ISOMETRY GROUPS OF PROPER CAT(0)-SPACES

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Abstract. Let X be a proper CAT(0)-space and let G be a closed subgroup of the isometry group $\text{Iso}(X)$ of X. We show that if G is non-elementary and contains a rank-one element then its second bounded cohomology group with coefficients in the regular representation is non-trivial. As a consequence, up to passing to an open subgroup of finite index, either G is a compact extension of a totally disconnected group or G is a compact extension of a simple Lie group of rank one.

1. Introduction

A geodesic metric space (X, d) is called *proper* if closed balls in X of finite radius are compact. A proper CAT(0)-metric space X can be compactified by adding the *visual boundary* ∂X. The isometry group $\text{Iso}(X)$ of X, equipped with the compact open topology, is a locally compact σ-compact topological group which acts as a group of homeomorphisms on ∂X. The *limit set* Λ of a subgroup G of $\text{Iso}(X)$ is the set of accumulation points in ∂X of an orbit of the action of G on X. The group G is called *elementary* if either its limit set consists of at most two points or if G fixes a point in ∂X.

For every $g \in \text{Iso}(X)$ the *displacement function* of g is the function $x \to d(x, gx)$. The isometry g is called *semisimple* if its displacement function assumes a minimum on X. If this minimum vanishes then g has a fixed point in X and is called *elliptic*, and otherwise g is called *axial*. If g is axial then the closed convex subset A of X on which the displacement function is minimal is isometric to a product space $C \times \mathbb{R}$ where g acts on each of the geodesics $\{x\} \times \mathbb{R}$ as a translation. Such a geodesic is called an *axis* for g. We refer to the books [3, 4, 7] for basic properties of CAT(0)-spaces and for references.

Call an axial isometry g of X *rank-one* if there is an axis γ for g which does not bound a flat half-plane. Here by a flat half-plane we mean a totally geodesic embedded isometric copy of an euclidean half-plane in X.

A *compact extension* of a topological group H is a topological group G which contains a compact normal subgroup K such that $H = G/K$ as topological groups. Extending earlier results for isometry groups of proper hyperbolic geodesic metric spaces [15, 19, 21] we show.

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Theorem 1. *Let X be a proper* CAT(0)*-space and let $G < \mathrm{Iso}(X)$ be a closed subgroup. Assume that G is non-elementary and contains a rank-one element. Then one of the following two possibilities holds.*

(1) *Up to passing to an open subgroup of finite index, G is a compact extension of a simple Lie group of rank one.*
(2) *G is a compact extension of a totally disconnected group.*

Caprace and Monod showed the following version of Theorem 1 (Corollary 1.7 of [12]): A CAT(0)-space X is called *irreducible* if it is not a non-trivial metric product. Let $X \neq \mathbb{R}$ be an irreducible proper CAT(0)-space with finite dimensional Tits boundary. Assume that the isometry group $\mathrm{Iso}(X)$ of X does not have a global fixed point in ∂X and that its action on X does not preserve a non-trivial closed convex subset of X. Then $\mathrm{Iso}(X)$ is either totally disconnected or an almost connected simple Lie group with trivial center.

We also note the following consequence (see Corollary 1.24 of [12]).

Corollary 1. *Let M be a closed Riemannian manifold of non-positive sectional curvature. If the universal covering $\tilde{M}$ of M is irreducible and if the isometry group of $\tilde{M}$ contains a parabolic element then M is locally symmetric.*

Our proof of Theorem 1 is different from the approach of Caprace and Monod and uses second bounded cohomology for locally compact topological groups G with coefficients in a *Banach module* for G. Such a Banach module is a separable Banach space E together with a continuous homomorphism of G into the group of linear isometries of E. For every such Banach module E for G and every $i \geq 1$, the group G naturally acts on the vector space $C_b(G^i, E)$ of continuous bounded maps $G^i \to E$. If we denote by $C_b(G^i, E)^G \subset C_b(G^i, E)$ the linear subspace of all G-invariant such maps, then the *second continuous bounded cohomology group* $H^2_{cb}(G, E)$ of G with coefficients E is defined as the second cohomology group of the complex

$$0 \to C_b(G, E)^G \xrightarrow{d} C_b(G^2, E)^G \xrightarrow{d} \dots$$

with the usual homogeneous coboundary operator d (see [20]). We write $H^2_b(G, E)$ to denote the second continuous bounded cohomology group of G with coefficients E where G is equipped with the discrete topology (but where the action of G on E is defined by a continuous homomorphism of G equipped with the usual topology into the group of linear isometries of E as before).

A closed subgroup G of $\mathrm{Iso}(X)$ is a locally compact and σ-compact topological group and hence it admits a left invariant locally finite Haar measure μ. In particular, for every $p \in (1, \infty)$ the separable Banach space $L^p(G, \mu)$ of functions on G which are p-integrable with respect to μ is a Banach module for G with respect to the isometric action of G by left translation. Extending an earlier result for isometry groups of proper hyperbolic spaces [15] (see also the work of Monod-Shalom [21], of Mineyev-Monod-Shalom [19], of Bestvina-Fujiwara [6] and of Caprace-Fujiwara [11] for closely related results) we obtain the following non-vanishing result for second bounded cohomology.

Theorem 2. *Let G be a closed non-elementary subgroup of the isometry group of a proper* CAT(0)*-space X with limit set $\Lambda \subset \partial X$. If G contains a rank-one element then $H_b^2(G, L^p(G,\mu)) \neq \{0\}$ for every $p \in (1,\infty)$. If G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then we also have $H_{cb}^2(G, L^p(G,\mu)) \neq 0$.*

As an application of Theorem 2 we obtain the following super-rigidity theorem.

Corollary 2. *Let G be a connected semi-simple Lie group with finite center, no compact factors and of rank at least 2. Let Γ be an irreducible lattice in G, let X be a proper* CAT(0)*-space and let $\rho : \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. Let $H < \mathrm{Iso}(X)$ be the closure of $\rho(\Gamma)$. If H is non-elementary and contains a rank-one element, then H is compact extension of a simple Lie group L of rank one, and up to passing to an open subgroup of finite index, ρ extends to a continuous homomorphism $G \to L$.*

Remark: As in [15], our proof of Theorem 2 also shows the following. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ which contains a rank-one element. If G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then the second bounded cohomology group $H_b^2(G, \mathbb{R})$ is infinite dimensional. However, this was proved by Bestvina and Fujiwara [6]. Moreover, the arguments in [15] show together with the geometric discussion in Sections 2-5 of this paper that if G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then $H_{cb}^2(G, \mathbb{R}) = 0$. Under the additional assumption that G acts on X cocompactly, this is due to Caprace and Fujiwara [11].

The organization of this note is as follows. In Section 2 we collect some geometric properties of a proper CAT(0)-space X needed in the sequel. In particular, we discuss contracting geodesics as introduced by Bestvina and Fujiwara [6].

In Section 3 we investigate for a fixed number $B > 0$ the space of all B-contracting geodesics in X. We construct a family of finite distance functions on the space of pairs of endpoints of such geodesics which are parametrized by the points in X and which are equivariant under the natural action of the isometry group of X. In Section 4 we use these distance functions to construct for a closed non-elementary subgroup G of $\mathrm{Iso}(X)$ with limit set Λ which contains a rank one element continuous bounded cocycles for G on a G-invariant closed subspace of the space of triples of pairwise distinct points in Λ with values in $L^p(G \times G, \mu \times \mu)$.

If the action of the group G on the complement of the diagonal in $\Lambda \times \Lambda$ is transitive, then this space of triples equals the entire space of triples of pairwise distinct points in Λ. In this case standard arguments can be used in Section 5 to complete the proof of Theorem 2.

The case when G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ uses a technically more difficult variant of the construction and is established in Section 6. The proof of Theorem 1 and of the corollaries is contained in Section 7.

2. Metric contraction in CAT(0)-spaces

In this section we collect some geometric properties of CAT(0)-spaces needed in the later sections. We use the books [3, 4, 7] as our main references.

2.1. **Shortest distance projections.** A proper CAT(0)-space has strong convexity properties which we summarize in this subsection.

In a complete CAT(0)-space X, any two points can be connected by a unique geodesic which varies continuously with the endpoints. The distance function is convex: If $\gamma, \zeta : J \to X$ are two geodesics in X parametrized on the same interval $J \subset \mathbb{R}$ then the function $t \to d(\gamma(t), \zeta(t))$ is convex. More generally, we call a function $f : X \to \mathbb{R}$ *convex* if for every geodesic $\gamma : J \to \mathbb{R}$ the function $t \to f(\gamma(t))$ is convex [3].

The *visual boundary* ∂X of X is defined to be the space of all geodesic rays issuing from a fixed point $x \in X$ equipped with the topology of uniform convergence on compact sets. This definition is independent of the choice of x. We denote the point in ∂X defined by a geodesic ray $\gamma : [0, \infty) \to X$ by $\gamma(\infty)$. We also say that γ *connects* x to $\gamma(\infty)$.

There is another description of the visual boundary of X as follows. Let $C(X)$ be the space of all continuous functions on X endowed with the topology of uniform convergence on bounded sets. Fix a point $y \in X$ and for $x, z \in X$ define

$$b_x(y, z) = d(x, z) - d(x, y).$$

Then we have

$$b_x(y, z) = -b_x(z, y) \text{ for all } y, z \in X \tag{1}$$

and

$$|b_x(y, z) - b_x(y, z')| \leq d(z, z') \text{ for all } z, z' \in X \tag{2}$$

and hence the function $b_x(y, \cdot) : z \to b_x(y, z)$ is one-Lipschitz and vanishes at y. Moreover, the function $b_x(y, \cdot)$ is convex. If $\tilde{y} \in X$ is another basepoint then we have

$$b_x(\tilde{y}, \cdot) = b_x(y, \cdot) + b_x(\tilde{y}, y). \tag{3}$$

The assignment $x \to b_x(y, \cdot)$ is an embedding of X into $C(X)$. A sequence $\{x_n\} \subset X$ *converges at infinity* if $d(x_n, y) \to \infty$ and if the functions $b_{x_n}(y, \cdot)$ converge in $C(X)$. The visual boundary ∂X of X can also be defined as the subset of $C(X)$ of all functions which are obtained as limits of functions $b_{x_n}(y, \cdot)$ for sequences $\{x_n\} \subset X$ which converge at infinity. In particular, the union $X \cup \partial X$ is naturally a closed subset of $C(X)$ (Chapter II.1 and II.2 of [3]). In this way, each $\xi \in \partial X$ corresponds to a *Busemann function* $b_\xi(y, \cdot)$ at ξ normalized at y. If $\gamma : [0, \infty) \to X$ is the geodesic ray which connects y to ξ then this Busemann function b_ξ satisfies $b_\xi(y, \gamma(t)) = -t$ for all $t \geq 0$.

From now on let X be a *proper* (i.e. complete and locally compact) CAT(0)-space. Then $X \cup \partial X$ is compact. A subset $C \subset X$ is *convex* if for $x, y \in C$ the geodesic connecting x to y is contained in C as well. For every closed convex set

$C \subset X$ and every $x \in X$ there is a unique point $\pi_C(x) \in C$ of smallest distance to x (Proposition II.2.4 of [7]). Now let $J \subset \mathbb{R}$ be a closed connected set and let $\gamma : J \to X$ be a geodesic arc. Then $\gamma(J) \subset X$ is closed and convex and hence there is a shortest distance projection $\pi_{\gamma(J)} : X \to \gamma(J)$. The projection point $\pi_{\gamma(J)}(x)$ of $x \in X$ is the unique minimum for the restriction of the function $b_x(y, \cdot)$ to $\gamma(J)$. By equality (3), this does not depend on the choice of the basepoint $y \in X$. The projection $\pi_{\gamma(J)} : X \to \gamma(J)$ is distance non-increasing.

For $\xi \in \partial X$ the function $t \to b_\xi(y, \gamma(t))$ is convex. Let $\overline{\gamma(J)}$ be the closure of $\gamma(J)$ in $X \cup \partial X$. If $b_\xi(y, \cdot)|\gamma(J)$ assumes a minimum then we can define $\pi_{\gamma(J)}(\xi) \subset \overline{\gamma(J)}$ to be the closure in $\overline{\gamma(J)}$ of the connected subset of $\gamma(J)$ of all such minima. If $b_\xi(y, \cdot)|\gamma(J)$ does not assume a minimum then by continuity the set J is unbounded and by convexity either $\lim_{t\to\infty} b_\xi(y, \gamma(t)) = \inf\{b_\xi(y, \gamma(s)) \mid s \in J\}$ or $\lim_{t\to-\infty} b_\xi(y, \gamma(t)) = \inf\{b_\xi(y, \gamma(s)) \mid s \in J\}$. In the first case we define $\pi_{\gamma(J)}(\xi) = \gamma(\infty) \in \partial X$, and in the second case we define $\pi_{\gamma(J)}(\xi) = \gamma(-\infty)$. Then for every $\xi \in \partial X$ the set $\pi_{\gamma(J)}(\xi)$ is a closed connected subset of $\overline{\gamma(J)}$ (which may contain points in both X and ∂X).

2.2. **Contracting geodesics.** A CAT(0)-space may have many totally geodesic embedded flat subspaces, but it may also have subsets with hyperbolic behavior. To give a precise description of such hyperbolic behavior, Bestvina and Fujiwara introduced a geometric property for geodesics in a CAT(0)-space (Definition 3.1 of [6]) which we repeat in the following definition. For the remainder of this note, geodesics are always defined on closed connected subsets of $\mathbb{R}$.

Definition 2.1. A geodesic arc $\gamma : J \to X$ is *B-contracting* for some $B > 0$ if for every closed metric ball K in X which is disjoint from $\gamma(J)$ the diameter of the projection $\pi_{\gamma(J)}(K)$ does not exceed B.

We call a geodesic *contracting* if it is B-contracting for some $B > 0$. Lemma 3.3 of [16] relates B-contraction for a geodesic γ to the diameter of the projections $\pi_{\gamma(\mathbb{R})}(\xi)$ where $\xi \in \partial X$.

Lemma 2.2. *Let $\gamma : \mathbb{R} \to X$ be a B-contracting geodesic. Then for every $\xi \in \partial X - \{\gamma(-\infty), \gamma(\infty)\}$ the projection $\pi_{\gamma(\mathbb{R})}(\xi)$ is a compact subset of $\gamma(\mathbb{R})$ of diameter at most $6B + 4$.*

Lemma 3.2 and 3.5 of [6] show that a connected subarc of a contracting geodesic is contracting and that a triangle containing a B-contracting geodesic as one of its sides is uniformly thin.

Lemma 2.3.
1. *Let $\gamma : J \to X$ be a B-contracting geodesic. Then for every closed connected subset $I \subset J$, the subarc $\gamma(I)$ of γ is $B + 3$-contracting.*
2. *Let $\gamma : J \to X$ be a B-contracting geodesic. Then for $x \in X$ and for every $t \in J$ the geodesic connecting x to $\gamma(t)$ passes through the $3B + 1$-neighborhood of $\pi_{\gamma(J)}(x)$.*

Note that by convexity of the distance function, if $\zeta_i : [a_i, b_i] \to X$ $(i = 1, 2)$ are two geodesic segments such that $d(\zeta_1(a_1), \zeta_2(a_2)) \leq R, d(\zeta_1(b_1), \zeta_2(b_2)) \leq R$ then the *Hausdorff distance* between the subsets $\zeta_1[a_1, b_1], \zeta_2[a_2, b_2]$ of X is at most R. Here the Hausdorff distance between closed (not necessarily compact) subsets A, B of X is the infimum of all numbers $R > 0$ such that A is contained in the R-neighborhood of B and B is contained in the R-neighborhood of A. (This number may be infinite).

A *visibility point* is a point $\xi \in \partial X$ with the property that any $\eta \in \partial X - \{\xi\}$ can be connected to ξ by a geodesic line. By Lemma 3.5 of [16], the endpoint of a contracting geodesic ray is a visibility point. Geodesic rays which abut at an endpoint of a contracting geodesic ray are themselves contracting.

Lemma 2.4. *For every $B > 0$ there is a number $C = C(B) > B$ with the following property. Let $\gamma : [0, \infty) \to X$ be a B-contracting ray and let $\xi \in \partial X - \gamma(\infty)$. Then every geodesic ζ connecting $\xi = \zeta(-\infty)$ to $\gamma(\infty) = \zeta(\infty)$ passes through the $9B+6$-neighborhood of every point $x \in \pi_{\gamma[0,\infty)}(\xi)$. If $t \in \mathbb{R}$ is such that $d(\zeta(t), x) \leq 9B+6$ then the geodesic ray $\zeta[t, \infty)$ is C-contracting.*

Proof. Let $\gamma : [0, \infty) \to X$ be a B-contracting geodesic ray and let $\xi \in \partial X - \gamma(\infty)$. Assume that $\gamma(s) \in \pi_{\gamma[0,\infty)}(\xi)$. By Lemma 2.2, the projection $\pi_{\gamma[0,\infty)}(\xi)$ is contained in $\gamma[s - 6B - 4, s + 6B + 4]$. Let $\zeta : \mathbb{R} \to X$ be a geodesic connecting ξ to $\gamma(\infty)$. Lemma 2.2 of [16] shows that for sufficiently large t we have

$$\pi_{\gamma[0,\infty)}(\zeta(-t)) \in \gamma[s - 6B - 5, s + 6B + 5].$$

Thus by Lemma 2.3, the geodesic ray $\zeta[-t, \infty)$ connecting $\zeta(-t)$ to $\gamma(\infty)$ passes through the $9B + 6$-neighborhood of $\gamma(s)$. Since ζ was an arbitrary geodesic connecting ξ to $\gamma(\infty)$, the first part of the lemma follows.

From this and Lemma 3.8 of [6], the second part of the lemma is immediate as well. Namely, let again ζ be a geodesic connecting ξ to $\gamma(\infty)$ and assume that ζ is parametrized in such a way that $d(\zeta(0), \pi_{\gamma[0,\infty)}(\xi)) \leq 9B + 6$. The geodesic ray $\zeta[0, \infty)$ is a locally uniform limit as $t \to \infty$ of the geodesics ζ_t connecting $\zeta(0)$ to $\gamma(t)$. By Lemma 3.8 of [6], there is a number $C > 0$ only depending on B such that each of the geodesics ζ_t is C-contracting. Now Lemma 3.3 of [16] shows that a limit of a sequence of C-contracting geodesics is C-contracting from which the lemma follows. ∎

The next observation is an extension of Lemma 2.3. For its formulation, define an *ideal geodesic triangle* to consist of three biinfinite geodesics $\gamma_1, \gamma_2, \gamma_3$ with $\gamma_i(\infty) = \gamma_{i+1}(-\infty)$ (where indices are taken modulo three). The points $\gamma_i(\infty)$ $(i = 1, 2, 3)$ are called the vertices of the ideal geodesic triangle. If $a, b \in \partial X$ are visibility points then for every $\xi \in \partial X - \{a, b\}$ there is an ideal geodesic triangle with vertices a, b, ξ. Note that such a triangle need not be unique.

Lemma 2.5. *Let $B > 0$ and let $\gamma : \mathbb{R} \to X$ be a B-contracting geodesic. Then for every ideal geodesic triangle T with side γ there is a point $x \in X$ whose distance to each of the sides of T does not exceed $9B + 6$. The diameter of the set of all such points does not exceed $54B + 36$.*

Proof. Let $\gamma : \mathbb{R} \to X$ be a B-contracting geodesic and let T be an ideal geodesic triangle with side γ and vertex $\xi \in \partial X - \{\gamma(\infty), \gamma(-\infty)\}$ opposite to γ. Assume that γ is parametrized in such a way that $\gamma(0) \in \pi_{\gamma(\mathbb{R})}(\xi)$. Let $c : [0, \infty) \to X$ be the geodesic ray connecting $c(0) = \gamma(0)$ to ξ. By Lemma 2.4 and by CAT(0)-comparison, the side α of T connecting ξ to $\gamma(\infty)$ is contained in the $9B+6$-tubular neighborhood of $c[0, \infty) \cup \gamma[0, \infty)$, and the side β of T connecting ξ to $\gamma(-\infty)$ is contained in the $9B + 6$-tubular neighborhood of $c[0, \infty) \cup \gamma(-\infty, 0]$. The distance between $\gamma(0) = c(0)$ and every side of T does not exceed $9B + 6$.

Since the projection $\pi_{\gamma(\mathbb{R})}$ is distance non-increasing and since $\pi_{\gamma(\mathbb{R})}(c[0, \infty)) = \gamma(0)$ (see [7] and the proof of Lemma 3.5 in [16]), if as before α, β are the sides of T connecting ξ to $\gamma(\infty), \gamma(-\infty)$, respectively, then

$$\pi_{\gamma(\mathbb{R})}(\alpha) \subset \gamma[-9B - 6, \infty) \text{ and } \pi_{\gamma(\mathbb{R})}(\beta) \subset \gamma(-\infty, 9B + 6].$$

Now if $x \in X$ is such that the distance between x and each side of T is at most $9B+6$ then using again that $\pi_{\gamma(\mathbb{R})}$ is distance non-increasing we conclude that $\pi_{\gamma(\mathbb{R})}(x) \in \gamma[-18B - 12, 18B + 12]$. But $d(x, \gamma(\mathbb{R})) \leq 9B + 6$ and hence $d(x, \gamma(0)) \leq 27B + 18$. This completes the proof of the lemma. □

2.3. **Isometries.** For an isometry g of X define the displacement function d_g of g to be the function $x \to d_g(x) = d(x, gx)$. An isometry g of X is called *semisimple* if d_g assumes a minimum in X. If g is semisimple and $\min d_g = 0$ then g is called *elliptic*. Thus an isometry is elliptic if and only if it fixes at least one point in X. A semisimple isometry g with $\min d_g > 0$ is called *axial*. By Proposition 3.3 of [3], an isometry g of X is axial if and only if there is a geodesic $\gamma : \mathbb{R} \to X$ such that $g\gamma(t) = \gamma(t+\tau)$ for every $t \in \mathbb{R}$ where $\tau = \min\, d_g > 0$ is the *translation length* of g. Such a geodesic is called an *oriented axis* for g. Note that the geodesic $t \to \gamma(-t)$ is an oriented axis for g^{-1}. The endpoint $\gamma(\infty)$ of γ is a fixed point for the action of g on ∂X which is called the *attracting fixed point*. The closed convex set $A \subset X$ of all points for which the displacement function of g is minimal is isometric to a product space $C \times \mathbb{R}$. For each $x \in C$ the set $\{x\} \times \mathbb{R}$ is an axis of g.

Bestvina and Fujiwara introduced the following notion to identify isometries of a CAT(0)-space with geometric properties similar to the properties of isometries in a hyperbolic geodesic metric space (Definition 5.1 of [6]).

Definition 2.6. An isometry $g \in \mathrm{Iso}(X)$ is called *B-rank-one* for some $B > 0$ if g is axial and admits a B-contracting axis.

We call an isometry g *rank-one* if g is B-rank-one for some $B > 0$.

The following statement is Theorem 5.4 of [6].

Proposition 2.7. *An axial isometry of X with axis γ is rank-one if and only if γ does not bound a flat half-plane.*

Let $G < \mathrm{Iso}(X)$ be a subgroup of the isometry group of X. The *limit set* Λ of G is the set of accumulation points in ∂X of one (and hence every) orbit of the action of G on X. The limit set is a compact non-empty G-invariant subset of ∂X. Call

G *non-elementary* if its limit set contains at least three points and if moreover G does not fix globally a point in ∂X.

A compact space is *perfect* if it does not have isolated points. The action of a group G on a topological space Λ is called *minimal* if every orbit is dense. A homeomorphism g of a space Λ is said to act with *north-south dynamics* if there are two fixed points $a \neq b \in \Lambda$ for the action of g such that for every neighborhood U of a, V of b there is some $k > 0$ such that $g^k(\Lambda - V) \subset U$ and $g^{-k}(\Lambda - U) \subset V$. The point a is called the *attracting fixed point* for g, and b is the repelling fixed point. The following is shown in [16] (see also [3] for a similar discussion).

Lemma 2.8. *Let $G < \mathrm{Iso}(X)$ be a non-elementary group which contains a rank-one element. Then the limit set Λ of G is perfect, and it is the smallest closed G-invariant subset of ∂X. The action of G on Λ is minimal. An element $g \in G$ is rank-one if and only if g acts on ∂X with north-south dynamics.*

For every proper metric space X, the isometry group $\mathrm{Iso}(X)$ of X can be equipped with a natural locally compact σ-compact metrizable topology, the so-called *compact open topology*. With respect to this topology, a sequence $(g_i) \subset \mathrm{Iso}(X)$ converges to some isometry g if and only if $g_i \to g$ uniformly on compact subsets of X. In this topology, a closed subset $A \subset \mathrm{Iso}(X)$ is compact if and only if there is a compact subset K of X such that $gK \cap K \neq \emptyset$ for every $g \in A$. In particular, the action of $\mathrm{Iso}(X)$ on X is proper. In the sequel we always equip subgroups of $\mathrm{Iso}(X)$ with the compact open topology.

Denote by Δ the diagonal in $\partial X \times \partial X$. We have (Lemma 6.1 of [15]).

Lemma 2.9. *Let $G < \mathrm{Iso}(X)$ be a closed subgroup with limit set Λ. Let $(a,b) \in \Lambda \times \Lambda - \Delta$ be the pair of fixed points of a rank one element of G. Then the G-orbit of (a,b) is a closed subset of $\Lambda \times \Lambda - \Delta$.*

The following technical observation is useful in Section 6.

Lemma 2.10. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group with limit set Λ. If G contains a rank-one element $g \in G$ with fixed points $a \neq b \in \Lambda$ and if G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then there is some $h \in G$ such that $hb \neq b$ and that the stabilizer in G of the pair of points $(b, hb) \in \Lambda \times \Lambda$ is compact.*

Proof. Let $g \in G$ be a rank-one element with attracting fixed point $a \in \Lambda$, repelling fixed point $b \in \Lambda$. Let $\gamma : \mathbb{R} \to X$ be an axis for g connecting b to a. Then γ is B-contracting for some $B > 0$.

Let $h \in G$ be such that $hb \neq b$. The rays $\gamma(-\infty, 0]$ and $h(\gamma(-\infty, 0])$ are B-contracting, with endpoints $b, hb \in \Lambda$. Now a biinfinite geodesic $\xi : \mathbb{R} \to X$ with the property that there are numbers $-\infty < s < t < \infty, C > 0$ such that the rays $\xi(-\infty, s], \xi[t, \infty)$ are C-contracting is C'-contracting for a number $C' > C$ only depending on C and on $[s,t]$. Therefore Lemma 2.4 implies that there is a number $B_0 > B$ such that each geodesic connecting b to hb is B_0-contracting. As a consequence, the set $A \subset X$ of all points which are contained in a geodesic connecting b to hb is closed and convex and isometric to $K_0 \times \mathbb{R}$ for a compact convex subset K_0 of X.

An isometry of X which stabilizes the pair of points (b, hb) preserves the closed convex set $K_0 \times \mathbb{R}$. Since K_0 is compact, each such isometry u is semi-simple. Moreover, if u is not elliptic then u is rank-one. Since G is a closed subgroup of $\mathrm{Iso}(X)$, this implies that either the stabilizer of (b, hb) in G is compact or it contains a rank one element.

Now assume that there is no $h \in G$ with $hb \neq b$ such that the stabilizer of (b, hb) in G is compact. Then each such stabilizer contains a rank-one element. The stabilizer G_b of b in G is a closed subgroup of G.

Let $h \in G$ with $hb \notin \{a, b\}$ and let again γ be an oriented axis for g connecting b to a. Assume that $\gamma(0) \in \pi_{\gamma(\mathbb{R})}(hb)$. Since γ is B-contracting, by Lemma 2.4 and convexity the ray $\gamma(-\infty, 0]$ is contained in the $9B + 6$-neighborhood of every geodesic connecting b to hb. By assumption and the above discussion, there is a rank-one element $u \in G_b$ with attracting fixed point hb and repelling fixed point b. By Lemma 2.8, u acts with north-south dynamics on Λ and hence we have $u^i a \to hb$ $(i \to \infty)$ and also $u^i g^{-k} a \to hb$ for all k. Let $\tau > 0$ be the translation length of g and let K be the closed $18B + 12 + 2\tau$-neighborhood of $\gamma(0)$. By the choice of the set K and the fact that u preserves a geodesic connecting b to hb which contains the ray $\gamma(-\infty, 0]$ in its $9B + 6$-neighborhood, for every $i > 0$ there is some $k(i) > 0$ such that $u^i g^{-k(i)} \gamma(0) \in K$. Since G is a closed subgroup of $\mathrm{Iso}(X)$ and $G_b < G$ is closed, up to passing to a subsequence the sequence $\{u^i g^{-k(i)}\} \subset G_b$ converges to an element $v \in G_b$ with $va = hb$.

As a consequence, for every $x \in Gb - \{b\}$ there is some $v \in G_b$ with $va = x$. This implies that the image of (a, b) under the action of the group G is dense in $\Lambda \times \Lambda - \Delta$. Namely, by Lemma 2.8, the G-orbit of b is dense in Λ. Thus it suffices to show that for every $u \in G$ and every $x \in Gb - \{b, ub\}$ there is some $v \in G$ with $v(a, b) = (x, ub)$. For this let $y = u^{-1}x$. Then $y \neq b$ and hence there is some $w \in G_b$ with $w(a) = y$. Then the isometry $v = uw$ satisfies $v(b) = u(b)$, $v(a) = x$.

Now by Lemma 2.9, the G-orbit of (a, b) is closed in $\Lambda \times \Lambda - \Delta$. The above discussion shows that it is also dense and therefore G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$. The lemma follows. $\square$

A free group with two generators is hyperbolic in the sense of Gromov [13]. In particular, it admits a Gromov boundary which can be viewed as a compactification of the group. The following result is contained in [6] (see also Proposition 5.8 of [16] and [11, 5]).

Lemma 2.11. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group which contains a rank-one element. Let $\Lambda \subset \partial X$ be the limit set of G. If G does not act transitively on $\Lambda \times \Lambda - \Delta$ then G contains a free subgroup Γ with two generators and the following properties.*

(1) *Every element $e \neq g \in \Gamma$ is rank-one.*
(2) *There is a Γ-equivariant embedding of the Gromov boundary of Γ into Λ.*

(3) *There are infinitely many elements* $u_i \in \Gamma$ $(i > 0)$ *with fixed points* a_i, b_i *such that for all* i *the* G*-orbit of* $(a_i, b_i) \in \Lambda \times \Lambda - \Delta$ *is distinct from the orbit of* $(b_j, a_j)(j > 0)$ *or* $(a_j, b_j)(j \neq i)$.

3. The space of B-contracting geodesics

In the previous section we introduced for some $B > 0$ a B-contracting geodesic in a proper CAT(0)-space X. In this section we consider in more detail the space of all such geodesics in X.

The main idea is as follows: Even though the geometry of a CAT(0)-space X may be very different from the geometry of a hyperbolic geodesic metric space, if X admits B-contracting geodesics then by Lemma 2.5, these geodesics have the same global geometric properties as geodesics in a δ-hyperbolic geodesic metric space where $\delta > 0$ only depends on B. As a consequence, given a fixed point $x \in X$, we can describe the position of two such geodesics γ, ζ relative to each other as seen from x by introducing a metric quantity which can be thought of being equivalent to the (oriented) sum of the Gromov distances at x of their endpoints in the case that the space X is hyperbolic.

We continue to use the assumptions and notations from Section 2. In the remainder of this section, a geodesic in X is always defined on a closed connected subset J of $\mathbb{R}$. For some $B > 0$ denote by $\mathcal{A}(B) \subset \partial X \times \partial X - \Delta$ the set of all pairs of points in ∂X which are connected by a B-contracting geodesic. We have.

Lemma 3.1. $\mathcal{A}(B)$ *is a closed subset of* $\partial X \times \partial X - \Delta$.

Proof. Let $\{(\xi_i, \eta_i)\} \subset \mathcal{A}(B)$ be a sequence which converges in $\partial X \times \partial X - \Delta$ to a point (ξ, η). For each i let γ_i be a B-contracting geodesic connecting ξ_i to η_i. We first claim that the geodesics γ_i pass through a fixed compact subset of X.

Namely, choose a point $x \in X$ and let $x_i = \pi_{\gamma_i(\mathbb{R})}(x)$. If the geodesics γ_i do not pass through a fixed compact subset of X then we have $d(x_i, x) \to \infty$. Since $X \cup \partial X$ is compact, after passing to a subsequence we may assume that $x_i \to \alpha \in \partial X$ as $i \to \infty$. On the other hand, the geodesic γ_i is B-contracting and therefore by Lemma 2.3 the geodesics connecting x to $\xi_i = \gamma_i(-\infty), \eta_i = \gamma_i(\infty)$ both pass through the $3B+1$-neighborhood of x_i. By CAT(0)-comparison, this implies that $\xi_i \to \alpha, \eta_i \to \alpha$ which contradicts the assumption that $\xi_i \to \xi, \eta_i \to \eta \neq \xi$.

Thus the geodesics γ_i pass through a fixed compact subset of X and therefore after passing to a subsequence we may assume that $\gamma_i \to \gamma$ locally uniformly where γ is a geodesic connecting ξ to η. The limit geodesic is B-contracting by Lemma 3.6 of [16]. $\square$

For a number $B > 0$, a point $x \in X$ and an ordered pair $(\zeta_1 : J_1 \to X, \zeta_2 : J_2 \to X)$ of oriented geodesics in X which share at most one endpoint in ∂X define a number $\tau_B(x, \zeta_1, \zeta_2) \geq 0$ as follows.

By convexity of the distance function, there are (perhaps empty) closed connected subsets $[a_1, b_1] \subset J_1, [a_2, b_2] \subset J_2$ such that

$$[a_i, b_i] = \{t \mid d(\zeta_i(t), \zeta_{i+1}(J_{i+1})) \le 6B + 2\}.$$

(Here $i = 1, 2$ and indices are taken modulo two. If ζ_1, ζ_2 have a common endpoint in ∂X then one of the numbers a_1, b_1 and one of the numbers a_2, b_2 may be infinite.)

If $[a_i, b_i] \neq \emptyset$ then let $s_i, t_i \in J_i \cup \{\pm\infty\}$ be such that

$$\pi_{\zeta_i(J_i)}(\zeta_{i+1}(a_{i+1})) = \zeta_i(s_i) \text{ and } \pi_{\zeta_i(J_i)}(\zeta_{i+1}(b_{i+1})) = \zeta_i(t_i)$$

($i = 1, 2$ and indices are taken modulo two) and let $x_i = \pi_{\zeta_i(J_i)}(x)$ ($i = 1, 2$).

If $s_i < t_i$ and if $x_i \in \zeta_i[a_i, b_i]$ for $i = 1, 2$ then define

$$\tau_B(x, \zeta_1, \zeta_2) = \min\{d(x_i, \zeta_i(a_i)), d(x_i, \zeta_i(b_i)) \mid i = 1, 2\}.$$

In all other cases define $\tau_B(x, \zeta_1, \zeta_2) = 0$. Note that $\tau_B(x, \zeta_1, \zeta_2)$ depends on the orientation of ζ_1, ζ_2 but not on the parametrization of ζ_1, ζ_2 defining a fixed orientation.

We collect some first easy properties of the function τ_B.

Lemma 3.2. *For any two geodesics ζ_1, ζ_2 in X and any $x \in X$ the following holds true.*

(1) $\tau_B(x, \zeta_1, \zeta_2) = \tau_B(x, \zeta_2, \zeta_1)$.
(2) *If $\hat{\zeta}_i$ equals the geodesic obtained from ζ_i by reversing the orientation then* $\tau_B(x, \hat{\zeta}_1, \hat{\zeta}_2) = \tau_B(x, \zeta_1, \zeta_2)$.
(3) $\tau_B(x, \zeta_1, \zeta_2) \le \tau_B(y, \zeta_1, \zeta_2) + d(x, y)$ *for all* $x, y \in X$.

Proof. The first and the second property in the lemma is obvious from the definition. To show the third property simply note that for a geodesic $\zeta : J \to X$ the projection $\pi_{\zeta(J)}$ is distance non-increasing. ☐

Moreover we observe.

Lemma 3.3. *Let $\zeta_i : J_i \to X$ be B-contracting geodesics ($i = 1, 2$) such that $\tau_B(x, \zeta_1, \zeta_2) > 0$. Then we have*

$$d(\pi_{\zeta_1(J_1)}(x), \pi_{\zeta_2(J_2)}(x)) \le 24B + 8.$$

Proof. If $\tau_B(x, \zeta_1, \zeta_2) > 0$ and if (after reparametrization) we have $\pi_{\zeta_i(J_i)}(x) = \zeta_i(0)$ ($i = 1, 2$) then by definition of the function τ_B there is a point on the geodesic ζ_2 whose distance to $\zeta_1(0)$ does not exceed $6B + 2$. This shows that

$$d(x, \zeta_2(J_2)) \le d(x, \zeta_1(J_1)) + 6B + 2.$$

By symmetry we conclude that

$$|d(x, \zeta_1(J_1)) - d(x, \zeta_2(J_2))| \le 6B + 2. \tag{4}$$

Thus if $t \in J_2$ is such that $d(\zeta_1(0), \zeta_2(t)) \le 6B + 2$ then

$$d(x, \zeta_2(t)) \le d(x, \zeta_2(J_2)) + 12B + 4. \tag{5}$$

On the other hand, by Lemma 2.3, the geodesic connecting x to $\zeta_2(t)$ passes through the $3B+1$-neighborhood of $\zeta_2(0)$ and hence

$$d(x,\zeta_2(t)) \geq d(x,\zeta_2(J_2)) + |t| - 6B - 2. \tag{6}$$

The two inequalities (5) and (6) together show that $|t| \leq 18B+6$ and therefore $d(\pi_{\zeta_1(J_1)}(x), \pi_{\zeta_2(J_2)}(x)) \leq 24B+8$ as claimed. □

We also have.

Lemma 3.4. *Let $\zeta_i : [0,\infty) \to X$ $(i=1,2)$ be two geodesic rays with the same endpoint $\zeta_1(\infty) = \zeta_2(\infty)$. Let $s \in [1,\infty)$ be such that*

$$p = \tau_B(\zeta_1(s), \zeta_1, \zeta_2) \geq 1.$$

Then $\tau_B(\zeta_1(s+t), \zeta_1, \zeta_2) \geq p + t - 12B - 4$ for all $t \geq 0$.

Proof. Let $\zeta_i : [0,\infty) \to X$ be geodesic rays in X $(i=1,2)$ with $\zeta_1(\infty) = \zeta_2(\infty)$. Let $s \in [0,\infty)$ be such that $\tau_B(\zeta_1(s), \zeta_1, \zeta_2) \geq 1$. Then the geodesic ray $\zeta_2[0,\infty)$ passes through the $6B+2$-neighborhood of $\zeta_1(s)$. If $s' \in [0,\infty)$ is such that $d(\zeta_1(s), \zeta_2(s')) \leq 6B+2$ then by convexity of the distance function we have

$$d(\zeta_1(s+t), \zeta_2(s'+t)) \leq 6B+2 \text{ for all } t \geq 0.$$

Now let $t \geq 0$ and let $\sigma \in \mathbb{R}$ be such that $\pi_{\zeta_2[0,\infty)}(\zeta_1(s+t)) = \zeta_2(\sigma)$. Then $d(\zeta_1(s+t), \zeta_2(\sigma)) \leq 6B+2$ and hence the triangle inequality shows that $\sigma \in [s'+t-12B-4, s'+t+12B+4]$. From this and the definition of the function τ_B the lemma follows. □

The next observation is the analog of the familiar ultrametric inequality for Gromov products in hyperbolic spaces.

Lemma 3.5. *There is a number $L > 0$ such that for every $B > 0$ and for all B-contracting geodesics $\zeta_i : J_i \to X$ $(i=1,2)$ we have*

$$\tau_B(x,\zeta_1,\zeta_3) \geq \min\{\tau_B(x,\zeta_1,\zeta_2), \tau_B(x,\zeta_2,\zeta_3)\} - LB.$$

Proof. Let $\zeta_i : J_i \to X$ be B-contracting geodesics and let $x \in X$. Number the geodesics ζ_i in such a way that $\tau_B(x,\zeta_1,\zeta_3) = \min\{\tau_B(x,\zeta_i,\zeta_{i+1}) \mid i = 1,2,3\}$. Assume also without loss of generality that $r_1 = \tau_B(x,\zeta_1,\zeta_2) \leq r_2 = \tau_B(x,\zeta_2,\zeta_3)$. If $r_1 = 0$ then there is nothing to show. So assume that $r_1 > 0$. By Lemma 3.3 we then have

$$d(\pi_{\zeta_2(J_2)}(x), \pi_{\zeta_j(J_j)}(x)) \leq 24B + 8\,(j=1,3) \tag{7}$$

and hence $d(\pi_{\zeta_1(J_1)}(x), \pi_{\zeta_3(J_3)}(x)) \leq 48B+16$. Since the lemma is only significant if r_1 is large, we successively increase a lower bound for r_1 by a controlled amount in the course of the proof so that all the geometric estimates are meaningful without explicit mentioning.

For simplicity parametrize the geodesics ζ_i in such a way that $\pi_{\zeta_i(J_i)}(x) = \zeta_i(0)$ $(i=1,2,3)$. By definition of the function τ_B there is a number $t_2 \geq 0$ such that $d(\zeta_1(r_1), \zeta_2(t_2)) \leq 6B+2$. By the distance estimate (7), we have

$$t_2 = d(\zeta_2(t_2), \pi_{\zeta_2(J_2)}(x)) \in [r_1 - 30B - 10, r_1 + 30B + 10]$$

and hence $d(\zeta_1(r_1),\zeta_2(r_1))\leq 36B+12$.

Now $r_2\geq r_1$ by assumption and therefore using once more the definition of the function τ_B we have $d(\zeta_2(r_1),\zeta_3(J_3))\leq 6B+2$. Thus if we write $R_0=42B+18$ then we have $d(\zeta_1(r_1),\zeta_3(J_3))\leq R_0$ and similarly $d(\zeta_1(-r_1),\zeta_3(J_3))\leq R_0$. Since $d(\zeta_1(0),\zeta_3(0))\leq 48B+16$ by the estimate (7) above, we conclude that for $R_1=48B+16+R_0$ there are numbers $s_3,t_3\geq r_1-R_1$ such that $[-s_3,t_3]\subset J_3$ and that

$$d(\zeta_1(-r_1),\zeta_3(-s_3))\leq R_0 \text{ and } d(\zeta_1(r_1),\zeta_3(t_3))\leq R_0. \tag{8}$$

By assumption, ζ_1 and ζ_3 are B-contracting. Let $\rho:[0,b]\to X$ be the geodesic connecting $\zeta_1(-r_1)=\rho(0)$ to $\zeta_3(t_3)=\rho(b)$. Let $z=\pi_{\zeta_1(J_1)}(\zeta_3(t_3))$. By the estimate (8) and the triangle inequality, the distance between z and $\zeta_3(t_3)$ is at most R_0, and the distance between z and $\zeta_1(r_1)$ is bounded from above by $2R_0$.

Since ζ_1 is B-contracting, by Lemma 2.3 and the remark thereafter, there is a number $T\leq b$ such that the Hausdorff distance between the subarc of ζ_1 connecting $\zeta_1(-r_1)$ to z and the arc $\rho[0,T]$ is at most $3B+1$. Moreover, we can choose T in such a way that $T\geq b-R_0$.

Similarly, since ζ_3 is B-contracting, if $w=\pi_{\zeta_3(J_3)}(\zeta_1(-r_1))$ then the distance between w and $\zeta_1(-r_1)$ is at most R_0. There is a number $S\leq R_0$ such that the Hausdorff distance between $\rho[S,b]$ and the subarc of ζ_3 connecting w to $\zeta_3(t_3)$ is at most $3B+1$.

As a consequence, there are two subarcs ζ_1' of $\zeta_1(J_1)$, ζ_3' of $\zeta_3(J_3)$ whose Hausdorff distance to the geodesic arc $\rho[S,T]$ is at most $3B+1$. Hence the Hausdorff distance between ζ_1' and ζ_3' is at most $6B+2$.

If r_1 is sufficiently large depending on B then there is a number $L>0$ and there is a subarc ζ_1',ζ_3' of $\zeta_1(J_1),\zeta_3(J_3)$ with the following property. The arc ζ_1',ζ_3' contains x_1,x_3 as an interior point, and the distance of x_1,x_3 to the endpoints of ζ_1',ζ_2' is at least r_1-LB. Moreover, the Hausdorff distance in X between ζ_1',ζ_3' is smaller than $6B+2$. This shows that

$$\tau_B(x,\zeta_1,\zeta_3)\geq r_1-LB\geq\tau_B(x,\zeta_1,\zeta_2)-LB$$

which completes the proof of the lemma. $\square$

For distinct pairs of points $(\xi_1,\eta_1),(\xi_2,\eta_2)\in\mathcal{A}(B)$ define

$$\tau_B(x,(\xi_1,\eta_1),(\xi_2,\eta_2))\geq 0$$

to be the infimum of the numbers $\tau_B(x,\zeta_1,\zeta_2)$ over all B-contracting geodesics ζ_i connecting ξ_i to η_i $(i=1,2)$. Clearly we have

$$\tau_B(x,\alpha_1,\alpha_2)=\tau_B(x,\alpha_2,\alpha_1) \text{ for all } x\in X,\alpha_1,\alpha_2\in\mathcal{A}(B).$$

Moreover, by Lemma 3.5, there is a number $L>0$ such that for all $\alpha_1,\alpha_2,\alpha_3\in\mathcal{A}(B)$ we have

$$\tau_B(x,\alpha_1,\alpha_3)\geq\min\{\tau_B(x,\alpha_1,\alpha_2),\tau_B(x,\alpha_2,\alpha_3)\}-LB.$$

Now we follow Section 7.3 of [13]. Namely, let $\chi > 0$ be sufficiently small that $\chi' = e^{\chi LB} - 1 < \sqrt{2} - 1$. Note that χ only depends on B. For this number χ and for $x \in X$, $\alpha_1, \alpha_2 \in \mathcal{A}(B) \times \mathcal{A}(B)$ define

$$\tilde{\delta}_x(\alpha_1, \alpha_2) = e^{-\chi\tau_B(x,\alpha_1,\alpha_2)}. \tag{9}$$

From Lemma 3.5 and Proposition 7.3.10 of [13] we obtain.

Corollary 3.6. *There is a family $\{\delta_x\}$ $(x \in X)$ of distances on $\mathcal{A}(B)$ with the following properties.*

(1) *The topology on $\mathcal{A}(B)$ defined by the distances δ_x is the restriction of the product topology on $\partial X \times \partial X - \Delta$. In particular, $(\mathcal{A}(B), \delta_x)$ is locally compact.*
(2) *The distances δ_x are invariant under the involution $\iota : (\xi, \eta) \to (\eta, \xi)$ of $\mathcal{A}(B)$ exchanging the two components of a point in $\mathcal{A}(B)$.*
(3) $(1 - 2\chi')\tilde{\delta}_x \leq \delta_x \leq \tilde{\delta}_x$ *for all $x \in X$.*
(4) $e^{-\chi d(x,y)} \leq \delta_y \leq e^{\chi d(x,y)}\delta_x$ *for all $x, y \in X$.*
(5) *The family $\{\delta_x\}$ is invariant under the action of* $\mathrm{Iso}(X)$ *on $\mathcal{A}(B) \times X$.*

Proof. The existence of a family $\{\delta_x\}$ of distance functions with the property stated in the third part of the corollary is immediate from Lemma 3.5 and Proposition 3.7.10 of [13]. The forth part follows from the construction of the distance δ_x from the functions $\tilde{\delta}_x$ and from the third part of Lemma 3.2. Invariance under the action of the isometry group and under the involution ι is an immediate consequence of invariance of the function τ_B.

We are left with showing that for a given $x \in X$ the distance δ_x induces the restriction of the product topology. By the definition of the distances δ_x, if $(\xi_i, \eta_i) \to (\xi, \eta)$ in $(\mathcal{A}(B), \delta_x)$ then there are B-contracting geodesics γ_i connecting ξ_i to η_i which have longer and longer subsegments contained in a tubular neighborhood of radius $6B + 2$ about some geodesic γ connecting ξ to η. Moreover, these segments all pass through a a fixed compact subset of X. By the definition of the topology on ∂X, this implies that $(\xi_i, \eta_i) \to (\xi, \eta)$ in $\partial X \times \partial X - \Delta$.

Continuity of the identity $\mathcal{A}(B) \subset \partial X \times \partial X - \Delta \to (\mathcal{A}(B), \delta_x)$ follows in the same way. Namely, by the first part of the proof of Lemma 3.1, if $(\xi_i, \eta_i) \subset \mathcal{A}(B)$, if $(\xi_i, \eta_i) \to (\xi, \eta) \in \partial X \times \partial X - \Delta$ with respect to the product topology and if γ_i is a B-contracting geodesic connecting ξ_i to η_i then up to passing to a subsequence, we may assume that the geodesics γ_i converge uniformly on compact sets to a B-contracting geodesic γ connecting ξ to η. By convexity, by Lemma 2.3 and by the definition of the function τ_B, this implies that $(\xi_i, \eta_i) \to (\xi, \eta)$ in $(\mathcal{A}(B), \delta_x)$ for every $x \in X$. ☐

Using Corollary 3.6, we obtain the following analog of Lemma 2.1 of [15] (with identical proof).

Lemma 3.7. *$\mathcal{A}(B) \times X$ admits a natural* $\mathrm{Iso}(X)$*-invariant ι-invariant distance function $\tilde{d}$ inducing the product topology. There is a number $c > 0$ such that for every $x \in X$, the restriction of $\tilde{d}$ to $\mathcal{A}(B) \times \{x\}$ satisfies*

$$c\delta_x(\alpha, \beta) \leq \tilde{d}((\alpha, x), (\beta, x)) \leq \delta_x(\alpha, \beta) \forall \alpha, \beta \in \mathcal{A}(B).$$

4. Continuous bounded cocycles

In this section we consider again X a proper CAT(0)-space X. Let G be a *closed* non-elementary subgroup of the isometry group of X with limit set Λ. Then G is a locally compact σ-compact topological group. Assume that G contains a rank one element. Let $T \subset \Lambda^3$ be the space of triples of pairwise distinct points in Λ. By Lemma 2.8, T is a locally compact uncountable topological G-space without isolated points. As in Section 3, for a number $B > 0$ denote by $\mathcal{A}(B) \subset \partial X \times \partial X - \Delta$ the set of pairs of distinct points in ∂X which can be connected by a B-contracting geodesic. Let moreover $T(B) \subset T$ be the set of triples $(a_1, a_2, a_3) \in T$ with the additional property that $(a_i, a_{i+1}) \in \mathcal{A}(B)$ ($1 \leq i \leq 3$ and where indices are taken modulo three). By Lemma 3.1, $T(B)$ is closed subset of T which is invariant under the diagonal action of G.

For a Banach-module E for G define an *E-valued continuous bounded two-cocycle* for the action of G on $T(B)$ to be a continuous *bounded* G-equivariant map $\omega : T(B) \to E$ which satisfies the following two properties.

(1) For every permutation σ of the three variables, the *anti-symmetry condition* $\omega \circ \sigma = \mathrm{sgn}(\sigma)\omega$ holds.
(2) For every quadruple (a_1, a_2, a_3, a_4) of distinct points in Λ such that $(a_i, a_j) \in \mathcal{A}(B)$ for $i \neq j$ the *cocycle equality*

$$\omega(a_2, a_3, a_4) - \omega(a_1, a_3, a_4) + \omega(a_1, a_2, a_4) - \omega(a_1, a_2, a_3) = 0 \tag{10}$$

is satisfied.

Every locally compact σ-compact topological group G admits a left invariant locally finite Haar measure μ. For $p \in (1, \infty)$ denote by $L^p(G \times G, \mu \times \mu)$ the Banach space of all functions on $G \times G$ which are p-integrable with respect to the product measure $\mu \times \mu$. The group G acts continuously and isometrically on $L^p(G \times G, \mu \times \mu)$ by left translation via $(gf)(h, u) = f(gh, gu)$.

Let $C_b(G \times G)$ be the space of continuous bounded functions on $G \times G$ and let $x_0 \in X$ be an arbitrary fixed point. The following result is a modification of Theorem 2.3 of [15].

Theorem 4.1. *Let X be a proper* CAT(0)*-space and let $G < \mathrm{Iso}(X)$ be a non-elementary closed subgroup. For $B > 0$, for every $p \in (1, \infty)$ and for every triple $(a, b, \xi) \in T(B)$ such that (a, b) is the pair of fixed points of a rank-one element of G there is a continuous map $\alpha : \mathcal{A}(B) \to C_b(G \times G)$ with the following properties.*

(1) *$g \circ \alpha(g\xi, g\eta) = \alpha(\xi, \eta) = -\alpha(\eta, \xi)$ for all $(\xi, \eta) \in \mathcal{A}(B)$ and all $g \in G$.*

(2) *For every* $(\xi,\eta)\in\mathcal{A}(B)$ *and all neighborhoods* A_1 *of* ξ, A_2 *of* η *in* $X\cup\partial X$ *the intersection of the support of* $\alpha(\xi,\eta)$ *with the set* $\{(g,h)\in G\times G\mid gx_0\in X-(A_1\cup A_2)\}$ *is compact.*

(3) *For every* $p\in(1,\infty)$ *the assignment*

$$\omega:(\sigma,\eta,\beta)\in T(B)\to\omega(\sigma,\eta,\beta)=\alpha(\sigma,\eta)+\alpha(\eta,\beta)+\alpha(\beta,\sigma)$$

is an $L^p(G\times G,\mu\times\mu)$*-valued continuous bounded two-cocycle for the action of* G *on* $T(B)$*.*

(4) $\omega(a,b,\xi)\neq 0$.

(5) *If* G *does not act transitively on* $\Lambda\times\Lambda-\Delta$ *and if* $(a_i,b_i)\in\mathcal{A}(B)$ $(i=1,\dots,k)$ *are pairs of fixed points of rank-one elements of* G *such that the* G*-orbits of* $(a_i,b_i),(a,b)$ *are disjoint then we can choose* α *in such a way that the support of* ω *does not contain* (a_i,b_i,σ) *for* $i=1,\dots,k$ *and all* σ*.*

Proof. Let $G<\mathrm{Iso}(X)$ be a closed non-elementary subgroup which contains a B-rank-one element for some $B>0$. We divide the proof of the theorem into five steps.

Step 1:

Let $x_0\in X$ be an arbitrary point and denote by G_{x_0} the stabilizer of x_0 in G. Then G_{x_0} is a compact subgroup of G, and the quotient space G/G_{x_0} is G-equivariantly homeomorphic to the orbit $Gx_0\subset X$ of x_0. Note that Gx_0 is a closed subset of X and hence it is locally compact. The group G acts on the locally compact space $\mathcal{A}(B)\times Gx_0$ as a group of homeomorphisms.

The $\mathrm{Iso}(X)$-invariant metric $\tilde d$ on $\mathcal{A}(B)\times X$ constructed in Lemma 3.7 induces a G-invariant metric on $\mathcal{A}(B)\times G/G_{x_0}$ which defines the product topology. Hence we obtain a G-invariant symmetrized product metric $\hat d$ on

$$V=\mathcal{A}(B)\times G/G_{x_0}\times G/G_{x_0}$$

by defining

$$\hat d((\xi,x,y),(\xi',x',y'))=\frac{1}{2}\big(\tilde d((\xi,x),(\xi',x'))+\tilde d((\xi,y),(\xi',y'))\big). \tag{11}$$

The topology defined on V by this metric is the product topology, in particular it is locally compact.

Since V is a locally compact G-space, the quotient space $W=G\backslash V$ admits a natural metric d_0 as follows. Let

$$P:V\to W$$

be the canonical projection and define

$$d_0(x,y)=\inf\{\hat d(\tilde x,\tilde y)\mid P\tilde x=x,P\tilde y=y\}. \tag{12}$$

The topology induced by this metric is the quotient topology for the projection P. In particular, W is a locally compact metric space. A set $U\subset W$ is open if and only if $P^{-1}(U)\subset V$ is open. In other words, open subsets of W correspond precisely to G-invariant open subsets of V. The projection P is open and distance non-increasing.

The distance $\tilde{d}$ on $\mathcal{A}(B) \times X$ is invariant under the involution $\iota : (\xi, \eta, x) \to (\eta, \xi, x)$ exchanging the two components of a point in $\mathcal{A}(B)$ and hence the same is true for the distance $\hat{d}$ on V. Since the action of G commutes with the isometric involution ι, the map ι descends to an isometric involution of the metric space (W, d_0) which we denote again by ι.

An open subset U of W is said to have *property* (R_1, R_2) for some $R_1, R_2 > 0$ if for every $((\xi, \eta), gx_0, hx_0) \in P^{-1}(U) \subset V$ the distance in X between gx_0, hx_0 and any geodesic in X connecting ξ to η is at most R_1 and if moreover $d(gx_0, hx_0) \le R_2$.

We claim that for every $w \in W$ there are numbers $R_1, R_2 > 0$ and there is a neighborhood of w in W which has property (R_1, R_2). Namely, let $v = ((\xi, \eta), gx_0, hx_0) \in P^{-1}(w)$. Then ξ can be connected to η by a B-contracting geodesic γ and therefore any geodesic connecting ξ to η is contained in the B-tubular neighborhood of γ. By the discussion in Section 3 (see the proof of Lemma 3.1), there is a neighborhood A of (ξ, η) in $\mathcal{A}(B)$ such that for all $(\xi', \eta') \in A$, any geodesic connecting ξ' to η' passes through a fixed compact neighborhood of gx_0. Thus by continuity, there are numbers $R_1 > 0, R_2 > 0$ and there is an open neighborhood U' of v in V such that for every $((\xi', \eta'), g'x_0, h'x_0) \in U'$ the distance between $g'x_0, h'x_0$ and any geodesic connecting ξ' to η' is at most R_1 and that moreover $d(g'x_0, h'x_0) \le R_2$. However, distances and geodesics are preserved under isometries and hence every point in $\tilde{U} = \cup_{g \in G} gU'$ has this property. Since $\tilde{U}$ is open, G-invariant and contains v, the set $\tilde{U}$ projects to an open neighborhood of w in W. This neighborhood has property (R_1, R_2).

Step 2:

In equation (12) in Step 1 above, we defined a distance d_0 on the space $W = G\backslash V$. With respect to this distance, the involution ι acts non-trivially and isometrically. Choose a small closed metric ball D in W which is disjoint from its image under ι. In Step 5 below we will construct explicitly such balls D, however for the moment, we simply assume that such a ball exists. By Step 1 above, we may assume that D has property (R_1, R_2) for some $R_1, R_2 > 0$.

Let $\mathcal{H}$ be the vector space of all Hölder continuous functions $f : W \to \mathbb{R}$ supported in D. An example of such a function can be obtained as follows.

Let z be an interior point of D and let $r > 0$ be sufficiently small that the closed metric ball $B(z, r)$ of radius r about z is contained in D. Choose a smooth function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi(t) = 1$ for $t \in (-\infty, r/2]$ and $\chi(t) = 0$ for $t \in [r, -\infty)$ and define $f(y) = \chi(d_0(z, y))$. Since the function $y \to d_0(z, y)$ on W is one-Lipschitz and χ is smooth, the function f on W is Lipschitz, does not vanish at z and is supported in D.

Since D is disjoint from $\iota(D)$ by assumption and since ι is an isometry, every function $f \in \mathcal{H}$ admits a natural extension to a Hölder continuous function f_0 on W supported in $D \cup \iota(D)$ whose restriction to D coincides with the restriction of f and which satisfies $f_0(\iota z) = -f_0(z)$ for all $z \in W$. The function $\hat{f} = f_0 \circ P : V \to \mathbb{R}$ is invariant under the action of G, and it is *anti-invariant* under the involution ι

of V, i.e. it satisfies $\hat{f}(\iota(v)) = -\hat{f}(v)$ for all $v \in V$ (here as before, $P : V \to W$ denotes the canonical projection).

Equip $\tilde{V} = \mathcal{A}(B) \times G \times G$ with the product topology. The group G acts on $G \times G$ by left translation, and it acts diagonally on $\tilde{V}$. There is a natural continuous projection $\Pi : \tilde{V} \to V$ which is equivariant with respect to the action of G and with respect to the action of the involution ι on $\tilde{V}$ and V. The function $\hat{f}$ on V lifts to a G-invariant ι-anti-invariant continuous function $\tilde{f} = \hat{f} \circ \Pi$ on $\tilde{V}$.

For $(\xi, \eta) \in \mathcal{A}(B)$ write

$$F(\xi, \eta) = \{(\xi, \eta, z) \mid z \in G \times G\}.$$

The sets $F(\xi, \eta)$ define a G-invariant foliation $\mathcal{F}$ of $\tilde{V}$. The leaf $F(\xi, \eta)$ of $\mathcal{F}$ can naturally be identified with $G \times G$. For all $(\xi, \eta) \in \mathcal{A}(B)$ and every function $f \in \mathcal{H}$ we denote by $f_{\xi,\eta}$ the restriction of the function $\tilde{f}$ to $F(\xi, \eta)$, viewed as a continuous bounded function on $G \times G$. For every $f \in \mathcal{H}$, all $(\xi, \eta) \in \mathcal{A}(B)$ and all $g \in G$ we then have $f_{g\xi,g\eta} \circ g = f_{\xi,\eta} = -f_{\eta,\xi}$. Since the functions $f_{\xi,\eta}$ are restrictions to the leaves of the foliation $\mathcal{F}$ of a globally continuous bounded function on $\tilde{V}$, the assignment

$$(\xi, \eta) \in \mathcal{A}(B) \to \alpha(\xi, \eta) = f_{\xi,\eta} \in C_b(G \times G)$$

is a continuous map of $\mathcal{A}(B)$ into $C_b(G \times G)$. By construction, it satisfies

$$g \circ \alpha(g\xi, g\eta) = \alpha(\xi, \eta) = -\alpha(\eta, \xi) \, \forall (\xi, \eta) \in \mathcal{A}(B), \forall g \in G.$$

Thus the map α fulfills the first requirement in the statement of the theorem. The second requirement is also satisfied since the set D is assumed to have property (R_1, R_2) and since moreover for every geodesic $\zeta : \mathbb{R} \to X$, for every open neighborhood A of $\zeta(\infty) \cup \zeta(-\infty)$ in $X \cup \partial X$ and for every $R > 0$ the intersection of the closed R-neighborhood of ζ with $X - A$ is compact.

Step 3:

For $f \in \mathcal{H}$ and for an ordered triple $(\xi, \eta, \beta) \in T(B)$ define

$$\omega(\xi, \eta, \beta) = f_{\xi,\eta} + f_{\eta,\beta} + f_{\beta,\xi} \in C_b(G \times G). \tag{13}$$

Since $f_{\xi,\eta} = -f_{\eta,\xi}$ for all $(\xi, \eta) \in \mathcal{A}(B)$, we have

$$\omega \circ \sigma = (\mathrm{sgn}(\sigma))\omega$$

for every permutation σ of the three variables. As a consequence, the cocycle condition for ω is also satisfied. The assignment $(\xi, \eta, \beta) \in T(B) \to \omega(\xi, \eta, \beta) \in C_b(G \times G)$ is continuous with respect to the compact open topology on $C_b(G \times G)$. Moreover, it is equivariant with respect to the natural action of G on the space $T(B)$ and on $C_b(G \times G)$. This means that ω is a continuous bounded cocycle for the action of G on $T(B)$ with values in $C_b(G \times G)$.

For the proof of the theorem, we have to show that $\omega(\xi, \eta, \beta) \in L^p(G \times G, \mu \times \mu)$ for every $p \in (1, \infty)$, with L^p-norm bounded from above by a constant which does not depend on (ξ, η, β). For this let $(\xi, \eta, \beta) \in T(B)$ and let $\gamma : \mathbb{R} \to X$ be a B-contracting geodesic connecting ξ to η. By Lemma 2.5 there is a point $y_0 \in X$ which is contained in the $\kappa_0 = \kappa_0(B) = 9B + 6$-neighborhood of every side of a geodesic triangle with vertices ξ, η, β and side γ. Via reparametrization of γ

we may assume that $d(\gamma(0), y_0) \leq \kappa_0$. Lemma 2.3, applied to the geodesic ray $\gamma[0, \infty)$ and a geodesic ζ connecting β to η shows that there is a subray of ζ whose Hausdorff distance to $\gamma[2\kappa_0, \infty)$ is bounded from above by $3B + 1$. Then by the definition of the distances δ_x on $\mathcal{A}(B)$ and by Lemma 3.4 and Corollary 3.6, there is a number $r_0 > 0$ depending on κ_0, R_1, R_2 such that if $t \geq 0$ and if $y \in X$ satisfies $d(\gamma(t), y) < \kappa_0 + R_1 + R_2$ then

$$\delta_y((\xi, \eta), (\beta, \eta)) \leq r_0 e^{-\chi t} \tag{14}$$

where $\chi > 0$ is as in Corollary 3.6. Moreover, by the definition (11) of the distance function $\hat{d}$ on V and by the estimate in Lemma 3.7 for the distance function $\tilde{d}$ on $\mathcal{A}(B) \times X$, we have

$$\begin{aligned} &\hat{d}\big((\xi, \eta, ux_0, hx_0), (\beta, \eta, ux_0, hx_0)\big) \\ &\leq \frac{1}{2}(\delta_{ux_0}((\xi, \eta), (\beta, \eta)) + \delta_{hx_0}((\xi, \eta), (\beta, \eta))) \leq r_0 e^{-\chi t} \end{aligned} \tag{15}$$

whenever $d(ux_0, \gamma(t)) \leq \kappa_0 + R_1$ and $((\xi, \eta), ux_0, hx_0)$ is contained in the support of $f_{\xi,\eta}$ or of $f_{\eta,\beta}$.

The function $\hat{f} = f_0 \circ P : V \to \mathbb{R}$ constructed in Step 2 from the function $f \in \mathcal{H}$ is Hölder continuous and ι-anti-invariant. Therefore by the estimate (15) there are numbers $\sigma > 0, r_1 > r_0$ only depending on the Hölder norm for f with the following property. Let $0 \leq t$ and let $u, h \in G$ be such that $d(ux_0, \gamma(t)) < \kappa_0 + R_1, d(hx_0, \gamma(t)) < \kappa_0 + R_1 + R_2$; then

$$|\hat{f}(\xi, \eta, ux_0, hx_0) + \hat{f}(\eta, \beta, ux_0, hx_0)| \leq r_1 e^{-\sigma\chi t}. \tag{16}$$

The function f is bounded in absolute value by a universal constant. Hence from the definition of the functions $f_{\xi,\eta}$ and $f_{\eta,\beta}$ and from the estimate (16) we obtain the existence of a constant $r > r_1$ (depending on the Hölder norm of f) such that

$$|(f_{\xi,\eta} + f_{\eta,\beta})(u, h)| \leq r e^{-\sigma\chi t} \tag{17}$$

whenever $d(ux_0, \gamma(t)) \leq \kappa_0 + R_1$.

Step 4:

Let $\nu = \mu \times \mu$ be the left invariant product measure on $G \times G$. Our goal is to show that for every $p \in (1, \infty)$, the cocycle ω defined in equation (13) above is in fact a bounded cocycle with values in $L^p(G \times G, \nu)$. For this we show that for every $(\xi, \eta, \beta) \in T(B)$ and for every $p > 1$ the L^p-norm of the function $\omega(\xi, \eta, \beta)$ with respect to the measure ν on $G \times G$ is uniformly bounded and that moreover the assignment $(\xi, \eta, \beta) \to \omega(\xi, \eta, \beta) \in L^p(G \times G, \nu)$ is continuous.

For a subset C of X write

$$C_{G,R_2} = \{(u, h) \in G \times G \mid ux_0 \in C, d(ux_0, hx_0) \leq R_2\}.$$

We claim that there is a number $m > 0$ such that for every subset C of X of diameter at most $2R_1 + 4\kappa_0 + 1$ the ν-mass of the set $N(C)_{G,R_2}$ is at most m. Namely, the set

$$D = \{(u, h) \in G \times G \mid d(ux_0, x_0) \leq 2R_1 + 4\kappa_0 + 1, d(ux_0, hx_0) \leq R_2\}$$

of $G \times G$ is compact and hence its ν-mass is finite, say this mass equals $m > 0$. On the other hand, if $C \subset X$ is a set of diameter at most $2R_1 + 4\kappa_0 + 1$ and if there is some $g\in G$ such that $gx_0\in C$ then any pair $(u,h)\in N(C)_{G,R_2}$ is contained in gD. Our claim now follows from the fact that ν is invariant under left translation.

As in Step 3 above, let $(\xi,\eta,\beta)\in T(B)$ and let T be an ideal geodesic triangle with B-contracting sides and vertices ξ,η,β. Let $y_0\in X$ be a point which is contained in the κ_0-neighborhood of every side of T. Let $\gamma:\mathbb{R}\to X$ be the side of T connecting ξ to η, parametrized in such a way that $d(y_0,\gamma(0))\leq\kappa_0$. Also, let $\rho:\mathbb{R}\to X$ be the side of T connecting β to η which is parametrized in such a way that $d(y_0,\rho(0))\leq\kappa_0$. Then $\gamma[0,\infty),\rho[0,\infty)$ are two sides of a geodesic triangle in X with vertices $\gamma(0),\rho(0),\eta$. Since $d(\gamma(0),\rho(0))\leq 2\kappa_0$, by convexity of the distance function we have $d(\gamma(t),\rho(t))\leq 2\kappa_0$ for all $t\geq 0$. In particular, the R_1-neighborhood of $\rho[0,\infty)$ is contained in the $R_1+2\kappa_0$-neighborhood of $\gamma[0,\infty)$.

By construction, if $N(C,r)$ denotes the r-neighborhood of a set C then the support of the function $f_{\xi,\eta}$ is contained in $N(\gamma(\mathbb{R}),R_1)_{G,R_2}$ and similarly for the functions $f_{\eta,\beta},f_{\beta,\xi}$. As a consequence, the support of the function ω defined in (13) above is the disjoint union of the three sets

$$N(\gamma[0,\infty),R_1+2\kappa_0)_{G,R_2},N(\gamma(-\infty,-0],R_1+2\kappa_0)_{G,R_2},\\ N(\rho(-\infty,-0],R_1+2\kappa_0)_{G,R_2}. \tag{18}$$

Moreover, there is a number $\tau>0$ only depending on R_1,R_2 and κ_0 such that the restriction of ω to $N(\gamma[\tau,\infty),R_1+2\kappa_0)_{G,R_2}$ coincides with the restriction of the function $f_{\xi,\eta}+f_{\eta,\beta}$ and similarly for the other two sets in the above decomposition of the support of ω. Since ω is uniformly bounded, to show that ω is contained in $L^p(G\times G,\nu)$ it is now enough to show that there is constant $c_p>0$ only depending on p and the Hölder norm of f such that

$$\int_{N(\gamma[\tau,\infty),R_1+2\kappa_0)_{G,R_2}}|f_{\xi,\eta}+f_{\eta,\beta}|^p d\nu<c_p.$$

However, this is immediate from the estimate (17) together with the control on the ν-mass of subsets of $N(\gamma[\tau,\infty),R_1+2\kappa_0)_{G,R_2}$. Namely, we showed that for every integer $k\geq 0$ the ν-mass of the set $N(\gamma[\tau+k,\tau+k+1],R_1+2\kappa_0)_{G,R_2}$ is bounded from above by a universal constant $m>0$. Moreover, for every $p\geq 1$ the value of the function $|f_{\xi,\eta}+f_{\beta,\eta}|^p$ on this set does not exceed $r^pe^{-p\sigma\chi(\tau+k)}$. Thus the inequality holds true with $c_p=r^p\sum_{k=0}^\infty e^{-p\sigma\chi(\sigma+k)}$.

Since the function $\tilde f$ on $\tilde V$ is globally continuous, the same consideration also shows that $\omega(\xi,\eta,\beta)\in L^p(G\times G,\nu)$ depends continuously on $(\xi,\eta,\beta)\in T(B)$. Namely, let $(\xi_i,\zeta_i,\eta_i)\subset T(B)$ be a sequence of triples of pairwise distinct points converging to a triple $(\xi,\eta,\beta)\in T(B)$. By the above consideration, for every $\epsilon>0$ there is a compact subset A of $G\times G$ such that $\int_{G\times G-A}|\omega(\xi_i,\eta_i,\beta_i)|^pd\nu\leq\epsilon$ for all sufficiently large $i>0$ and that the same holds true for $\omega(\xi,\eta,\zeta)$. Let χ_A be the characteristic function of A. By continuity of the function $\tilde f$ on $\tilde V$ and compactness, the functions $\chi_A\omega(\xi_i,\eta_i,\beta_i)$ converge as $i\to\infty$ in $L^p(G\times G,\nu)$ to $\chi_A\omega(\xi,\eta,\zeta)$. Since $\epsilon>0$ was arbitrary, the required continuity follows.

By construction, the assignment $(\xi,\eta,\beta)\to\omega(\xi,\eta,\beta)$ is equivariant under the action of G on the space $T(B)$ and on $L^p(G\times G,\nu)$ and satisfies the cocycle equality

(10). In other words, ω defines a continuous $L^p(G \times G, \nu)$-valued bounded cocycle for the action of G on $T(B)$ as required.

Step 5:

Let $g \in G$ be a B-rank one isometry, let $a \neq b \in \partial X$ be the attracting and repelling fixed point for the action of g on ∂X, respectively, and let $\xi \in \Lambda - \{a, b\}$ be such that $(a, b, \xi) \in T(B)$. We have to show that we can find a cocycle ω as in (13) above with $\omega(a, b, \xi) \neq 0$.

For this let γ be a B-contracting oriented axis for g and let $x_0 \in \pi_{\gamma(\mathbb{R})}(\xi)$ be the basepoint for the above construction. The orbit of x_0 under the infinite cyclic subgroup of G generated by g is contained in the geodesic γ. Since $g^j x_0 \to a, g^{-j} x_0 \to b$ $(j \to \infty)$, there are numbers $k < \ell$, $R_1 > 2\kappa_0$ such that the $R_1 + \kappa_0$-neighborhood of the geodesic connecting a to ξ and the $R_1 + 2\kappa_0$-neighborhood of the geodesic connecting b to ξ contains at most one of the points $g^k x_0, g^\ell x_0$ and that moreover the distance between $g^k x_0, g^\ell x_0$ is at least $4\kappa_0$. Choose $R_2 > 2d(g^k x_0, g^\ell x_0)$.

We claim that there is no $h \in G$ with $hg^k x_0 = g^k x_0, hg^\ell x_0 = g^\ell x_0$ and $h(a) = b, h(b) = a$. Namely, any isometry h which exchanges a and b and fixes a point on the axis γ of g, say the point $\gamma(t)$, maps the geodesic ray $\gamma[t, \infty)$ to the geodesic ray $\gamma(-\infty, t]$. Thus a fixed point of h on γ is unique. In particular, the projection of $(a, b, g^k x_0, g^\ell x_0)$ into the space W is not fixed by the involution ι.

The discussion at the end of Step 1 above shows that we can find a function $f \in \mathcal{H}$ supported in a ball $D \subset W$ about the projection of $(a, b, g^k x_0, g^\ell x_0)$ with property (R_1, R_2) whose lift $\tilde{f}$ to $\tilde{V}$ does not vanish at (a, b, g^k, g^ℓ). By the choice of R_1, this means that $f_{a,b}(g^k, g^\ell) \neq 0$ and $f_{b,\xi}(g^k, g^\ell) = f_{\xi,a}(g^k, g^\ell) = 0$. In other words, the cocycle ω constructed as above from f does not vanish at (a, b, ξ).

By Lemma 2.9 the G-orbit of a pair of fixed points of a rank-one element of G is closed in $\Lambda \times \Lambda - \Delta$. If the G-orbits of $(a_i, b_i), (a, b)$ $(i = 1, \dots, k)$ are mutually disjoint where $(a_i, b_i) \in \mathcal{A}(B)$ are pairs of fixed points of such rank-one elements then we can choose the function f in such a way that the support of its lift to V does not intersect the leaves of the foliation $\mathcal{F}$ determined by the G-orbits of (a_i, b_i). Thus we can find some cocycle ω which satisfies the fifth property of the theorem as well. This completes the proof of the theorem. □

5. Second continuous bounded cohomology

Let X be a proper CAT(0)-space with isometry group $\mathrm{Iso}(X)$. In this section we use Theorem 4.1 to construct nontrivial second bounded cohomology classes for closed non-elementary subgroups G of $\mathrm{Iso}(X)$ with limit set Λ which contain a rank-one element and act transitively on the complement of the diagonal in $\Lambda \times \Lambda$.

We use the arguments from Section 3 of [15] (see also [19, 21] for earlier results along the same line). Namely, let $\mathcal{P}(\Lambda)$ be the space of all probability measures on Λ, equipped with the weak*-topology. Denote moreover by $\mathcal{P}_{\geq 3}(\Lambda) \subset \mathcal{P}(\Lambda)$ the set of all probability measures which are not concentrated on at most two points. We first show.

Lemma 5.1. *Let $G < \mathrm{Iso}(X)$ be a non-elementary closed subgroup with limit set Λ. If G contains a rank-one element and acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then the action of G on $\mathcal{P}_{\geq 3}(\Lambda)$ is tame with compact point stabilizers.*

Proof. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group with limit set Λ which acts transitively on the complement of the diagonal Δ in $\Lambda \times \Lambda$ and contains a rank one element $g \in G$. Then there is a number $B > 0$ and for every pair $(\xi, \eta) \in \Lambda \times \Lambda - \Delta$ there is a B-contracting geodesic $\gamma : \mathbb{R} \to X$. This geodesic is the image of an axis of g under an element of G.

Let $T \subset \Lambda^3$ be the space of triples of pairwise distinct points in Λ. If $\gamma : \mathbb{R} \to X$ is a B-contracting geodesic then every other geodesic connecting $\gamma(-\infty)$ to $\gamma(\infty)$ is contained in the B-tubular neighborhood of γ. Thus by Lemma 2.5, for every triple $(a, b, c) \in T$ there is a point $x_0 \in X$ whose distance to any of the sides of any geodesic triangle in X with vertices a, b, c is at most $10B + 6$. The set $K(a, b, c)$ of all points with this property is clearly closed. Lemma 2.5 shows that it is moreover of uniformly bounded diameter. In other words, $K(a, b, c)$ is compact and hence it has a unique center $\Phi(a, b, c) \in X$ where a center of a compact set $K \subset X$ is a point $x \in X$ such that the radius of the smallest closed ball about x containing K is minimal (see p.10 in [4]).

This construction defines a map $\Phi : T \to X$ which is equivariant with respect to the action of G. Moreover, it is continuous. Namely, if $(a_i^1, a_i^2, a_i^3) \to (a^1, a^2, a^3)$ in T then by the discussion in the proof of Lemma 3.1 there is a compact neighborhood A of $K(a^1, a^2, a^3)$ such that for all sufficiently large i, every geodesic connecting a pair of points (a_i^j, a_i^{j+1}) passes through A. Since X is proper by assumption, up to passing to a subsequence we may assume that the compact sets $K(a_i^1, a_i^2, a_i^3)$ of uniformly bounded diameter converge in the Hausdorff topology for compact subsets of X to a compact set K. On the other hand, up to passing to a subsequence and reparametrization, a sequence of geodesics γ_i^j connecting a_i^j to a_i^{j+1} converge as $i \to \infty$ locally uniformly to a geodesic γ^j connecting a^j to a^{j+1}. This implies that $K \subset K(a^1, a^2, a^3)$.

However, by invariance under the action of G, every geodesic connecting a^j to a^{j+1} is a limit as $i \to \infty$ of a sequence of geodesics connecting a_i^j to a_i^{j+1} and therefore $K = K(a, b, c)$. Since the map which associates to a compact subset of X its center is continuous with respect to the Hausdorff topology on compact subsets of X, we conclude that the map Φ is in fact continuous. Since the action of G on X is proper, Lemma 3.4 of [1] then shows that the action of G on T is proper as well.

Now the group $G < \mathrm{Iso}(X)$ is closed and hence the stabilizer in G of a point in X is compact. By equivariance, the point stabilizer in G of some $\mu \in \mathcal{P}_{\geq 3}(\Lambda)$ is compact as well. This completes the proof of the lemma. $\square$

The next easy consequence of a result of Adams and Ballmann [2] will be important for the proof of Theorem 1. For later reference, recall that the closure of a normal subgroup of a topological group G is normal, and the closure of an amenable subgroup of G is amenable (Lemma 4.1.13 of [23]).

Lemma 5.2. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup which contains a rank-one element. Then a closed normal amenable subgroup N of G is compact, and N fixes the limit set of G pointwise.*

Proof. Let $G < \mathrm{Iso}(X)$ be a closed non-elementary group which contains a rank-one element and let $N \lhd G$ be a closed normal amenable subgroup. Since N is amenable, either N fixes a point $\xi \in \partial X$ or N fixes a flat $F \subset X$ [2].

Assume first that N fixes a point $\xi \in \partial X$. Since N is normal in G, for every $g \in G$ the point $g\xi$ is a fixed point for $gNg^{-1} = N$. On the other hand, by Lemma 2.8, the closure in ∂X of every orbit for the action of G contains the limit set Λ of G, and the action of G on Λ is minimal. Therefore by continuity, N fixes Λ pointwise. Then Lemma 5.1 shows that N is compact and hence has a fixed point $x \in X$ by convexity.

If N fixes a flat $F \subset X$ then we argue in the same way. Namely, the image of F under an isometry of X is a flat. Let $a \neq b \in \Lambda$ be the attracting and repelling fixed points, respectively, of a rank one element g of G. Then a, b are visibility points in ∂X and hence there is no flat in X whose boundary in ∂X contains one of the points a, b. Thus the boundary $\partial F \subset \partial X$ of F is contained in $\partial X - \{b\}$ and consequently $g^k\partial F \to \{a\}$ $(k \to \infty)$. But by the argument in the previous paragraph, N fixes $g^k\partial F = \partial(g^kF)$ and therefore N fixes a by continuity. In other words, N fixes a point in ∂X. The first part of this proof then shows that indeed N is compact. This shows the lemma. □

Remark: Caprace and Monod (Theorem 1.6 of [12], see also [11]) found geometric conditions which guarantee that an amenable normal subgroup of a non-elementary group G of isometries of X is trivial. This however need not be true under the above more general assumptions. A simple example is a space of the form $X = \mathbf{H}^2 \times X_2$ where $\mathbf{H}^2$ is the hyperbolic plane and where X_2 is a compact CAT(0)-space whose isometry group H is non-trivial group. Then any axial isometry of $\mathbf{H}^2$ acts as a rank-one isometry on X. The compact group H is a normal subgroup of the isometry group of X.

As in [15] we use Lemma 5.1 and Lemma 5.2 to show.

Proposition 5.3. *Let $G < \mathrm{Iso}(X)$ be a non-elementary closed subgroup with limit set Λ. If G contains a rank one element and acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then $H^2_{cb}(G, L^p(G, \mu)) \neq 0$ for every $p \in (1, \infty)$.*

Proof. A *strong boundary* for a locally compact topological group G is a standard Borel space (B, ν) with a probability measure ν and a measure class preserving amenable action of G which is *doubly ergodic* (we refer to [20] for a detailed explanation of the significance of a strong boundary). A strong boundary exists for every locally compact topological group G [17].

Let $G < \mathrm{Iso}(X)$ be a non-elementary closed subgroup with limit set Λ. Assume that G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$. Since the action of G on its strong boundary (B, ν) is amenable, there is a G-equivariant measurable Furstenberg map $\varphi : (B, \nu) \to \mathcal{P}(\Lambda)$ [23]. By ergodicity of the action of G on (B, ν), either the set of all $x \in B$ with $\varphi(x) \in \mathcal{P}_{\geq 3}(\Lambda)$ has full mass or vanishing mass.

Assume that this set has full mass. By Lemma 5.1, the action of G on $\mathcal{P}_{\geq 3}(\Lambda)$ is tame, with compact point stabilizers. Thus φ induces a G-invariant map $(B, \nu) \to \mathcal{P}_{\geq 3}(\Lambda)/G$ which is almost everywhere constant by ergodicity. Therefore by changing the map φ on a set of measure zero, we can assume that φ is an equivariant map $(B, \nu) \to G/G_\mu$ where G_μ is the stabilizer of a point in $\mathcal{P}_{\geq 3}(\Lambda)$ and hence it is compact. Since the action of G on (B, ν) is amenable, the group G is amenable (see p.108 of [23]). By assumption, G does not fix globally a point in ∂X. Then the group G fixes a flat F in X [2]. The boundary ∂F of F is closed and G-invariant and hence by Lemma 2.8 it contains the limit set Λ of G. But this means that there is a rank-one element in G with an axis contained in F which is impossible.

As a consequence, the image under φ of ν-almost every $x \in B$ is a measure supported on at most two points. By Lemma 5.1, the action of G on the space of triples of pairwise distinct points is proper and hence the assumptions in Lemma 23 of [19] are satisfied. We can then use Lemma 23 of [19] as in the proof of Lemma 3.4 of [21] to conclude that the image under φ of almost every $x \in B$ is supported in a single point. In other words, φ is a G-equivariant Borel map of (B, ν) into Λ. Note that since the action of G on Λ is minimal, by equivariance the support of the measure class $\varphi_*(\nu)$ is all of Λ.

Let μ be a Haar measure of G. By invariance under the action of G, there is some $B > 0$ such that $(a, b) \in \mathcal{A}(B)$ for all $(a, b) \in \Lambda \times \Lambda - \Delta$. Thus by Theorem 4.1, for every $p \in (1, \infty)$ there is a nontrivial bounded continuous $L^p(G \times G, \mu \times \mu)$-valued cocycle ω on the space of triples of pairwise distinct points in Λ. Then the $L^p(G \times G, \mu \times \mu)$-valued $\nu \times \nu \times \nu$-measurable bounded cocycle $\omega \circ \varphi^3$ on $B \times B \times B$ is non-trivial on a set of positive measure. Since B is a strong boundary for G, this cocycle then defines a *non-trivial* class in $H^2_{cb}(G, L^p(G \times G, \mu \times \mu))$ (see [20]). On the other hand, the isometric G-representation space $L^p(G \times G, \mu \times \mu)$ is a direct integral of copies of the isometric G-representation space $L^p(G, \mu)$ and therefore by Corollary 2.7 of [21] and Corollary 3.4 of [22], if $H^2_{cb}(G, L^p(G, \mu)) = \{0\}$ then also $H^2_{cb}(G, L^p(G \times G, \mu \times \mu)) = \{0\}$. This shows the proposition. □

6. Second bounded cohomology

In this section we investigate non-elementary closed subgroups of $\mathrm{Iso}(X)$ with limit set Λ which contain a rank-one element and which do not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$. Such a group G is a locally compact σ-compact group which admits a Haar measure μ. Our goal is to show that for every $p \in (1, \infty)$ the second bounded cohomology group $H^2_b(G, L^p(G, \mu))$ is infinite dimensional.

Unlike in Section 5, for this we can not use Theorem 4.1 directly since the cocycle constructed in this theorem may not be defined on the entire space of triples of pairwise distinct points in Λ. Instead we use the strategy from the proof of Theorem 4.1 to construct explicitly for every $p \in (1,\infty)$ bounded cocycles for G with values in $L^p(G,\mu)$ which define an infinite dimensional subspace of $H^2_b(G, L^p(G,\mu))$. Unfortunately, our construction does not yield continuous bounded cocycles, so we only obtain information on the group $H^2_b(G, L^p(G,\mu))$. (The construction of Bestvina and Fujiwara [6] does not yield continuous bounded cocycles with real coefficients either. However, these cocycles are integer-valued, and by an observation of Caprace and Fujiwara [11], such cocycles define in fact continuous bounded cohomology classes.)

A *twisted $L^p(G,\mu)$-valued quasi-morphism* for a closed subgroup G of $\mathrm{Iso}(X)$ is a map $\psi : G \to L^p(G,\mu)$ such that

$$\sup_{g,h} \|\psi(g) + g\psi(h) - \psi(gh)\|_p < \infty$$

where $\| \|_p$ is the L^p-norm for functions on G.

Every unbounded twisted $L^p(G,\mu)$-valued quasi-morphism for G defines a second bounded cohomology class in $H^2_b(G, L^p(G,\mu))$ which vanishes if and only if there is a cocycle $\rho : G \to L^p(G,\mu)$ (i.e. ρ satisfies the cocycle equation $\rho(g)+g\rho(h)-\rho(gh) = 0$) such that $\psi - \rho$ is bounded (compare the discussion in [14]). We use twisted quasimorphisms to complete the proof of Theorem 2 from the introduction.

Proposition 6.1. *Let $G < \mathrm{Iso}(X)$ be a closed non-elementary subgroup with limit set Λ which contains a rank-one element. If G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then for every $p \in (1,\infty)$ the second bounded cohomology group $H^2_b(G, L^p(G,\mu))$ is infinite dimensional.*

Proof. Let $G < \mathrm{Iso}(X)$ be a closed subgroup with limit set $\Lambda \subset \partial X$ which contains a rank-one element and which does not act transitively on the complement of the diagonal Δ in $\Lambda \times \Lambda$.

By Lemma 2.11 there are infinitely many rank-one elements $g \in G$ with attracting and repelling fixed points $a, b \in \Lambda$ and the additional property that there is no $u \in G$ with $u(a,b) = (b,a)$. Thus let $g \in G$ be such a rank-one element with fixed points $a \neq b \in \Lambda$ and let $B_0 > 0$ be such that every geodesic in X connecting b to a is B_0-contracting. Such a number exists by Lemma 2.4. By Lemma 2.10, we can find some $h \in G$ such that $hb \neq b$ and that the stabilizer $\mathrm{Stab}(b, hb)$ in G of the pair of points (b, hb) is compact.

By the consideration in the proof of Lemma 2.10, there is a number $B > B_0$ depending on B_0 and h such that every geodesic connecting b to hb is B-contracting. In particular, the set of all points in X which are contained in a geodesic connecting b to hb is isometric to $K \times \mathbb{R}$ where K is compact. Since $\mathrm{Stab}(b, hb)$ is compact, there is a geodesic γ connecting b to hb which is fixed pointwise by $\mathrm{Stab}(b, hb)$. Let γ^{-1} be the geodesic obtained by reversing the orientation of γ (note that for all of our constructions, only the orientation of a geodesic but not an explicit parametrization plays any role). By invariance under isometries, for every $k \in \mathbb{Z}$, every geodesic connecting b to $g^k hb$ is B-contracting.

For this number $B > 0$ let $C = C(B) > 0$ be as in Lemma 2.4. The G-orbit of b consists of visibility points. Thus for $u \neq v \in G$ with $ub \neq vb$ there is a (parametrized) geodesic ξ connecting ub to vb. The geodesic $vh^{-1}\gamma$ connects $vh^{-1}b$ to vb and hence by Lemma 2.4, the geodesic ξ passes through the $9B+6$-neighborhood of every point in $\pi_{vh^{-1}\gamma(\mathbb{R})}(ub)$ (this also holds true if $ub = vh^{-1}b$).

Let $b(v,\xi) \in \mathbb{R} \cup \{-\infty\}$ be the infimum of all numbers $t \in \mathbb{R}$ such that $\xi(t)$ is contained in the $9B+6$-neighborhood of a point in $\pi_{vh^{-1}\gamma(\mathbb{R})}(ub)$; then the geodesic ray $\xi(b(v,\xi),\infty)$ is C-contracting. This ray only depends on the geodesic line ξ and on the coset $[v]$ of v in $G/\mathrm{Stab}(b,hb)$. If $ub = b$ and if $v = g^k h$ then we have $b(v,\xi) = -\infty$. Define similarly a number $a(u,\xi) \in \mathbb{R} \cup \{\infty\}$ using the above procedure for the inverse of the geodesic ξ and the geodesic $uh^{-1}\gamma$. The resulting ray $\xi(-\infty, a(u,\xi))$ depends on the geodesic ξ and on $[u] \in G/\mathrm{Stab}(b,hb)$.

Define

$$b(\xi) = \inf\{b(\tilde v,\xi) \mid \tilde v \in G, \tilde v(b) = v(b)\}$$
$$a(\xi) = \sup\{a(\tilde u,\xi) \mid \tilde u \in G, \tilde u(b) = u(b)\}.$$

Then the rays $\xi(-\infty, a(\xi)), \xi(b(\xi),\infty)$ are C-contracting.

Let $\mathcal{G}(C)$ be the set of all (oriented) geodesics $\eta : \mathbb{R} \to X$ such that there is an open connected relatively compact (perhaps empty) set $(a(\eta), b(\eta)) \subset \mathbb{R}$ with the property that $\eta(\mathbb{R} - (a(\eta), b(\eta)))$ is C-contracting. For $\eta \in \mathcal{G}(C)$ we consider the subarc $\eta(a(\eta), b(\eta))$ to be part of the structure of η. Thus the same geodesic with two distinct subarcs removed defines two distinct points in $\mathcal{G}(C)$. Sometimes we write $(\eta, (a(\eta), b(\eta))) \in \mathcal{G}(C)$ to describe explicitly the data which make out a point in $\mathcal{G}(C)$. The group G naturally acts on $\mathcal{G}(C)$ from the left.

The above construction associates to any ordered pair of points $(\sigma, \eta) \in Gb \times Gb - \Delta$ and every geodesic ξ connecting σ to η a (possibly empty) subarc $\xi(a(\xi), b(\xi))$ of ξ in such a way that $(\xi, (a(\xi), b(\xi))) \in \mathcal{G}(C)$. The assignment

$$\Pi : \xi \to \Pi(\xi) = (\xi, (a(\xi), b(\xi))) \in \mathcal{G}(C)$$

satisfies the following properties.

(1) Coarse independence on the connecting geodesics: Let ξ, ζ be any two geodesic lines connecting σ to η. Then the Hausdorff distance between the rays $\xi(b(\xi),\infty), \zeta(b(\zeta),\infty)$ and between the rays $\xi(-\infty, a(\xi))$, $\zeta(-\infty, a(\zeta))$ is bounded from above by $18B + 12$.
(2) Invariance under change of orientation: If $\hat\xi$ is the geodesic obtained from ξ by reversal of orientation then $\hat\xi(b(\xi),\infty) = \xi(-\infty, a(\xi))$ and $\hat\xi(-\infty, a(\xi)) = \xi(b(\xi),\infty)$ (as subsets of X).
(3) Invariance under the action of G: For $q \in G$ we have $\Pi(q\xi) = q\Pi(\xi)$.

Recall from Section 3 the definition of the function τ_C which associates to a point $y \in X$ and two (finite or infinite) geodesics ζ_1, ζ_2 with at most one common endpoint in ∂X a number $\tau_C(x,\zeta_1,\zeta_2)\geq 0$. For $x\in X$ and for two geodesics $\gamma_1,\gamma_2\in\mathcal{G}(C)$ define

$$\tau_{C\mathrm{rel}}(x,\gamma_1,\gamma_2)=$$
$$\max\{\tau_C(x,\gamma_1[b(\gamma_1),\infty),\gamma_2[b(\gamma_2),\infty)),\tau_C(x,\gamma_1(-\infty,a(\gamma_1)],\gamma_2(-\infty,a(\gamma_2)]).$$

Note that if the empty subarc is associated to each of the geodesics γ_1,γ_2 then $\tau_{C\mathrm{rel}}(x,\gamma_1,\gamma_2)=\tau_C(\gamma_1,\gamma_2)$.

Let $\mathcal{A}(b)$ be the union of all ordered pairs of distinct point in Gb with the G-translates of $(a,b),(b,a)$. The group G naturally acts on $\mathcal{A}(b)$ from the left. For $(\sigma_1,\eta_1),(\sigma_2,\eta_2)\in\mathcal{A}(b)$ and $x\in X$ define

$$\tau_{C\mathrm{rel}}(x,(\sigma_1,\eta_1),(\sigma_2,\eta_2))=\inf\tau_{C\mathrm{rel}}(x,\gamma_1,\gamma_2)$$

where the infimum is taken over all elements $\Pi(\gamma_1),\Pi(\gamma_2)\in\mathcal{G}(C)$ defined by all geodesics γ_1,γ_2 connecting σ_1 to η_1 and σ_2 to η_2. By construction, we have

$$\tau_{C\mathrm{rel}}(x,(\sigma_1,\eta_1),(\sigma_2,\eta_2))=\tau_{C\mathrm{rel}}(x,(\sigma_2,\eta_2),(\sigma_1,\eta_1))$$

for all $(\sigma_1,\eta_1),(\sigma_2,\eta_2)\in\mathcal{A}(b)$.

Lemma 3.5 shows that the function $\tau_{C\mathrm{rel}}$ on $\mathcal{A}(b)$ satisfies the ultrametric inequality. Thus as in Section 3, for each $x\in X$ we can use the function $\tau_{C\mathrm{rel}}(x,\cdot,\cdot)$ to define a distance $\delta_x^{C\mathrm{rel}}$ on $\mathcal{A}(b)$. The family $\{\delta_x^{C\mathrm{rel}}\}$ is invariant under the natural action of G on $X\times\mathcal{A}(b)$, and it is invariant under the natural involution ι defined by $\iota(\sigma,\eta)=(\eta,\sigma)$. In the sequel we always equip $\mathcal{A}(b)$ with the topology induced by one (and hence each) of these distance functions.

Since to each geodesic in X connecting a to b or connecting b to g^khb we associated the empty subarc, for each $x\in X$ the points $(b,g^khb)\in\mathcal{A}(b)$ converge as $k\to\infty$ in $(\mathcal{A}(b),\delta_x^{C\mathrm{rel}})$ to (b,a). In particular, the point $(b,a)\in\mathcal{A}(b)$ is not isolated for $\delta_x^{C\mathrm{rel}}$. We use the distances $\delta_x^{C\mathrm{rel}}$ as in Lemma 3.7 to construct a G-invariant distance ρ on $\mathcal{A}(b)\times X$ with the properties stated in Lemma 3.7. Then ρ induces the product topology on $\mathcal{A}(b)\times X$.

We now use the strategy from the proof of Theorem 4.1. Namely, let $x_0\in X$ be a point on an axis for the rank-one element $g\in G$. Let G_{x_0} be the stabilizer of x_0 in G and let $V(b)=\mathcal{A}(b)\times G/G_{x_0}=\mathcal{A}(b)\times Gx_0$. The group G acts on $V(b)$ as a group of isometries with respect to the restriction of the distance ρ. Define $W=G\backslash V(b)$ and let $P:V(b)\to W$ be the canonical projection. The distance ρ on $V(b)$ induces a distance $\hat\rho$ on W by defining $\hat\rho(x,y)=\inf\{\rho(\tilde x,\tilde y)\mid P\tilde x=x,P\tilde y=y\}$. Note that we have $\hat\rho(x,y)>0$ for $x\neq y$ by the definition of the distance ρ and the fact that the distances $\{\delta_x^{C\mathrm{rel}}\}$ depend uniformly Lipschitz continuously on $x\in X$.

The isometric involution ι of $(\mathcal{A}(b)\times X,\rho)$ decends to an isometric involution on W again denoted by ι. Since there is no $u\in G$ with $u(a,b)=(b,a)$, we can find an open neighborhood D of $w=P((b,a),x_0)\in V(b)$ which is disjoint from its image under ι. We choose D to be contained in the image under the projection P of the set $\mathcal{A}(b)\times K$ where K is the closed ball of radius 1 about x_0 in $Gx_0\subset X$.

Let $C_b(Gx_0)$ be the Banach space of continuous bounded functions on $Gx_0\subset X$. As in the proof of Theorem 4.1, we use the induced distance on W to construct from a Hölder continuous function f supported in D with $f(w)>0$ a G-invariant

ι-anti-invariant uniformly bounded continuous map $\tilde{\alpha} : \mathcal{A}(b) \to C_b(Gx_0)$ which lifts to a bounded continuous map $\alpha : \mathcal{A}(b) \to C_b(G)$ with the equivariance properties as stated in this theorem.

By invariance and the definition of the distances $\delta_x^{C\mathrm{rel}}$, if $z \in G$ is such that zx_0 is contained in the support of the function $\tilde{\alpha}(\sigma, \eta)$ then there is a geodesic $\xi \in \mathcal{G}(C)$ connecting σ to η so that zx_0 is contained in a tubular neighborhood of $\xi(-\infty, a(\xi)) \cup \xi(b(\xi), \infty)$ of uniformly bounded radius.

Let A be a small compact neighborhood of b in $X \cup \partial X$ which does not contain the attracting fixed point a of g. For $u \in G$ with $ub \neq b$ define a function $\Psi_\alpha(u) : G \to \mathbb{R}$ by

$$\Psi_\alpha(u)(w) = \alpha(b, ub)(w)$$

if $wx_0 \in X - (A \cup uA)$ and let $\Psi_\alpha(u)(w) = 0$ otherwise. If $ub = b$ then define $\Psi_\alpha(u) \equiv 0$. By the construction of the function α, for every $u \in G$ the function $\Psi_\alpha(u)$ is measurable and supported in a compact subset of G. Moreover, it is pointwise uniformly bounded independent of u.

For every compact subset K_0 of G there is a compact subset C of G containing the support of each of the functions $\Psi_\alpha(u)$ $(u \in K_0)$. In particular, we have $\Psi_\alpha(u) \in L^p(G, \mu)$ for every $p > 1$, and for every compact subset K_0 of G the set $\{\Psi_\alpha(u) \mid u \in K_0\} \subset L^p(G, \mu)$ is bounded.

We claim that Ψ_α is unbounded. For this note that as $k \to \infty$ we have $(b, g^k hb) \to (b, a)$ in $\mathcal{A}(b)$ and $g^k hA \to \{a\}$. In particular, if ξ is a B-contracting axis for g containing the point x_0 then $X - A - g^k hA$ contains longer and longer subsegments of ξ which uniformly fellow-travel the geodesic $g^k\gamma$ connecting b to $g^k hb$. Now the function $\alpha(b, a)$ is invariant under the action of the rank-one element g and its support contains the point x_0. This implies that $\alpha(b, a)$ is *not* integrable. But then for $p > 1$ the L^p-norm of the functions $\Psi_\alpha(g^k h)$ tends to infinity as $k \to \infty$.

Define a function $\omega : G^3 \to L^p(G, \mu)$ by

$$\omega(u, uw, uh) = \omega(e, w, v) = \Psi_\alpha(w) + w\Psi_\alpha(v) - \Psi_\alpha(wv)$$

if ub, uwb, uvb are pairwise distinct and let $\omega(u, uw, uv) = 0$ otherwise. Then ω is invariant under the diagonal action of G, and we have $\omega \circ \sigma = \mathrm{sgn}(\sigma)\omega$ for every permutation of the three variables. Moreover, ω satisfies the cocycle identity

$$\omega(v, w, z) - \omega(u, w, z) + \omega(u, v, z) - \omega(u, v, w) = 0.$$

This is immediate if the points ub, vb, wb, zb are pairwise distinct. If two of these points coincide, say if $ub = vb$, then $\omega(v, w, z) = \omega(u, w, z)$ and $\omega(u, v, z) = 0 = \omega(u, v, w)$ and hence in this case the cocycle equality holds as well. In other words, for every $p \in (1, \infty)$, ω is an $L^p(G, \mu)$-valued 2-cocycle for G.

We claim that the image of ω is uniformly bounded. For this we argue as in the proof of Theorem 4.1. Namely, by assumption, if $\Psi_\alpha(v)(w) \neq 0$ then the point wx_0 is contained in a uniformly bounded neighborhood of a geodesic $\tilde{v}h^{-1}\gamma$ for some $\tilde{v} \in G$ with $\tilde{v}b = vb$ and hence by the definition of the subarc $\xi(a(\xi), b(\xi))$ of a geodesic ξ connecting b to vb and the estimate 3.4, for every $z \in G$ with $zb \neq vb$ the L^p-norm of the restriction of $\alpha(b, vb) + \alpha(vb, zb)$ to a tubular neighborhood of $\xi[b(\xi), \infty)$ is uniformly bounded. By symmetry and the properties of the support of the functions $\alpha(b, vb)$, this implies as in the proof of Theorem 4.1 that ω is a bounded cocycle.

As a consequence, ω defines a bounded $L^p(G,\mu)$-valued cocycle for G and hence an element in $H_b^2(G, L^p(G,\mu))$. We are left with showing that the cocycles constructed in this way define an infinite dimensional subspace of $H_b^2(G, L^p(G,\mu))$.

For this recall from Lemma 2.11 that if G does not act transitively on its limit set then G contains a free subgroup Γ with two generators consisting of rank one elements which contains elements from infinitely many conjugacy classes of G. Using the above notations, we may assume that Γ is generated by g, h. If ω is any $L^p(G,\mu)$-valued bounded cocycle which defines a trivial cohomology class for Γ then there is a bounded function $\rho : \Gamma \to L^p(G,\mu)$ such that

$$\omega(e, v, w) = \rho(v) + v\rho(w) - \rho(vw).$$

By construction, if there is a bounded function $\rho : \Gamma \to L^p(G,\mu)$ such that $\omega(e, v, w) = \rho(v) + v\rho(w) - \rho(vw)$ for all $v, w \in \Gamma$ then there is an unbounded function $\Psi_\alpha : \Gamma \to L^p(G,\mu)$ such that

$$\rho(v) + v\rho(w) - \rho(vw) = \Psi_\alpha(v) + v\Psi_\alpha(w) - \Psi_\alpha(vw)$$

whenever b, vb, wb are pairwise distinct. In other words, $\rho - \Psi_\alpha$ is the restriction to the set of all $v \in \Gamma$ with $vb \neq b$ of a $L^p(G,\mu)$-valued one-cocycle, i.e. a function $\beta : \Gamma \to L^p(G,\mu)$ which satisfies

$$\beta(v) + v\beta(w) - \beta(vw) \equiv 0.$$

Since Ψ_α is unbounded, the same holds true for $\Psi_\alpha - \rho$ and therefore the one-cocycle determined in this way is non-trivial.

Now a one-cocycle is determined by its values on a generating set. On the other hand, since the G-orbit of any pair of fixed points of rank-one elements in G is a closed subset of $\Lambda \times \Lambda - \Delta$, as in the proof of the fifth property in Theorem 4.1 we conclude from this that there are indeed infinitely many linearly independent distinct such classes which pairwise can not be obtained from each other by adding a bounded function. This shows the proposition. $\square$

Remark: The construction in the proof of Proposition 6.1 does not yield continuous bounded cohomology classes since in general the topology on $\mathcal{A}(b)$ induced by the distance functions $\delta_x^{C\mathrm{rel}}$ does not coincide with the restriction of the product topology. However, the twisted quasi-morphisms constructed in the course of the proof take on uniformly bounded values on every compact subset of the group G.

7. Structure of the isometry group

In this section we use the results from Section 5 and Section 6 to complete the proof of Theorem 1 from the introduction.

Proposition 7.1. *Let X be a proper* CAT(0)*-space and let $G < \mathrm{Iso}(X)$ be a closed subgroup which contains a rank-one element. Then one of the following three possibilities holds.*

(1) *G is elementary.*
(2) *G contains an open subgroup G' of finite index which is a compact extension of a simple Lie group of rank 1.*
(3) *G is a compact extension of a totally disconnected group.*

Proof. Let G be a closed subgroup of the isometry group $\mathrm{Iso}(X)$ of a proper CAT(0)-space X. Then G is locally compact. Assume that G is non-elementary and contains a rank-one element. Then by Lemma 5.2, the maximal normal amenable subgroup N of G is compact, and the quotient $L = G/N$ is a locally compact σ-compact group. Moreover, N acts trivially on the limit set Λ of G.

By the solution to Hilbert's fifth problem (see Theorem 11.3.4 in [20]), after possibly replacing L by an open subgroup of finite index (which we denote again by L for simplicity), the group L splits as a direct product $L = H \times Q$ where H is a semisimple connected Lie group with finite center and without compact factors and Q is totally disconnected. If H is trivial then G is a compact extension of a totally disconnected group.

Now assume that H is nontrivial. Let $H_0 < G$ and $Q_0 < G$ be the preimage of H, Q under the projection $G \to L$. Then H_0 is not compact and the limit set $\Lambda_0 \subset \Lambda$ of $H_0 < G$ is nontrivial. Since Q commutes with H and the group N acts trivially on Λ, the group Q_0 acts trivially on Λ_0 (this is discussed in the proof of Proposition 4.3 of [15], and the proof given there is valid in our situation as well). In particular, if Λ_0 consists of a single point then G is elementary. As a consequence, if G is non-elementary then Q_0 fixes at least two points in ∂X.

We show that $\Lambda_0 = \Lambda$. Since $\Lambda_0 \subset \Lambda$ is closed, by Lemma 2.9 it suffices to show that the fixed points of every rank-one element of G are contained in Λ_0. Thus let $g \in G$ be a rank-one element. By Lemma 2.8, g acts with north-south dynamics on ∂X with attracting fixed point $a \in \Lambda$ and repelling fixed point $b \in \Lambda$. If $a \notin \Lambda_0$ then there is a point $\xi \in \Lambda_0 - \{a, b\}$. Write $g = g_0 q$ with $g_0 \in H_0, q \in Q_0$. Since g_0 and q commute up to a compact normal subgroup which fixes $\Lambda \supset \Lambda_0$ pointwise, we have $g_0^k \xi = g_0^k q^k \xi = g^k \xi \to a$ $(k \to \infty)$. But $g_0^k \xi \in \Lambda_0$ for all $k > 0$ and therefore by compactness we have $a \in \Lambda_0$. Since a was an arbitrary fixed point of a rank-one element in G we conclude that $\Lambda_0 = \Lambda$ and hence Q_0 fixes the limit set of G pointwise. However, since G is non-elementary by assumption, in this case the argument in the proof of Lemma 5.2 shows that Q_0 is compact and hence Q is trivial.

To summarize, if G is non-elementary then up to passing to an open subgroup of finite index, either G is a compact extension of a totally disconnected group or G is a compact extension of a semisimple Lie group H with finite center and without compact factors.

We are left with showing that if the group G is a compact extension of a semisimple Lie group H with finite center and without compact factors then H is simple and of rank 1. For this assume first that G acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$. Then Proposition 5.3 shows that $H^2_{cb}(G, L^2(G, \mu)) \neq \{0\}$. By Corollary 8.5.2 of [20], this implies that $H^2_{cb}(H, L^2(H, \mu)) \neq \{0\}$ as well. By the super-rigidity result for bounded cohomology of Burger and Monod [8], we conclude that H is simple of rank one (see [21, 15] for details on this argument).

Now assume that G does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$. Let Γ be an irreducible lattice in H and let Γ_0 be the preimage of Γ under the projection $G \to H = G/N$. We may assume that Γ_0 contains a rank-one element. Moreover, Γ_0 is non-elementary since this is the case for G. Let μ be a (bi-invariant) Haar measure on the compact normal subgroup N and let ν_0, ν be a Haar measure on Γ_0, Γ. Then there is a continuous linear map

$$L^2(\Gamma_0, \nu_0) \to L^2(\Gamma, \nu)$$

obtained by mapping a square integrable function $f_0 : \Gamma_0 \to \mathbb{R}$ to the function $f : \Gamma \to \mathbb{R}$ defined by

$$f(g) = \int_N f_0(gn) d\mu(n).$$

Note that this map is equivariant under the natural left action of G. Via this map, the space $L^2(\Gamma, \nu)$ is a coefficient module for Γ_0.

The subgroup Γ_0 of G is closed and hence $H_b^2(\Gamma_0, L^2(\Gamma_0, \nu_0)) \neq \{0\}$ by Proposition 6.1. Composition with the coefficient map $L^2(\Gamma_0, \nu_0) \to L^2(\Gamma, \nu)$ shows that we have $H_b^2(\Gamma_0, L^2(\Gamma, \nu)) \neq \{0\}$ as well (see the discussion in [20] for details). This means that there are (not necessarily continuous) unbounded twisted quasi-morphisms on Γ_0 with values in $L^2(\Gamma, \nu)$ which do not admit a cocycle at bounded distance. By the remark following Proposition 6.1, we may assume that the restriction of such a quasi-morphism φ_0 to each compact subset of Γ_0 is uniformly bounded. By the defining inequality for a twisted quasi-morphism, this implies that there is a universal constant $c > 0$ such that $\|\varphi_0(h_1) - \varphi_0(h_2)\|_p \leq c$ whenever h_1, h_2 project to the same element of Γ. Then the quasi-morphism φ_0 determines (non-uniquely) an unbounded $L^2(\Gamma, \nu)$-valued twisted quasi-morphism φ on Γ by defining $\varphi(g) = \varphi_0(h)$ for some $h \in \Gamma_0$ which projects to g. This twisted quasi-morphism is not at bounded distance from a cocycle since this was not the case for φ_0. Therefore we have

$$H_{cb}^2(\Gamma, L^2(\Gamma, \mu)) \neq 0.$$

As before, the Burger-Monod super-rigidity result for cohomology [9] shows that H is simple of rank one (compare Theorem 14.2.2 of [20]). □

Now we are ready for the proof of the corollary from the introduction (which is immediate from Corollary 1.24 of [12]). For this recall that a simply connected complete Riemannian manifold $\tilde{M}$ of non-positive sectional curvature is called *irreducible* if $\tilde{M}$ does not split as a non-trivial product. We have.

Corollary 7.2. *Let M be a closed Riemannian manifold of non-positive sectional curvature. If the universal covering $\tilde{M}$ of M is irreducible and if* $\mathrm{Iso}(\tilde{M})$ *contains a parabolic element then M is locally symmetric.*

Proof. Let M be a closed Riemannian manifold of non-positive sectional curvature with irreducible universal covering $\tilde{M}$. The fundamental group $\pi_1(M)$ of M acts cocompactly on the Hadamard space $\tilde{M}$ as a group of isometries. By the celebrated rank-rigidity theorem (we refer to [3] for a discussion and for references), either $\pi_1(M)$ contains a rank-one element or M is locally symmetric of higher rank.

Now assume that $\pi_1(M)$ contains a rank-one element. By Lemma 5.2, the amenable radical N of $\mathrm{Iso}(\tilde M)$ fixes a point $x \in X$, and it fixes the limit set of $\pi_1(M)$ pointwise. Since the action of $\pi_1(M) < \mathrm{Iso}(\tilde M)$ on $\tilde M$ is cocompact, the limit set Λ of $\pi_1(M)$ is the entire ideal boundary $\partial\tilde M$ of $\tilde M$. Then N fixes every geodesic ray issuing from x. This implies that N is trivial.

By Theorem 1, either the isometry group of $\tilde M$ is an almost connected simple Lie group G of rank one or $\mathrm{Iso}(\tilde M)$ is totally disconnected. However, in the first case $\pi_1(M)$ is necessarily a cocompact lattice in $G = \mathrm{Iso}(\tilde M)$ since the action of $\mathrm{Iso}(\tilde M)$ on $\tilde M$ is proper and cocompact. Moreover, the dimension of the symmetric space G/K associated to G coincides with the cohomological dimension of any of its uniform lattices and hence it coincides with the dimension of M. But then the action of G on $\tilde M$ is open. Since this action is also closed, the action is transitive and hence $\tilde M$ is a symmetric space.

We are left with showing that if $\mathrm{Iso}(\tilde M)$ contains a parabolic element then the isometry group of $\tilde M$ is not totally disconnected. Assume to the contrary that $\mathrm{Iso}(\tilde M)$ is totally disconnected. Since the action of $\mathrm{Iso}(\tilde M)$ on $\tilde M$ is cocompact, there is no non-trivial closed convex $\mathrm{Iso}(\tilde M)$-invariant subset of $\tilde M$. Since $\mathrm{Iso}(\tilde M)$ is totally disconnected, by Theorem 5.1 of [12], point stabilizers of $\mathrm{Iso}(\tilde M)$ are open. Then Corollary 3.3 of [10] implies that every element of $\mathrm{Iso}(\tilde M)$ with vanishing translation length is elliptic. This is a contradiction to the assumption that $\mathrm{Iso}(\tilde M)$ contains a parabolic isometry. □

Finally we show Corollary 2 from the introduction.

Corollary 7.3. *Let G be a semi-simple Lie group with finite center, no compact factors and rank at least 2. Let $\Gamma < G$ be an irreducible lattice, let X be a proper* $\mathrm{Cat}(0)$*-space and let $\rho : \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. If $\rho(\Gamma)$ is non-elementary and contains a rank-one element then there is closed subgroup H of* $\mathrm{Iso}(X)$ *which is a compact extension of a simple Lie group L of rank one and there is a surjective homomorphism $\rho : G \to L$.*

Proof. Let $\Gamma < G$ be an irreducible lattice and let $\rho : \Gamma \to \mathrm{Iso}(X)$ be a homomorphism. Let $H < \mathrm{Iso}(X)$ be the closure of $\rho(\Gamma)$. If $\rho(\Gamma)$ is non-elementary and contains a rank-one element then the same is true for H. Then there is a non-trivial bounded cohomology class in $H^2_b(H, L^2(H,\mu))$ which induces via ρ a non-trivial bounded cohomology class in $H^2_b(\Gamma, L^2(H,\mu))$. Since Γ is discrete, we have $H^2_{cb}(\Gamma, L^2(H,\mu)) \neq 0$. By the constructions of Burger and Monod (see [20]), via inducing we deduce that the second bounded cohomology group $H^2_{cb}(G, L^{[2]}(G/\Gamma, L^2(H,\mu)))$ does not vanish, where $L^{[2]}(G/\Gamma, L^2(H,\mu))$ denotes the Hilbert G-module of all measurable maps $G/\Gamma \to L^2(H,\mu)$ with the additional property that for each such map φ the function $x \to \|\varphi(x)\|$ is square integrable on G/Γ with respect to the projection of the Haar measure. Here the G-action is determined by the homomorphism ρ.

By the results of Monod and Shalom [22], if G is simple then there is a ρ-equivariant map $G/\Gamma \to L^2(H,\mu)$. However, this implies that H is compact which is impossible since H contains a rank-one element.

If $G = G_1 \times G_2$ for semi-simple Lie groups G_1, G_2 with finite center and no compact factor then the results of Burger and Monod [8, 9] show that via possibly exchanging G_1 and G_2 we may assume that there is a G_1-equivariant map $G/\Gamma \to L^2(H,\mu)$ for the restriction of ρ to G_1. Since Γ is irreducible by assumption, the action of G_1 on G/Γ is ergodic [23]. By Lemma 5.2, the amenable radical N of H is compact and we deduce as in [21] that there is a continuous homomorphism $\psi : G \to H$. Since G is connected, the image $\psi(G) = H/N$ is connected and hence by Proposition 7.1, H/L is a simple Lie group of rank one. □

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