BOWEN'S CONSTRUCTION FOR THE TEICHMÜLLER FLOW

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Abstract. Let $\mathcal{Q}$ be a connected component of a stratum in the moduli space of abelian or quadratic differentials for a non-exceptional Riemann surface S of finite type. We show that the probability measure on $\mathcal{Q}$ in the Lebesgue measure class which is invariant under the Teichmüller flow is obtained by Bowen's construction.

1. Introduction

Given a non-exceptional surface S of genus $g \geq 0$ with $m \geq 0$ punctures, the *Teichmüller flow* Φ^t acts on a component $\mathcal{Q}$ of a stratum in the moduli space of area one abelian or quadratic differentials for S. This flow on $\mathcal{Q}$ has many properties which resemble the properties of an Anosov flow. For example, there is a pair of transverse invariant foliations, and there is an invariant mixing Borel probability measure λ in the Lebesgue measure class which is absolutely continuous with respect to these foliations, with conditional measures which are uniformly expanded and contracted by the flow [M82, V86]. The measure λ is even exponentially mixing, i.e. exponential decay of correlations for Hölder observables holds true [AGY06, AR09, AG10, H11].

The entropy h of the Lebesgue measure λ is the supremum of the topological entropies of the restriction of Φ^t to compact invariant subsets of $\mathcal{Q}$ (see [H10b] for a proof of this fact for the main stratum and note that the argument carries over without changes to all strata). The entropy of λ also is the supremum of the entropies of all Φ^t-invariant Borel probability measures on the component. Moreover, λ is the unique invariant measure of maximal entropy [BG07, H11].

An Anosov flow Ψ^t on a compact manifold M also admits a unique Borel probability measure μ of maximal entropy. This measure can be obtained as follows [B73, Mar04]. Every periodic orbit γ of Ψ^t of prime period $\ell(\gamma) > 0$ supports a unique Ψ^t-invariant Borel measure $\delta(\gamma)$ of total mass $\ell(\gamma)$. If $h > 0$ is the topological entropy of Ψ^t then μ is the (unique) weak limit of the sequence of measures

$$e^{-hR} \sum_{\ell(\gamma) \leq R} \delta(\gamma)$$

Date: December 2, 2012.
Keywords: Strata, Teichmueller flow, periodic orbits, equidistribution
AMS subject classification: 37C40, 37C27, 30F60
Research partially supported by a grant of the DFG.

as $R \to \infty$. In particular, the number of periodic orbits of period at most R is asymptotic to e^{hR}/hR as $R \to \infty$.

The goal of this paper is to show that for any connected component $\mathcal{Q}$ of a stratum of abelian or quadratic differentials the Φ^t-invariant Lebesgue measure λ on $\mathcal{Q}$ can be obtained in the same way.

For a precise formulation, we say that a family $\{\mu_i\}$ of finite Borel measures on the moduli space $\mathcal{H}(S)$ of area one abelian differentials or on the moduli space $\mathcal{Q}(S)$ of area one quadratic differentials *converges weakly* to λ if for every continuous function f on $\mathcal{H}(S)$ or on $\mathcal{Q}(S)$ with compact support we have

$$\int f d\mu_i \to \int f d\lambda.$$

Let $\Gamma(\mathcal{Q})$ be the set of all periodic orbits for Φ^t contained in $\mathcal{Q}$. For $\gamma \in \Gamma(\mathcal{Q})$ let $\ell(\gamma) > 0$ be the prime period of γ and denote by $\delta(\gamma)$ the Φ^t-invariant Lebesgue measure on γ of total mass $\ell(\gamma)$.

Theorem. *For every component $\mathcal{Q}$ of a stratum in the moduli space of abelian or quadratic differentials, the measures*

$$\mu_R = e^{-hR} \sum_{\gamma \in \Gamma(\mathcal{Q}), \ell(\gamma) \leq R} \delta(\gamma)$$

converge as $R \to \infty$ weakly to the Lebesgue measure on $\mathcal{Q}$.

In other words, periodic orbits in components of strata are equidistributed.

The theorem implies that as $R \to \infty$, the number of periodic orbits in $\mathcal{Q}$ of period at most R is asymptotically not smaller than e^{hR}/hR. However, since the closure in $\mathcal{Q}(S)$ of a component $\mathcal{Q}$ of a stratum is non-compact, it does not yield a precise asymptotic growth rate for all periodic orbits in $\mathcal{Q}$. Namely, there may be a set of periodic orbits in $\mathcal{Q}$ whose growth rate exceeds h and which eventually exit every compact subset of $\mathcal{Q}(S)$.

However, a precise counting result is an immediate consequence of the theorem and deep results of Eskin and Mirzakhani [EM08]. They showed that the asymptotic growth rate of periodic orbits for the Teichmüller flow which lie deeply in the cusp of moduli space is strictly smaller than the entropy h of the Lebesgue measure. More recently, the corresponding result for components of strata was also established [EMR12, H11]. Together this yields

Corollary. *For any component $\mathcal{Q}$ of a stratum in the moduli space of area one abelian or quadratic differentials, the number of periodic orbits for the Teichmüller flow on $\mathcal{Q}$ of period at most R is asymptotic to e^{hR}/hR as $R \to \infty$.*

The proof of the above theorem uses ideas which were developed by Margulis for hyperbolic flows (see [Mar04] for an account with comments). This strategy is by now standard, and the main task is to overcome the difficulty of absence of hyperbolicity for the Teichmüller flow in the thin part of moduli space and the absence of nice product coordinates near a boundary point of a stratum.

Relative homology coordinates [V90] define local product structures for strata. As the main technical tool of this paper, we construct dynamically controlled product coordinates on neighborhoods of recurrent points in $\mathcal{Q}$.

These coordinates do not extend in a straightforward way to points in the boundary of the stratum. To overcome this difficulty we note that symmetric complex functions can be used to construct coordinates on neighborhoods in $\overline{\mathcal{Q}}$ of boundary points of $\mathcal{Q}$. This leads to a construction of parametrizations of such neighborhoods with a finite unions of sets with local product structures intersecting only in the boundary of $\mathcal{Q}$.

Absence of hyperbolicity in the thin part of moduli space is dealt with using the curve graph similar to the strategy developed in [H10b]. The curve graph is also used to establish a strong version of the Anosov closing lemma. Integration of the Hodge norm as discussed in [ABEM12] and some standard ergodic theory yield the necessary measure estimates.

2. Laminations and the curve graph

Let S be an oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g - 3 + m \geq 2$, i.e. that S is not a sphere with at most four punctures or a torus with at most one puncture.

The *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete finite volume hyperbolic metrics on S under the action of the group of diffeomorphisms of S which are isotopic to the identity. The fibre bundle $\tilde{\mathcal{Q}}(S)$ over $\mathcal{T}(S)$ of all *marked holomorphic quadratic differentials* of area one can be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the *Teichmüller metric* $d_{\mathcal{T}}$. Each such differential is holomorphic on the complement of the punctures and has at most a simple pole at each puncture. The *Teichmüller flow* Φ^t on $\tilde{\mathcal{Q}}(S)$ commutes with the action of the *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving self-homeomorphisms of S. Therefore this flow descends to a flow on the quotient orbifold $\mathcal{Q}(S) = \tilde{\mathcal{Q}}(S)/\mathrm{Mod}(S)$, again denoted by Φ^t.

2.1. **Geodesic laminations.**

A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination ν is called *minimal* if each of its half-leaves is dense in ν. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*.

Every geodesic lamination ν consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of ν either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components. A geodesic lamination ν *tightly fills up* S if its complementary components are topological discs or once punctured monogons, i.e. once punctured discs bounded by a single leaf of ν. Note that this definition deviates from the standard definition of filling which only requires that a geodesic lamination decomposes S into discs and once punctured discs.

The set $\mathcal{L}$ of all geodesic laminations on S can be equipped with the restriction of the *Hausdorff topology* for compact subsets of S. With respect to this topology, the space $\mathcal{L}$ is compact.

The projectivized tangent bundle $PT\nu$ of a geodesic lamination ν is a compact subset of the projectivized tangent bundle PTS of S. The geodesic lamination ν is *orientable* if there is an continuous orientation of the tangent bundle of ν. This is equivalent to stating that there is a continuous section $PT\nu \to T^1S|\nu$ where T^1S denotes the unit tangent bundle of S.

Definition 2.1. A *large geodesic lamination* is a geodesic lamination ν which tightly fills up S and can be approximated in the Hausdorff topology by simple closed geodesics.

A minimal geodesic lamination ν can be approximated in the Hausdorff topology by simple closed geodesics (Lemma 4.2.15 of [CEG87]) and hence if ν tightly fills up S then ν is large. Moreover, the set of all large geodesic laminations is closed with respect to the Hausdorff topology and hence it is compact.

The *topological type* of a large geodesic lamination ν is a tuple

$$(m_1, \dots, m_\ell; -m) \text{ where } 1 \leq m_1 \leq \dots \leq m_\ell, \sum_i m_i = 4g - 4 + m$$

such that the complementary components of ν which are topological discs are m_i+2-gons. Let

$$\mathcal{LL}(m_1, \dots, m_\ell; -m)$$

be the space of all large geodesic laminations of type $(m_1, \dots, m_\ell; -m)$ equipped with the restriction of the Hausdorff topology for compact subsets of S. A geodesic lamination is called *complete* if it is large of type $(1, \dots, 1; -m)$. The complementary components of a complete geodesic lamination are all trigons or once punctured monogons.

A *measured geodesic lamination* is a geodesic lamination ν equipped with a translation invariant transverse measure ξ such that the ξ-weight of every compact arc in S with endpoints in $S - \nu$ which intersects ν nontrivially and transversely is positive. We say that ν is the *support* of the measured geodesic lamination. The geodesic lamination ν is *uniquely ergodic* if up to scale, ξ is the only transverse measure with support ν.

The space $\mathcal{ML}$ of measured geodesic laminations equipped with the weak*-topology admits a natural continuous action of the multiplicative group $(0, \infty)$. The quotient under this action is the space $\mathcal{PML}$ of *projective measured geodesic laminations* which is homeomorphic to the sphere $S^{6g-7+2m}$.

Every simple closed geodesic c on S defines a measured geodesic lamination. The geometric intersection number between simple closed curves on S extends to a continuous function ι on $\mathcal{ML} \times \mathcal{ML}$, the *intersection form.* We say that a pair $(\xi, \mu) \in \mathcal{ML} \times \mathcal{ML}$ of measured geodesic laminations *jointly fills up S* if for every measured geodesic lamination $\eta \in \mathcal{ML}$ we have $\iota(\eta,\xi)+\iota(\eta,\mu)>0$. This is equivalent to stating that every complete simple (possibly infinite) geodesic on S intersects either the support of ξ or the support of μ transversely.

2.2. **The curve graph.** The *curve graph* $\mathcal{C}(S)$ of S is the locally infinite metric graph whose vertices are the free homotopy classes of essential simple closed curves on S, i.e. curves which are neither contractible nor freely homotopic into a puncture. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The mapping class group $\mathrm{Mod}(S)$ of S acts on $\mathcal{C}(S)$ as a group of simplicial isometries.

The curve graph $\mathcal{C}(S)$ is a hyperbolic geodesic metric space [MM99] and hence it admits a *Gromov boundary* $\partial\mathcal{C}(S)$. For $c\in\mathcal{C}(S)$ there is a complete distance function δ_c on $\partial\mathcal{C}(S)$ of uniformly bounded diameter, and there is a number $\rho>0$ such that

$$\delta_c \le e^{\rho d(c,a)}\delta_a \text{ for all } c,a\in\mathcal{C}(S).$$

The group $\mathrm{Mod}(S)$ acts on $\partial\mathcal{C}(S)$ as a group of homeomorphisms.

Let $\kappa_0>0$ be a *Bers constant* for S, i.e. κ_0 is such that for every complete hyperbolic metric on S of finite volume there is a pants decomposition of S consisting of pants curves of length at most κ_0. Define a map

$$\Upsilon_{\mathcal{T}}:\mathcal{T}(S)\to\mathcal{C}(S) \tag{1}$$

by associating to $x\in\mathcal{T}(S)$ a simple closed curve of x-length at most κ_0. Then there is a number $c>0$ such that

$$d_{\mathcal{T}}(x,y)\ge d(\Upsilon_{\mathcal{T}}(x),\Upsilon_{\mathcal{T}}(y))/c-c \tag{2}$$

for all $x,y\in\mathcal{T}(S)$ ([MM99] and see the discussion in [H10a]).

For a number $L>1$, a map $\gamma:[0,s)\to\mathcal{C}(S)$ ($s\in(0,\infty]$) is an *L-quasi-geodesic* if for all $t_1,t_2\in[0,s)$ we have

$$|t_1-t_2|/L-L\le d(\gamma(t_1),\gamma(t_2))\le L|t_1-t_2|+L.$$

A map $\gamma:[0,\infty)\to\mathcal{C}(S)$ is called an *unparametrized L-quasi-geodesic* if there is an increasing homeomorphism $\varphi:[0,s)\to[0,\infty)$ ($s\in(0,\infty]$) such that $\gamma\circ\varphi$ is an L-quasi-geodesic. We say that an unparametrized quasi-geodesic is *infinite* if its image set has infinite diameter. The following important result was established in [MM99].

Theorem 2.2. *There is a number $p>1$ such that the image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic is an unparametrized p-quasi-geodesic.*

Choose a smooth function $\sigma:[0,\infty)\to[0,1]$ with $\sigma[0,\kappa_0]\equiv 1$ and $\sigma[2\kappa_0,\infty)\equiv 0$. For each $x\in\mathcal{T}(S)$, the number of essential simple closed curves c on S whose x-length $\ell_x(c)$ (i.e. the length of a geodesic representative in its free homotopy class) does not exceed $2\kappa_0$ is bounded from above by a constant not depending on x, and the diameter of the subset of $\mathcal{C}(S)$ containing these curves is uniformly bounded as well. Thus we obtain for every $x \in \mathcal{T}(S)$ a finite Borel measure μ_x on $\mathcal{C}(S)$ by defining

$$\mu_x = \sum_{c \in \mathcal{C}(S)} \sigma(\ell_x(c))\Delta_c$$

where Δ_c denotes the Dirac mass at c. The total mass of μ_x is bounded from above and below by a universal positive constant, and the diameter of the support of μ_x in $\mathcal{C}(S)$ is uniformly bounded as well. Moreover, the measures μ_x depend continuously on $x \in \mathcal{T}(S)$ in the weak*-topology. This means that for every bounded function $f : \mathcal{C}(S) \to \mathbb{R}$ the function $x \to \int f d\mu_x$ is continuous.

For $x \in \mathcal{T}(S)$ define a distance δ_x on $\partial\mathcal{C}(S)$ by

$$\delta_x(\xi, \zeta) = \int \delta_c(\xi, \zeta) d\mu_x(c)/\mu_x(\mathcal{C}(S)). \tag{3}$$

The distances δ_x are equivariant with respect to the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ and $\partial\mathcal{C}(S)$. Moreover, there is a constant $\kappa > 0$ such that

$$\delta_x \leq e^{\kappa d_{\mathcal{T}}(x,y)}\delta_y \text{ and } \kappa^{-1}\delta_y \leq \delta_{\Upsilon_{\mathcal{T}}(y)} \leq \kappa\delta_y \tag{4}$$

for all $x, y \in \mathcal{T}(S)$ (see p.230 and p.231 of [H09b]).

An area one quadratic differential $z \in \tilde{\mathcal{Q}}(S)$ is determined by a pair (μ, ν) of measured geodesic laminations which jointly fill up S and such that $\iota(\mu, \nu) = 1$. The laminations μ, ν are called *vertical* and *horizontal*, respectively. Namely, Levitt [L83] constructed from a measured foliation on S a measured geodesic lamination, and the measured geodesic lamination determines the measured foliation up to Whitehead moves. On the other hand, a pair $(\hat{\mu}, \hat{\nu})$ of measured foliations is the pair consisting of the horizontal and the vertical measured foliation for a quadratic differential q on S if and only if the corresponding measured geodesic laminations jointly fill up S.

For $z \in \tilde{\mathcal{Q}}(S)$ let $W^u(z) \subset \tilde{\mathcal{Q}}(S)$ be the set of all quadratic differentials whose horizontal projective measured geodesic laminations coincide with the horizontal projective measured geodesic lamination of z. The space $W^u(z)$ is called the *unstable* manifold of z, and these unstable manifolds define the *unstable foliation* W^u of $\tilde{\mathcal{Q}}(S)$. The *strong unstable manifold* $W^{su}(z) \subset W^u(z)$ is the set of all quadratic differentials whose horizontal measured geodesic laminations coincide with the horizontal measured geodesic lamination of z. These sets define the *strong unstable* foliation W^{su} of $\tilde{\mathcal{Q}}(S)$. The *flip* $\mathcal{F} : q \to \mathcal{F}(q) = -q$ exchanges the vertical and the horizontal measured lamination of a quadratic differential q. The image of the unstable (or the strong unstable) foliation of $\tilde{\mathcal{Q}}(S)$ under the flip $\mathcal{F}$ is the *stable* foliation W^s (or the *strong stable foliation* W^{ss}).

By the Hubbard-Masur theorem [HM79], for each $z \in \tilde{\mathcal{Q}}(S)$ the restriction to $W^u(z)$ of the canonical projection

$$P : \tilde{\mathcal{Q}}(S) \to \mathcal{T}(S)$$

is a homeomorphism. Thus the Teichmüller metric lifts to a complete path metric d^u on $W^u(z)$ (i.e. a distance function so that any two points can be connected by a minimal geodesic). Denote by d^{su} the restriction of this distance function to $W^{su}(z)$. Then $d^s = d^u \circ \mathcal{F}, d^{ss} = d^{su} \circ \mathcal{F}$ are distance functions on the leaves of the stable and strong stable foliation, respectively. For $z \in \tilde{\mathcal{Q}}(S)$ and $r > 0$ let moreover $B^i(z,r) \subset W^i(z)$ be the closed ball of radius r about z with respect to d^i $(i = u, su, s, ss)$.

Let

$$\tilde{\mathcal{A}} \subset \tilde{\mathcal{Q}}(S) \tag{5}$$

be the set of all marked quadratic differentials q such that the unparametrized quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)$ $(t \in [0,\infty))$ is infinite. Then $\tilde{\mathcal{A}}$ is the set of all quadratic differentials whose vertical measured geodesic lamination fills up S (i.e. its support decomposes S into ideal polygons and once punctured polygons, see [H06] for a comprehensive discussion of this result of Klarreich [Kl99]). There is a natural $\mathrm{Mod}(S)$-equivariant surjective map

$$F : \tilde{\mathcal{A}} \to \partial\mathcal{C}(S)$$

which associates to a point $z \in \tilde{\mathcal{A}}$ the endpoint of the infinite unparametrized quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)$ $(t \in [0,\infty))$.

Call a marked quadratic differential $z \in \tilde{\mathcal{Q}}(S)$ *uniquely ergodic* if the support of its vertical measured geodesic lamination is uniquely ergodic and fills up S. A uniquely ergodic quadratic differential is contained in the set $\tilde{\mathcal{A}}$ [H06, Kl99]. We have (Section 3 of [H09b])

Lemma 2.3. (1) *The map $F : \tilde{\mathcal{A}} \to \partial\mathcal{C}(S)$ is continuous and closed.*
(2) *If $z \in \tilde{\mathcal{Q}}(S)$ is uniquely ergodic then the sets $F(B^{su}(z,r) \cap \tilde{\mathcal{A}})$ $(r > 0)$ form a neighborhood basis for $F(z)$ in $\partial\mathcal{C}(S)$.*

For $z \in \tilde{\mathcal{A}}$ and $r > 0$ let

$$D(z,r)$$

be the closed ball of radius r about $F(z)$ with respect to the distance function δ_{Pz}. As a consequence of Lemma 2.3, if $z \in \tilde{\mathcal{Q}}(S)$ is uniquely ergodic then for every $r > 0$ there are numbers $r_0 < r$ and $\beta > 0$ such that

$$F(B^{su}(z,r_0) \cap \tilde{\mathcal{A}}) \subset D(z,\beta) \subset F(B^{su}(z,r) \cap \tilde{\mathcal{A}}). \tag{6}$$

3. Strata

As in Section 2, for a closed oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures let $\tilde{\mathcal{Q}}(S)$ be the bundle of marked area one holomorphic quadratic differentials with at most simple poles at the punctures over the Teichmüller space $\mathcal{T}(S)$ of marked complex structures on S.

A tuple $(m_1, \dots, m_\ell)$ of positive integers $1 \leq m_1 \leq \cdots \leq m_\ell$ with $\sum_i m_i = 4g - 4 + m$ defines a *stratum* $\tilde{\mathcal{Q}}(m_1, \dots, m_\ell; -m)$ in $\tilde{\mathcal{Q}}(S)$. This stratum consists of all marked area one quadratic differentials with m simple poles and ℓ zeros of order $m_1, \dots, m_\ell$ which are not squares of holomorphic one-forms. The stratum is a real hypersurface in a complex manifold of dimension $2g - 2 + m + \ell$.

The closure in $\tilde{\mathcal{Q}}(S)$ of a stratum is a union of components of strata. Strata are invariant under the action of the mapping class group $\mathrm{Mod}(S)$ of S and hence they project to strata in the moduli space $\mathcal{Q}(S) = \tilde{\mathcal{Q}}(S)/\mathrm{Mod}(S)$ of quadratic differentials on S with at most simple poles at the punctures. We denote the projection of the stratum $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m)$ by $\mathcal{Q}(m_1,\dots,m_\ell;-m)$.

The strata in moduli space need not be connected, but their connected components have been identified by Lanneau [L08]. A stratum in $\mathcal{Q}(S)$ has at most three connected components. The entropy h of the invariant Lebesgue measure on a component $\mathcal{Q}$ of a stratum $\mathcal{Q}(m_1,\dots,m_\ell;-m)$ just equals the dimension $2g-2+m+\ell$ [M82, V86], i.e. we have

$$h = 2g-2+m+\ell. \tag{7}$$

Similarly, if $m=0$ then we let $\tilde{\mathcal{H}}(S)$ be the bundle of marked area one holomorphic one-forms over Teichmüller space $\mathcal{T}(S)$ of S. For a tuple $k_1 \leq \dots \leq k_\ell$ of positive integers with $\sum_i k_i = 2g-2$, the stratum $\tilde{\mathcal{H}}(k_1,\dots,k_\ell)$ of marked area one holomorphic one-forms on S with ℓ zeros of order k_i $(i=1,\dots,\ell)$ is a real hypersurface in a complex manifold of dimension $2g-1+\ell$. It projects to a stratum $\mathcal{H}(k_1,\dots,k_\ell)$ in the moduli space $\mathcal{H}(S)$ of area one holomorphic one-forms on S. Strata of holomorphic one-forms in moduli space need not be connected, but the number of connected components of a stratum is at most three [KZ03]. Moreover, as before, the entropy of the invariant Lebesgue measure on a component of a stratum $\mathcal{H}(k_1,\dots,k_\ell)$ coincides with the dimension $2g-1+\ell$, i.e. we have

$$h = 2g-1+\ell. \tag{8}$$

Recall from Section 2 the definition of the strong stable, the stable, the unstable and the strong unstable foliation W^{ss}, W^s, W^u, W^{su} of $\tilde{\mathcal{Q}}(S)$. Let $\tilde{\mathcal{Q}}$ be a component of a stratum $\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;-m)$ of marked quadratic differentials or of a stratum $\tilde{\mathcal{H}}(m_1/2,\dots,m_\ell/2)$ of marked abelian differentials. Using period coordinates, one sees that every $q\in\tilde{\mathcal{Q}}$ has a connected neighborhood U in $\tilde{\mathcal{Q}}$ with the following properties [V90]. For $u\in U$ let $[u^v]$ (or $[u^h]$) be the vertical (or the horizontal) projective measured geodesic lamination of u. Then $\{[u^v] \mid u\in U\}$ is homeomorphic to an open ball in $\mathbb{R}^{h-1}$ (where $h>0$ is as in equation (7,8)). Moreover, for $q\in U$ the set

$$\{u\in U \mid [u^v]=[q^v]\} = W^s_{\tilde{\mathcal{Q}},\mathrm{loc}}(q) \subset W^s(q)$$

is a smooth connected local submanifold of U of (real) dimension h which is called the *local stable manifold* of q in $\tilde{\mathcal{Q}}$ (see [V90]). Similarly we define the *local unstable manifold* $W^u_{\tilde{\mathcal{Q}},\mathrm{loc}}(q)$ of q in $\tilde{\mathcal{Q}}$. If two such local stable (or unstable) manifolds intersect then their union is again a local stable (or unstable) manifold. The maximal connected set containing q which is a union of intersecting local stable (or unstable) manifolds is the *stable manifold* $W^s_{\tilde{\mathcal{Q}}}(q)$ (or the *unstable manifold* $W^u_{\tilde{\mathcal{Q}}}(q)$) of q in $\tilde{\mathcal{Q}}$. Note that $W^i_{\tilde{\mathcal{Q}}}(q)\subset W^i(q)$ $(i=s,u)$. A stable (or unstable) manifold is invariant under the action of the Teichmüller flow Φ^t.

Remark: There may be a component $\tilde{\mathcal{Q}}$ of a stratum and some $\tilde{q}\in\tilde{\mathcal{Q}}$ such that $W^s(\tilde{q})\cap\tilde{\mathcal{Q}}$ has infinitely many components.

The stable and unstable manifolds define smooth foliations $W^s_{\tilde{\mathcal{Q}}}, W^u_{\tilde{\mathcal{Q}}}$ of $\tilde{\mathcal{Q}}$ which are called the *stable* and *unstable* foliations of $\tilde{\mathcal{Q}}$, respectively. Define the *strong stable* foliation $W^{ss}_{\tilde{\mathcal{Q}}}$ (or the *strong unstable* foliation $W^{su}_{\tilde{\mathcal{Q}}}$) of $\tilde{\mathcal{Q}}$ by requiring that the leaf $W^{ss}_{\tilde{\mathcal{Q}}}(q)$ (or $W^{su}_{\tilde{\mathcal{Q}}}(q)$) through q is the subset of $W^s_{\tilde{\mathcal{Q}}}(q)$ (or of $W^u_{\tilde{\mathcal{Q}}}(q)$) of all marked quadratic differentials whose vertical (or horizontal) measured geodesic lamination equals the vertical (or horizontal) measured geodesic lamination of q. The strong stable foliation of $\tilde{\mathcal{Q}}$ is transverse to the unstable foliation of $\tilde{\mathcal{Q}}$.

The foliations $W^i_{\tilde{\mathcal{Q}}}$ $(i = ss, s, su, u)$ are invariant under the action of the stabilizer $\mathrm{Stab}(\tilde{\mathcal{Q}})$ of $\tilde{\mathcal{Q}}$ in $\mathrm{Mod}(S)$, and they project to Φ^t-invariant singular foliations $W^i_{\mathcal{Q}}$ of $\mathcal{Q} = \tilde{\mathcal{Q}}/\mathrm{Stab}(\tilde{\mathcal{Q}})$.

3.1. **Orbifold coordinates.** In this technical subsection we describe for every component $\mathcal{Q}$ of a stratum in the moduli space of quadratic differentials and for every point $q \in \mathcal{Q}$ a basis of neighborhoods of q in $\mathcal{Q}$ with local product structures. The material is well known to the experts but a bit difficult to find in the literature. In the course of the discussion we introduce some notations which will be used throughout.

We begin with a discussion of the structure of $\tilde{\mathcal{Q}}(S)$. For $\tilde{q} \in \tilde{\mathcal{Q}}(S)$ and $z \in W^s(\tilde{q})$ there is a neighborhood V of $\tilde{q}$ in $W^{su}(\tilde{q})$ and there is a homeomorphism

$$\zeta_z : V \to \zeta_z(V) \subset W^{su}(z) \tag{9}$$

with $\zeta_z(\tilde{q}) = z$ which is determined by the requirement that $\zeta_z(u) \in W^s(u)$. We call ζ_z a *holonomy map* for the strong unstable foliation along the stable foliation.

To be more precise, since $z \in W^s(\tilde{q})$, the vertical measured geodesic lamination $\tilde{q}^v$ of $\tilde{q}$ and the horizontal measured lamination z^h of z jointly fill up S. Since jointly filling up S is an open condition for pairs of measured laminations, there is a neighborhood Z of $[\tilde{q}^v]$ in $\mathcal{PML}$ such that for every $[\nu] \in Z$ and every representative ν of the projective class $[\nu]$ the laminations ν and z^h jointly fill up S. But this just means that there is a point in $W^{su}(z)$ whose projective vertical measured lamination equals $[\nu]$.

Similarly, for $\tilde{q} \in \tilde{\mathcal{Q}}(S)$ and $z \in W^u(\tilde{q})$ there is a neighborhood Y of $\tilde{q}$ in $W^{ss}(\tilde{q})$ and there is a homeomorphism

$$\theta_z : Y \to \theta_z(Y) \subset W^{ss}(z) \tag{10}$$

with $\theta_z(\tilde{q}) = z$ which is determined by the requirement that $\theta_z(u) \in W^u(u)$. We call θ_z a *holonomy map* for the strong stable foliation along the unstable foliation. The holonomy maps are equivariant under the action of the mapping class group and hence they project to locally defined holonomy maps in $\mathcal{Q}(S)$ which are denoted by the same symbols.

Recall from Section 2 the definition of the intrinsic path-metrics d^i on the leaves of the foliation W^i $(i = s, u)$. These path metrics are invariant under the action of the mapping class group and hence they project to path metrics on the leaves of W^i in $\mathcal{Q}(S)$ which we denote by the same symbols. For $q \in \mathcal{Q}(S), z \in W^i(q)$ and any preimage $\tilde{q}$ of q in $\tilde{\mathcal{Q}}(S)$, the distance $d^i(q, z)$ is the shortest length of a path in

$W^i(\tilde q)$ connecting $\tilde q$ to a preimage of z. Let moreover d^{ss}, d^{su} be the restrictions of d^s, d^u to distances on the leaves of the strong stable and strong unstable foliation of $\tilde{\mathcal{Q}}(S)$ and $\mathcal{Q}(S)$.

Let

$$\Pi : \tilde{\mathcal{Q}}(S) \to \mathcal{Q}(S)$$

be the canonical projection. For $q \in \mathcal{Q}(S)$ and $r > 0$ let

$$B^i(q,r)$$

be the closed ball of radius r about q in $W^i(q)$ $(i = ss, su, s, u)$ with respect to the metric d^i. Call such a ball $B^i(q,r)$ a *metric orbifold ball* centered at q if there is a lift $\tilde q \in \tilde{\mathcal{Q}}(S)$ of q with the following properties.

(1) The closed ball $B^i(\tilde q, r) \subset (W^i(\tilde q), d^i)$ about $\tilde q$ of the same radius is contractible and precisely invariant under the stabilizer $\mathrm{Stab}(\tilde q)$ of $\tilde q$ in $\mathrm{Mod}(S)$.
(2) $B^i(q,r) = B^i(\tilde q, r)/\mathrm{Stab}(\tilde q)$ which means that the restriction of the map Π to $B^i(\tilde q, r)$ factors through a homeomorphism $B^i(\tilde q, r)/\mathrm{Stab}(\tilde q) \to B^i(q,r)$.

We also say that $B^i(q,r)$ is an *orbifold quotient* of $B^i(\tilde q, r)$. Note that every metric orbifold ball $B^i(q,r) \subset W^i(q)$ is contractible. If $B^i(q,r)$ is a metric orbifold ball and if $s \leq r$ then the same holds true for $B^i(q,s)$. There is also an obvious notion of an orbifold ball which is not necessarily metric. If $\mathrm{Stab}(\tilde q)$ is trivial then the restriction of the projection Π to $B^i(\tilde q, r)$ is a homeomorphism.

For every point $q \in \mathcal{Q}(S)$ there is a number

$$a(q) > 0$$

such that the balls $B^i(q, a(q))$ are metric orbifold balls $(i = ss, su)$ and that for any preimage $\tilde q$ of q in $\tilde{\mathcal{Q}}(S)$ and any $z \in B^{ss}(\tilde q, a(q))$ (or $z \in B^{su}(\tilde q, a(q))$) the holonomy map ζ_z (or θ_z) is defined on $B^{su}(\tilde q, a(q))$ (or on $B^{ss}(\tilde q, a(q))$).

Now let

$$W_1 \subset B^{ss}(q, a(q)), W_2 \subset B^{su}(q, a(q))$$

be Borel sets and let $\tilde W_1 \subset B^{ss}(\tilde q, a(q))$, $\tilde W_2 \subset B^{su}(\tilde q, a(q))$ be the preimages of W_1, W_2 in $B^{ss}(\tilde q, a(q)), B^{su}(\tilde q, a(q))$. Then $\tilde W_1, \tilde W_2$ are precisely invariant under $\mathrm{Stab}(\tilde q)$. Define

$$V(\tilde W_1, \tilde W_2) = \cup_{z \in \tilde W_1} \zeta_z \tilde W_2 \text{ and } V(W_1, W_2) = \Pi V(\tilde W_1, \tilde W_2).$$

Note that the map $\xi : \tilde W_1 \times \tilde W_2 \to V(\tilde W_1, \tilde W_2)$ defined by $\xi(z,u) = \zeta_z(u)$ is a homeomorphism. If W_1, W_2 are path connected and contain the point q then the set $V(\tilde W_1, \tilde W_2)$ is path connected, and $V(W_1, W_2)$ is path connected as well.

Similarly, define

$$Y(\tilde W_1, \tilde W_2) = \cup_{u \in \tilde W_2} \theta_u \tilde W_1 \text{ and } Y(W_1, W_2) = \Pi Y(\tilde W_1, \tilde W_2).$$

Then there is a continuous function

$$\sigma : V(B^{ss}(\tilde q, a(q)), B^{su}(\tilde q, a(q))) \to \mathbb{R} \tag{11}$$

which vanishes on $B^{ss}(\tilde q, a(q)) \cup B^{su}(\tilde q, a(q))$ and such that

$$Y(\tilde W_1, \tilde W_2) = \{\Phi^{\sigma(z)} z \mid z \in V(\tilde W_1, \tilde W_2)\}.$$

In particular, for every number $\kappa > 0$ there is a number $r(q, \kappa) > 0$ such that the restriction of the function σ to $V(B^{ss}(\tilde{q}, r(q, \kappa)), B^{su}(\tilde{q}, r(q, \kappa)))$ assumes values in $[-\kappa, \kappa]$.

For $t_0 > 0$ define

$$V(\tilde{W}_1, \tilde{W}_2, t_0) = \cup_{-t_0 \leq s \leq t_0} \Phi^s V(\tilde{W}_1, \tilde{W}_2) \tag{12}$$
$$\text{and } V(W_1, W_2, t_0) = \Pi V(\tilde{W}_1, \tilde{W}_2, t_0).$$

Then for sufficiently small t_0, say for all $t_0 \leq t(q)$, the following properties are satisfied.

a) $V(W_1, W_2, t_0)$ is homeomorphic to $V(\tilde{W}_1, \tilde{W}_2)/\mathrm{Stab}(\tilde{q}) \times [-t_0, t_0]$.
b) Every connected component of the intersection of an orbit of Φ^t with $V(W_1, W_2, t_0)$ is an arc of length $2t_0$.

We call a set $V(W_1, W_2, t_0)$ as in (12) which satisfies the assumptions a),b) a set with a *local product structure*. Note that every point $q \in \mathcal{Q}(S)$ has a neighborhood in $\mathcal{Q}(S)$ with a local product structure, e.g. the set $V(B^{ss}(q, r), B^{su}(q, r), t)$ for $r \in (0, a(q))$ and $t \in (0, t(q))$. Moreover, the neighborhoods of q with a local product structure form a basis of neighborhoods.

The above discussion can be applied to strata as follows.

A connected component $\mathcal{Q}$ of a stratum $\mathcal{Q}(m_1, \dots, m_\ell; -m)$ or of a stratum $\mathcal{H}(m_1/2, \dots, m_\ell/2)$ is locally closed in $\mathcal{Q}(S)$ (here we identify an abelian differential with its square). This means that for every $q \in \mathcal{Q}$ there exists an open neighborhood V of q in $\mathcal{Q}(S)$ such that $V \cap \mathcal{Q}$ is a closed subset of V.

Using period coordinates [V90], one obtains that for every point $q \in \mathcal{Q}$ there is a number $a_{\mathcal{Q}}(q) \leq a(q)$ and a number $t_{\mathcal{Q}}(q) \leq t(q)$ with the following property. For $r \leq a_{\mathcal{Q}}(q)$ let

$$B^{ss}_{\mathcal{Q}}(q, r), B^{su}_{\mathcal{Q}}(q, r)$$

be the component containing q of the intersection $B^{ss}(q, r) \cap \mathcal{Q}, B^{su}(q, r) \cap \mathcal{Q}$ (note that the intersection $B^{ss}(q, r) \cap \mathcal{Q}$ may not be closed and may have infinitely many components). Then $V(B^{ss}_{\mathcal{Q}}(q, r), B^{su}_{\mathcal{Q}}(q, r), t_{\mathcal{Q}}(q))$ is a neighborhood of q in $\mathcal{Q}$ (Proposition 6.1 of [V90] which uses the definitions on p.128 of that paper, in particular formula 5.1).

We say that a Borel set $Z \subset \mathcal{Q}$ has a *local product structure* if there is some $q \in Z$ and if there are Borel sets

$$W_1 \subset B^{ss}_{\mathcal{Q}}(q, a_{\mathcal{Q}}(q)), W_2 \subset B^{su}_{\mathcal{Q}}(q, a_{\mathcal{Q}}(q))$$

and a number $t_0 < t(q)$ such that $Z = V(W_1, W_2, t_0)$.

The Φ^t-invariant Borel probability measure λ on $\mathcal{Q}$ in the Lebesgue measure class admits a natural family of conditional measures $\lambda^{ss}, \lambda^{su}$ on strong stable and strong unstable manifolds. The conditional measures λ^i are well defined up to a universal constant, and they transform under the Teichmüller geodesic flow Φ^t via

$$d\lambda^{ss} \circ \Phi^t = e^{-ht} d\lambda^{ss} \text{ and } d\lambda^{su} \circ \Phi^t = e^{ht} d\lambda^{su}.$$

Let $\mathcal{F} : \mathcal{Q}(S) \to \mathcal{Q}(S)$ be the flip $q \to \mathcal{F}(q) = -q$ and let dt be the Lebesgue measure on the flow lines of the Teichmüller flow. Any given choice of conditional measures λ^{su} on the strong unstable manifolds determines a choice of conditional measures λ^{ss} on the strong stable manifolds by the requirement that $\mathcal{F}_*\lambda^{su} = \lambda^{ss}$. The measure which can be written with respect to a local product structure in the form

$$d\lambda^{ss} \times d\lambda^{su} \times dt$$

is invariant under the Teichmüller flow and contained in the Lebesgue measure class. This implies that there is a unique choice of conditionals λ^{su} such that

$$d\lambda = d\lambda^{ss} \times d\lambda^{su} \times dt,$$

i.e. that the measure on the right hand side of the equation is a probability measure. The measures λ^u on unstable manifolds defined by $d\lambda^u = d\lambda^{su} \times dt$ are invariant under holonomy along strong stable manifolds.

Let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ and let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a preimage of a point $q \in \mathcal{Q}$. Since $\mathrm{Stab}(\tilde{q})$ is a finite group, there is a unique $\mathrm{Stab}(\tilde{q})$-invariant Lebesgue measure $\tilde{\lambda}$ on

$$V(B^{ss}_{\mathcal{Q}}(\tilde{q}, a_{\mathcal{Q}}(q)), B^{su}_{\mathcal{Q}}(\tilde{q}, a_{\mathcal{Q}}(q)), t_{\mathcal{Q}}(q)) = \tilde{V}$$

which projects to λ on $V = \tilde{V}/\mathrm{Stab}(\tilde{q})$. Similarly, the conditional measures λ^i of λ lift to $\mathrm{Stab}(\tilde{q})$-invariant conditional measures $\tilde{\lambda}^i$ on the leaves of the corresponding foliations on $\tilde{V}$. The natural homeomorphism

$$\begin{aligned} &B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q)) \times B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q)) \times [-t_{\mathcal{Q}}(q), t_{\mathcal{Q}}(q)] \\ &\quad \to V(B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q)), B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q)), t_{\mathcal{Q}}(q)) = \tilde{V} \end{aligned}$$

maps the measure λ_0 which is defined by $d\lambda_0 = d\tilde{\lambda}^{ss} \times d\tilde{\lambda}^{su} \times dt$ to a measure on $\tilde{V}$ of the form $e^{\varphi}\tilde{\lambda}$ where φ is a continuous function on $\tilde{V}$ which vanishes on $\cup_{t\in[-t_{\mathcal{Q}}(q), t_{\mathcal{Q}}(q)]}\Phi^t B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q))$ (see [V86]).

Call a point $q \in \mathcal{Q}$ a *smooth point* if the stabilizer of some (and hence every) preimage $\tilde{q} \in \tilde{\mathcal{Q}}$ consists of mapping classes which preserve the entire component $\tilde{\mathcal{Q}}$ pointwise. If q is smooth then the restriction of the projection Π to the set $V(B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q)) \times B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q)) \times [-t_{\mathcal{Q}}(q), t_{\mathcal{Q}}(q)])$ is a homeomorphism. By continuity of the function φ we have the following.

Lemma 3.1. *Let $q \in \mathcal{Q}$ be a smooth point. The for every $\epsilon > 0$ there is a number $a(q, \epsilon) \in (0, a_{\mathcal{Q}}(q))$ with the following property. For every $a \le a(q, \epsilon)$ the holonomy maps define a homeomorphism*

$$\Psi : B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a) \times B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a) \times [-t_{\mathcal{Q}}(q), t_{\mathcal{Q}}(q)] \to V(B^{ss}_{\mathcal{Q}}(q, a), B^{su}_{\mathcal{Q}}(q, a), t_0)$$

whose Jacobian with respect to the measure $\lambda^{ss} \times \lambda^{su} \times dt$ and the measure λ is contained in the interval $[(1+\epsilon)^{-1}, 1+\epsilon]$.

3.2. **Product coordinates near boundary points.** Throughout this subsection we consider a component $\mathcal{Q}$ of a stratum $\mathcal{Q}(m_1,\dots,m_\ell;-0)$, i.e. a stratum of holomorphic quadratic differentials without poles on a surface of genus $g\geq 2$. The goal is to construct coordinates for the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ near boundary points which have properties similar to product coordinates near interior points of $\mathcal{Q}$.

As before, it will be more convenient to work in the preimage $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ and its closure in $\tilde{\mathcal{Q}}(S)$. Note that boundary points of $\tilde{\mathcal{Q}}$ are contained in strata of the form $\tilde{\mathcal{Q}}(n_1,\dots,n_s;0)$ for some $s\leq \ell$ and such that each n_i $(i\leq s)$ can be written in the form $n_i=\sum_j m_{i_j}$.

The next lemma is geared at understanding the structure of a stratum near a boundary point. As an example, assume that $\tilde q\in\tilde{\mathcal{Q}}(2,m_3,\dots,m_\ell;0)$ is a boundary point of a component $\tilde{\mathcal{Q}}$ of the stratum $\tilde{\mathcal{Q}}(1,1,m_3,\dots,m_\ell;0)$. Let x_0 be the zero of order two for $\tilde q$ which is the collision of two simple zeros for nearby points in $\tilde{\mathcal{Q}}$. Then for any point $z\in\tilde{\mathcal{Q}}$ close to $\tilde q$ there is a short saddle connection in z connecting two simple zeros a_1,a_2 near x_0. If we pick one of these zeros, say a_1, then this saddle connection defines a direction at a_1 which is contained in some closed half-plane bounded by two vertical separatrices.

The simple idea is now to associate to the direction of a short saddle connection the vertical sector containing it and to use this information for the construction of coordinates. In the example described in the previous paragraph, if $\tilde q$ does not have vertical saddle connections then the double zero of $\tilde q$ which is the collision of the two simple zeros of nearby points in $\tilde{\mathcal{Q}}$ is contained in a quadrangle complementary component of the vertical measured geodesic lamination of $\tilde q$. The direction of the short saddle connection of a nearby point determines one of the two possible subdivisions of this quadrangle into two triangles. That this information can be used for the construction of coordinates will be made precise in the next lemma.

For its formulation, remember that a large geodesic lamination is a *topological* object which may not be the support of a transverse measure. Namely, such a large geodesic lamination may have isolated non-closed leaves which can not be contained in the support of a transverse measure. For example, if $\tilde q\in\mathcal{Q}(n_1,\dots,n_s;0)$ is a quadratic differential without vertical saddle connection and if we add a diagonal leaf to the support of the vertical measured lamination of $\tilde q$ then we obtain a large geodesic lamination with an isolated leaf.

Call a large geodesic lamination μ an *extension* of a geodesic lamination ν if μ can be obtained from ν by adding finitely many isolated leaves which subdivide some of the complementary regions of ν. Note that if ν decomposes S into discs then the number of different extensions of ν is bounded from above by a number only depending on the topological type of S.

Lemma 3.2. *Let $\tilde q\in\tilde{\mathcal{Q}}(S)-\tilde{\mathcal{Q}}$ be a boundary point of $\tilde{\mathcal{Q}}$ without vertical saddle connection. Let ν be the support of the vertical measured geodesic lamination of $\tilde q$. Then there is a set $L\subset\mathcal{LL}(m_1,\dots,m_\ell;0)$ of extensions of ν with the following property. If $(\tilde q_i)\subset\tilde{\mathcal{Q}}\subset\tilde{\mathcal{Q}}(m_1,\dots,m_\ell;0)$ is a sequence of quadratic differentials without vertical saddle connection which converges to $\tilde q$ then the supports of the* *vertical measured geodesic laminations of* $\tilde{q}_i$ *converge in the Hausdorff topology to a point in* L.

Proof. To simplify the notations, assume that there are $u \geq 2$ zeros $x_1^i, \dots, x_u^i$ of the differentials $\tilde{q}_i$ which merge to a single zero x_0 for $\tilde{q}$, and that these are the only zeros which collide in $\tilde{q}$. It will be clear that our argument is local and hence it applies to the case that more than one zero of $\tilde{q}$ is obtained by merging zeros of the differentials $\tilde{q}_i$.

By reordering, assume that for each i the order of the zero x_b^i equals m_{j_b} $(1 \leq b \leq u)$. Then the order of the zero x_0 of $\tilde{q}$ equals $n = \sum_b m_{j_b}$.

For each i and each $b \in \{2, \dots, u\}$ connect the zeros x_1^i and x_b^i of $\tilde{q}_i$ by a shortest geodesic s_b^i. Then the lengths of the geodesics s_b^i tend to zero as $i \to \infty$. Moreover, for large i the geodesic s_b^i is unique. Namely, otherwise there is a geodesic loop on $\tilde{q}_i$ whose length tends to zero with i. However, such a loop is homotopically nontrivial and hence this implies that the underlying Riemann surfaces degenerate to a stable curve which was ruled out by our assumptions.

Each of the geodesics s_b^i is a concatenation of saddle connections of $\tilde{q}_i$. The endpoints of each of these saddle connection are contained in the set $\{x_1^i, \dots, x_u^i\}$. In particular, if s_b^i consists of more than one saddle connection then exactly one of these saddle connections begins at x_1^i, and this saddle connection is one of the arcs s_j^i. Thus the union of the geodesics s_b^i $(b = 2, \dots, u)$ defines a tree T_i in S rooted in x_1^i whose vertex set is the set $\{x_1^i, \dots, x_u^i\}$. As $i \to \infty$, the rooted trees T_i collapse to the single zero x_0 of $\tilde{q}$. Since the number of vertices of the tree T_i does not exceed ℓ and is independent of i, by passing to a subsequence we may assume that for all i, j there is a homeomorphism of T_i onto T_j which maps the vertex x_b^i to the vertex x_b^j. We use this homeomorphism to identify the trees T_i, T_j in the sequel. Note that for sufficiently large i the tree T_i does not depend on the choice of the basepoint x_1^i. Namely, otherwise there are two of the vertices x_s^i which can be connected by two distinct geodesics of very short length which is impossible.

For each i let e_i be an edge of the tree T_i, chosen in such a way that for $j \neq i$ the edges e_i, e_j are mapped to each other by the homeomorphism $T_i \to T_j$. Then e_i is a saddle connection for $\tilde{q}_i$ connecting a zero in the distinguished set, say the zero x_a^i, to a zero x_b^i. Since $\tilde{q}_i$ does not have vertical saddle connections, locally near x_a^i the interior of the saddle connection e_i is contained in the interior of an euclidean sector based at x_a^i of angle π which is bounded by two vertical separatrices α_1^i, α_2^i of $\tilde{q}_i$ issuing from x_a^i. The union $\alpha_i = \alpha_1^i \cup \alpha_2^i$ is a smooth vertical geodesic line passing through x_a^i, i.e. a geodesic which is a limit in the compact open topology of a sequence of geodesic segments not passing through a singular point.

Similarly, there are two vertical separatrices β_1^i, β_2^i issuing from the zero x_b^i so that the sum of the angles at x_a^i, x_b^i of the (local) strip bounded by $\alpha_1^i, e_i, \beta_1^i$ equals π and that the same holds true for the angle sum of the (local) strip bounded by $\alpha_2^i, e_i, \beta_2^i$. The vertical length of e_i is positive. The union $\beta_i = \beta_1^i \cup \beta_2^i$ is a smooth vertical geodesic line passing through x_b^i.

Equip S with the marked hyperbolic metric defined by the conformal structure of $\tilde{q}$. For each i lift the singular euclidean metric on S defined by $\tilde{q}_i$ to a $\pi_1(S)$-invariant singular euclidean metric on the universal covering $\mathbf{H}^2$ of S. Let $\tilde{e}_i$ be a lift of the saddle connection e_i. Since $\tilde{e}_i$ is not vertical, the leaves of the vertical foliation of $\tilde{q}_i$ which pass through $\tilde{e}_i$ define a strip of positive transverse measure in $\mathbf{H}^2$. The boundary of this strip consists of the two lifts $\tilde{\alpha}_i, \tilde{\beta}_i$ of the smooth vertical geodesics α_i, β_i which pass through the endpoints of $\tilde{e}_i$. Up to normalization, up to changing the lifts $\tilde{e}_i$ of the saddle connections e_i and passing to a subsequence, as $i \to \infty$ the vertical geodesics $\tilde{\alpha}_i, \tilde{\beta}_i$ converge in the compact open topology to vertical geodesics $\tilde{\alpha}, \tilde{\beta}$ for the singular euclidean metric defined by $\tilde{q}$. The geodesics $\tilde{\alpha}, \tilde{\beta}$ pass through a preimage $\tilde{x}_0$ of the distinguished zero x_0 of $\tilde{q}$. By construction, the geodesics $\tilde{\alpha}, \tilde{\beta}$ coincide in a neighborhood of $\tilde{x}_0$. Since $\tilde{q}$ does not have vertical saddle connections, the geodesics $\tilde{\alpha}, \tilde{\beta}$ coincide.

An interior point of the edge e_i divides the tree T_i into two connected components T_i^1, T_i^2. Let J_i^v be the set of vertices of the tree T_i^v. At $\tilde{x}_0$, the angle enclosed at a fixed side of $\tilde{\alpha} = \tilde{\beta}$ can be calculated as

$$\pi(1 + \sum_{b \in J_i^v} m_{j_b}).$$

In particular, the geodesic $\tilde{\alpha}$ encloses an angle of at least 2π at each of its sides.

Since this reasoning is valid for each of the segments e_i, we conclude that locally near $\tilde{x}_0$ the union of the limiting geodesics constructed as above for all edges of the tree T_i divide a neighborhood of $\tilde{x}_0$ in $\mathbf{H}^2$ into sectors bounded by vertical separatrices which are determined by the trees T_i and whose angles are prescribed by the orders of the zeros x_b^i of $\tilde{q}_i$.

The singular euclidean metrics on $\mathbf{H}^2$ defined by the differentials $\tilde{q}_i, \tilde{q}$ are uniformly quasi-isometric to the hyperbolic metric. Thus each biinfinite geodesic for $\tilde{q}_i, \tilde{q}$ is contained in a uniformly bounded neighborhood of a unique hyperbolic geodesic. In particular, each biinfinite geodesic for one of these singular euclidean metrics has two well defined endpoints in the ideal boundary $\partial\mathbf{H}^2$ of $\mathbf{H}^2$.

Now let $\tilde{\gamma}_i, \tilde{\gamma}$ be the hyperbolic geodesic with the same endpoints in $\partial\mathbf{H}^2$ as $\tilde{\alpha}_i, \tilde{\alpha}$. Then $\tilde{\gamma}_i \to \tilde{\gamma}$ locally uniformly. By the explicit construction of a measured geodesic lamination from a measured foliations in [L83], for each i the projection γ_i to S of the geodesic $\tilde{\gamma}_i$ is contained in the support ν_i of the vertical measured geodesic lamination of $\tilde{q}_i$. As a consequence, the projection γ to S of the geodesic $\tilde{\gamma}$ is contained in a limit in the Hausdorff topology of the geodesic laminations ν_i. On the other hand, since at $\tilde{x}_0$ the vertical geodesic $\tilde{\alpha}$ encloses an angle of at least 2π at each of its sides, the geodesic γ is not contained in the support ν of the vertical measured geodesic lamination of $\tilde{q}$ [L83].

The geodesic lines constructed above from the tree T_i are pairwise disjoint and disjoint from the support ν of the vertical measured geodesic lamination of $\tilde{q}$. They decompose the complementary regions of ν into polygons as prescribed by the orders of the zeros of the differentials $\tilde{q}_i$ and by the tree T_i.

By assumption, $\tilde{q}$ does not have vertical saddle connections and therefore the closure of any geodesic line disjoint from ν contains ν. As a consequence, whenever the trees T_i are constant along the sequence in the sense described above, then the supports of the vertical measured geodesic laminations ν_i of $\tilde{q}_i$ converge in the Hausdorff topology to a large geodesic lamination μ of the same topological type as the laminations ν_i. This shows the lemma. □

We use Lemma 3.2 to analyze the structure of strong unstable manifolds in $\mathcal{Q}$ near boundary points.

Proposition 3.3. *There is a number $k_0 > 0$ and for each $q \in \overline{\mathcal{Q}} - \mathcal{Q}$ there is a compact neighborhood of q in $\overline{\mathcal{Q}} \cap W^{su}(q)$ of the form*

$$W^{su}_{\mathcal{Q},\mathrm{loc}}(q) = \cup_{i=1}^{k} A_i. \tag{13}$$

for some $k \le k_0$. Here for each $i \le k$, the set A_i contains q, and it is the closure of an open connected subset U_i of $\mathcal{Q} \cap W^{su}(q)$ which is diffeomorphic to an open ball.

Proof. Let $\tilde{Q}$ be a component of the preimage of $\mathcal{Q}$ in $\tilde{\mathcal{Q}}(S)$. Let $q \in \overline{\mathcal{Q}} - \mathcal{Q}$ and let $\tilde{q}$ be a lift of q to the closure of $\tilde{\mathcal{Q}}$. Assume first that $\tilde{q}$ does not have vertical saddle connections.

Let ν be the support of the vertical measured geodesic lamination $\tilde{q}^v$ of $\tilde{q}$. By Lemma 3.2, since $\tilde{q}$ does not have vertical saddle connections, there is a number $k_0 > 0$ only depending on the genus of S, there is a number $k \le k_0$ and there is a set $L = \{\xi_1, \dots, \xi_k\} \subset \mathcal{LL}(m_1, \dots, m_\ell; 0)$ of extensions of ν with the following property. If $\tilde{q}_i \subset \tilde{\mathcal{Q}}$ is a sequence of quadratic differentials without vertical saddle connection which converges to $\tilde{q}$ then any accumulation point in the Hausdorff topology of the supports of the vertical measured geodesic laminations of $\tilde{q}_i$ is contained in L. The set L is necessarily invariant under the action of the stabilizer $\mathrm{Stab}(\tilde{q})$ of $\tilde{q}$ in $\mathrm{Mod}(S)$.

We now use a construction which is carried out in detail in [H09a]. Namely, the topological shape of a geodesic lamination can be described by collapsing almost parallel strands to a single arc. If done correctly, the result is a *train track* on S. To formalize the idea that such a train track η "looks like" the lamination, let g be a complete hyperbolic metric on S of finite volume and represent the large lamination ξ_j as a geodesic lamination for this metric. For a small number $\epsilon > 0$ we say that a train track η_j *ϵ-follows* ξ_j if the projectivized tangent bundle of the graph obtained by replacing the branches of η_j by geodesic arcs in the same homotopy class with fixed endpoints is ϵ-close to the projectivized tangent bundle of ξ_j in the Hausdorff topology for closed subsets of the projectivized tangent bundle of S.

Now for each j the geodesic lamination ξ_j is obtained from ν by adding some diagonal leaves. This makes sense for train tracks as well. Namely, for a small number $\epsilon > 0$, a train track η_j which *carries* ξ_j (i.e. ξ_j can be mapped into η_j by a map of class C^1 whose differential is nonsingular on the leaves of ξ_j) and ϵ-follows ξ_j contains a subtrack τ which carries ν and ϵ-follows ν, and η_j is obtained from τ by subdividing complementary components (see [H09a]). For sufficiently small ϵ, a carrying map $\xi_j \to \eta_j$ is surjective and defines a bijection of the complementary components of ξ_j onto the complementary components of η_j. We also may assume that the set $\eta_1, \dots, \eta_k$ is invariant under the action of $\mathrm{Stab}(\tilde{q})$.

The set of all large geodesic laminations $\zeta \in \mathcal{LL}(m_1, \dots, m_\ell; 0)$ carried by η_j is open and closed in the Hausdorff topology [H09a]. Thus by the choice of the set L, the support of the vertical measured geodesic lamination of every quadratic differential $z \in \tilde{\mathcal{Q}}$ without vertical saddle connection which is sufficiently close to $\tilde{q}$ is carried by one of the train tracks η_j. Since the set of differentials without vertical saddle connection is dense in $\tilde{\mathcal{Q}}$, by continuity the train tracks $\eta_1, \dots, \eta_k$ carry the support of the vertical measured geodesic lamination of every quadratic differential $z \in \tilde{\mathcal{Q}}$ which is sufficiently close to $\tilde{q}$. The dimension of the space of transverse measures on η_j equals precisely the dimension of the unstable foliation of $\mathcal{Q}$ [H11] (this is implicitly also contained in [PH92]).

The same argument also applies in the case that $\tilde{q}$ has vertical saddle connections. Namely, in this case there is a train track τ which decomposes S into complementary regions of the type prescribed by the stratum $\tilde{\mathcal{Q}}_0$ of $\tilde{q}$ and which carries the vertical measured geodesic lamination of each quadratic differential $z \in \tilde{\mathcal{Q}}_0$ which is sufficiently close to $\tilde{q}$ [H11].

The above discussion shows that there is a uniformly bounded number of train tracks $\eta_1, \dots, \eta_k$ obtained from τ by subdividing complementary regions and such that the union of these train tracks carries the vertical measured geodesic lamination of each quadratic differential $z \in \tilde{\mathcal{Q}}$ which is sufficiently close to $\tilde{q}$.

For each j let C_j be an closed contractible neighborhood of $\tilde{q}^v$ in the subset of C of all projective measured geodesic laminations which are carried by η_j. Let $\tilde{q}^h$ be the horizontal measured geodesic lamination of $\tilde{q}$. Then a neighborhood of $\tilde{q}$ in $W^{su}(\tilde{q})$ can be identified with a neighborhood of the vertical geodesic lamination $\tilde{q}^v$ of $\tilde{q}$ in the space C of all measured geodesic laminations μ with $\iota(\mu, \tilde{q}^h) = 1$ (note that here we do not keep track of strata). As a consequence, for each j and up to decreasing C_j, the set C_j can be identified with a closed subset A_j of $W^{su}(\tilde{q})$ which contains $\tilde{q}$. The above discussion shows that $A = \cup_j A_j$ is a neighborhood of $\tilde{q}$ in $W^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}) \cap \tilde{\mathcal{Q}}$. Moreover, we may assume that A is $\mathrm{Stab}(\tilde{q})$-invariant. This completes the proof of the proposition. □

4. Absolute continuity

Let again $\mathcal{Q}$ be a connected component of a stratum in $\mathcal{Q}(S)$. Then $\mathcal{Q}$ is invariant under the Teichmüller flow Φ^t. For a periodic orbit $\gamma \subset \mathcal{Q}$ for Φ^t, the Lebesgue measure supported in γ is a Φ^t-invariant Borel measure $\sigma(\gamma)$ on $\mathcal{Q}$ whose total mass equals the prime period $\ell(\gamma)$ of γ. If we denote for $R > 0$ by $\Gamma(R)$ the set of all periodic orbits for Φ^t of period at most R which are contained in $\mathcal{Q}$ then we obtain a finite Φ^t-invariant Borel measure μ_R on $\mathcal{Q}$ by defining

$$\mu_R = e^{-hR} \sum_{\gamma \in \Gamma(R)} \sigma(\gamma). \tag{14}$$

Let μ be any weak limit of the measures μ_R as $R \to \infty$. Then μ is a Φ^t-invariant Borel measure on $\mathcal{Q}(S)$ supported in the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ (which may a priori be zero or locally infinite). The purpose of this section is to show

Proposition 4.1. *The measure μ on $\overline{\mathcal{Q}}$ satisfies $\mu \leq \lambda$.*

This means that $\mu(U) \leq \lambda(U)$ for every open relative compact subset U of $\overline{\mathcal{Q}}$. In particular, the measure μ is finite and absolutely continuous with respect to the Lebesgue measure, and it gives full mass to $\mathcal{Q}$. Note that this statement is stronger than claiming that $\mu(U) \leq \lambda(U)$ for every open relative compact subset of $\mathcal{Q}$. Namely, it also says that no mass can accumulate on $\overline{\mathcal{Q}} - \mathcal{Q}$.

The strategy of proof is the strategy developed by Margulis (see [Mar04]). Namely, to control the measure of a sufficiently small neighborhood of a point $q \in \mathcal{Q}$ we use a neighborhood U of q with a local product structure and mixing properties of the Teichmüller flow to establish that each periodic point in U of sufficiently large period $T > 0$ defines an intersection component of $U \cap \Phi^T U$ of controlled measure.

In the case of an Anosov flow, to obtain sufficient control of the measure of a connected component of the intersection of a set V with a local product structure with its image under the time-T-map of the flow, one first arranges that V is defined by two small compact balls B_1, B_2 in a strong stable and strong unstable manifold, respectively. Then one decreases slightly the balls B_i to compact balls $B_i' \subset B_i$ of almost the same volume which are contained in the interior of B_i, and one considers the corresponding set $V' \subset V$ with a local product structure.

By uniform hyperbolicity, for sufficiently large $T > 0$ only depending on the hyperbolicity constant and the choice of V, V', if there is a point $x \in V'$ which is mapped by the time-T-map of the flow back into V', then the image of the ball B_1' under the holonomy map which moves it into the strong stable manifold of x is entirely contained in V independent of the position of x in V'. Reversing time shows that there is a neighborhood of x in the strong unstable manifold of x which is contained in V and which is mapped by the time T-map onto a ball in a strong unstable manifold contained in V of large volume. This allows to conclude that the volume of the intersection component containing x of V with its image under the time-T-map equals e^{-T} times the mass of V for the measure of maximal entropy up to a multiplicative constant which is arbitrarily close to one depending on V and V'. Now each such intersection component can contain at most one periodic orbit, and together with mixing properties of the flow this results in the requested upper counting control.

The Teichmüller flow is not hyperbolic, and adopting this strategy requires a significant amount of care. Our first task is to construct neighborhoods of recurrent points in strong unstable manifolds which have the above expansion property provided that the return time is big enough depending on the recurrent point. This part of the argument does not use specific properties of the Teichmüller flow (the curve graph can be replaced by a more abstract way to measure hyperbolicity) and may be useful in other context as well.

We moreover face the difficulty that due to nontrivial point stabilizers for the mapping class group, the moduli space of quadratic differentials is simply connected, and periodic orbits for the Teichmüller flow do not correspond to free homotopy classes in moduli space. Therefore we have to be careful with detecting and separating periodic orbits. To this end we work most of the time in the universal covering of the component. Finally an analysis of dynamical properties of the Teichmüller flow near the boundary of strata is needed to show that concentration of measure near boundary points does not occur. We carry out this program in the remainder of this section.

A point $q \in \mathcal{Q}$ is called *forward recurrent* (or *backward recurrent*) if it is contained in its own ω-limit set (or in its own α-limit set) under the action of Φ^t. A point $q \in \mathcal{Q}$ is *recurrent* if it is forward and backward recurrent. The set $\mathcal{R} \subset \mathcal{Q}$ of recurrent points is a Φ^t-invariant Borel subset of $\mathcal{Q}$. It follows from the work of Masur [M82] that a recurrent point $q \in \mathcal{R}$ has uniquely ergodic vertical and horizontal measured geodesic laminations whose supports fill up S. As a consequence, the preimage $\tilde{\mathcal{R}}$ of $\mathcal{R}$ in $\tilde{\mathcal{Q}}(S)$ is contained in the set $\tilde{\mathcal{A}}$ defined in (5) of Section 2.

Using the notations from Section 2, there is a number $p > 1$ such that for every $q \in \tilde{\mathcal{Q}}(S)$ the map $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)$ is an unparametrized p-quasi-geodesic in the curve graph $\mathcal{C}(S)$. If q is a lift of a recurrent point in $\mathcal{Q}(S)$ then this unparametrized quasi-geodesic is of infinite diameter (see [Kl99] for details on this).

Recall from (3) of Section 2 the definition of the distances δ_x ($x \in \mathcal{T}(S)$) on $\partial\mathcal{C}(S)$ and of the sets $D(q,r) \subset \partial\mathcal{C}(S)$ ($q \in \tilde{\mathcal{A}}, r > 0$). The following lemma is a version of Lemma 2.1 of [H10b] which is going to be used as a substitute for hyperbolicity.

Lemma 4.2. *There are numbers $\alpha_0 > 0, \beta > 0, b > 0$ with the following property. Let $q \in \tilde{\mathcal{R}}$ and for $s > 0$ write $\sigma(s) = d(\Upsilon_{\mathcal{T}}(Pq), \Upsilon_{\mathcal{T}}(P\Phi^s q))$; then*

$$\beta e^{-b\sigma(s)}\delta_{P\Phi^s q} \leq \delta_{Pq} \leq \beta^{-1}e^{-b\sigma(s)}\delta_{P\Phi^s q} \text{ on } D(\Phi^s q, \alpha_0).$$

The map $F : \tilde{\mathcal{A}} \to \partial\mathcal{C}(S)$ defined in Section 2 is equivariant for the action of the mapping class group on $\tilde{\mathcal{A}} \subset \tilde{\mathcal{Q}}(S)$ and on $\partial\mathcal{C}(S)$. In particular, for $q \in \tilde{\mathcal{A}}$ and $r > 0$ the set $D(q,r) \subset \partial\mathcal{C}(S)$ is invariant under $\mathrm{Stab}(q)$, and the same holds true for $F^{-1}D(q,r)$.

Let $\tilde{\mathcal{Q}} \subset \tilde{\mathcal{Q}}(S)$ be a component of the preimage of $\mathcal{Q}$ and let $\mathrm{Stab}(\tilde{\mathcal{Q}}) < \mathrm{Mod}(S)$ be the stabilizer of $\tilde{\mathcal{Q}}$ in $\mathrm{Mod}(S)$. The Φ^t-invariant Borel probability measure λ on $\mathcal{Q}$ in the Lebesgue measure class lifts to a $\mathrm{Stab}(\tilde{\mathcal{Q}})$-invariant locally finite measure on $\tilde{\mathcal{Q}}$ which we denote again by λ. The conditional measures $\lambda^{ss}, \lambda^{su}$ of λ on the leaves of the strong stable and strong unstable foliation of $\mathcal{Q}$ lift to a family of locally finite Borel measures on the leaves of the strong stable and strong unstable foliation $W^{ss}_{\tilde{\mathcal{Q}}}, W^{su}_{\tilde{\mathcal{Q}}}$ of $\tilde{\mathcal{Q}}$, respectively, which we denote again by $\lambda^{ss}, \lambda^{su}$ (see the discussion in Section 3.1).

The next observation is used to overcome the difficulty that the Teichmüller flow is not hyperbolic. In its formulation, the number $\alpha_0 > 0$ is the constant from Lemma 4.2.

Lemma 4.3. *For every $\epsilon > 0$, for every $\tilde q \in \tilde{\mathcal{Q}} \cap \tilde{\mathcal{R}}$ and for all compact neighborhoods $W_1 \subset W_2$ of $\tilde q$ in $W^{su}_{\tilde{\mathcal{Q}}}(\tilde q)$ there are compact neighborhoods $K \subset C \subset W_1$ of $\tilde q$ in $W^{su}_{\tilde{\mathcal{Q}}}(\tilde q)$ with the following properties.*

1. *K, C are precisely invariant under* $\mathrm{Stab}(\tilde q)$.
2. *There are numbers $0 < r_1 < r_2 < \alpha_0/2$ such that*
$$K = \overline{W_1 \cap F^{-1}D(\tilde q, r_1)},\ C = \overline{W_1 \cap F^{-1}D(\tilde q, r_2)}.$$
3. *$\lambda^{su}(K)(1+\epsilon) \geq \lambda^{su}(C)$.*
4. *If $z \in K \cap \tilde{\mathcal{A}}$ then $\overline{F^{-1}D(z,(r_2-r_1)/2) \cap W_2} \subset C$.*

Proof. Let $q \in \mathcal{Q}$ be a recurrent point and let $\tilde q \in \tilde{\mathcal{Q}}$ be a lift of q. Let $W_1 \subset W_2 \subset W^{su}_{\tilde{\mathcal{Q}}}(\tilde q)$ be compact neighborhoods of $\tilde q$. Choose $r > 0$ such that
$$B^{su}_{\tilde{\mathcal{Q}}}(\tilde q, 2r) \subset W_1 \subset W^{su}_{\tilde{\mathcal{Q}}}(\tilde q)$$
is precisely invariant under $\mathrm{Stab}(\tilde q)$ and projects to a metric orbifold ball in $W^{su}_{\mathcal{Q}}(q)$.

By Lemma 2.3, the map $F : \tilde{\mathcal{A}} \to \partial\mathcal{C}(S)$ is continuous and closed, and the sets $F(B^{su}(\tilde q, \nu) \cap \tilde{\mathcal{A}})$ ($\nu > 0$) form a neighborhood basis of $F\tilde q$ in $\partial\mathcal{C}(S)$. Thus there is a number $u_0 > 0$ such that
$$D(\tilde q, u_0) \cap F(W_2 \cap \tilde{\mathcal{A}}) \subset F(B^{su}_{\tilde{\mathcal{Q}}}(\tilde q, r) \cap \tilde{\mathcal{A}}).$$

For $u \leq u_0$ let $K_u \subset W^{su}_{\tilde{\mathcal{Q}}}(\tilde q)$ be the closure of the set
$$F^{-1}(D(\tilde q, u)) \cap W_2.$$
Then K_u is a closed neighborhood of $\tilde q$ in $W^{su}_{\tilde{\mathcal{Q}}}(\tilde q)$ which is contained in W_1 and is precisely invariant under $\mathrm{Stab}(\tilde q)$. Moreover, $K_t \subset K_u$ for $t < u$, and Lemma 2.3 shows that $\cap_{u>0} K_u = \{\tilde q\}$. Since the conditional measure λ^{su} on $W^{su}_{\tilde{\mathcal{Q}}}(\tilde q)$ is Borel regular, for every $\epsilon > 0$ there are numbers $r_1 < r_2 < u_0$ so that
$$\lambda^{su}(K_{r_1}) \geq \lambda^{su}(K_{r_2})(1+\epsilon)^{-1}.$$
If we define $K = K_{r_1}$ and $C = K_{r_2}$ then the sets $K \subset C$ have all properties required in the lemma. This shows the lemma. ∎

Remark: Since $\tilde{\mathcal{A}}$ is dense in $\tilde{\mathcal{Q}}(S)$ and the map $F : \tilde{\mathcal{A}} \to \partial\mathcal{C}(S)$ is continuous and closed, the sets $K \subset C \subset W^{su}_{\tilde{\mathcal{Q}}}(\tilde q)$ have dense interior. Moreover, we may assume that their boundaries have vanishing Lebesgue measures.

Let again $\tilde{\mathcal{Q}} \subset \tilde{\mathcal{Q}}(S)$ be a component of the preimage of $\mathcal{Q}$. For $q \in \mathcal{Q}$ let $\tilde q \in \tilde{\mathcal{Q}}$ be a preimage of q and let $|\mathrm{Stab}(q)|$ be the cardinality of the quotient of $\mathrm{Stab}(\tilde q)$ by the normal subgroup of all elements of $\mathrm{Stab}(\tilde q)$ which fix $\tilde{\mathcal{Q}}$ pointwise (for example, the hyperelliptic involution acts trivially on the preimage of a hyperelliptic component of a stratum, see [KZ03, L08]. This does not depend on the choice of $\tilde q$. We note

Lemma 4.4. *The set $\mathcal{S} = \{q \in \mathcal{Q} \mid |\mathrm{Stab}(q)| = 1\}$ is an open dense Φ^t-invariant submanifold of $\mathcal{Q}$.*

Proof. The mapping class group preserves the Teichmüller metric on $\mathcal{T}(S)$ and hence an element $h \in \mathrm{Mod}(S)$ which stabilizes a quadratic differential $\tilde{q} \in \tilde{\mathcal{Q}}(S)$ fixes pointwise the Teichmüller geodesic with initial cotangent $\tilde{q}$. Therefore the set $\mathcal{S}$ is Φ^t-invariant, moreover it is clearly open. Since the Teichmüller flow on $\mathcal{Q}$ has dense orbits, either $\mathcal{S}$ is empty or dense. However, $\mathrm{Mod}(S)$ acts properly discontinuously on $\mathcal{T}(S)$ and consequently the first possibility is ruled out by the fact that the conjugacy class of an element of $\mathrm{Mod}(S)$ which fixes an entire component of the preimage of $\mathcal{Q}$ does not contribute towards $|\mathrm{Stab}(q)|$. □

For a control of the measure μ we use a variant of an argument of Margulis [Mar04]. Namely, for numbers $R_1 < R_2$ let $\Gamma(R_1, R_2)$ be the set of all periodic orbits of Φ^t which are contained in $\mathcal{Q}$, with prime periods in the interval (R_1, R_2). For an open or closed subset V of $\overline{\mathcal{Q}}$ and numbers $R_1 < R_2$ define

$$H(V, R_1, R_2) = \sum_{\gamma \in \Gamma(R_1,R_2)} \int_\gamma \chi(V)$$

where $\chi(V)$ is the characteristic function of V.

To obtain control on the quantities $H(V, R_1, R_2)$ we use a tool from [ABEM12]. Namely, every leaf $W^{ss}(q)$ of the strong stable foliation of $\mathcal{Q}(S)$ can be equipped with the *Hodge distance* d_H (or, rather, the modified Hodge distance, [ABEM12]). This Hodge distance is defined by a norm on the tangent space of $W^{ss}(q)$ (with a suitable interpretation). In particular, closed d_H-balls of sufficiently small finite radius are compact, and balls about a given point q define a neighborhood basis of q in $W^{ss}(q)$. We also obtain a Hodge distance on the leaves of the strong unstable foliation as the image under the flip $\mathcal{F}$ of the Hodge distance on the leaves of the strong stable foliation. These Hodge distances restrict to Hodge distances on the leaves of the foliations $W^{ss}_{\mathcal{Q}}, W^{su}_{\mathcal{Q}}$ which we denote by the same symbol d_H.

The following result is Theorem 8.12 of [ABEM12],

Theorem 4.5. *There is a number $c_H > 0$ such that*

$$d_H(\Phi^t q, \Phi^t q') \le c_H d_H(q, q'). \tag{15}$$

for all $q \in \mathcal{Q}(S), q' \in W^{ss}(q)$ and all $t \ge 0$.

The following lemma is the main technical tool for applying the strategy of Margulis to the non-uniformly hyperbolic setting at hand. Observe that property (4) assures that each periodic orbit contributes with a weight estimated in (3) to the volume of intersection, so this gives the upper bound on the number of such orbits. The setup is more involved than in the case of Anosov flows (which is already technically complicated, see [Mar04]) due to the difficulty of lack of uniform hyperbolicity.

Lemma 4.6. *Let $q \in \mathcal{Q}$ be a recurrent point with $|\mathrm{Stab}(q)| = 1$ and let V be a neighborhoood of q in $\mathcal{Q}$. Then for every $\epsilon > 0$ there are closed neighborhoods $Z_1 \subset Z_2 \subset Z_3 \subset V_0 \subset V$ of q in $\mathcal{Q}$ with dense interior and there is a number $t_0 > 0$ such that for all sufficiently large $R > 0$ the following properties are satisfied.*

(1) V_0 *is connected and has a local product structure. There are compact sets* $K^i \subset C^i \subset W^i_{\mathcal{Q},\mathrm{loc}}(q)$ *with dense interior so that* K^i *is contained in the interior of* C^i $(i = ss, su)$*, and there is a number* $\omega > 0$ *such that*

$$Z_1 = V(K^{ss}, K^{su}, t_0), Z_2 = V(K^{ss}, C^{su}, t_0),$$
$$Z_3 = V(C^{ss}, C^{su}, t_0(1+\omega)).$$

(2) $\lambda(Z_3) \le \lambda(Z_1)(1+\epsilon)$.

(3) *Let* $z \in Z_1$ *and assume that* $\Phi^\tau z = z$ *for some* $\tau \in (R - t_0, R + t_0)$*. Let* $\hat{E}$ *be the component containing* z *of the intersection* $\Phi^\tau V_0 \cap V_0$ *and let* $E = \hat{E} \cap \Phi^\tau Z_2 \cap Z_3$*. Then*

$$\lambda(E) \in [e^{-hR}\lambda(Z_1)/(1+\epsilon), e^{-hR}\lambda(Z_1)(1+\epsilon)],$$

and the length of the connected orbit subsegment of $(\cup_{t\in\mathbb{R}}\Phi^t z) \cap Z_1$ *containing* z *equals* $2t_0$*.*

(4) *The* Φ^t*-orbit through* z *is the only periodic orbit for* Φ^t *of period* $\sigma \in (R - t_0, R + t_0)$ *which intersects* E*.*

Proof. Let $q \in \mathcal{Q}$ be recurrent with $|\mathrm{Stab}(q)| = 1$ and let V be a neighborhood of q in $\mathcal{Q}$. For $\epsilon > 0$ choose $\omega > 0$ sufficiently small that $(1+\omega)^{12} < 1+\epsilon$. Using the notations from Subsection 3.1, there are numbers

$$a_0 < a_{\mathcal{Q}}(q), t_0 < \min\{t_{\mathcal{Q}}(q)/4(1+\omega), \log((1+\omega)/h)\}$$

such that

$$V_0 = V(B^{ss}_{\mathcal{Q}}(q, a_0), B^{su}_{\mathcal{Q}}(q, a_0), t_0(1+\omega)) \subset V$$

is a connected set with a local product structure.

Let $\tilde{q} \in \tilde{\mathcal{Q}}(S)$ be a preimage of q. By construction (see the discussion in Section 3.1), the set

$$\tilde{V}_0 = V(B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_0), B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_0), t_0(1+\omega))$$

is precisely invariant under $\mathrm{Stab}(\tilde{q})$. In particular, since $|\mathrm{Stab}(q)| = 1$, the set $\tilde{V}_0$ is mapped homeomorphically onto V_0 by the projection $\tilde{\mathcal{Q}} \to \mathcal{Q}$.

Since periodic orbits for Φ^t are in bijection with conjugacy classes of pseudo-Anosov elements of $\mathrm{Mod}(S)$, up to making a_0 smaller we may assume that the following holds true. For every $r > 8t_0$, every component of the intersection $\Phi^r V_0 \cap V_0$ is intersected by at most one periodic orbit for the Teichmüller flow with prime period contained in the interval $[r - 2t_0, r + 2t_0]$, and if such an orbit exists then its intersection with $\Phi^r V_0 \cap V_0$ is connected. As a consequence, Property 4) stated in the lemma holds true for subsets Z_i of this set V_0 with the properties claimed in the lemma.

The following construction is used to estimate volumes. It is identical with the usual construction for Anosov flows. Property c) below is obvious for Anosov flows and has to be established in the situation at hand.

As in (10) of Section 3, for $z \in \tilde{V}_0$ let $\theta_z : B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_0) \to W^{ss}_{\tilde{\mathcal{Q}},\mathrm{loc}}(z)$ be defined by the requirement that $\theta_z(u) \in W^u_{\tilde{\mathcal{Q}},\mathrm{loc}}(u)$ for all u. Similarly, as in (9) of Section 3, for $z \in \tilde{V}_0$ let $\zeta_z : B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_0) \to W^{su}_{\tilde{\mathcal{Q}},\mathrm{loc}}(z)$ be defined by $\zeta_z(u) \in W^{s}_{\tilde{\mathcal{Q}},\mathrm{loc}}(u)$. We claim that for sufficiently small $a_1 < a_0$ and for every

$$z \in \tilde{V}_1 = V(B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_1), B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_1), t_0)$$

the following holds true (where V_1 denotes the projection of $\tilde{V}_1$ to $\mathcal{Q}$).

a) The Jacobian of the embedding $\theta_z : B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_1) \to W^{ss}_{\tilde{\mathcal{Q}},\mathrm{loc}}(z)$ and of the embedding $\zeta_z : B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_1) \to W^{su}_{\tilde{\mathcal{Q}},\mathrm{loc}}(z)$ with respect to the measures λ^{ss} and λ^{su}, respectively, is contained in the interval $[(1+\omega)^{-1}, 1+\omega]$.
b) The restriction to $\tilde{V}_1$ of the function σ defined in (11) takes values in the interval $[-\log((1+\omega)/h), \log((1+\omega)/h)]$.
c) If $z \in V(B^{ss}_{\mathcal{Q}}(q, a_1), B^{su}_{\mathcal{Q}}(q, a_1))$ and if $t > 8t_0$ is such that

$$\Phi^t z \in V(B^{ss}_{\mathcal{Q}}(q, a_1), B^{su}_{\mathcal{Q}}(q, a_1))$$

then $\Phi^t(V_1 \cap W^s_{\mathcal{Q},\mathrm{loc}}(z)) \subset V_0$ and $\Phi^{-t}(V_1 \cap W^u_{\mathcal{Q},\mathrm{loc}}(z)) \subset V_0$.

Here and in the sequel, for $z \in V_1$ we denote by $V_1 \cap W^i_{\mathcal{Q},\mathrm{loc}}(z)$ the connected component containing z of the intersection $V_1 \cap W^i_{\mathcal{Q}}(z)$ and similarly for $\tilde{V}_1$ $(i = s, u)$.

To verify the claim, note first that property b) holds true for sufficiently small $a_1 > 0$ since σ is continuous and Φ^t-invariant and equals one at q (note that we want to keep t_0 fixed and only adjust a_1).

The measures λ^s (or λ^u) are invariant under holonomy along the strong unstable (or the strong unstable) foliation, and we have $d\lambda^s = d\lambda^{ss} \times dt$ and $d\lambda^u = d\lambda^{su} \times dt$. As a consequence, the Jacobians of the maps θ_z, ζ_z are controlled by the function σ and therefore property a) is fulfilled for sufficiently small a_1.

By property b) above and by Theorem 4.5, property c) is fulfilled if we choose $a_1 << a_0$ small enough so that for some $r > 0$ the following is satisfied. For every $u \in V_1$ the diameter of $\theta_u(B^{ss}_{\mathcal{Q}}(q, a_1))$ with respect to the Hodge distance does not exceed r, the Hodge distance between $\theta_u(B^{ss}_{\mathcal{Q}}(q, a_1))$ and the boundary of $\theta_u(B^{ss}_{\mathcal{Q}}(q, a_0))$ is not smaller than $c_H r$, the diameter of $\zeta_u(B^{su}_{\mathcal{Q}}(q, a_1))$ is not smaller than $c_H r$ and the Hodge distance between $\zeta_u(B^{su}_{\mathcal{Q}}(q, a_1))$ and the boundary of $\zeta_u(B^{su}_{\mathcal{Q}}(q, a_0))$ is not smaller than $c_H r$.

Since $h \geq 1$, Property b) implies the following. For all closed sets $A^i \subset B^i_{\mathcal{Q}}(q, a_1)$ $(i = ss, su)$ and for every $z \in V(B^{ss}_{\mathcal{Q}}(q, a_1), B^{su}_{\mathcal{Q}}(q, a_1))$ we have

$$V(A^{ss}, A^{su}, t_0(1+\omega)^{-1}) \subset V(\theta_z(A^{ss}), \zeta_z(A^{su}), t_0) \subset V(A^{ss}, A^{su}, t_0(1+\omega)). \tag{16}$$

The estimate (16) together with property a) also implies

$$\lambda(V(\theta_z(A^{ss}), \zeta_z(A^{su}), t_0))/2t_0\lambda^{ss}(A^{ss})\lambda^{su}(A^{su}) \in [(1+\omega)^{-4}, (1+\omega)^4]. \tag{17}$$

After estimating measures, we have to control components of intersections. To this end we use the hyperbolicity of the curve graph as our main tool.

Recall first from the estimate (4) in Section 2 that there is a number $\kappa > 0$ such that for any two points $u, x \in \mathcal{T}(S)$ with $d_{\mathcal{T}}(u,x) \leq 1$ the distances δ_u, δ_x on $\partial\mathcal{C}(S)$ are e^κ-bilipschitz equivalent.

Let again $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in $\tilde{\mathcal{Q}}(S)$ and let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q. Using the above numbers $a_1 < a_0$, choose closed neighborhoods $K^{ss} \subset C^{ss} \subset B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_1) \subset B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_0)$ of $\tilde{q}$ whose images under the flip $\mathcal{F}$ satisfy the properties in Lemma 4.3 for some numbers $0 < r_1 < r_2 < \alpha_0/2e^\kappa$ where $\alpha_0 > 0$ is as in Lemma 4.2. Choose also closed neighborhoods $\tilde{K}^{su} \subset \tilde{C}^{su} \subset B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_1) \subset B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_0)$ of $\tilde{q}$ with the properties in Lemma 4.3 for some numbers $0 < \tilde{r}_1 < \tilde{r}_2 < \alpha_0/2e^\kappa$.

By the choice of the set V_0, for any two points $u, z \in V(C^{ss}, \tilde{C}^{su}, t_0(1+\omega))$ the distances δ_{Pu} and δ_{Pz} are e^κ-bilipschitz equivalent. As a consequence, for all $u \in V(C^{ss}, \tilde{C}^{su}, t_0(1+\omega))$ the δ_{Pu}-diameter of $F(\mathcal{F}C^{ss} \cap \mathcal{A})$ and $F(\tilde{C}^{su} \cap \mathcal{A})$ does not exceed $\alpha_0/2$. Let

$$\rho_0 \in (0, \min\{(r_2 - r_1)/2, (\tilde{r}_2 - \tilde{r}_1)/2\}).$$

By assumption, q is recurrent (i.e. forward and backward recurrent) and hence by Lemma 4.2, applied to both $\tilde{q}$ and $-\tilde{q} = \mathcal{F}(\tilde{q})$, there is a number $R_0 = R_0(\rho_0, q) > 0$ so that for every $R \geq R_0$ and for every $z \in B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_1)$ with $d_{\mathcal{T}}(P\Phi^R z, P\Phi^R \tilde{q}) \leq 1$ we have

$$\begin{aligned} &\delta_{P\Phi^R z} \leq \rho_0 \delta_{Pz}/\alpha_0 \text{ on } F(\mathcal{F}C^{ss} \cap \tilde{\mathcal{A}}) \text{ and} \\ &\delta_{P\Phi^R z} \geq \alpha_0 \delta_{Pz}/\rho_0 \text{ on } D(\Phi^R \tilde{q}, \alpha_0). \end{aligned} \tag{18}$$

Moreover, there is a mapping class $h \in \mathrm{Stab}(\tilde{\mathcal{Q}})$ and a number $R_1 > R_0$ such that $\Phi^{R_1}\tilde{q}$ is an interior point of $hV(K^{ss}, \tilde{K}^{su})$.

By equivariance under the action of the mapping class group, for every $u \in hV(C^{ss}, \tilde{C}^{su})$ the δ_{Pu}-diameter of $F(hV(C^{ss}, \tilde{C}^{su}) \cap \tilde{\mathcal{A}})$ is smaller than $\alpha_0/2$. In particular, the $\delta_{P\Phi^{R_1}\tilde{q}}$-diameter of $F(h\tilde{C}^{su} \cap \tilde{\mathcal{A}})$ is smaller than $\alpha_0/2$. The second part of inequality (18) then implies that the $\delta_{P\tilde{q}}$-diameter of $F(h\tilde{C}^{su} \cap \tilde{\mathcal{A}})$ does not exceed ρ_0. Thus by Property c) above, by the choice of ρ_0 and by part 4) of Lemma 4.3, we have

$$F(h\tilde{C}^{su} \cap \tilde{\mathcal{A}}) \subset F(\tilde{C}^{su} \cap \tilde{\mathcal{A}}).$$

Define

$$\begin{aligned} K^{su} &= \overline{\{x \in W^{su}_{\tilde{\mathcal{Q}},\mathrm{loc}}(\tilde{q}) \cap \tilde{\mathcal{A}} \mid F(x) \in F(h\tilde{K}^{su} \cap \tilde{\mathcal{A}})\}} \text{ and} \\ C^{su} &= \overline{\{x \in W^{su}_{\tilde{\mathcal{Q}},\mathrm{loc}}(\tilde{q}) \cap \tilde{\mathcal{A}} \mid F(x) \in F(h\tilde{C}^{su} \cap \tilde{\mathcal{A}})\}}. \end{aligned}$$

Then $\tilde{q}$ is an interior point of K^{su} (as a subset of $W^{su}_{\tilde{\mathcal{Q}},\mathrm{loc}}(\tilde{q})$) because by assumption, $\Phi^{R_1}\tilde{q}$ is an interior point of $hV(K^{ss}, \tilde{K}^{su})$. Moreover, since by assumption a non-trivial element of $\mathrm{Stab}(\tilde{q})$ fixes $\tilde{\mathcal{Q}}$ pointwise, the sets K^{su}, C^{su} are precisely invariant under $\mathrm{Stab}(\tilde{q})$.

The conditional measures λ^{su} are invariant under holonomy along the strong stable foliation and transform under the Teichmüller flow by $\lambda^{su} \circ \Phi^t = e^{ht}\lambda^{su}$.

Moreover, $\lambda^{su}(\tilde{K}^{su}) \geq \lambda^{su}(\tilde{C}^{su})(1+\omega)^{-1}$ and hence properties a) and b) above and the definition of the function σ imply that

$$\lambda^{su}(K^{su}) \geq \lambda^{su}(C^{su})(1+\omega)^{-3}.$$

Define

$$\tilde{Z}_1 = V(K^{ss}, K^{su}, t_0),\ \tilde{Z}_2 = V(K^{ss}, C^{su}, t_0),\ \tilde{Z}_3 = V(C^{ss}, C^{su}, t_0(1+\omega))$$

and let Z_i be the projection of $\tilde{Z}_i$ to $\mathcal{Q}$. Note that we have $Z_1 \subset Z_2 \subset Z_3$ and

$$\lambda(Z_1) \geq \lambda(Z_3)(1+\omega)^{-8}$$

by the choice of K^{ss}, C^{ss}, by the estimate in a) above, by invariance of λ under the flow Φ^t (which implies that $\lambda(\tilde{Z}_3) \leq \lambda V(C^{ss}, C^{su}, t_0)(1+\omega)^2$) and by the fact that $\tilde{Z}_i$ is mapped homeomorphically onto Z_i for $i = 1, 2, 3$. Moreover, each of the sets Z_i is closed with dense interior. Since $(1+\omega)^8 < 1+\epsilon$, this means that the sets $Z_1 \subset Z_2 \subset Z_3$ satisfy properties (1) and (2) stated in the lemma.

Let $R > R_1 + t_0$ and let $z \in Z_1$ be a periodic point for Φ^t of period $r \in [R - t_0, R + t_0]$. Since every orbit of Φ^t which intersects Z_1 also intersects $V(K^{ss}, K^{su})$ we may assume that $z \in V(K^{ss}, K^{su})$. Let $\hat{E}$ be the component containing z of the intersection $\Phi^r V_0 \cap V_0$ and let

$$E = \hat{E} \cap \Phi^r Z_2 \cap Z_3 \subset Z_3.$$

We claim that

$$\lambda(E) \in [e^{-hr}\lambda(Z_1)(1+\omega)^{-10}, e^{-hr}\lambda(Z_1)(1+\omega)^{10}]. \tag{19}$$

To see that this is indeed the case, let $\tilde{z} \in \tilde{Z}_1$ be a lift of z. By the choice of the set C^{su} and by the first part of the estimate (18), the $\delta_{P\Phi^r\tilde{z}}$-diameter of the set $F(\mathcal{F}\Phi^r C^{ss} \cap \tilde{\mathcal{A}})$ does not exceed ρ_0. In particular, since $z \in Z_1$ and Property c) above holds true, we have

$$\Phi^r(W^s_{\mathcal{Q},\mathrm{loc}}(z) \cap Z_2) \subset E \tag{20}$$

and similarly

$$\Phi^{-r}(W^u_{\mathcal{Q},\mathrm{loc}}(z) \cap Z_1) \subset E. \tag{21}$$

Let $D \subset C^{ss}$ be such that

$$\theta_z(D) = \Phi^r(W^s_{\mathcal{Q},\mathrm{loc}}(z) \cap Z_2) \cap \theta_z(C^{ss}).$$

Then by the estimate (16) and by (21), we have

$$Q_1 = V(\theta_z(D), \zeta_z(K^{su}), t_0(1+\omega)^{-1}) \subset E \subset V(\theta_z(D), \zeta_z(C^{su}), t_0(1+\omega)) = Q_2.$$

Now by the estimate (17) and the fact that Φ^r preserves the stable foliation and contracts the measures λ^s by the factor e^{-hr}, we conclude that

$$\lambda(Q_1) \geq e^{-hr}\lambda^{ss}(K^{ss})\lambda^{su}(K^{su})/2t_0(1+\omega)^6$$

and similarly

$$\lambda(Q_2) \leq e^{-hr}\lambda^{ss}(K^{ss})\lambda^{su}(C^{su})(1+\omega)^6/2t_0.$$

Together with the estimate (17) this implies the estimate (19).

Since $r \in [R-t_0, R+t_0]$ and $e^{ht_0} \le 1+\epsilon$ we conclude that the measure of the set E is contained in the interval

$$[e^{-hR}\lambda(Z_1)(1+\omega)^{-12}, e^{-hR}\lambda(Z_1)(1+\omega)^{12}].$$

Moreover, the Lebesgue measure of the component containing z of the orbit segment $\{\Phi^t z \mid -t_0 < t < t_0\} \cap Z_2$ equals $2t_0$. Since $(1+\omega)^{12} \le 1+\epsilon$, this shows property (3) stated in the lemma. □

The next corollary implies that $\mu \le \lambda$ on the open Φ^t-invariant subset of $\mathcal{Q}$ of full Lebesgue measure of all points q with $|\mathrm{Stab}(q)| = 1$.

Corollary 4.7. *For every recurrent point $q \in \mathcal{Q}$ with $|\mathrm{Stab}(q)| = 1$, for every neighborhood V of q in $\mathcal{Q}$ and for every $\epsilon > 0$ there is a number $t_0 > 0$ and there is an open neighborhood $U \subset V$ of q such that*

$$\limsup_{R\to\infty} H(U, R-t_0, R+t_0)e^{-hR} \le 2t_0\lambda(U)(1+\epsilon).$$

Proof. We use the strategy of the proof of Lemma 6.1 of [Mar04]. The idea is to estimate the number of segments of periodic orbits in a sufficiently nice neighborhood of a recurrent point $q \in \mathcal{Q}$ with $|\mathrm{Stab}(q)| = 1$ using control of the measure of the intersection component for the time-T-map where T is the period of the orbit.

For this let V be any neighborhood of q in $\mathcal{Q}$ and let $\epsilon \in (0,1)$. Let $\omega > 0$ be sufficiently small that $(1+\omega)^4 < 1+\epsilon$. Let $Z_1 \subset Z_2 \subset Z_3 \subset V$ be the closed neighborhoods of q in $\mathcal{Q}$ constructed in Lemma 4.6 for ω and let $t_0 > 0$ be the number from that lemma.

Let $z \ne z' \in Z_1$ be periodic points of prime periods $r, s \in [R-t_0, R+t_0]$. By property 4) in the statement of Lemma 4.6, the components containing z, z' of the intersection $\Phi^R Z_2 \cap Z_3$ are disjoint. Thus by the third part of Lemma 4.6 there are at most

$$\lambda(\Phi^R Z_2 \cap Z_3)e^{hR}(1+\omega)/\lambda(Z_1)$$

such intersection arcs which are subarcs of periodic orbits of prime period in $[R-t_0, R+t_0]$. However, since the Lebesgue measure λ is mixing for the Teichmüller flow [M82, V86], for sufficiently large R we have

$$\lambda(\Phi^R Z_2 \cap Z_3) \le \lambda(Z_2)\lambda(Z_3)(1+\omega) \le \lambda(Z_1)^2(1+\omega)^3.$$

From this we deduce that

$$H(Z_1, R-t_0, R+t_0)e^{-hR} \le 2t_0\lambda(Z_1)(1+\omega)^4$$

for all sufficiently large $R > 0$. This shows the lemma. □

Now we are ready for the proof of Proposition 4.1.

Proof of Proposition 4.1. Let μ be a weak limit of the measures μ_R as $R \to \infty$. Then μ is a (a priori locally infinite) Φ^t-invariant Borel measure supported in the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$. This measure is moreover invariant under the flip $\mathcal{F} : q \to -q$.

By Corollary 4.7 it suffices to show the following. Let $A \subset \overline{\mathcal{Q}}$ be a closed Φ^t-invariant set of vanishing Lebesgue measure. Then every $q \in A$ has a neighborhood U in $\overline{\mathcal{Q}}$ such that for every $\epsilon > 0$ we have $\mu(A \cap U) < \epsilon$.

First let $q \in A \cap \mathcal{Q}$. Let $c_H > 0$ be as in Lemma 4.5. Choose compact balls $B^i \subset C^i \subset W^i_{\mathcal{Q},\mathrm{loc}}(q)$ about q for the Hodge distance of radius $r_1 > 0, r_2 > 2c_H r_1 > 0$ $(i = ss, su)$ and numbers $t_0 > 0, \delta > 0$ such that $V_3 = V(C^{ss}, C^{su}, t_0(1+\delta))$ is a set with a local product structure. This implies in particular that for every preimage $\tilde{q}$ of q in $\tilde{\mathcal{Q}}(S)$ the component $\tilde{V}_3$ of the preimage of V_3 containing $\tilde{q}$ is precisely invariant under $\mathrm{Stab}(\tilde{q})$. Then

$$V_0 = V(B^{ss}, B^{su}, t_0(1-\delta)) \subset V(C^{ss}, C^{su}, t_0(1+\delta)) = V_3$$

are closed neighborhoods of q in $\mathcal{Q}$. Let moreover

$$V_1 = V(B^{ss}, B^{su}, t_0) \subset V_2 = V(B^{ss}, C^{su}, t_0).$$

We may assume that for one (and hence every) component $\tilde{V}_3$ of the preimage of V_3 in $\tilde{\mathcal{Q}}(S)$ the diameter of the projection $P\tilde{V}_3$ of $\tilde{V}_3$ to $\mathcal{T}(S)$ does not exceed one.

As in the proof of Lemma 4.6, we require that moreover the following holds true.

$(*)$ If $z \in V(B^{ss}, B^{su})$ and if $t > 8t_0$ is such that $\Phi^t z \in V(B^{ss}, B^{su})$ then $\Phi^t(V_1 \cap W^s_{\mathcal{Q},\mathrm{loc}}(z)) \subset V_3$ and moreover $\Phi^{-t}(V_0 \cap W^u_{\mathcal{Q},\mathrm{loc}}(z)) \subset V_2$.

That this requirement can be met follows from Theorem 4.5 and the discussion in the proof of Lemma 4.6.

Now let $q \in \overline{\mathcal{Q}} - \mathcal{Q}$. If $m = 0$, i.e. if the differentials in the component $\mathcal{Q}$ do not have poles, then we choose compact sets $B^{ss} \subset C^{ss} \subset W^{ss}(q)$ and $B^{su} \subset C^{su} \subset W^{su}(q)$ containing q which are the intersections of closed balls for the Hodge norm with the set $W^i_{\mathcal{Q},\mathrm{loc}}(q)$ as in Proposition 3.3. We require that property $(*)$ above holds true (with a slight abuse of notation). We then define

$$V_0 = V(B^{ss}, B^{su}, t_0(1-\delta)) \cap \overline{\mathcal{Q}} \subset V_3 = V(C^{ss}, C^{su}, t_0(1+\delta)) \cap \overline{\mathcal{Q}}$$

and note that V_0 contains a neighborhood of q in $\overline{\mathcal{Q}}$. Define also

$$V_1 = V(B^{ss}, B^{su}, t_0) \cap \overline{\mathcal{Q}} \subset V_2 = V(B^{ss}, C^{su}, t_0) \cap \overline{\mathcal{Q}}.$$

If $m > 0$ then consider the surface S_0 obtained from S as double cover with a simple branch point at each pole of the differentials in $\mathcal{Q}$ and with no other branch point. The differentials in $\mathcal{Q}$ lift to differentials on S_0 which are invariant under the automorphism φ of S_0 defining the cover and which are contained in a fixed component $\mathcal{Q}_0$ of some stratum. The differential $\tilde{q}$ lifts to a differential $\hat{q}$ in the boundary of $\mathcal{Q}_0$ where the double cover map φ may be degenerate. The discussion in the previous paragraph can now be used to construct product coordinates near $\hat{q}$ whose restriction to the set of φ-invariant differentials has the required properties.

Let $u \in V_1$ and let $r > 0$ be such that $\Phi^r u = u$. Let Y be the connected component containing u of the intersection $V_3 \cap \Phi^r(V_2)$. By the property $(*)$, we have $Y \supset \Phi^r(V_1 \cap W^s_{\mathcal{Q},\mathrm{loc}}(u))$. Moreover, the connected component containing u of the intersection $V_3 \cap \Phi^r(V_2 \cap W^u_{\mathcal{Q},\mathrm{loc}}(u))$ contains the component containing u of the intersection $W^u_{\mathcal{Q},\mathrm{loc}}(u) \cap V_0$. Thus as in the proof of Lemma 4.6, we observe that the Lebesgue measure of the component containing u of the intersection $\Phi^r V_2 \cap V_3$ is bounded from below by χe^{-hr} where $\chi > 0$ is a fixed constant which only depends on V_1, V_2, V_3. Moreover, the number of periodic points $z \in V_1$ of period $s \in [r - t_0, r + t_0]$ such that the intersection components $\Phi^r V_2 \cap V_3, \Phi^s V_2 \cap V_3$ containing u, z are *not* disjoint is bounded from above by the cardinality of $\mathrm{Stab}(\tilde{q})$ where $\tilde{q}$ is a preimage of q in $\tilde{\mathcal{Q}}(S)$.

For $q, z \in \overline{\mathcal{Q}}$ and $t > 0$ write $q \approx_t z$ if there are lifts $\tilde{q}, \tilde{z}$ of q, z to $\tilde{\mathcal{Q}}(S)$ such that

$$d(P\Phi^s \tilde{q}, P\Phi^s \tilde{z}) < 1 \text{ for } 0 \leq s \leq t.$$

Write moreover $q \sim_u z$ if there are lifts $\tilde{q}, \tilde{z}$ of q, z to $\tilde{\mathcal{Q}}(S)$ such that

$$d(\tilde{q}, \tilde{z}) < 1, d(P\Phi^u \tilde{q}, P\Phi^u \tilde{z}) < 1.$$

Note that if $y \approx_t z$ then also $y \sim_t z$. For a subset D of $\overline{\mathcal{Q}}$ define

$$U_t(D) = \{z \mid z \approx_t y \text{ for some } y \in D\} \text{ and}$$

$$Y_u(D) = \{z \mid z \sim_u y \text{ for some } y \in D\}. \tag{22}$$

Then $U_t(D)$ and $Y_u(D)$ are open neighborhoods of D.

For $j > 0$ define

$$Z_j = U_j(A \cap V_1) \cap V_1 \text{ and} \tag{23}$$

$$W_{j,k} = Y_k(Z_j) \cap V_1. \tag{24}$$

Then for all $k > 0, j > 0$, the set Z_j is an open neighborhood of $A \cap V_1$ in V_1, and $W_{j,k}$ is an open neighborhood of Z_j in V_1. Moreover, for all j we have $Z_j \supset Z_{j+1}$, and $\cap_j Z_j \supset A \cap V_1$.

If $z \in \cap_j Z_j - A$ then there is some $y \in A$ and there are lifts $\tilde{z}, \tilde{y}$ of z, y to $\tilde{\mathcal{Q}}(S)$ such that $d(P\Phi^t(\tilde{z}), P\Phi^t(\tilde{y})) \leq 1$ for all $t \geq 0$. However, up to removing from $\cap_j Z_j$ a set of vanishing Lebesgue measure, this implies that $z \in W^{ss}_{\mathcal{Q},\mathrm{loc}}(y)$ [M82, V86]. But $\lambda(A) = 0$ and therefore $\lambda(\cap_j Z_j) = \lambda(A \cap V_1) = 0$ by absolute continuity of λ with respect to the stable foliation. Now λ is Borel regular and therefore the Lebesgue measures of the sets Z_j tend to zero as $j \to \infty$.

Similarly, we infer that $\lambda(Z_j) = \limsup_{k \to \infty} \lambda(W_{j,k})$. Thus for every $\kappa > 0$ there are numbers $j_0 = j_0(\kappa) > 0$ and $k_0 = k_0(\kappa) > j_0$ such that we have $\lambda(W_{j,k}) < \kappa$ for all $j \geq j_0, k \geq k_0$.

Now let $R > k_0 + 2t_0$ and let $w \in V_1 \cap Z_{j_0}$ be a periodic point for Φ^t of prime period $r \in [R - t_0, R + t_0]$. Let Z be the component of $\Phi^r V_2 \cap V_3$ containing w. Then every point in Z is contained in $W_{j_0,R}$. By the above discussion, the Lebesgue measure of this intersection component is bounded from below by χe^{-hR} where $\chi > 0$ is a universal constant. Moreover, the number of periodic points $u \neq z$ for which these intersection components are not disjoint is uniformly bounded. In particular, there is a number $\beta > 0$ not depending on R, j_0 such that the number of such periodic points of prime period in $[R - t_0, R + t_0]$ is bounded from above by βe^{hR} times the Lebesgue measure of $W_{j_0,R}$, i.e. by $\beta e^{hR} \kappa$. This implies that we have $\mu(Z_{j_0}) \leq 2t_0 \beta \kappa$. Since $\kappa > 0$ was arbitrary, we conclude that $\mu(A \cap V_1) = 0$. Proposition 4.1 follows.

5. Proof of the theorem

In this section we complete the proof of the theorem from the introduction. We continue to use the assumptions and notations from Sections 2-5.

As before, let $\mathcal{Q} \subset \mathcal{Q}(S)$ be a component of a stratum, equipped with the Φ^t-invariant Lebesgue measure λ. Let $\mathcal{S} \subset \mathcal{Q}$ be the open dense Φ^t-invariant subset of full Lebesgue measure of all points q with $|\mathrm{Stab}(q)| = 1$. Then $\mathcal{S}$ is a manifold.

Let $q \in \mathcal{S}$ and let $U \subset \mathcal{S}$ be an open relative compact contractible neighborhood of q. For $n > 0$ define a *periodic* (U,n)*-pseudo-orbit* for the Teichmüller flow Φ^t on $\mathcal{Q}$ to consist of a point $x \in U$ and a number $t \in [n,\infty)$ such that $\Phi^t x \in U$. We denote such a periodic pseudo-orbit by (x,t). A periodic (U,n)-pseudo-orbit (x,t) determines up to homotopy a closed curve beginning and ending at x which we call a *characteristic curve* (compare Section 4 of [H10b]). This characteristic curve is the concatenation of the orbit segment $\{\Phi^s x \mid 0 \leq s \leq t\}$ with a smooth arc in U which is parametrized on $[0,1]$ and connects the endpoint $\Phi^t x$ of the orbit segment with the starting point x.

Recall from Section 4 the definition of a recurrent point for the Teichmüller flow on $\mathcal{Q}$. Lemma 4.4 of [H10b] shows

Lemma 5.1. *There is a number $L > 0$ and for every recurrent point $q \in \mathcal{S}$ there is an open relative compact contractible neighborhood V of q in $\mathcal{S}$ and there is a number $n_0 > 0$ depending on V with the following property. Let (x,t_0) be a periodic (V,n_0)-pseudo-orbit and let γ be a lift to $\tilde{\mathcal{Q}}(S)$ of a characteristic curve of the pseudo-orbit. Then the curve $t \to \Upsilon_{\mathcal{T}}(P\gamma(t))$ is an infinite unparametrized L-quasi-geodesic in $\mathcal{C}(S)$.*

Remark: Lemma 4.4 of [H10b] is formulated for $\mathcal{Q}(S)$ rather than for a component of a stratum. However, the statement and its proof immediately carry over to the result formulated in Lemma 5.1.

For a point $q \in \mathcal{Q}$ choose a preimage $\tilde{q} \in \tilde{\mathcal{Q}}(S)$ of q and for $t > 0$ define

$$\beta(q,t) = d(\Upsilon_{\mathcal{T}}(P\tilde{q}), \Upsilon_{\mathcal{T}}(P\Phi^t\tilde{q})).$$

Note that $\beta(q,t)$ depends on the choice of the map $\Upsilon_{\mathcal{T}}$ (and on the choice of the lift $\tilde{q}$). Let $\zeta > 0$ be the maximal diameter in the curve graph of the set of all simple closed curves on S whose length with respect to a fixed hyperbolic metric does not exceed the Bers constant. Then for all q and all $s,t > 0$ we have

$$\beta(q,s+t) \leq \beta(q,s) + \beta(\Phi^s q,t) + \zeta. \tag{25}$$

By Lemma 3.3 of [H10a], there is a number $a > \zeta$ and a *continuous* function $\tilde{\beta} : \mathcal{Q} \times [0,\infty) \to \mathbb{R}$ such that $|\tilde{\beta}(q,t) - \beta(q,t)| \leq a$ for all $(q,t) \in \mathcal{Q} \times [0,\infty)$. In particular, the values $\liminf_{t\to\infty} \frac{1}{t}\beta(q,t)$ and $\limsup_{t\to\infty} \frac{1}{t}\beta(q,t)$ are independent of any choices made and coincide with the corresponding values for $\tilde{\beta}$. We use this observation to show

Lemma 5.2. *There is a number $c > 0$ such that for λ-almost every $q \in \mathcal{Q}$ we have*

$$\lim_{t\to\infty} \frac{1}{t}\beta(q,t) = c.$$

Proof. It suffices to show the lemma for the continuous function $\tilde{\beta}$.

By the choice of $a > 0$ and by the triangle inequality (25), we have

$$\tilde{\beta}(q, s+t) \leq \tilde{\beta}(q,s) + \tilde{\beta}(\Phi^s q, t) + 4a$$

for all $q \in \mathcal{Q}, s, t \in \mathbb{R}$. Therefore the subadditive ergodic theorem shows that for λ-almost all $q \in \mathcal{Q}$ the limit $\lim_{t\to\infty} \frac{1}{t}\tilde{\beta}(q,t)$ exists and is independent of q. We are left with showing that this limit is positive.

By Lemma 2.4 of [H10a], there is a number $r > 0$ such that for every $z \in \tilde{\mathcal{Q}}(S)$ and all $t \geq s \geq 0$ we have

$$d(\Upsilon_{\mathcal{T}}(Pz), \Upsilon_{\mathcal{T}}(P\Phi^t z)) \geq d(\Upsilon_{\mathcal{T}}(Pz), \Upsilon_{\mathcal{T}}(P\Phi^s z)) - r. \tag{26}$$

Let $q \in \mathcal{Q}$ be a periodic point for Φ^t. There is a number $b > 0$ such that for every lift $\tilde{q}$ of q to $\tilde{\mathcal{Q}}(S)$ the map $t \to \Upsilon_{\mathcal{T}}(P\Phi^t\tilde{q})$ is a *parametrized* biinfinite b-quasi-geodesic in $\mathcal{C}(S)$ [H10a]. Thus by inequality (2) and continuity of Φ^t we can find an open neighborhood $U \subset \mathcal{Q}$ of q and a number $T > 0$ such that

$$\tilde{\beta}(u,T) \geq 4r + 4a \text{ for all } u \in U$$

where $r > 0$ is as in (26).

Let $z \in \mathcal{Q}$, let $n > k > 0$ and let $(j_i)_{0\leq i\leq k}$ be a sequence of positive integers such that $j_k \leq n-1$ and that $j_{i+1} - j_i \geq 1$ and $\Phi^{j_i T} z \in U$ for all i. Then the estimates (25) and (26) together with the choice of U imply that $\tilde{\beta}(z, nT) \geq kr$.

The measure λ is Φ^T-invariant and ergodic, and $\lambda(U) > 0$. Thus by the Birkhoff ergodic theorem, the proportion of time a typical orbit for the map Φ^T spends in U is positive. This means that there is a number $\chi > 0$ so that for a typical point $z \in \mathcal{Q}$ and large enough n, the cardinality of the set of all numbers $j < n$ with $\Phi^{jT} z \in U$ is not smaller than χn. Then $\beta(n,z) \geq \chi n r$ for all large enough n from which the lemma follows. □

The next proposition is the main remaining step in the proof of the theorem from the introduction.

Proposition 5.3. *For every recurrent point $q \in \mathcal{S}$, for every neighborhood V of q in $\mathcal{S}$ and for every $\epsilon > 0$ there is an open neighborhood $U \subset V$ of q in $\mathcal{S}$ and a number $t_0 > 0$ such that*

$$\liminf_{R\to\infty} H(U, R - t_0 - \epsilon, R + t_0 + \epsilon)e^{-hR} \geq 2t_0\lambda(U)(1-\epsilon).$$

Proof. Let $q \in \mathcal{S}$ be recurrent and let U be an open neighborhood of q which satisfies the conclusion of Lemma 5.1 for some $n_0 > 0$. Let $\epsilon > 0$. With the notations from Subsection 3.1, let $a_0 < a_{\mathcal{Q}}(q), t_0 < \min\{t_{\mathcal{Q}}(q), \log(1+\epsilon)/2h, \epsilon/4\}$ be such that $U_0 = V(B^{ss}_{\mathcal{Q}}(q,a_0), B^{su}_{\mathcal{Q}}(q,a_0), t_0) \subset U$. By Lemma 3.1 there is a number $a_1 < a_0$ which is sufficiently small that for every $z \in V = V(B^{ss}_{\mathcal{Q}}(q,a_1), B^{su}_{\mathcal{Q}}(q,a_1), t_0)$ the Jacobian at z of the homeomorphism

$$V(B^{ss}_{\mathcal{Q}}(q,a_1), B^{su}_{\mathcal{Q}}(q,a_1), t_0) \to B^{ss}_{\mathcal{Q}}(q,a_1) \times B^{su}_{\mathcal{Q}}(q,a_1) \times [-t_0, t_0]$$

with respect to the measures λ and $\lambda^{ss} \times \lambda^{su} \times dt$ is contained in the interval $[(1+\epsilon)^{-1}, (1+\epsilon)]$. We may assume that any two points in a component $\tilde{V}$ of the preimage of V can be connected in $\tilde{V}$ by a piecewise smooth curve whose projection to $\mathcal{T}(S)$ is of length at most $\epsilon/2$.

Let $\alpha_0 > 0$ be as in Lemma 4.2. Let $\tilde{q}$ be a lift of q to a component $\tilde{\mathcal{Q}}$ of the preimage of $\mathcal{Q}$ in $\tilde{\mathcal{Q}}(S)$. Recall from Section 2 the definition of the map $F : \tilde{\mathcal{A}} \to \partial\mathcal{C}(S)$. Since q is recurrent, the horizontal and the vertical measured geodesic laminations of $\tilde{q}$ are uniquely ergodic [M82]. Let

$$Z_1 \subset Z_2 \subset Z_3 \subset V$$

be neighborhoods of q as in Lemma 4.6 and let $\tilde{Z}_1 \subset \tilde{Z}_2 \subset \tilde{Z}_3 \subset \tilde{V}$ be components of lifts of $Z_1 \subset Z_2 \subset Z_3 \subset V$ to $\tilde{\mathcal{Q}}$ which contain $\tilde{q}$. By decreasing V if necessary and using Lemma 4.3 as in the proof of Lemma 4.6, we may assume that in addition to the properties stated in Lemma 4.6, the following holds true.

i) For every $u \in \tilde{Z}_3$ the δ_{Pu}-diameter of $F(\tilde{Z}_3 \cap \tilde{\mathcal{A}})$ and of $F(\mathcal{F}\tilde{Z}_3 \cap \tilde{\mathcal{A}})$ is not bigger than α_0.
ii) There is a number $\rho > 0$ with the following property. If $z \in \tilde{Z}_1$ and if $C \subset B^{su}_{\tilde{\mathcal{Q}}}(z,a_1)$ (or $C \subset B^{ss}_{\tilde{\mathcal{Q}}}(z,a_1)$) is an open neighborhood of z such that the δ_{Pz}-diameter of $F(C \cap \tilde{\mathcal{A}})$ (or of $F(\mathcal{F}(C) \cap \tilde{\mathcal{A}})$) is not bigger than ρ then $C \subset \tilde{Z}_3$ and the Φ^t-orbit of every point of C intersects $\tilde{Z}_3$ in an arc of length $2t_0$.

Let $\Pi : \tilde{\mathcal{Q}} \to \mathcal{Q}$ be the canonical projection. By Lemma 5.2 and Lemma 4.2, there is a number $T > 0$ and there is a Borel subset $Z_0 \subset Z_1 \cap \Pi(\tilde{\mathcal{A}})$ with

$$\lambda(Z_0) > \lambda(Z_1)/(1+\epsilon)$$

such that for every $z \in \tilde{Z}_0 = \tilde{Z}_1 \cap \Pi^{-1}(Z_0)$ and every $t \geq T$ we have

$$\delta_{Pz} \leq \rho \delta_{P\Phi^t z}/e^{\kappa} \text{ on } D(\Phi^t z, \alpha_0)$$

where $\kappa > 0$ is as in the estimate (4) and $\rho > 0$ is as in ii) above.

We may assume that $Z_0 = V(K^{ss}, A_0, t_0)$ for some Borel set $A_0 \subset K^{su}$. In particular, we conclude as in the proof of Lemma 4.6 (see the estimate (19)) that (with some a-priori adjustment of the constant ϵ) the following holds true. Let $z \in Z_0$ and let $t \geq T$ be such that $\Phi^t z \in Z_1$. Let $\hat{E}$ be the connected component containing $\Phi^t z$ of the intersection $\Phi^t V \cap V$. Then the Lebesgue measure of the intersection $\Phi^t Z_2 \cap Z_3 \cap \hat{E}$ is not bigger than

$$e^{-ht}\lambda(Z_1)(1+\epsilon)^3 \leq e^{-ht}\lambda(Z_0)(1+\epsilon)^4.$$

On the other hand, since the Lebesgue measure is mixing, for sufficiently large $t > T$ we have

$$\lambda(\Phi^t Z_0 \cap Z_0) \geq \lambda(Z_0)^2/(1+\epsilon).$$

Together this implies that the number of such intersection components is at least

$$e^{ht}\lambda(Z_0)/(1+\epsilon)^5.$$

Next we claim that for sufficiently large $n \geq T$ and for a point $z \in Z_0$ with $\Phi^n z \in Z_1$ there is a periodic orbit for the flow Φ^t which intersects Z_3 in an arc of length at least $2t_0$ and whose period is contained in the interval $[n-\epsilon, n+\epsilon]$. To this end let $n_1 > \max\{n_0, T\}$ where $n_0 > 0$ is as in Lemma 5.1; then the conclusion of Lemma 5.1 is satisfied for every periodic (Z_1, n_1)-pseudo-orbit beginning at a point $z \in Z_0 \subset V$.

Let $u \in Z_0$ be such that $\Phi^n u \in Z_1$ for some $n > n_1 + \epsilon$. Up to replacing n by $R = n + \tau$ for some $\tau \in [-2t_0, 2t_0] \subset [-\epsilon/2, \epsilon/2]$ we may assume that $u \in V(K^{ss}, K^{su}), \Phi^R u \in V(K^{ss}, K^{su})$. Let γ be a characteristic curve of the periodic (Z_1, n_1)-pseudo-orbit (u, R) which we obtain by connecting $\Phi^R u \in Z_1$ with $u \in Z_0$ by a smooth arc contained in Z_1.

Let $\tilde\gamma$ be a lift of γ to $\tilde{\mathcal{Q}}$ with starting point $\tilde\gamma(0) \in \tilde Z_0$. Then $\tilde\gamma$ is invariant under a mapping class $g \in \mathrm{Mod}(S)$ whose conjugacy class defines the homotopy class of γ in $\mathcal{S}$ (note that this makes sense since $\mathcal{S}$ is a manifold). A fundamental domain for the action of g on $\tilde\gamma$ projects to a piecewise smooth arc in $\mathcal{T}(S)$ of length at most $R + \epsilon/2 < n + \epsilon$.

By Lemma 5.1 and the choice of Z_0, R the curve $t \to \Upsilon_{\mathcal{T}}(P\tilde\gamma(t))$ is an unparametrized L-quasi-geodesic in $\mathcal{C}(S)$ of infinite diameter. Up to perhaps a uniformly bounded modification (which may be necessary since the definition of the map $\Upsilon_{\mathcal{T}}$ involved some choices), the quasi-geodesic $\Upsilon_{\mathcal{T}}(P\tilde\gamma)$ is invariant under the mapping class $g \in \mathrm{Mod}(S)$, and g acts on $\Upsilon_{\mathcal{T}}(P\tilde\gamma)$ as a translation. As a consequence, g acts on $\mathcal{C}(S)$ with unbounded orbits and hence it is pseudo-Anosov. By invariance of $\tilde\gamma$ under g, the attracting fixed point for the action of g on $\partial\mathcal{C}(S)$ is just the endpoint of $\Upsilon_{\mathcal{T}}(P\tilde\gamma)$.

Since g is pseudo-Anosov, there is a closed orbit ζ for Φ^t in $\mathcal{Q}(S)$ which is the projection of a g-invariant flow line $\tilde\zeta$ for Φ^t in $\tilde{\mathcal{Q}}(S)$. The length of the orbit is at most $R + \epsilon/2 < n + \epsilon$. The image under the map $\Upsilon_{\mathcal{T}} P$ of the orbit $\tilde\zeta$ in $\tilde{\mathcal{Q}}(S)$ is an unparametrized p-quasi-geodesic in $\mathcal{C}(S)$ which connects the two fixed points for the action of g on $\partial\mathcal{C}(S)$.

Assume that the characteristic curve γ is parametrized on $[0, R+1]$ with $\gamma(0) = u$. As in the proof of Theorem 4.3 of [H10b], we claim that for every $i > 0$ we have

$$\delta_{P\tilde\gamma(0)}(F(\tilde\gamma(0)), F(\tilde\gamma(iR+i))) < \rho$$

(note that this makes sense since the points $\tilde\gamma(iR+i)$ are lifts of recurrent points in $\mathcal{Q}(S)$ by assumption). To see this we proceed by induction on i. The case $i = 1$ follows from the definition and from (4) above, so assume that the claim is known for all $j \leq i-1$ and some $i \geq 0$. By equivariance under the action of the mapping class group we have

$$\delta_{P\tilde{\gamma}(R+1)}(F\tilde{\gamma}(R), F\tilde{\gamma}(R+1)) \leq e^{\kappa}\rho, \tag{27}$$

moreover the distances $\delta_{P\tilde{\gamma}(R)}, \delta_{P\tilde{\gamma}(R+1)}$ are e^{κ}-bilipschitz equivalent.

Now $F(\tilde{\gamma}(jR+j)) \in D(\tilde{\gamma}(R+1),\rho)$ for all $j \in \{1,\dots,i\}$ by the induction hypothesis and therefore

$$\delta_{P\tilde{\gamma}(R)}(F(\tilde{\gamma}(R)), F(\tilde{\gamma}(jR+j))) \leq 2e^{\kappa}\rho. \tag{28}$$

On the other hand, by the choice of ρ and the choice of R and the fact that $\tilde{\gamma}(0) \in \tilde{Z}_0$ we obtain that

$$\delta_{P\tilde{\gamma}(R)}(F\tilde{\gamma}(R), F\tilde{\gamma}(jR+j)) \geq \delta_{P\tilde{\gamma}(0)}(F\tilde{\gamma}(0), F\tilde{\gamma}(jR+j))/2e^{\kappa}. \tag{29}$$

Together this implies the above claim.

As a consequence, the attracting fixed point ξ for the action of the pseudo-Anosov element g on $\partial\mathcal{C}(S)$ is contained in the ball $D(\tilde{\gamma}(0),\rho)$, moreover it is contained in the closure of the set $F(W^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}) \cap \tilde{\mathcal{A}}) \subset F(\tilde{\mathcal{A}} \cap \tilde{\mathcal{Q}})$. The same argument also shows that the repelling fixed point of g is contained in the intersection of $D(-\tilde{\gamma}(0),\rho)$ with the closure of $F(\mathcal{F}W^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}) \cap \tilde{\mathcal{A}}) \subset F(\tilde{\mathcal{A}} \cap \tilde{\mathcal{Q}})$. Since the map F is closed we conclude that the axis of g is contained in the closure of $\tilde{\mathcal{Q}}$. Since $\tilde{\gamma}(0) \in Z_1$, by property ii) above, this axis passes through the lift $\tilde{Z}_3$ of Z_3 containing $\tilde{q}$. In other words, the projection of this axis to $\overline{\mathcal{Q}}$ passes through Z_3, and, in particular, it is contained in $\mathcal{Q}$. Moreover, it intersects the component of $\Phi^R Z_1 \cap Z_3$ which contains $\Phi^R u$. This implies the length of the axis is contained in $[R-\epsilon/2, R+\epsilon/2] \subset [n-\epsilon, n+\epsilon]$.

To summarize, there is an injective assignment which associates to every $n > n_1$ and to every connected component of the intersection $\Phi^n Z_1 \cap Z_1$ containing points in $\Phi^n Z_0 \cap Z_1$ a subarc of length $2t_0$ of the intersection with Z_3 of a periodic orbit for Φ^t whose period is contained in $[n-\epsilon, n+\epsilon]$. Together with the above discussion, this completes the proof of the proposition. ☐

We use Proposition 5.3 to complete the proof of our theorem from the introduction.

Theorem 5.4. *The Lebesgue measure on every stratum $\mathcal{Q}$ is obtained from Bowen's construction.*

Proof. By Proposition 4.1, the measure μ is absolutely continuous with respect to the Lebesgue measure λ, with Radon Nikodym derivative at most one, moreover μ is Φ^t-invariant by construction. Thus by ergodicity, it suffices to show the following.

Let $q \in \mathcal{Q}$ be birecurrent and let $\epsilon > 0$. For $R > 0$ let $\Gamma(R)$ be the set of all periodic orbits of Φ^t in $\mathcal{Q}$ of period at most R. Then there is a compact neighborhood K of q in $\mathcal{Q}$ and there is a number $n > 0$ such that for every $N > n$ the measure

$$\mu_N = e^{-hN} \sum_{\gamma \in \Gamma(R)} \delta(\gamma)$$

assigns the mass

$$\mu_N(K) \geq (1-\epsilon)\lambda(K)$$

to K. However, this holds true by Proposition 5.3. The theorem is proven. □

Acknowledgement: This work was carried out in fall 2007 while I participated in the program on Teichmüller theory and Kleinian groups at the MSRI in Berkeley. I thank the organizers for inviting me to participate, and I thank the MSRI for its hospitality. I also thank Juan Souto for raising the question which is answered in this note.

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