Some topological properties of surface bundles

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Abstract. We describe the second integral cohomology group of a surface bundle as the group of Chern classes of fiberwise holomorphic complex line bundles and use this to obtain some new information on this group.

1. Introduction

A *surface bundle over a surface* is a smooth closed 4-manifold E which fibers over a closed oriented surface B of genus $h \geq 0$, with fiber a closed oriented surface S_g of genus $g \geq 0$.

Assume that $g, h \geq 2$. Then the monodromy of the bundle determines a homomorphism ρ of the fundamental group $\pi_1(B)$ of B into the *mapping class group* Γ_g of S_g, that is, the group of isotopy classes of orientation preserving diffeomorphisms of S_g. Thus the geometry and topology of surface bundles over surfaces is intimately related to properties of the mapping class group.

Natural topological invariants of such surface bundles E are the *Euler characteristic* $\chi(E)$ and the *signature* $\sigma(E)$. For the Euler characteristic we have

$$\chi(E) = \chi(B)\chi(S_g) = (2h-2)(2g-2).$$

The signature can be computed as follows.

The tangent bundle ν of the fibers of the surface bundle, called the *vertical tangent bundle* in the sequel, is a real two-dimensional oriented smooth subbundle of the tangent bundle TE of E. Hence it can be equipped with the structure of a complex line bundle.

Choose a smooth Riemannian metric on TE and let $\nu^\perp$ be the orthogonal complement of ν in TE for this metric. Then the differential of the projection $\Pi : E \to B$ maps each fiber of $\nu^\perp$ isomorphically onto a fiber of TB and hence as a smooth vector bundle, $\nu^\perp$ is isomorphic to the bundle $\Pi^* TB$. Now TB can be equipped with the structure of a complex line bundle as well, and

$$TE = \nu \oplus \Pi^* TB$$

is the direct sum of two complex line bundles (this is a decomposition of TE as a smooth vector bundle). In particular, the first and second Chern class of TE are defined, and they are independent of the choices made.

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By Hirzebruch's signature theorem, the signature $\sigma(E)$ of E equals

$$\sigma(E) = \frac{1}{3}p_1(E)$$

where $p_1(E)$ is the first Pontryagin number of E. We then have

$$\sigma(E) = \frac{1}{3}(c_1(E)^2 - 2c_2(E)) = \frac{1}{3}(c_1(E)^2 - 2\chi(E)) \tag{1}$$

where as is customary, $c_1(E)^2$ and $c_2(E)$ denote Chern numbers of E.

As $TE = \nu \oplus \Pi^* TB$, the total Chern class of TE equals

$$\begin{aligned} c(TE) &= (1 + c_1(\nu)) \cup (1 + c_1(\Pi^* TB)) \\ &= 1 + c_1(\nu) + c_1(\Pi^* TB) + c_1(\nu) \cup c_1(\Pi^* TB) \end{aligned}$$

and hence since $c_1(\Pi^* TB) \cup c_1(\Pi^* TB)(E) = 0$ we have

$$3\sigma(E) = c_1(\nu) \cup c_1(\nu)(E). \tag{2}$$

Let $\mathcal{M}_g$ be the *moduli space of complex curves of genus* g, that, is the moduli space of complex structures on S_g up to biholomorphic equivalence, and let $\mathcal{U} \to \mathcal{M}_g$ be the *universal curve* whose fiber over a point $X \in \mathcal{M}_g$ is just the complex curve X. Let us now assume that E is a *Kodaira fibration*, that is, E is a complex manifold, and the complex structures of the fibers vary nontrivially. This is equivalent to stating that there is a complex structure on the base B, and there is a nonconstant holomorphic map $\varphi : B \to \mathcal{M}_g$ such that $E = \varphi^* \mathcal{U}$. By the classification of complex surfaces, a Kodaira fibration is of general type and hence projective and Kähler.

The Miaoka inequality for complex surfaces Y of general type states that $c_1^2(Y) \leq 3c_2(Y)$. Therefore by equation (1) which is valid for all closed oriented 4-manifolds, we have

$$3|\sigma(Y)| \leq |\chi(Y)|.$$

Equality holds if and only if Y is a quotient of the ball. On the other hand, Kapovich [**Ka98**] showed that no surface bundle over a surface is a quotient of the ball and hence we always have $3|\sigma(E)| < |\chi(E)|$ for all Kodaira fibrations E. It is also known that the signature of a Kodaira fibration does not vanish.

For surface bundles over surfaces which do not admit a complex structure, much less is known about the relation between signature and Euler characteristic. The most general result to date seems to be a theorem of Kotschick [**K98**]. Using Seibert Witten invariants, he showed

$$2|\sigma(E)| \leq |\chi(E)| \tag{3}$$

for all surface bundles over surfaces. If E admits an Einstein metric, then the stronger inequality

$$3|\sigma(E)| < |\chi(E)|$$

holds true, generalizing the Miaoka inequality for Kodaira fibrations. The following conjecture was formulated among others in [**K98**].

Conjecture. $3|\sigma(E)| \leq |\chi(E)|$ *holds true for all surface bundles over surfaces.*

Perhaps the motivation for this conjecture stems from the general belief that aspherical smooth closed 4-manifolds should admit an Einstein metric.

The conjecture can be viewed as a twisted higher dimensional version of the Milnor-Wood inequality for the Euler number of a flat circle bundle over a closed oriented surface. Namely, call a circle bundle $H \to M$ over a manifold M (or any CW-complex) *flat* [**M58, W71**] if the following holds true. Let $\mathrm{Top}^+(S^1)$ be the group of orientation preserving homeomorphisms of the circle S^1. We require that there is a homomorphism $\eta : \pi_1(M) \to \mathrm{Top}^+(S^1)$ such that

$$H = \tilde{M} \times S^1/\pi_1(M)$$

where $\tilde{M}$ is the universal covering of M and $\pi_1(M)$ acts on $\tilde{M} \times S^1$ via

$$(x,t)g = (xg, \eta(g)^{-1}(t)).$$

The same definition applies if M is a good orbifold, that is, M is the quotient of a smooth simply connected manifold $\tilde{M}$ by the action of a group of diffeomorphisms which acts properly discontinuously, but not necessarily freely.

The celebrated Milnor Wood inequality bounds the absolute value of the Euler number (or first Chern class) of a complex line bundle with flat circle subbundle over a closed surface by the absolute value of the Euler characteristic of the surface [**M58, W71**].

Now as was pointed out by Morita [**Mo88**], the circle subbundle of the vertical tangent bundle of a surface bundle $\Pi : E \to B$ over an arbitrary smooth base B is flat. In this vein, the conjecture predicts a twisted higher dimensional analog of the Milnor Wood inequality.

The goal of this article is to provide a geometric perspective on the topology of surface bundles over a surface. We begin with summarizing some constructions of Kodaira fibrations in Section 2. In Section 3 we give a geometric proof of Morita's theorem (see also Chapter 5 of [**FM12**]).

A *section* of a surface bundle $\Pi : E \to B$ is a smooth map $\sigma : B \to E$ such that $\Pi \circ \sigma = \mathrm{Id}$. In Section 4 we study the self-intersection number of a section of a surface bundle over a surface and compute all such self-intersection numbers for the trivial bundle. We also point out that Morita's theorem yields an elementary and purely topological proof of the following extension of Proposition 1 of [**BKM13**] (see also [**Bow11**]) which was originally established using Seiberg Witten invariants.

Proposition. *Let $E \to B$ be a surface bundle over a surface. Let Σ be closed surface and let $f : \Sigma \to E$ be a smooth map; then*

$$|c_1(f^*\nu)(\Sigma)| \leq |\chi(\Sigma)|.$$

In Section 5 we describe the second integral cohomology group of a surface bundle over a smooth base in an explicit way as the group of first Chern classes of complex line bundles. We apply this discussion to show an analog of a result of Morita who computed the cohomology of a surface bundle with rational coefficients (Proposition 3.1 of [**Mo87**]). The following is also related to the work of Harer [**H83**] who computed for $g \geq 5$ the second homology group of $\mathcal{M}_g$ with integral coefficients (see also the more recent and more complete account in [**KS03**]).

Theorem. *Let $E \to B$ be a surface bundle with fiber genus g. If E admits a section then there exists an integral class $e \in H^2(E, \mathbb{Z})$ such that $(2g-2)e = c_1(\nu)$, and*

$$H^p(E, \mathbb{Z}) \equiv H^p(B, \mathbb{Z}) \oplus H^{p-1}(B; H^1(S_g, \mathbb{Z})) \oplus eH^{p-2}(B, \mathbb{Z})$$

In view of the result of Chen and Salter [**CS18**] that the pullback of the universal curve to any finite orbifold cover of $\mathcal{M}_g$ does not admit a section, it seems that most surface bundles do not admit sections.

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2. Constructions

In this section we review some constructions of Kodaira fibrations from the literature.

The best known construction method of Kodaira fibrations goes back to an idea of Atiyah and Kodaira. Their examples are branched covers over a product of two complex curves, and they fiber in two different ways. A variation of this idea was used by Bryan and Donagi [**BD02**] to show that for any integers $h, n \geq 2$, there exists a connected algebraic surface $X_{h,n}$ of signature $\sigma(X_{h,n}) = \frac{4}{3}h(h-1)(n^2-1)n^{2h-3}$ that admits two smooth fibrations $\theta_1 : X_{h,n} \to C$ and $\theta_2 : X_{h,n} \to D$ with base and fiber genus (b_i, f_i) equal to

$$(b_1, f_1) = (h, h(hn-1)n^{2h-2}+1) \text{ and}$$
$$(b_2, f_2) = (h(h-1)n^{2h-2}+1, hn)$$

respectively. Note that the smallest fibre genus of the surfaces in the above family equals 4.

Taking $n = h = 2$, we conclude that there is a surface bundle with fiber genus 4 and base genus 9 with

$$\frac{\sigma(E)}{\chi(E)} = \frac{16}{96} = \frac{1}{6}.$$

According to my knowledge, this is the example with the largest known ratio between signature and Euler characteristic.

Complete intersections provide a more indirect way to construct Kodaira fibrations (see [**Ar17**] for a recent discussion). To be more specific, call a Kodaira fibration with fiber S_g of genus g *generic* if the fundamental group $\pi_1(B)$ of the base surjects onto a finite index subgroup of the mapping class group Γ_g by the monodromy homomorphism ρ. Generic Kodaira fibrations can be constructed as follows.

The action of a diffeomorphism of S_g on the first homology group $H_1(S_g, \mathbb{Z})$ preserves the intersection form and only depends on the isotopy class of the diffeomorphism. Thus there exists a surjective [**FM12**] homomorphism

$$\Psi : \Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z}).$$

For every $n \geq 3$, the kernel of the induced homomorphism $\Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ is torsion free and determines the *fine moduli space of genus g curves with level n structure* $\mathcal{M}_g[n]$, which is a complex manifold.

Let $\mathcal{M}_g[n]^*$ be the Satake compactification of $\mathcal{M}_g[n]$. If $g \geq 3$, then the boundary $\mathcal{M}_g[n]^* - \mathcal{M}_g[n]$ has complex codimension at least 2. Therefore a curve $C \subset \mathcal{M}_g[n]^*$ given as an intersection of general ample divisors lies entirely in $\mathcal{M}_g[n]$. By the weak Lefschetz theorem, the inclusion $C \to \mathcal{M}_g[n]$ induces a surjection of

fundamental groups. As the fundamental group of $\mathcal{M}_g[n]$ is a finite index subgroup of the mapping class group, the restriction to C of the universal curve defines a generic Kodaira fibration.

The Atiyah Kodaira examples which are branched covers over the product of two complex curves are not generic. Namely, if $E \to B$ is a generic Kodaira fibration, then the image of the monodromy group under the homomorphism Ψ is a Zariski dense subgroup of $Sp(2g, \mathbb{R})$. However, if $E \to B$ is an Atiyah Kodaira example, then the group $\Psi(\rho(\pi_1(B)))$ fixes a symplectic plane in $H^1(B, \mathbb{R})$ and hence by duality, $\Psi(\rho(\pi_1(B)))$ is not Zariski dense in $Sp(2g, \mathbb{R})$. Interestingly, Flapan [**Fl17**] used complete intersections to construct Kodaira fibrations with fiber of genus 3 which also have this property. She also classified all $\mathbb{Q}$-algebraic subgroups of $Sp(6, \mathbb{R})$ which arise as the smallest algebraic group containing the image of the monodromy group of a Kodaira fibration with fiber of genus 3.

On the other hand, Bregman [**Br18**] established that variations of the Atiyah Kodaira construction may in some sense be universal for Kodaira fibrations whose monodromy fixes a symplectic plane in $H^1(B, \mathbb{R})$. He showed that if the dimension of the fixed point set of the mondromy of a Kodaira fibration E acting on the holomorphic one-forms of a fixed fiber equals d for some $1 \leq d \leq 2$, then there exists a genus d curve D and a ramified covering $F : E \to D \times B$ inducing an isomorphism on first cohomology with rational coefficients.

There are also explicit constructions of surface bundles over surfaces with non-trivial signature which do not admit a complex structure [**B12**]. The fiber genus of such a surface bundle is at least 4. The article [**EKKOS02**] constructs surface bundles over surfaces with positive signature for any fiber genus $g \geq 3$. In contrast, the signature of a surface bundle over a surface with fiber genus 2 always vanishes. This follows from the fact that the second cohomology group $H^2(\Gamma_g, \mathbb{Z})$ is isomorphic to $H_2(\Gamma_g, \mathbb{Z})/\text{torsion}$ (see p.158 of [**FM12**]), on the other hand we have $H_2(\Gamma_2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ [**KS03**].

3. Flat circle bundles

Consider the universal curve $\Pi : \mathcal{U} \to \mathcal{M}_g$ over the moduli space $\mathcal{M}_g$ of genus g curves. Its fiber over a point $X \in \mathcal{M}_g$ is just the complex curve X. The tangent bundle ν of the fibers of this bundle is a holomorphic complex line bundle on the complex orbifold $\mathcal{U}$.

The following observation (which is due to Morita [**Mo88**]) is based on some facts which were probably already known to Nielsen.

Proposition 3.1. *The circle subbundle of the bundle $\nu \to \mathcal{U}$ is flat.*

Proof. Let $\Gamma_{g,1}$ be the mapping class group of a surface of genus g with one marked point (puncture), and denote by $\Theta : \Gamma_{g,1} \to \Gamma_g$ the homomorphism induced by the puncture forgetful map. This homomorphism fits into the *Birman exact sequence* [**Bi74, FM12**]

$$1 \to \pi_1(S_g) \to \Gamma_{g,1} \xrightarrow{\Theta} \Gamma_g \to 1.$$

Via this sequence, the group $\Gamma_{g,1}$ is the orbifold fundamental group of the universal curve.

We claim that the group $\Gamma_{g,1}$ admits an action on the circle S^1 by orientation preserving homeomorphisms, where we view S^1 as the ideal boundary $\partial\mathbb{H}^2$ of the

hyperbolic plane $\mathbb{H}^2$. Namely, let $x \in S_g$ be a fixed point. The group $\Gamma_{g,1}$ can be viewed as the group of isotopy classes of orientation preserving diffeomorphisms of the surface S_g preserving x. Isotopies are also required to fix x. Any orientation preserving diffeomorphism f of S_g which fixes x induces an automorphism $f_* \in \mathrm{Aut}(\pi_1(S_g, x))$, and since the group of diffeomorphisms of S_g isotopic to the identity is contractible, the isotopy class of f is uniquely determined by the induced map f_*.

The group $PSL(2,\mathbb{R})$ is just the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2$, or, equivalently, it is the group of biholomorphic automorphisms of the unit disk $D \subset \mathbb{C}$. The choice of a hyperbolic structure on S_g then determines the conjugacy class of an embedding $\pi_1(S_g) \to PSL(2,\mathbb{R})$, with discrete cocompact image.

Since the group $PSL(2,\mathbb{R})$ acts simply transitively on the unit tangent bundle $T^1\mathbb{H}^2$ of the hyperbolic plane, we can choose an identification of $PSL(2,\mathbb{R})$ with $T^1\mathbb{H}^2$ which maps the identity to the point $0 \in D = \mathbb{H}^2$. We also may assume that 0 is a preimage of the point $x \in S_g$. With these identifications, the group $\pi_1(S_g, x)$ determines an embedding $\pi_1(S_g, x) \to PSL(2,\mathbb{R})$, unique up to conjugation with the central subgroup $SO(2) \subset PSL(2,\mathbb{R})$, that is, the stabilizer of the basepoint 0. Fix once and for all such an embedding.

A diffeomorphism f of S_g fixing x is a bilipschitz map for the hyperbolic structure of S_g. Thus f can be lifted to a $\pi_1(S_g, x)$-equivariant bilipschitz map $\tilde{f} : \mathbb{H}^2 = D \to D$ which fixes 0. This means that the map $\tilde{f}$ satisfies $\tilde{f}(\psi y) = f_*(\psi)(\tilde{f}(y))$ for all $y \in D$ and all $\psi \in \pi_1(S_g, x) \subset PSL(2,\mathbb{R})$. Equivariance and the requirement that $\tilde{f}(0) = 0$ determines the lift $\tilde{f}$ completely.

Now any bilipschitz map of the hyperbolic plane which fixes the point 0 maps geodesic rays beginning at 0 to uniform quasi-geodesic rays issuing from the same point. Such a uniform quasi-geodesic ray is at uniformly bounded distance from a geodesic ray, and this geodesic ray is unique if its starting point is required to be the fixed point 0. As the ideal boundary $\partial D = S^1$ of the hyperbolic plane is just the set of geodesic rays issuing from 0, this shows that the map $\tilde{f}$ induces a homeomorphism $\Upsilon(f) \in \mathrm{Top}^+(S^1)$. Here preservation of orientation of $\Upsilon(f)$ follows from preservation of orientation of f. The homeomorphism $\Upsilon(f)$ only depends on the isotopy class of f provided that such an isotopy fixes the point x.

By construction, if u is another orientation preserving bilipschitz homemorphism of S_g fixing x, then $\Upsilon(u \circ f) = \Upsilon(u) \circ \Upsilon(f)$. As a consequence, the assignment $f \to \Upsilon(f)$ which associates to a diffeomorphism f of S_g fixing x the homeomorphism $\Upsilon(f) \in \mathrm{Top}^+(S^1)$ descends to a homomorphism $\hat{\Upsilon} : \Gamma_{g,1} \to \mathrm{Top}^+(S^1)$, unique up to conjugation in $\mathrm{Top}^+(S^1)$. This homomorphism then defines a flat circle bundle $H \to \mathcal{U}$. We claim that this circle bundle is (up to equivalence) the circle subbundle of the vertical tangent bundle of the surface bundle $\mathcal{U}$.

Namely, consider the space $\mathcal{T}(S_g^1)$ of all discrete faithful orientation preserving homomorphisms $\rho : \pi_1(S_g, x) \to PSL(2,\mathbb{R})$. The group $PSL(2,\mathbb{R})$ acts on this space by conjugation. Each orbit of this action is a fiber of the bundle $\mathcal{T}(S_g^1) \to \mathcal{T}(S_g)$ where $\mathcal{T}(S_g)$ denotes the Teichmüller space of marked complex structures on the closed oriented surface S_g of genus g. The quotient of $\mathcal{T}(S_g^1)$ by the action of the central circle group $SO(2) = S^1$ is the Teichmüller space $\mathcal{T}(S_{g,1})$ of all marked complex structures on an oriented surface $S_{g,1}$ of genus g with one marked point.

The circle bundle $PSL(2,\mathbb{R}) = T^1\mathbb{H}^2 \to \mathbb{H}^2$ has a $PSL(2,\mathbb{R})$-equivariant identification with $\mathbb{H}^2 \times S^1$ where the action of $PSL(2,\mathbb{R})$ on $S^1 = \partial\mathbb{H}^2$ is described as follows.

For a unit tangent vector $u \in T^1\mathbb{H}^2$ let γ_u be the geodesic ray with initial velocity u. The projection of the identity in $PSL(2,\mathbb{R})$ is a fixed basepoint $0 \in D = \mathbb{H}^2$. Identify $\partial\mathbb{H}^2$ with the fiber of the unit tangent bundle over this basepoint by associating to a unit tangent vector $v \in T_0\mathbb{H}^2$ the endpoint $\gamma_v(\infty)$ of the geodesic ray γ_v. For each $\alpha \in PSL(2,\mathbb{R})$, the differential $d\alpha(0)$ of α at the basepoint 0 then induces the homeomorphism $\gamma_v(\infty) \to \gamma_{d\alpha(v)}(\infty)$ of $\partial\mathbb{H}^2 = S^1$. The thus defined map $PSL(2,\mathbb{R}) \to \mathrm{Top}^+(S^1)$ is equivariant with respect to the action of $PSL(2,\mathbb{R})$ on itself by conjugation.

Now let $f : (S_g, x) \to (S_g, x)$ be an arbitrary diffeomorphism which fixes the point x. Taking a quotient by the action of $(0,\infty)$ on the tangent bundle of S_g by scaling shows that its differential induces an isomorphism of the circle bundle T^1S_g covering the base map f. On the other hand, using the above construction, the map f induces a second isomorphism of T^1S_g as follows. Lift f to a diffeomorphism $\tilde{f}$ of $\mathbb{H}^2$ which fixes 0 and is equivariant with respect to the action of $\pi_1(S_g, x)$ and its image under the automorphism f_*. For any point $y \in \mathbb{H}^2$, map a unit tangent vector $v \in T^1_y\mathbb{H}^2$ to the unit tangent vector $w \in T_{\tilde{f}(y)}\mathbb{H}^2$ such that $\gamma_w(\infty) = \Upsilon(f)\gamma_v(\infty)$. As this construction depends continuously on v and is equivariant with respect to the action of $\pi_1(S_g, x)$, it descends to an isomorphism of T^1S_g covering f.

The circle subbundle of the vertical tangent bundle of the universal curve $\mathcal{U}$ is the quotient of the vertical tangent bundle of $\mathcal{T}(S_{g,1})$, that is of $\mathcal{T}(S^1_g)$, by the action of $\Gamma_{g,1}$ via the tangent map of isotopy classes of diffeomorphisms fixing the basepoint x. Thus to show that this circle bundle is indeed the flat bundle defined by the homomorphism $\hat{\Upsilon} : \Gamma_{g,1} \to \mathrm{Top}^+(S^1)$, it suffices to show that for any diffeomorphism $f : (S_g, x) \to (S_g, x)$, the isomorphism of T^1S_g induced by df is homotopic to the isomorphism induced by $\Upsilon(f)$.

To show that this is indeed the case lift as before the diffeomorphism f to an f_*-equivariant diffeomorphism $\tilde{f}$ of $\mathbb{H}^2$ fixing 0. We deform equivariantly the tangent map $d\tilde{f}$ of $\tilde{f}$ as follows.

For a number $r > 0$ and a point $y \in \mathbb{H}^2$, identify the fiber of the unit tangent bundle of $\mathbb{H}^2$ at y with the boundary $\partial B(y,r)$ of the ball of radius r about y using the exponential map of the hyperbolic plane. The image $\tilde{f}(\partial B(y,r))$ bounds a disk containing $\tilde{f}(y)$. Use the inverse of the exponential map at $\tilde{f}(y)$ to map this circle onto the fiber of the unit tangent bundle of $\mathbb{H}^2$ at $\tilde{f}(y)$. Doing this simultaneously for all $y \in \mathbb{H}^2$ defines a continuous map $\tilde{\zeta}_r : T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$ which is equivariant with respect to the action of $\pi_1(S_g)$ and hence descends to a continuous map $\zeta_r : T^1S_g \to T^1S_g$ covering f. Clearly the maps ζ_r depend continuously on r, and as $r \to 0$, they converge to the map induced by df. Thus for all r, the map ζ_r is homotopic to the map induced by df.

Now by construction, as $r \to \infty$ the maps ζ_r converge to the map induced by $\Upsilon(f)$. As this construction is moreover equivariant with respect to isotopy, this shows that the action of $\Gamma_{g,1}$ on the vertical tangent bundle of the universal covering of the universal curve $\mathcal{U}$ coincides with the action defined by the homomorphism $\hat{\Upsilon}$. Hence the flat circle bundle on $\mathcal{U}$ constructed from the homomorphism $\hat{\Upsilon} : \Gamma_{g,1} \to \mathrm{Top}^+(S^1)$ indeed equals the circle subbundle of the vertical tangent bundle of $\mathcal{U}$. This is what we wanted to show. $\square$

Let now $\Pi : E \to B$ be a surface bundle over an arbitrary smooth base B, with fibre S_g of genus $g \geq 2$. Any such surface bundle can be obtained as a pull-back of the universal curve $\mathcal{U} \to \mathcal{M}_g$ by a smooth (in the sense of orbifolds) map $f : B \to \mathcal{M}_g$. Thus we may assume that the fibres of E are equipped with a complex structure varying smoothly over the base. As a consequence, the vertical tangent bundle ν of E is a smooth complex line bundle over E.

Since $\mathcal{M}_g$ is a classifying space (in the orbifold sense) for its (orbifold) fundamental group, the homotopy class of a map $f : B \to \mathcal{M}_g$ is uniquely determined by the induced homomorphism $f_* = \rho : \pi_1(B) \to \Gamma_g$. Here as before, Γ_g is the mapping class group of S_g. Furthermore, homotopic maps give rise to homeomorphic surface bundles, so the bundle E is determined by ρ. We refer to [**Mo87**] for more details about these well known facts.

Let as before $\Theta : \Gamma_{g,1} \to \Gamma_g$ be the natural surjective homomorphism. Since E is the pull-back of the universal curve under the map f, there exists an exact diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S_g) & \longrightarrow & \pi_1(E) & \longrightarrow & \pi_1(B) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(S_g) & \longrightarrow & \Theta^{-1}(\rho(\pi_1(B))) & \longrightarrow & \rho(\pi_1(B)) & \longrightarrow & 1. \end{array} \tag{4}$$

As a consequence, there exists a homomorphism $\pi_1(E) \to \Theta^{-1}(\rho(\pi_1(B))) \subset \Gamma_{g,1}$ whose restriction to the subgroup $\pi_1(S_g)$ is an isomorphism. By naturality under pull-back, in the case that B is a closed surface we obtain

Corollary 3.2. *Let $\Pi : E \to B$ be a surface bundle over a surface. Then $TE = \nu \oplus \Pi^* TB$ is a sum of two complex line bundles whose circle subbundles are flat.*

Proof. We observed before that $TE = \nu \oplus \Pi^* TB$, and by Proposition 3.1, the circle subbundle of ν is a pull-back of a flat bundle and hence flat. On the other hand, as B is a surface of genus $h \geq 2$, the circle subbundle of the tangent bundle TB of B is flat as well and hence the same holds true for the circle subbundle of the pull-back $\Pi^* TB$. □

4. Selfintersection numbers of sections

A *section* of a surface bundle $\Pi : E \to B$ is a smooth map $f : B \to E$ so that $\Pi \circ f = \mathrm{Id}$. The image $f(B)$ of a section f is a cycle in E which defines a homology class $[f(B)] \in H_k(E, \mathbb{Z})$ where $k = \dim(B)$. In the case that B is a surface, the self-intersection number $[f(B)]^2$ of this class is defined. Our next goal is to shed some light on this self-intersection number from a geometric point of view.

Let as before ν be the vertical line bundle of E, with first Chern class $c_1(\nu)$. Equivalently, $c_1(\nu)$ is the Euler class of the oriented 2-dimensional real oriented vector bundle ν. We note

Lemma 4.1. $[f(B)]^2 = c_1(\nu)(f(B))$ *for any section $f : B \to E$.*

Proof. Since $f(B)$ is a smoothly embedded surface in E, the self-intersection number of $f(B)$ equals the Euler number of the pull-back to B of the oriented normal bundle of $f(B)$ in E, that is, it equals the evaluation of the Euler class of this normal bundle on the homology class $[f(B)]$.

As f is a section, $f(B)$ is everywhere transverse to the fibers of $E \to B$. Thus this oriented normal bundle is isomorphic to the restriction of the vertical tangent bundle ν of E. □

It was shown by Milnor [**M58**] and Wood [**W71**] that the Euler number $e(H)$ of a flat circle bundle $H \to B$ over a closed oriented surface B of genus $h \geq 2$ and the Euler characteristic $\chi(B)$ of B satisfy the inequality

$$|e(H)| \leq |\chi(B)|.$$

In view of this result, the conjecture stated in the introduction can be viewed as a higher dimensional analog of the Milnor Wood inequality.

By a result of Thom, for any compact CW-complex X, any homology class $\alpha \in H_2(X, \mathbb{Z})$ can be represented by a map from a closed surface into X, and if α is not a two-torsion class, then the surface can be chosen to be orientable. As a consequence of Proposition 3.1, we obtain

Corollary 4.2. *Let $\beta \in H_2(E, \mathbb{Z})$ be represented by a map $f : \Sigma \to E$ where Σ is a closed oriented surface. Then $|c_1(\nu)(\beta)| \leq |\chi(\Sigma)|$.*

Proof. By Proposition 3.1, the pull-back by f of the circle subbundle of ν is a flat circle bundle over Σ. By naturality, we have

$$|c_1(\nu)(\beta)| = |f^*(c_1(\nu))(\Sigma)| \leq |\chi(\Sigma)|$$

by the Milnor Wood inequality. □

As an immediate consequence, we obtain the following result of Baykur, Korkmaz and Monden (Proposition 1 of [**BKM13**]) and Bowden [**Bow11**], bypassing the use of Seiberg-Witten invariants used to derive this statement in [**BKM13**] and [**Bow11**].

Corollary 4.3. *Let $f : B \to E$ be a section of a surface bundle $E \to B$; then $|[f(B)]^2| \leq |\chi(B)|$.*

Proof. The section is defined by a smooth map $B \to E$ and hence the corollary follows from Lemma 4.1 and Corollary 4.2. □

Theorem 15 of [**BKM13**] shows that for every $g \geq 2, h \geq 1$ and every integer $k \in [-2h+2, 2h-2]$ there is a surface bundle with fibre S_g and base of genus h which admits a section of self-intersection number k. We complement this result by analyzing self-intersection numbers of sections of the trivial bundle.

Proposition 4.4. *Let $E \to B$ be the trivial surface bundle with fibre genus $g \geq 2$ and base genus h. If $h < g$ then every section of E has self-intersection number zero. If $h \geq g$ then for each integer k with $h - 1 \geq |k|(g-1)$ there is a section of self-intersection number $2k(g-1)$, and no other self-intersection numbers occur.*

Proof. Let B be a surface of genus $h \geq 1$ and let $E = B \times S_g \to B$ be the trivial surface bundle. Then a section $f : B \to E$ is just a smooth map of the form $x \to (x, \Phi(x))$ where $\Phi : B \to S_g$ is smooth. Let $d \in \mathbb{Z}$ be the degree of Φ. We claim that the self-intersection number of f equals $d(2-2g)$.

To see that this is the case, denoting as before by ν the vertical tangent bundle, we have $c_1(\nu)(f(B)) = \Phi^* c_1(TS_g)(B) = d(2-2g)$. By Lemma 4.1, the self-intersection number of the section f coincides with this Euler number.

This shows that the self-intersection number of a section of the trivial bundle is a multiple of $2g-2$. The proposition now follows from the fact that for a surface B of genus h and every $k \in \mathbb{Z}$, there exists a smooth map $\Phi : B \to S_g$ of degree k if and only if $|k| \leq \frac{h-1}{g-1}$. Then $x \in B \to (x, \Phi(x))$ is a section of $E \to B$ with self-intersection number $k(2-2g)$.

To show that the condition on k is sufficient for the existence of a map $B \to S_g$ of degree k, note that if $k \leq \frac{h-1}{g-1}$ is positive, then a map $B \to S_g$ of degree k can be constructed as follows. Let $\psi : \Sigma \to S_g$ be an unbranched cover of degree k. The Euler characteristic of Σ fulfills $|\chi(\Sigma)| = k|\chi(S_g)| \leq |\chi(B)|$. Thus there is a degree one map $\zeta : B \to \Sigma$ which pinches a subsurface of B of genus g' to a point, where $g' = h-1-k(g-1) \geq 0$ [**Ed79**]. The composition $\psi \circ \zeta : B \to S_g$ has degree k. A map of degree $-k$ can be taken as a composition of this map with an orientation reversing diffeomorphism of B.

On the other hand, by a result of Edwards [**Ed79**], any map $B \to S_g$ of degree $k \geq 1$ is homotopic to the composition of a pinch $B \to B'$ with a branched cover $B' \to S_g$. A nontrivial pinch collapses a subsurface of B bounded by an essential separating simple closed curve to a point and hence it strictly decreases the genus. Thus the genus q of B' is not bigger than the genus h of B.

Now if $B' \to S_g$ is a branched cover of degree k and if b is the total number of branch points, counted with multiplicity, then by the Hurwitz formula [**FK80**],

$$2q-2 = b + k(2g-2).$$

This implies that $|k| \leq \frac{q-1}{g-1}$ and hence $|k| \leq \frac{h-1}{g-1}$. As any map $B \to S_g$ can be precomposed with an orientation reversing diffeomorphism of B to yield a map of nonnegative degree, this completes the proof of the proposition. □

5. Cohomology of surface bundles

This final section collects some results on the second cohomology group of a surface bundle over a base B which is an arbitrary smooth closed manifold. We also give some additional information in the case that B is a surface.

The cohomology with rational coefficients of a surface bundle over a smooth base was computed by Morita. The following is Proposition 3.1 of [**Mo87**].

Proposition 5.1. *Let $\Pi : E \to B$ be a surface bundle over a smooth base B, with fiber S_g. Let $k = \mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ where p is a prime not dividing $2g-2$. Then the homomorphism $\Pi^* : H^*(B,k) \to H^*(E,k)$ is injective, and for all q we have*

$$H^q(E,k) \cong H^q(B,k) \oplus H^{q-1}(B, H^1(S_g,k)) \oplus c_1(\nu) H^{q-2}(B,k).$$

In general, we can not hope that the proposition passes on to cohomology with integral coefficients. The reason is that for a surface bundle $E \to B$ with fiber of genus $g \geq 2$, the fiber inclusion $\iota : S_g \to E$ may not induce a surjection $\iota^* : H^2(E,\mathbb{Z}) \to H^2(S_g,\mathbb{Z}) = \mathbb{Z}$. Namely, the Euler class $e \in H^2(S_g,\mathbb{Z})$ of the tangent bundle of S_g has a $2g-2$-th root, but there may not exist such a root for the class $c_1(\nu)$ where as before, $c_1(\nu) \in H^2(E,\mathbb{Z})$ denotes the first Chern class of the vertical tangent bundle of E.

In the following observation, the component $H^1(B, H^1(S_g,\mathbb{Q})) \subset H^2(E,\mathbb{Q})$ is as in Proposition 5.1.

PROPOSITION 5.2. *Let $E \to B$ be a surface bundle over a surface. Then there exists an embedding $H^1(B, H^1(S_g, \mathbb{Z})) \to H^2(E, \mathbb{Z})$ which induces an isomorphism $H^1(B, H^1(S_g, \mathbb{Z})) \otimes \mathbb{Q} \to H^1(B, H^1(S_g, \mathbb{Q})) \subset H^2(E, \mathbb{Q})$.*

PROOF. The standard Leray spectral sequence for the fiber bundle $\Pi : E \to B$ starts with a finite good cover $\mathcal{U} = \{U_i \mid 1 \leq i \leq k\}$ of B consisting of open sets $U_i \in \mathcal{U}$ with the following properties.

(1) Each set $U_i \in \mathcal{U}$ is diffeomorphic to an open disk $D \subset \mathbb{R}^2$.
(2) Each intersection $U_i \cap U_j$ or $U_i \cap U_j \cap U_k$ is contractible or empty.
(3) The intersection of any four distinct of the sets U_i is empty.

Such a covering can be constructed from a triangulation T of B as follows.

For each vertex x of T, choose a disk neighborhood D_x of x so that the closures of these disks are pairwise disjoint. We also require that each edge e of T intersects a disk D_x if and only if the edge is incident on x, and in this case, the intersection of e with D_x is a connected subarc of e. Furthermore, we require that a two-simplex f intersects a disk D_x if and only if x is a vertex of f, and in this case, $D_x \cap f$ is a disk. Call these disks of vertex type.

For each edge e of T, choose a disk D_e containing a neighborhood of $e - \cup_x D_x$. This can be done in such a way that the disks D_e are pairwise disjoint, that for a vertex x of T, the intersection $D_e \cap D_x$ is empty or a disk, and that the later holds true if and only if e is incident on x. Call such a disk of edge type. The union of the disks of vertex and edge type covers a neighborhood of the one-skeleton of T, and the intersection of any two of these disks either is a disk or empty. The intersection of any three of the disks is empty.

Finally choose a disk for each two-simplex f of T which is contained in f and covers $f - \cup_e D_e - \cup_x D_x$. Call these disks of face type. They can be chosen in such a way that the resulting family of disks covers B and that furthermore, if the intersection of any three of the disks is non-empty, then these disks are of distinct type. The resulting cover $\mathcal{U}$ is called a *good cover* of B. The restriction of E to U_i is trivial for all $U_i \in \mathcal{U}$.

The good cover $\mathcal{U}$ determines a first quadrant double chain complex $K^{p,q}$ with $0 \leq p \leq 2$ which can be used to compute $H^2(E, \mathbb{Z})$ using the Leray spectral sequence for the sheaf $\mathcal{F}$ of locally constant $\mathbb{Z}$-valued functions on E. Leray's theorem (see Theorem 14.18 of [**BT82**]) shows that the E_2-term of the spectral sequence with coefficients $\mathbb{Z}$ has the form

$$E_2^{p,q} = H^p(\mathcal{U}, H^q(S_g, \mathbb{Z})).$$

This spectral sequence converges to $H^*(E, \mathbb{Z})$ by the generalized Mayer-Vietoris principle, because $\Pi^{-1}(\mathcal{U})$ is a cover of E, see p.169 and Theorem 15.11 of [**BT82**] for details on these facts.

Since $K^{p,q}$ is trivial for $p \geq 3$, for $r \geq 2$ and every $k \geq 2$ the differential $d_r : E_k^{1,1} \to E_k^{1+r,2-r}$ vanishes. Since the spectral sequence converges to $H^*(E, \mathbb{Z})$, this implies that indeed we have an embedding $H^1(\mathcal{U}, H^1(S_g, \mathbb{Z})) = H^1(B, H^1(S_g, \mathbb{Z})) \to H^2(E, \mathbb{Z})$.

We claim that the image group is precisely the subgroup of $H^2(E, \mathbb{Z})$ whose cup product with $C = \Pi^* H^2(B, \mathbb{Z}) \oplus c_1(\nu) H^0(B, \mathbb{Z})$ vanishes. Note to this end that C is a free subgroup of $H^2(E, \mathbb{Z})$ of rank two, and the restriction of the cup product to C is non-degenerate. Now the degree two part of the E_2-term of the spectral sequence decomposes as $E_2 = E_2^{2,0} \oplus E_2^{1,1} \oplus E_2^{0,2}$. The cup product defines

a homomorphism

$$E_2^{2,0} \otimes E_2^{1,1} \to E_2^{3,1} = 0,$$

and similarly, the cup product defines a homomorphism

$$E_2^{1,1} \otimes E_2^{0,2} \to E_2^{1,3} = 0.$$

As this argument is also valid with coefficients in $\mathbb{Q}$, and cup product is natural with respect to taking tensor product with $\mathbb{Q}$, this completes the proof. □

Remark 5.3. We do not know an example of a surface bundle for which the conclusion of Proposition 5.2 is violated. In particular, by [**H83**], it holds true for the universal curve, in fact we have $H^1(\Gamma_g, H^1(S_g, \mathbb{Z})) = 0$ for all g.

The final goal of this article is to give a geometric interpretation of the subgroup $H^1(B, H^1(S_g, \mathbb{Z}))$ of $H^2(E, \mathbb{Z})$ for a surface bundle over a surface $\Pi : E \to B$ and prove the theorem from the introduction. We begin with some results which hold true for an arbitrary surface bundle $E \to B$ over a smooth base. Assume as before that $E \to B$ is obtained by a smooth map $B \to \mathcal{M}_g$. This means that each of the fibers of E has a complex structure varying smoothly over the base.

Abel's theorem shows that the *Picard group* $\mathrm{Pic}(X, 2g-2)$ of all complex line bundles of degree $2g-2$ over a Riemann surface X can be identified with the *Jacobian* $\mathcal{J}(X)$ of X as follows [**FK80, GH78**].

Choose a geometric symplectic basis $a_1, b_1, \dots, a_g, b_g$ of $H_1(S_g, \mathbb{Z})$. This means that a_i, b_i are oriented non-separating simple closed curves in S_g so that a_i, b_i intersect in precisely one point, and $a_i \cap a_j = a_i \cap b_j = b_i \cap b_j = \emptyset$ for all $i \neq j$. This choice then determines a basis $\omega_1, \dots, \omega_g$ of the g-dimensional [**FK80**] complex vector space $H^{1,0}(X, \mathbb{C})$ of holomorphic one-forms on X so that $\omega_i(a_j) = \delta_{ij}$. The imaginary parts of ω_i are linearly independent over $\mathbb{R}$ and hence the one-forms $\omega_1, \dots, \omega_g$ determine a lattice $\Lambda(X)$ in $\mathbb{C}^g$, obtained by integration over the geometric symplectic basis $a_1, b_1, \dots, a_g, b_g$ of $H_1(S_g, \mathbb{Z})$. The quotient of $\mathbb{C}^g$ by this lattice then is the Jacobian $\mathcal{J}(X)$ of X. The fundamental group A of $\mathcal{J}(X)$ is isomorphic to the integral homology group $H_1(S_g, \mathbb{Z}) = \mathbb{Z}^{2g}$ of S_g, and hence using duality provided by the symplectic form, to the group $H^1(S_g, \mathbb{Z})$.

If X varies in a smooth family, then the holomorphic one-forms $\omega_i = \omega_i(X)$ on X defined by $\omega_i(X_i)(a_j) = \delta_{i,j}$ also vary smoothly. This means that there exists a smooth fiber bundle $\Theta : W \to B$ whose fiber over X is just the Jacobian $\mathcal{J}(X)$ of X.

The bundle W is naturally a quotient of the *Hodge bundle*, the complex vector bundle $Z \to B$ whose fiber at a point $X \in B$ equals the complex vector space of holomorphic one-forms on X. This bundle is in general not trivial as a complex vector bundle. However, it is flat as a real vector bundle with symplectic fiber. Namely, the action of the mapping class group of S_g on the first cohomology of S_g defines a homomorphism $\rho : \Gamma_g \to Sp(2g, \mathbb{Z})$, and the Hodge bundle is the bundle

$$Z = \tilde{B} \times H^1(S_g, \mathbb{R})/\pi_1(B)$$

where the action of $\pi_1(B)$ is defined by $(x, Y)g = (xg, \rho(g)^{-1}Y)$.

The right action of $H^1(S_g, \mathbb{R})$ by translation commutes with the action by ρ and hence there is a quotient bundle $W = Z/H^1(S_g, \mathbb{Z}) \to B$ whose fiber at X just equals the Jacobian $J(X)$ of X.

The Jacobian $\mathcal{J}(X)$ parameterizes divisors of degree 0 on X up to linear equivalence, that is, up to adding a divisor of a meromorphic function. Thus $\mathcal{J}(X)$ can

be viewed as the subgroup of the Picard group of X parameterizing holomorphic line bundles of degree zero. The group structure is given by the tensor product, with the trivial line bundle as the neutral element.

A *section* of the bundle $\Theta : W \to B$ is a smooth map $\sigma : B \to W$ such that $\Theta \circ \sigma = \mathrm{Id}$. Such a section then determines a splitting of the extension

$$1 \to A \to \pi_1(W) \xrightarrow{\Theta_*} \pi_1(B) \to 1, \tag{5}$$

that is, for some $x \in B$ it defines a homomorphism $\sigma_* : \pi_1(B, x) \to \pi_1(W, \sigma(x))$ such that $\Theta_* \circ \sigma_* = \mathrm{Id}$.

The following is a topological analog of a well known statement on group extensions as discussed in Proposition IV.2.1 of [**Bro82**]. Namely, if G is a discrete group and if A is any G-module, then A-conjugacy classes of splittings of the split extension

$$1 \to A \to A \rtimes G \to G \to 1 \tag{6}$$

are in 1-1-correspondence with the elements of $H^1(G, A)$.

In the topological setting, a conjugacy class of an element in the fundamental group $\pi_1(Y, y)$ of a path connected topological space Y is just a free homotopy class of loops in Y. Being able to move the basepoint continuously is the main difference to the setting of discrete groups. With this in mind, the next observation gives a topological interpretation of the sequence (6) in our setting. Here the G-module A is just the integral cohomology group $H^1(S_g, \mathbb{Z})$ with the monodromy action of $\pi_1(B)$ defined by the representation ρ.

Proposition 5.4. *Homotopy classes of sections $B \to W$ form a group which is isomorphic to $H^1(B, H^1(\pi_1(S_g), \mathbb{Z}))$.*

Proof. Let $\sigma : B \to W$ be a section. Then for some basepoint $x \in B$, the induced homomorphism $\sigma_* : \pi_1(B, x) \to \pi_1(W, \sigma(x))$ defines a splitting of the extension (5).

Let as before $Z \to B$ be Hodge bundle with fiber $H^1(S_g, \mathbb{R})$, viewed as an abelian group. Recall that we have $W = Z/H^1(S_g, \mathbb{Z})$. For each $x \in B$, there is a natural action of $H^1(S_g, \mathbb{R})$ on the fiber W_x of W over x. We claim that if η is another section of W then σ and η are homotopic if and only if there exists a section ρ of the bundle Z so that $\eta = \rho(\sigma)$, where the action of ρ is fiber preserving.

Namely, if ρ is any section of Z, then using the fiberwise group structure (or, alternatively, the fact that the fiber of Z is contractible), there is a smooth fiber preserving homotopy h_t of $\rho = h_1$ to the section h_0 of Z which associates to $x \in B$ the neutral element in $H^1(S_g, \mathbb{R}) = Z_x$. Then $s \to h_s\sigma$ is a fiber preserving homotopy between σ and $\rho\sigma$.

On the other hand, let us assume that η is homotopic to σ. Let h_t be a fiber preserving homotopy connecting $h_0 = \sigma$ to $h_1 = \eta$. Choose a point $x \in B$ and a preimage $q \in Z_x$ of $\sigma(x)$ in the fiber Z_x of Z at x. The path $t \to h(t, x) = h_t(x)$ lifts to a path $\tilde{h}(t, x)$ in Z_x beginning at q. We can write $\tilde{h}(1, x) = \beta(x) + q$ for some $\beta(x) \in H^1(S_g, \mathbb{R})$ (here we write the group multiplication additively). Now if $u \in Z_x$ is another preimage of $\sigma(x)$ in Z_x then $u = m + q$ for some $m \in A$, identified with the lattice in $H^1(S_g, \mathbb{R})$ defined by the complex structure on the fiber E_x of E. The path $t \to \tilde{h}(t, x) + m$ is the lift of $t \to h(t, x)$ through u. Thus the difference of the endpoints $\beta(x) = \tilde{h}(1, x) - \tilde{h}(0, x) \in H^1(S_g, \mathbb{R})$ does not depend on the choice of the preimage q of $\sigma(x)$ and hence only depends on h. Furthermore, the map

$x \to \beta(x)$ is continuous and hence defines a section of Z with $\eta = \beta(\sigma)$. This is what we wanted to show.

Let now $C^\infty(W)$ and $C^\infty(Z)$ be the sheaf of smooth sections of W and Z, respectively. Since W has a fiberwise structure of an abelian group, these are sheaves of abelian groups. Similarly we define the sheaf $C^\infty(H^1(S_g,\mathbb{Z}))$ of smooth sections of the fiber bundle with fiber the group $A = H^1(S_g,\mathbb{Z})$ (this is meant to be the twisted bundle). We then obtain a short exact sequence of sheaves

$$0 \to C^\infty(H^1(S_g,\mathbb{Z})) \to C^\infty(Z) \to C^\infty(W) \to 0.$$

It follows from the above discussion that $H^0(C^\infty(W))/\xi H^0(C^\infty(Z))$ can naturally be identified with the group of homotopy classes of sections of W where ξ is the projection map $H^0(C^\infty(Z)) \to H^0(C^\infty(W))$. On the other hand, the short exact sequence of sheaves induces a long exact cohomology sequence

$$\cdots \to H^0(C^\infty(Z)) \to H^0(C^\infty(W)) \to H^1(C^\infty(H^1(S_g,\mathbb{Z}))) \to H^1(C^\infty(Z)) \to \cdots .$$

Since the sheaf $C^\infty(Z)$ is the sheaf of smooth sections of a flat vector bundle $Z \to B$, it is fine and hence acyclic. To see this note that a morphism of $C^\infty(Z)$ is a smooth section of the bundle $Z^* \otimes Z$ over B whose fiber over x equals the vector space of endomorphisms of Z_x, that is, it equals the vector space $Z_x^* \otimes Z_x$. This vector space has a distinguished real one-dimensional subspace consisting of constant multiples of the identity, and these one-dimensional subspaces define a trivial one-dimensional real subbundle L of $Z^* \otimes Z$. A smooth section of L can be identified with a smooth function on B. The identity morphism corresponds to the real number 1 in this interpretation.

Now if $\mathcal{U} = \{U_i\}$ is a locally finite covering of B, then there is a subordinate partition of unity, and using the identification of the fiber of the bundle L with $\mathbb{R}$, this partition of unity defines a partition of unity for the sheaf $C^\infty(Z)$, showing that this sheaf is fine and hence acyclic. Thus we obtain the short exact sequence

$$H^0(C^\infty(Z)) \to H^0(C^\infty(W)) \to H^1(C^\infty(H^1(S_g,\mathbb{Z}))) \to 0.$$

But $H^1(C^\infty(H^1(S_g,\mathbb{Z}))) = H^1(B, H^1(S_g,\mathbb{Z}))$ by de Rham's theorem which completes the proof of the proposition. □

Let again $\sigma : B \to W$ be a smooth section. Then by Abel's theorem, for every $x \in B$, the value $\sigma(x)$ of σ at x can be thought of as a holomorphic line bundle of degree 0 on the fiber E_x of E at x depending smoothly on x. Thus σ defines a fiberwise holomorphic line bundle $L(\sigma) \to E$.

For our next observation, let us denote by $\mathcal{F}$ the sheaf of smooth functions on E whose restriction to a fiber is holomorphic, and let $\mathcal{F}^*$ be the subsheaf of functions which vanish nowhere. These are sheaves of abelian groups.

Lemma 5.5. *The cohomology group $H^1(E,\mathcal{F}^*)$ parameterizes smooth complex line bundles on E whose restrictions to a fiber are holomorphic.*

Proof. A smooth complex line bundle L on E whose restriction to a fiber is holomorphic is defined by some good cover $\mathcal{U} = \{U_i \mid i\}$ of E with the property that for each i, the intersection of U_i with a fiber is a disk or empty, and smooth trivializations of L on each of the open sets $U_i \in \mathcal{U}$ whose restrictions to the intersections of U_i with a fiber are holomorphic.

Then transition functions for L on $U_i \cap U_j$ are smooth $\mathbb{C}^*$-valued functions on $U_i \cap U_j$ whose restrictions to a fiber are holomorphic. Thus these functions define a one-cocycle for $\mathcal{U}$ with values in $\mathcal{F}^*$, and then they define a class in $H^1(E, \mathcal{F}^*)$.

Vice versa, each one-cocycle for $\mathcal{U}$ with values in $\mathcal{F}^*$ defines a smooth fiberwise holomorphic line bundle on E by gluing the trivial bundle over the sets U_i with the transition functions on $U_i \cap U_j$ defined by the cocycle. Passing to cohomology yields the lemma. □

Smooth line bundles on E are defined by classes in the cohomology group $H^1(E, (C^\infty)^*)$ where C^∞ is the sheaf of smooth functions on E and $(C^\infty)^*$ is the sheaf of smooth functions vanishing nowhere. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C^\infty \xrightarrow{\exp} (C^\infty)^* \longrightarrow 0$$

then induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^1(E, C^\infty) \longrightarrow H^1(E, (C^\infty)^*) \xrightarrow{\delta} H^2(E, \mathbb{Z}) \longrightarrow \cdots .$$

Since the sheaf C^∞ is fine, this sequence describes explicitly the parameterization of the group of isomorphism classes of smooth line bundles on E by their Chern classes, that is, by the group $H^2(E, \mathbb{Z})$.

Our next goal is to verify that homotopic sections of the bundle W define smoothly equivalent line bundles, or, equivalently, line bundles with the same Chern class, and that this Chern class is just the cohomology class in $H^1(B, H^1(S_g, \mathbb{Z}))$ corresponding to this homotopy class by Proposition 5.4.

To this end note that since the second cohomology of E is representable, each class $\alpha \in H^2(E, \mathbb{Z})$ is the Chern class of a smooth complex line bundle, obtained as the pull-back of the tautological line bundle under a smooth map $f : E \to \mathbb{C}P^N$ for some sufficiently large N which defines α. Homotopic maps define isomorphic line bundles.

Now if $L \to E$ is a smooth complex line bundle, then the *degree* of L can be defined as the evaluation of its Chern class $c_1(L)$ on one (and hence on any) fiber. Denote as before by $c_1(\nu)$ the Chern class of the vertical cotangent bundle. By Proposition 5.2, the subgroup $H^1(B, H^1(S_g, \mathbb{Z}))$ of $H^2(E, \mathbb{Z})$ is contained in the kernel of the homomorphism $\alpha \to \alpha \cup c_1(\nu)$. The restriction of this homomorphism to $\Pi^* H^2(B, \mathbb{Z})$ is injective. The next proposition provides the connection between the constructions in this section.

Proposition 5.6. *Let $E \to B$ be a surface bundle over a surface. Then the cohomology group $H^1(B, H^1(S_g, \mathbb{Z})) \subset H^2(E, \mathbb{Z})$ parameterizes isomorphism classes of fiberwise holomorphic line bundles L on E of degree 0 whose Chern class $c_1(L)$ satisfies $c_1(L) \cup c_1(\nu) = 0$.*

Proof. By the above discussion, a smooth section σ of the bundle W defines on the one hand an equivalence class of a complex line bundle $L(\sigma)$ on E whose restrictions to a fiber is holomorphic of degree 0. On the other hand, by Proposition 5.4 and Proposition 5.2, it defines a cohomology class in $H^1(B, H^1(S_g, \mathbb{Z})) \subset H^2(E, \mathbb{Z})$. We have to show that this cohomology class is just the first Chern class of $L(\sigma)$. As smooth line bundles on E with the same Chern class are equivalent, this implies that homotopic sections of W define smoothly equivalent line bundles on E, a fact which can also be verified directly.

Consider again the sheaf $\mathcal{F}$ of smooth functions on E which are fiberwise holomorphic, the subsheaf $\mathcal{F}^*$ of functions in $\mathcal{F}$ which vanish nowhere, and the sheaf $\mathbb{Z}$ of locally constant $\mathbb{Z}$-valued functions.

The short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{F} \xrightarrow{\exp} \mathcal{F}^* \longrightarrow 0 \tag{7}$$

induces a long exact sequence in cohomology. Since the sheaf C^∞ of smooth functions on E is fine, the inclusions $\mathcal{F} \to C^\infty$ and $\mathcal{F}^* \to (C^\infty)^*$ then determine an exact commutative diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\exp} & H^1(E,\mathcal{F}^*) & \longrightarrow & H^2(E,\mathbb{Z}) & \longrightarrow & H^2(E,\mathcal{F}) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\mathrm{Id}} & & \downarrow & & \\ \cdots & \xrightarrow{\exp} & H^1(E,(C^\infty)^*) & \xrightarrow{\delta} & H^2(E,\mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \tag{8}$$

We claim that the homomorphism η is surjective. By exactness and since the diagram commutes, this follows if we can show that $H^2(E,\mathcal{F}) = 0$. However, if E_x is any fiber of E then we have $H^2(E_x,\mathcal{O}) = 0$ where as usual, $\mathcal{O}$ is the sheaf of holomorphic functions on E_x. Namely, by Serre duality, this cohomology group can be identified with the space of holomorphic two-forms on E_x, and this space vanishes since the complex dimension of E_x equals one. On the other hand, the sheaf of smooth functions on B is fine and hence the Leray spectral sequence shows that indeed, $H^2(E,\mathcal{F}) = H^0(B, H^2(E_x,\mathcal{O})) = 0$.

Since $H^1(E,C^\infty) = 0$, the homomorphism δ is an isomorphism. This yields that every smooth line bundle on E is smoothly equivalent to a line bundle whose restriction to a fiber is holomorphic. It also follows that the homomorphism η maps $H^1(E,\mathcal{F}^*)/\exp(H^1(E,\mathcal{F}))$ isomorphically onto $H^1(E,(C^\infty)^*)$. Hence associating to an element in this group its Chern class is an isomorphism.

On the other hand, for each $x \in B$ the vector space $H^1(E_x,\mathcal{O})$ is just the space of holomorphic one-forms on the fiber E_x of E over x by Serre duality. That is, $H^1(E_x,\mathcal{F})$ is the fiber at x of the Hodge bundle $Z \to B$. Thus using the fact that the sheaf of smooth sections of Z is fine (see the discussion in the proof of Proposition 5.4 for details), the Leray spectral sequence shows that $H^1(E,\mathcal{F}) = H^0(Z)$, the vector space of smooth sections of Z.

Now consider the part

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^1(E,\mathbb{Z}) & \xrightarrow{exp} & H^1(E,\mathcal{F}) & \longrightarrow & H^1(E,\mathcal{F}^*) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle\mathrm{Id}} & & \downarrow & & \downarrow{\scriptstyle\eta} & & \\ \cdots & \longrightarrow & H^1(E,\mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & H^1(E,(C^\infty)^*) & \longrightarrow & \cdots \end{array} \tag{9}$$

of the above exact diagram. It shows that the kernel of η can be identified with $H^0(Z)/\exp H^1(E,\mathbb{Z})$. By Proposition 5.4 and its proof, this subgroup is precisely the group of sections of the bundle W which are homotopic to the trivial section. As a consequence, homotopic sections of W define line bundles with the same Chern class and hence line bundles which are smoothly equivalent. Furthermore, associating to a homotopy class of a section of W the Chern class of the line bundle it defines is an isomorphism of the space of all homotopy classes of sections onto $H^1(B, H^1(S_g,\mathbb{Z})) \subset H^2(E,\mathbb{Z})$. $\square$

REMARK 5.7. The assumption that $E \to B$ is a surface bundle over a surface was only used through the conclusion of Proposition 5.2.

REMARK 5.8. The proof of Proposition 5.6 also shows that any smooth complex line bundle on E is smoothly equivalent to a line bundle whose restriction to each fiber is holomorphic.

We use similar ideas to show

PROPOSITION 5.9. *Let $E \to B$ be a surface bundle which admits a section. Then there exists a cohomology class $e \in H^2(E, \mathbb{Z})$ with $(2g-2)e = c_1(\nu)$, and for all q we have*

$$H^q(E, \mathbb{Z}) = H^q(B, \mathbb{Z}) \oplus H^{q-1}(B, H^1(S_g, \mathbb{Z})) \oplus eH^{q-2}(B, \mathbb{Z}).$$

PROOF. Let $f : B \to E$ be a section of the surface bundle $\Pi : E \to B$. Assume as before that each fiber E_x of E is equipped with a complex structure varying smoothly with x. Then for each $x \in B$, the point $f(x) \in E_x$ can be thought of as a divisor in E_x defining a complex line bundle L_x of degree 1 on E_x. As these line bundles depend smoothly on x, they fit together to a fiberwise holomorphic line bundle L of fiberwise degree one.

Let $c_1(L) \in H^2(E, \mathbb{Z})$ be the Chern class of L. Consider the inclusion $\iota : E_x \to E$. As the fiberwise degree of L equals one, we know that $\iota^* c_1(L)$ is a generator of $H^2(E_x, \mathbb{Z})$. Thus the spectral sequence argument in the proof of Proposition 3.1 of [**Mo87**] applies to compute the cohomology of E with coefficients in $\mathbb{Z}$ (this argument only uses surjectivity of ι^* for the coefficient ring under consideration), yielding the formula in Proposition 5.1 but with coefficients $\mathbb{Z}$. □

REMARK 5.10. Although the existence of a section for a surface bundle $E \to B$ is simply equivalent to stating that the induced homomorphism $\pi_1(B) \to \Gamma_g$ lifts to a homomorphism $\pi_1(B) \to \Gamma_{g,1}$, we do not know how to characterize this property in purely topological terms of the surface bundle. In fact, if $E \to B$ is a surface bundle over a surface, then E is bordant to a surface bundle over a surface which admits a section, see [**H83**].

Proposition 5.9 describes a correspondence between line bundles on a surface bundle over a surface $\Pi : E \to B$, their Chern classes and their Poincaré dual. This can be extended as follows. Namely, a section $f : B \to E$ can be thought of as a section in the bundle over B whose fiber consists of all effective divisors of degree 1 on the fiber of E. This viewpoint generalizes as follows.

An effective divisor on a Riemann surface X of degree $k \geq 1$ is just a weighted collection of points on X with positive weights which sum up to k. Thus there is a natural topology on the total space $\mathcal{D}^k$ of all effective divisors of degree k on the fibers of E defined as follows. Let $V \to B$ be the fiber product of k copies of the fiber of E. There is a natural smooth fiber preserving free action of the symmetric group in k variables on V. Then $\mathcal{D}^k$ can be identified with the quotient of this action and hence it inherits from V the quotient topology. By abuse of notation, we denote again by Π the projection $\mathcal{D}^k \to B$.

Let us assume that there exists a section $\psi : B \to \mathcal{D}^k$. Associate to this section the fiberwise holomorphic line bundle $L(\psi)$ whose restriction of a fiber E_x is dual to the divisor $\psi(x)$, and associate to $L(\psi)$ its Chern class $c_1(L(\psi)) \in H^2(E, \mathbb{Z})$.

Now the section ψ of $\mathcal{D}^k$ defines a cycle in E which can be seen as follows. The projection of the fat diagonal of V is submanifold N of $\mathcal{D}^k$ of fiberwise positive real

codimension 2. Thus by transversality, we may assume that ψ is transverse to this submanifold. Then there are (at most) finitely many points $x_1, \dots, x_m$ such that $\psi(x_i) \in N$, and for each i, the image of x_i in E_{x_i} consists of $m-1$ distinct points, with precisely one point of multiplicity 2.

Choose a triangulation of B containing the points x_i as vertices. For any point $x \notin \{x_1, \dots, x_m\}$, the preimage of x in E_x defined by ψ (that is, the union of all points of $\psi(x)$) consists of precisely m points moving smoothly with the base. Thus each two-simplex of the triangulation has precisely k preimages in E, and the same holds true for all one-simplices. The preimages of the points x_i consist of only $m-1$ points. It follows from this construction that the union of these triangles is a surface $\Sigma \subset E$. The orientation of B induces an orientation of Σ. The restriction of the projection Π to Σ is a branched cover, ramified precisely at the points x_i. Thus Σ defines a homology class $\beta(\psi) \in H_2(E, \mathbb{Z})$. We have

Proposition 5.11. *The class $\beta(\psi)$ is Poincaré dual to $c_1(L(\psi))$.*

Proof. Let us recall how to construct from the embedded surface Σ which is transverse to the fibers of E a line bundle whose restriction to a fiber is holomorphic. Namely, for a point $x \in \Sigma$, choose a neighborhood U of x in E so that $U \cap \Sigma$ is a smooth disk. There exists a smooth $\mathbb{C}$-valued function f on U whose restriction to a fiber is holomorphic and with nowhere vanishing derivative, so that $U \cap \Sigma$ is the level set of level zero. Choose a covering of Σ by such sets, with corresponding functions. On the intersections of these sets, the quotients of these functions do not vanish. Thus these functions define a cocycle whose cohomology class defines a line bundle. This line bundle has a smooth section which is fiberwise holomorphic and vanishes precisely on Δ. In particular, this line bundle coincides with the line bundle $L(\psi)$.

Now if the section ψ intersects the fat diagonal N of $\mathcal{D}^k$, then at the finitely many intersection points with N, choose the function so that it has a double zero at that point and proceed as before.

Since every class in $H_2(E, \mathbb{Z})$ can be represented by a smooth map $f : M \to E$ where M is a closed oriented surface of some genus $h \geq 0$, for the proof of the proposition it now suffices to show the following.

Assume without loss of generality that $f(M)$ intersects Σ transversely in finitely many points $y_1, \dots, y_p \in E - \cup_j \Pi^{-1}(x_j)$, with intersection index $\sigma(y_i) \in \pm 1$. We have to show that $c_1(L(\psi))(f(M)) = \sum_i \sigma(y_i)$.

As the line bundle $L(\psi)$ is trivial on $E - \Sigma$, the pull-back of $L(\psi)$ under f is a complex line bundle on M with a section which vanishes precisely to first order at the points y_i, and the index of this zero is ± 1 depending on whether the intersection is positive or negative. On the other hand, $c_1(f^* L(\psi))(M)$ equals the number of zeros of a section of $f^* L(\psi)$, counted with sign and multiplicities, provided that this section is transverse to the zero section. Together this means that $c_1(f^* L(\psi))) = \sum_i \sigma(y_i)$. Since f was an aribtrary map of a closed oriented surface M into E, we conclude that indeed, for any second homology class $\alpha \in H_2(E, \mathbb{Z})$ we have $\alpha \cdot \Sigma = c_1(L(\psi))(\alpha)$ as claimed. $\square$

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