angles of Q_i at $x_0, g_i x_0$ do not exceed $\pi/2$. On the other hand, by equivariance under the action of g_i , the sum of the angles of Q_i at x_0 and at $g_i x_0$ is not smaller than the angle at x_0 between the tangent of the geodesic arc ζ_i and the negative $-g_i \zeta'_i(r_i)$ of the tangent of the geodesic arc $g_i^{-1} \zeta_i$. Now $\zeta'_i(0) \rightarrow \gamma'(0)$, $g_i^{-1} \zeta'_i(r_i) \rightarrow \gamma'(0)$ and therefore this angle converges to π as $i \rightarrow \infty$. Thus we may assume that the minimum of the two angles of Q_i at $x_0, g_i x_0$ is bigger than $3\pi/8$ for all i .



Let $\Delta_i \subset \overline{\mathcal{T}(S)}$ be the ruled triangle with vertices $x_0, x_i, g_i x_0$ which we obtain by connecting the vertex $g_i x_0$ to each point on the opposite side by a geodesic (see the figure). A geodesic for the Weil-Petersson metric connecting a point in $\mathcal{T}(S)$ to a point in $\overline{\mathcal{T}(S)}$ is entirely contained in $\mathcal{T}(S)$ except possibly for its endpoint [18, 19]. Thus $\Delta_i - x_i$ is a smooth embedded surface in $\mathcal{T}(S)$ whose intrinsic Gauß curvature is defined. The Gauß curvature at a point $x \in \Delta_i$ does not exceed the maximum of the sectional curvatures of the Weil-Petersson metric at x (compare the proof of Lemma 5.2 and see [8] for more and references).

Let $\epsilon > 0$ be such that $x_0 \in \mathcal{T}(S)_{2\epsilon}$ and let $A \subset T^1 \mathcal{M}(S)$ be the projection to $T^1 \mathcal{M}(S)$ of the set of all unit tangent vectors in $T^1 \mathcal{T}(S)$ with foot-point in $\mathcal{T}(S)_\epsilon$. Then A is compact. Let χ be the characteristic function of A . Since v is a density

point for ν by assumption and since there is a neighborhood of v in $T^1\mathcal{M}(S)$ which is entirely contained in A , we have $\int \chi d\nu > 0$. Let $r > 0$ be sufficiently small that the sectional curvature of the restriction of the Weil-Petersson metric to the r -neighborhood of $\mathcal{T}(S)_\epsilon$ is bounded from above by a negative constant $-b < 0$. Such a number exists by invariance under the action of the mapping class group and cocompactness.

By CAT(0)-comparison, there is a number $s > 0$ with the following property. Let $R, S > 0$ and let $\zeta : [0, R] \rightarrow \overline{\mathcal{T}(S)}$, $\eta : [0, S] \rightarrow \overline{\mathcal{T}(S)}$ be any two geodesics issuing from $\eta(0) = \zeta(0) = x_0$ which enclose an angle at least $\pi/4$. Then η does not pass through the r -neighborhood of $\zeta[s, R]$. For this number s and for any number $\delta < r/3$ let $T = T(\delta) > 0$ be such that $\chi(\Phi^T v) = 1$ and

$$\int_s^T \chi(\Phi^s v) ds \geq \pi/b\delta.$$

Such a number exists since by the Birkhoff ergodic theorem we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \chi(\Phi^s v) ds = \int \chi d\nu > 0.$$

Let $i > 0$ be sufficiently large that $t_i > T + 2r$, and that $d_{WP}(\zeta_i(r_i), \gamma(t_i)) \leq \delta/2$. By convexity, we then have $d_{WP}(\zeta_i(t), \gamma(t)) \leq \delta$ for all $t \in [0, r_i]$. Let η_i be the side of Δ_i connecting x_i to $g_i x_0$. If c is a geodesic arc in Δ_i of length at most r issuing from a point in $\zeta_i[s, r_i]$ then c does not intersect the side of Δ_i connecting x_i to x_0 and hence either it ends on a point in η_i or it can be extended. Thus if the distance between $\zeta_i(T)$ and η_i is bigger than δ then by convexity the ruled triangle Δ_i contains an embedded strip of width $\delta < r/3$ with the arc $\zeta_i[s, T]$ as one of its sides. This strip is the union of all geodesic arcs of length δ in Δ_i with respect to the intrinsic metric which issue from a point in $\zeta_i[s, T]$ and which are perpendicular to ζ_i . If c is such a geodesic arc issuing from a point $\zeta_i(t)$ where $t \geq s$ is such that $\gamma'_i(t) \in A$, then the Gauss curvature of Δ_i on each point of c does not exceed $-b$. By the choice of T and by volume comparison, the Lebesgue measure of the set of all points on such geodesic arcs is not smaller than π/b . Since the Gauss curvature of Δ_i is negative, this implies that the integral of the Gauss curvature of Δ_i over this strip is smaller than $-\pi$. However, this violates the Gauss-Bonnet theorem (compare [10] for more details for this argument, and see also the proof of Lemma 5.2).

As a consequence, the geodesic η_i passes through the δ -neighborhood of the point $\zeta_i(T)$. This implies that for sufficiently large i the isometry g_i is not elliptic. Namely, we have $d_{WP}(x_i, \zeta_i(T)) \geq d_{WP}(x_0, x_i) - T$ and $d_{WP}(x_0, g_i x_0) \geq t_i - \delta$ and hence if $t_i > 2T + 2\delta$ then

$$(5.1) \quad \begin{aligned} d_{WP}(x_i, g_i x_0) &\geq d_{WP}(x_i, \zeta_i(T)) + t_i - T - 2\delta \\ &\geq d_{WP}(x_i, x_0) + t_i - 2T - 2\delta > d_{WP}(x_i, x_0). \end{aligned}$$

On the other hand, if g_i is elliptic then we have $d_{WP}(x_i, x_0) = d_{WP}(g_i x_i, g_i x_0) = d_{WP}(x_i, g_i x_0)$ which contradicts inequality (5.1). Thus g_i is axial for all sufficiently large i . Let γ_i be an oriented axis for g_i .

We next observe that g_i is pseudo-Anosov for all sufficiently large i . Namely, using the above notation, there is a number $s_i > 0$ such that

$$d_{WP}(\zeta_i(T), \eta_i(s_i)) < \delta.$$

By CAT(0)-comparison, the angle at $g_i x_0$ between the geodesics ζ_i and η_i tends to zero as $i \rightarrow \infty$. Thus for large i , the angle at $g_i x_i$ of the triangle with vertices $x_i, g_i x_i, g_i x_0$ is not smaller than $\pi/4$.

If the distance between the axis γ_i of g_i and $\eta_i(s_i)$ is smaller than δ then the axis γ_i of g_i passes through the $3\delta < r$ -neighborhood of a point in $\mathcal{T}(S)_\epsilon$ and hence it passes through a point in $\mathcal{T}(S)$. As a consequence, g_i is pseudo-Anosov (see the remark after Lemma 3.4).

On the other hand, if the distance between γ_i and $\eta_i(s_i)$ is bigger than δ then let $\tau > T$ be such that $\int_T^\tau \chi(\Phi^s v) ds \geq \pi/b\delta$ and $\chi(\Phi^\tau v) = 1$. Apply the above consideration to the ruled triangle $\hat{\Delta}_i$ with vertices $x_i, g_i x_i, g_i x_0$ which we obtain by connecting x_i to each point on the opposite side by a geodesic segment and to the subarc $\eta_i[s_i, s_i + \tau - T]$ of η_i . We conclude that $d_{WP}(\eta_i(s_i), \gamma_i) < \delta$ and once again, γ_i passes through a point in $\mathcal{T}(S)$ and g_i is pseudo-Anosov.

By the above consideration, for large i the axis γ_i of g_i passes through the δ -neighborhood of $\zeta_i(T)$ where $T = T(\delta)$ only depends on δ . The same argument shows that this axis also passes through the δ -neighborhood of $\zeta_i(r_i - \tilde{T})$ where once more $\tilde{T} > 0$ only depends on δ (assume without loss of generality that $-v$ is a density point for the image of ν under the flip $w \rightarrow -w$ and use the fact that two orbit segments of the same finite length are uniformly close if their initial points are close enough). In particular, the translation length of g_i is contained in the interval $[t_i - T - \tilde{T}, t_i + 1]$.

As a consequence, $(g_i) \subset \text{Mod}(S)$ is a sequence of pseudo-Anosov elements which satisfies the conditions 1), 2) above. Therefore the normalized Lebesgue measures on the projections to $T^1\mathcal{M}(S)$ of the unit tangent lines of the axes of the elements g_i converge weakly to ν . This completes the proof of the proposition. \square

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