

BOWEN'S CONSTRUCTION FOR THE TEICHMÜLLER FLOW

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ABSTRACT. Let \mathcal{Q} be a connected component of a stratum in the space of quadratic differentials for a non-exceptional Riemann surface S of finite type. We show that the probability measure on \mathcal{Q} in the Lebesgue measure class which is invariant under the Teichmüller flow is obtained by Bowen's construction.

1. INTRODUCTION

The *Teichmüller flow* Φ^t acts on components of strata in the moduli space of area one abelian or quadratic differentials for a non-exceptional surface S of finite type. This flow has many properties which resemble the properties of an Anosov flow. For example, there is a pair of transverse invariant foliations, and there is an invariant mixing Borel probability measure λ in the Lebesgue measure class which is absolutely continuous with respect to these foliations, with conditional measures which are uniformly expanded and contracted by the flow [M82, V86]. This measure is even exponentially mixing, i.e. exponential decay of correlations for Hölder observables holds true [AGY06, AR09].

The entropy h of the Lebesgue measure λ is the supremum of the topological entropies of the restriction of Φ^t to compact invariant sets [H10b]. For strata of abelian differentials, λ is the unique invariant measure of maximal entropy [BG07].

The goal of this note is to extend further the analogy between the Teichmüller flow on components of strata and Anosov flows. An Anosov flow Ψ^t on a compact manifold M admits a unique Borel probability measure μ of maximal entropy. This measure can be obtained as follows [B73]. Every periodic orbit γ of Ψ^t of prime period $\ell(\gamma) > 0$ supports a unique Ψ^t -invariant Borel measure $\delta(\gamma)$ of total mass $\ell(\gamma)$. If $h > 0$ is the topological entropy of Ψ^t then μ is the (unique) weak limit of the sequence of measures

$$e^{-hR} \sum_{\ell(\gamma) \leq R} \delta(\gamma)$$

as $R \rightarrow \infty$. In particular, the number of periodic orbits of period at most R is asymptotic to e^{hR}/hR as $R \rightarrow \infty$.

For any connected component \mathcal{Q} of a stratum of abelian or quadratic differentials the Φ^t -invariant Lebesgue measure λ on \mathcal{Q} can be obtained in the same way. For a precise formulation, we say that a family $\{\mu_i\}$ of finite Borel measures on the

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moduli space $\mathcal{H}(S)$ of area one abelian differentials or on the moduli space $\mathcal{Q}(S)$ of area one quadratic differentials *converges weakly* to λ if for every continuous function f on $\mathcal{H}(S)$ or on $\mathcal{Q}(S)$ with compact support we have

$$\int f d\mu_i \rightarrow \int f d\lambda.$$

Let $\Gamma(\mathcal{Q})$ be the set of all periodic orbits for Φ^t contained in \mathcal{Q} . For $\gamma \in \Gamma(\mathcal{Q})$ let $\ell(\gamma) > 0$ be the prime period of γ and denote by $\delta(\gamma)$ the Φ^t -invariant Lebesgue measure on γ of total mass $\ell(\gamma)$. We show

Theorem. *For every component \mathcal{Q} of a stratum in the moduli space of abelian or quadratic differentials the measures*

$$\mu_R = e^{-hR} \sum_{\gamma \in \Gamma(\mathcal{Q}), \ell(\gamma) \leq R} \delta(\gamma)$$

converge as $R \rightarrow \infty$ weakly to the Lebesgue measure on \mathcal{Q} .

The theorem implies that as $R \rightarrow \infty$, the number of periodic orbits in \mathcal{Q} of period at most R is asymptotically not smaller than e^{hR}/hR . However, since the closure in $\mathcal{Q}(S)$ of a component \mathcal{Q} of a stratum is non-compact, we do not obtain a precise asymptotic growth rate for all periodic orbits in \mathcal{Q} . Namely, there may be a set of periodic orbits in \mathcal{Q} whose growth rate exceeds h and which eventually exit every compact subset of $\mathcal{Q}(S)$. For periodic orbits in the open principal stratum, Eskin and Mirzakhani [EM08] showed that the asymptotic growth rate of periodic orbits for the Teichmüller flow which lie deeply in the cusp of moduli space is strictly smaller than the entropy h , and they calculate the asymptotic growth rate of all periodic orbits. Eskin, Mirzakhani and Rafi [EMR10] also announced the analogous result for any component of any stratum.

The proof of the above theorem uses ideas which were developed by Margulis for hyperbolic flows (see [Mar04] for an account with comments). This strategy is by now standard, and the main task is to overcome the difficulty of absence of hyperbolicity for the Teichmüller flow in the thin part of moduli space and the absence of nice product coordinates near a boundary point of a stratum.

Absence of hyperbolicity in the thin part of moduli space is dealt with using the curve graph similar to the strategy developed in [H10b]. Integration of the Hodge norm as discussed in [ABEM10] and some standard ergodic theory is also used.

Relative homology coordinates [V90] define local product structures for strata. These coordinates do not extend in a straightforward way to points in the boundary of the stratum. In the case of the principal stratum, however, product coordinates about boundary points can be obtained by simply writing a quadratic differential as a pair of its vertical and horizontal measured geodesic lamination. Our approach is to show that there is a similar picture for strata. To this end, we use coordinates for strata based on train tracks which will be used in other contexts as well. The construction of these coordinates is carried out in Sections 3 and 4.

The tools developed in Sections 3 and 4 are used in Section 5 to show that a weak limit μ of the measures μ_R is absolutely continuous with respect to the Lebesgue measure, with Radon Nikodym derivative bounded from above by one. In Section

6 the proof of the theorem is completed. Section 2 summarizes some properties of the curve graph and geodesic laminations used throughout the paper.

2. LAMINATIONS AND THE CURVE GRAPH

Let S be an oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g - 3 + m \geq 2$, i.e. that S is not a sphere with at most four punctures or a torus with at most one puncture. The *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete finite volume hyperbolic metrics on S under the action of the group of diffeomorphisms of S which are isotopic to the identity. The fibre bundle $\mathcal{Q}^1(S)$ over $\mathcal{T}(S)$ of all *marked holomorphic quadratic differentials* of area one can be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the *Teichmüller metric* $d_{\mathcal{T}}$. We assume that each quadratic differential $q \in \mathcal{Q}^1(S)$ has a pole of first order at each of the punctures, i.e. we include the information on the number of poles of the differential in the number of punctures of S . The *Teichmüller flow* Φ^t on $\mathcal{Q}^1(S)$ commutes with the action of the *mapping class group* $\text{Mod}(S)$ of all isotopy classes of orientation preserving self-homeomorphisms of S . Therefore this flow descends to a flow on the quotient orbifold $\mathcal{Q}(S) = \mathcal{Q}^1(S)/\text{Mod}(S)$, again denoted by Φ^t .

2.1. Geodesic laminations. A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination ν is called *minimal* if each of its half-leaves is dense in ν . Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. Every geodesic lamination ν consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of ν either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components. A geodesic lamination ν *fills up* S if its complementary components are topological discs or once punctured monogons, i.e. once punctured discs bounded by a single leaf of ν .

The set \mathcal{L} of all geodesic laminations on S can be equipped with the restriction of the *Hausdorff topology* for compact subsets of S . With respect to this topology, the space \mathcal{L} is compact.

The projectivized tangent bundle $PT\nu$ of a geodesic lamination ν is a compact subset of the projectivized tangent bundle PTS of S . The geodesic lamination ν is *orientable* if there is a continuous orientation of the tangent bundle of ν . This is equivalent to stating that there is a continuous section $PT\nu \rightarrow T^1S$ where T^1S denotes the unit tangent bundle of S .

Definition 2.1. A *large geodesic lamination* is a geodesic lamination ν which fills up S and can be approximated in the Hausdorff topology by simple closed geodesics.

Note that a minimal geodesic lamination ν can be approximated in the Hausdorff topology by simple closed geodesics and hence if ν fills up S then ν is large. Moreover, the set of all large geodesic laminations is closed with respect to the Hausdorff topology and hence it is compact.

The *topological type* of a large geodesic lamination ν is a tuple

$$(m_1, \dots, m_\ell; -m) \text{ where } 1 \leq m_1 \leq \dots \leq m_\ell, \sum_i m_i = 4g - 4 + m$$

such that the complementary components of ν which are topological discs are m_i+2 -gons. Let

$$\mathcal{LL}(m_1, \dots, m_\ell; -m)$$

be the space of all large geodesic laminations of type $(m_1, \dots, m_\ell; -m)$ equipped with the restriction of the Hausdorff topology for compact subsets of S . A geodesic lamination is called *complete* if it is large of type $(1, \dots, 1; -m)$. The complementary components of a complete geodesic lamination are all trigons or once punctured monogons.

A *measured geodesic lamination* is a geodesic lamination ν equipped with a translation invariant transverse measure ξ such that the ξ -weight of every compact arc in S with endpoints in $S - \nu$ which intersects ν nontrivially and transversely is positive. We say that ν is the *support* of the measured geodesic lamination. The geodesic lamination ν is *uniquely ergodic* if ξ is the only transverse measure with support ν up to scale.

The space \mathcal{ML} of measured geodesic laminations equipped with the weak*-topology admits a natural continuous action of the multiplicative group $(0, \infty)$. The quotient under this action is the space \mathcal{PML} of *projective measured geodesic laminations* which is homeomorphic to the sphere $S^{6g-7+2m}$.

Every simple closed geodesic c on S defines a measured geodesic lamination. The geometric intersection number between simple closed curves on S extends to a continuous function ι on $\mathcal{ML} \times \mathcal{ML}$, the *intersection form*. We say that a pair $(\xi, \mu) \in \mathcal{ML} \times \mathcal{ML}$ of measured geodesic laminations *jointly fills up* S if for every measured geodesic lamination $\eta \in \mathcal{ML}$ we have $\iota(\eta, \xi) + \iota(\eta, \mu) > 0$. This is equivalent to stating that every complete simple (possibly infinite) geodesic on S intersects either the support of ξ or the support of μ transversely.

2.2. The curve graph. The *curve graph* $\mathcal{C}(S)$ of S is the locally infinite metric graph whose vertices are the free homotopy classes of essential simple closed curves on S , i.e. curves which are neither contractible nor freely homotopic into a puncture. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The mapping class group $\text{Mod}(S)$ of S acts on $\mathcal{C}(S)$ as a group of simplicial isometries.

The curve graph $\mathcal{C}(S)$ is a hyperbolic geodesic metric space [MM99] and hence it admits a *Gromov boundary* $\partial\mathcal{C}(S)$. For $c \in \mathcal{C}(S)$ there is a complete distance function δ_c on $\partial\mathcal{C}(S)$ of uniformly bounded diameter, and there is a number $\rho > 0$ such that

$$\delta_c \leq e^{\rho d(c,a)} \delta_a \text{ for all } c, a \in \mathcal{C}(S).$$

The group $\text{Mod}(S)$ acts on $\partial\mathcal{C}(S)$ as a group of homeomorphisms.

Let $\kappa_0 > 0$ be a *Bers constant* for S , i.e. κ_0 is such that for every complete hyperbolic metric on S of finite volume there is a pants decomposition of S consisting of pants curves of length at most κ_0 . Define a map

$$(1) \quad \Upsilon_{\mathcal{T}} : \mathcal{T}(S) \rightarrow \mathcal{C}(S)$$

by associating to $x \in \mathcal{T}(S)$ a simple closed curve of x -length at most κ_0 . Then there is a number $c > 0$ such that

$$(2) \quad d_{\mathcal{T}}(x, y) \geq d(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(y))/c - c$$

for all $x, y \in \mathcal{T}(S)$ (see the discussion in [H10a]).

For a number $L > 1$, a map $\gamma : [0, s) \rightarrow \mathcal{C}(S)$ ($s \in (0, \infty]$) is an L -*quasi-geodesic* if for all $t_1, t_2 \in [0, s)$ we have

$$|t_1 - t_2|/L - L \leq d(\gamma(t_1), \gamma(t_2)) \leq L|t_1 - t_2| + L.$$

A map $\gamma : [0, \infty) \rightarrow \mathcal{C}(S)$ is called an *unparametrized L -quasi-geodesic* if there is an increasing homeomorphism $\varphi : [0, s) \rightarrow [0, \infty)$ ($s \in (0, \infty]$) such that $\gamma \circ \varphi$ is an L -quasi-geodesic. We say that an unparametrized quasi-geodesic is *infinite* if its image set has infinite diameter. There is a number $p > 1$ such that the image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic is an unparametrized p -quasi-geodesic [MM99].

Choose a smooth function $\sigma : [0, \infty) \rightarrow [0, 1]$ with $\sigma[0, \kappa_0] \equiv 1$ and $\sigma[2\kappa_0, \infty) \equiv 0$. For each $x \in \mathcal{T}(S)$, the number of essential simple closed curves c on S whose x -length $\ell_x(c)$ (i.e. the length of a geodesic representative in its free homotopy class) does not exceed $2\kappa_0$ is bounded from above by a constant not depending on x , and the diameter of the subset of $\mathcal{C}(S)$ containing these curves is uniformly bounded as well. Thus we obtain for every $x \in \mathcal{T}(S)$ a finite Borel measure μ_x on $\mathcal{C}(S)$ by defining

$$\mu_x = \sum_{c \in \mathcal{C}(S)} \sigma(\ell_x(c)) \Delta_c$$

where Δ_c denotes the Dirac mass at c . The total mass of μ_x is bounded from above and below by a universal positive constant, and the diameter of the support of μ_x in $\mathcal{C}(S)$ is uniformly bounded as well. Moreover, the measures μ_x depend continuously on $x \in \mathcal{T}(S)$ in the weak*-topology. This means that for every bounded function $f : \mathcal{C}(S) \rightarrow \mathbb{R}$ the function $x \rightarrow \int f d\mu_x$ is continuous.

For $x \in \mathcal{T}(S)$ define a distance δ_x on $\partial\mathcal{C}(S)$ by

$$(3) \quad \delta_x(\xi, \zeta) = \int \delta_c(\xi, \zeta) d\mu_x(c) / \mu_x(\mathcal{C}(S)).$$

The distances δ_x are equivariant with respect to the action of $\text{Mod}(S)$ on $\mathcal{T}(S)$ and $\partial\mathcal{C}(S)$. Moreover, there is a constant $\kappa > 0$ such that

$$(4) \quad \delta_x \leq e^{\kappa d_{\mathcal{T}}(x, y)} \delta_y \text{ and } \kappa^{-1} \delta_y \leq \delta_{\Upsilon_{\mathcal{T}}(y)} \leq \kappa \delta_y$$

for all $x, y \in \mathcal{T}(S)$ (see p.230 and p.231 of [H09b]).

An area one quadratic differential $z \in \mathcal{Q}^1(S)$ is determined by a pair (μ, ν) of measured geodesic laminations which jointly fill up S and such that $\iota(\mu, \nu) = 1$. The laminations μ, ν are called *vertical* and *horizontal*, respectively. For $z \in \mathcal{Q}^1(S)$ let

$W^u(z) \subset \mathcal{Q}^1(S)$ be the set of all quadratic differentials whose horizontal projective measured geodesic lamination coincides with the horizontal projective measured geodesic lamination of z . The space $W^u(z)$ is called the *unstable manifold* of z , and these unstable manifolds define the *unstable foliation* W^u of $\mathcal{Q}^1(S)$. The *strong unstable manifold* $W^{su}(z) \subset W^u(z)$ is the set of all quadratic differentials whose horizontal measured geodesic lamination coincides with the horizontal measured geodesic lamination of z . These sets define the *strong unstable foliation* W^{su} of $\mathcal{Q}^1(S)$. The image of the unstable (or the strong unstable) foliation of $\mathcal{Q}^1(S)$ under the flip $\mathcal{F} : q \rightarrow \mathcal{F}(q) = -q$ is the *stable foliation* W^s (or the *strong stable foliation* W^{ss}).

By the Hubbard-Masur theorem, for each $z \in \mathcal{Q}^1(S)$ the restriction to $W^u(z)$ of the canonical projection

$$P : \mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$$

is a homeomorphism. Thus the Teichmüller metric lifts to a complete distance function d^u on $W^u(z)$. Denote by d^{su} the restriction of this distance function to $W^{su}(z)$. Then $d^s = d^u \circ \mathcal{F}$, $d^{ss} = d^{su} \circ \mathcal{F}$ are distance functions on the leaves of the stable and strong stable foliation, respectively. For $z \in \mathcal{Q}^1(S)$ and $r > 0$ let moreover $B^i(z, r) \subset W^i(z)$ be the closed ball of radius r about z with respect to d^i ($i = u, su, s, ss$).

Let

$$(5) \quad \tilde{\mathcal{A}} \subset \mathcal{Q}^1(S)$$

be the set of all marked quadratic differentials q such that the unparametrized quasi-geodesic $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t q)$ ($t \in [0, \infty)$) is infinite. Then $\tilde{\mathcal{A}}$ is the set of all quadratic differentials whose vertical measured geodesic lamination fills up S (i.e. its support fills up S , see [H06] for a comprehensive discussion of this result of Klarreich [Kl99]). There is a natural $\text{Mod}(S)$ -equivariant surjective map

$$F : \tilde{\mathcal{A}} \rightarrow \partial\mathcal{C}(S)$$

which associates to a point $z \in \tilde{\mathcal{A}}$ the endpoint of the infinite unparametrized quasi-geodesic $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t q)$ ($t \in [0, \infty)$).

Call a marked quadratic differential $z \in \mathcal{Q}^1(S)$ *uniquely ergodic* if the support of its vertical measured geodesic lamination is uniquely ergodic and fills up S . A uniquely ergodic quadratic differential is contained in the set $\tilde{\mathcal{A}}$ [H06, Kl99]. We have (Section 3 of [H09b])

- Lemma 2.2.** (1) *The map $F : \tilde{\mathcal{A}} \rightarrow \partial\mathcal{C}(S)$ is continuous and closed.*
(2) *If $z \in \mathcal{Q}^1(S)$ is uniquely ergodic then the sets $F(B^{su}(z, r) \cap \tilde{\mathcal{A}})$ ($r > 0$) form a neighborhood basis for $F(z)$ in $\partial\mathcal{C}(S)$.*

For $z \in \tilde{\mathcal{A}}$ and $r > 0$ let

$$D(z, r)$$

be the closed ball of radius r about $F(z)$ with respect to the distance function δ_{Pz} . As a consequence of Lemma 2.2, if $z \in \mathcal{Q}^1(S)$ is uniquely ergodic then for every $r > 0$ there are numbers $r_0 < r$ and $\beta > 0$ such that

$$(6) \quad F(B^{su}(z, r_0) \cap \tilde{\mathcal{A}}) \subset D(z, \beta) \subset F(B^{su}(z, r) \cap \tilde{\mathcal{A}}).$$

3. TRAIN TRACKS

In this section we establish some properties of train tracks on an oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$ which will be used in Section 4 to construct coordinates near boundary points of strata.

A *train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. A simple closed curve component of τ contains a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with 0, 1 or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. We always identify train tracks which are isotopic. Throughout we use the book [PH92] as the main reference for train tracks.

A train track is called *generic* if all switches are at most trivalent. For each switch v of a generic train track τ which is not contained in a simple closed curve component, there is a unique half-branch b of τ which is incident on v and which is *large* at v . This means that every germ of an arc of class C^1 on τ which passes through v also passes through the interior of b . A half-branch which is not large is called *small*. A branch b of τ is called *large* (or *small*) if each of its two half-branches is large (or small). A branch which is neither large nor small is called *mixed*.

Remark: As in [H09], all train tracks are assumed to be generic. Unfortunately this leads to a small inconsistency of our terminology with the terminology found in the literature.

A *trainpath* on a train track τ is a C^1 -immersion $\rho : [k, \ell] \rightarrow \tau$ such that for every $i < \ell - k$ the restriction of ρ to $[k + i, k + i + 1]$ is a homeomorphism onto a branch of τ . More generally, we call a C^1 -immersion $\rho : [a, b] \rightarrow \tau$ a *generalized trainpath*. A trainpath $\rho : [k, \ell] \rightarrow \tau$ is *closed* if $\rho(k) = \rho(\ell)$ and if the extension ρ' defined by $\rho'(t) = \rho(t)$ ($t \in [k, \ell]$) and $\rho'(\ell + s) = \rho(k + s)$ ($s \in [0, 1]$) is a trainpath.

A generic train track τ is *orientable* if there is a consistent orientation of the branches of τ such that at any switch s of τ , the orientation of the large half-branch incident on s extends to the orientation of the two small half-branches incident on s . If C is a complementary polygon of an oriented train track then the number of sides of C is even. In particular, a train track which contains a once punctured monogon component, i.e. a once punctured disc with one cusp at the boundary, is not orientable (see p.31 of [PH92] for a more detailed discussion).

A train track or a geodesic lamination η is *carried* by a train track τ if there is a map $F : S \rightarrow S$ of class C^1 which is homotopic to the identity and maps η into τ in such a way that the restriction of the differential of F to the tangent space of η vanishes nowhere; note that this makes sense since a train track has a tangent line

everywhere. We call the restriction of F to η a *carrying map* for η . Write $\eta \prec \tau$ if the train track η is carried by the train track τ . Then every geodesic lamination ν which is carried by η is also carried by τ .

A train track *fills up* S if its complementary components are topological discs or once punctured monogons. Note that such a train track τ is connected. Let $\ell \geq 1$ be the number of those complementary components of τ which are topological discs. Each of these discs is an $m_i + 2$ -gon for some $m_i \geq 1$ ($i = 1, \dots, \ell$). The *topological type* of τ is defined to be the ordered tuple $(m_1, \dots, m_\ell; -m)$ where $1 \leq m_1 \leq \dots \leq m_\ell$; then $\sum_i m_i = 4g - 4 + m$. If τ is orientable then $m = 0$ and m_i is even for all i . A train track of topological type $(1, \dots, 1; -m)$ is called *maximal*. The complementary components of a maximal train track are all trigons, i.e. topological discs with three cusps at the boundary, or once punctured monogons.

A *transverse measure* on a generic train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: for every trivalent switch s of τ , the sum of the weights of the two small half-branches incident on s equals the weight of the large half-branch. The space $\mathcal{V}(\tau)$ of all transverse measures on τ has the structure of a cone in a finite dimensional real vector space, and it is naturally homeomorphic to the space of all measured geodesic laminations whose support is carried by τ . The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We call such a transverse measure μ *positive*, and we write $\mu > 0$ (see [PH92] for more details).

A *subtrack* σ of a train track τ is a subset of τ which is itself a train track. Then σ is obtained from τ by removing some of the branches, and we write $\sigma < \tau$. If b is a small branch of τ which is incident on two distinct switches of τ then the graph σ obtained from τ by removing b is a subtrack of τ . We then call τ a *simple extension* of σ . Note that formally to obtain the subtrack σ from $\tau - b$ we may have to delete the switches on which the branch b is incident.

Lemma 3.1. (1) *A simple extension τ of a recurrent non-orientable connected train track σ is recurrent. Moreover,*

$$\dim \mathcal{V}(\sigma) = \dim \mathcal{V}(\tau) - 1.$$

(2) *An orientable simple extension τ of a recurrent orientable connected train track σ is recurrent. Moreover,*

$$\dim \mathcal{V}(\sigma) = \dim \mathcal{V}(\tau) - 1.$$

Proof. If τ is a simple extension of a train track σ then σ can be obtained from τ by the removal of a small branch b which is incident on two distinct switches s_1, s_2 . Then s_i is an interior point of a branch b_i of σ ($i = 1, 2$).

If σ is connected, non-orientable and recurrent then there is a trainpath $\rho_0 : [0, t] \rightarrow \tau - b$ which begins at s_1 , ends at s_2 and such that the half-branch $\rho_0[0, 1/2]$ is small at $s_1 = \rho_0(0)$ and that the half-branch $\rho_0[t - 1/2, t]$ is small at $s_2 = \rho_0(t)$. Extend ρ_0 to a closed trainpath ρ on $\tau - b$ which begins and ends at s_1 . This is possible since σ is non-orientable, connected and recurrent. There is a closed trainpath $\rho' : [0, u] \rightarrow \tau$ which can be obtained from ρ by replacing the trainpath ρ_0 by the branch b traveled through from s_1 to s_2 . The counting measure of

ρ' on τ satisfies the switch condition and hence it defines a transverse measure on τ which is positive on b . On the other hand, every transverse measure on σ defines a transverse measure on τ . Thus since σ is recurrent and since the sum of two transverse measures on τ is again a transverse measure, the train track τ is recurrent as well. Moreover, we have $\dim\mathcal{V}(\tau) \geq \dim\mathcal{V}(\sigma) + 1$.

Let p be the number of branches of τ . Label the branches of τ with the numbers $\{1, \dots, p\}$ so that the number p is assigned to b . Let e_1, \dots, e_p be the standard basis of \mathbb{R}^p and define a linear map $A : \mathbb{R}^p \rightarrow \mathbb{R}^p$ by $A(e_i) = e_i$ for $i \leq p-1$ and $A(e_p) = \sum_i \nu(i)e_i$ where ν is the weight function on $\{1, \dots, p\}$ defined by the trainpath ρ_0 . The map A is a surjection onto a linear subspace of \mathbb{R}^p of codimension one, moreover A preserves the linear subspace V of \mathbb{R}^p defined by the switch conditions for τ . In particular, the corank of $A(V)$ is at most one. The image under A of the cone of all nonnegative weights on the branches of τ satisfying the switch conditions is contained in the cone of all nonnegative weights on $\tau - b = \sigma$ satisfying the switch conditions for σ . Therefore the dimension of the space of transverse measures on σ equals the space of transverse measures on τ minus one. This implies $\dim(\mathcal{V}(\tau)) = \dim(\mathcal{V}(\sigma)) + 1$ and completes the proof of the first part of the lemma. The second part follows in exactly the same way. \square

As a consequence we obtain

- Corollary 3.2.** (1) $\dim\mathcal{V}(\tau) = 2g - 2 + m + \ell$ for every non-orientable recurrent train track τ of topological type $(m_1, \dots, m_\ell; -m)$.
(2) $\dim\mathcal{V}(\tau) = 2g - 1 + \ell$ for every orientable recurrent train track τ of topological type $(m_1, \dots, m_\ell; 0)$.

Proof. The disc components of a non-orientable recurrent train track τ of topological type $(m_1, \dots, m_\ell; -m)$ can be subdivided in $4g - 4 + m - \ell$ steps into trigons by successively adding small branches. A successive application of Lemma 3.1 shows that the resulting train track η is maximal and recurrent. Since for every maximal recurrent train track η we have $\dim\mathcal{V}(\eta) = 6g - 6 + 2m$ (see [PH92]), the first part of the corollary follows.

To show the second part, let τ be an orientable recurrent train track of type $(m_1, \dots, m_\ell; 0)$. Then m_i is even for all i . Add a branch b_0 to τ which cuts some complementary component of τ into a trigon and a second polygon with an odd number of sides. The resulting train track η_0 is not recurrent since a trainpath on η_0 can only pass through b_0 at most once. However, we can add to η_0 another small branch b_1 which cuts some complementary component of η_0 with at least 4 sides into a trigon and a second polygon such that the resulting train track η is non-orientable and recurrent. The inward pointing tangent of b_1 is chosen in such a way that there is a trainpath traveling both through b_0 and b_1 . The counting measure of any simple closed curve which is carried by η gives equal weight to the branches b_0 and b_1 . But this just means that $\dim\mathcal{V}(\eta) = \dim\mathcal{V}(\tau) + 1$ (see the proof of Lemma 3.1 for a detailed argument). By the first part of the corollary, we have $\dim\mathcal{V}(\eta) = 2g - 2 + \ell + 2$ which completes the proof. \square

Definition 3.3. A train track τ of topological type $(m_1, \dots, m_\ell; -m)$ is *fully recurrent* if τ carries a large minimal geodesic lamination $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m)$.

Note that by definition, a fully recurrent train track is connected and fills up S . The next lemma gives some first property of a fully recurrent train track τ . For its proof, recall that there is a natural homeomorphism of $\mathcal{V}(\tau)$ onto the subspace of \mathcal{ML} of all measured geodesic laminations carried by τ .

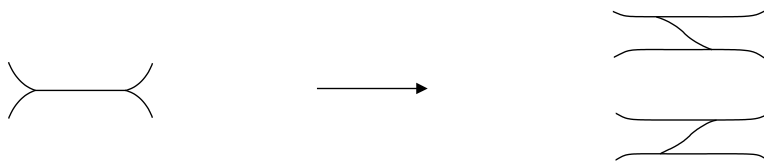
Lemma 3.4. *A fully recurrent train track τ of topological type $(m_1, \dots, m_\ell; -m)$ is recurrent.*

Proof. A fully recurrent train track τ of type $(m_1, \dots, m_\ell; -m)$ carries a large geodesic lamination $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m)$. The carrying map $\nu \rightarrow \tau$ induces a bijection between the complementary components of τ and the complementary components of ν . In particular, a carrying map $\nu \rightarrow \tau$ is necessarily surjective. The third paragraph in the proof of Lemma 2.3 of [H09] now shows that τ is recurrent. \square

There are two simple ways to modify a fully recurrent train track τ to another fully recurrent train track. Namely, if b is a mixed branch of τ then we can *shift* τ along b to a new train track τ' . This new train track carries τ and hence it is fully recurrent since it carries every geodesic lamination which is carried by τ [PH92, H09].

Similarly, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure A below. A (right or left) split τ' of a train track τ is carried by τ . If τ is of topological type $(m_1, \dots, m_\ell; -m)$, if $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m)$ is minimal and is carried by τ and if e is a large branch of τ , then there is a unique choice of a right or left split of τ at e such that the split track η carries ν . In particular, η is fully recurrent. Note however that there may be a split of τ at e such that the split track is not fully recurrent any more (see Section 2 of [H09] for details).

Figure A



The following simple observation is used to identify fully recurrent train tracks.

- Lemma 3.5.**
- (1) *Let e be a large branch of a fully recurrent non-orientable train track τ . Then no component of the train track σ obtained from τ by splitting τ at e and removing the diagonal of the split is orientable.*
 - (2) *Let e be a large branch of a fully recurrent orientable train track τ . Then the train track σ obtained from τ by splitting τ at e and removing the diagonal of the split is connected.*

Proof. Let τ be a fully recurrent non-orientable train track of topological type $(m_1, \dots, m_\ell; -m)$. Let e be a large branch of τ and let v be a switch on which the branch e is incident. Let σ be the train track obtained from τ by splitting τ at e and removing the diagonal branch of the split. Then the train tracks τ_1, τ_2 obtained from τ by a right and left split at e , respectively, are simple extensions of σ .

If σ is connected and orientable then the train tracks τ_1, τ_2 are not recurrent since no transverse measure can give positive weight to the diagonal of the split (compare the discussion in the proof of Lemma 3.1). However, since τ is fully recurrent, it can be split at e to a fully recurrent and hence recurrent train track. This is a contradiction.

Now assume that σ is disconnected and contains an orientable connected component σ_1 . Let $b_i \in \tau_i - \sigma$ be a diagonal of the split connecting τ to τ_i ($i = 1, 2$). If $\rho_i : [0, m] \rightarrow \tau_i$ is a trainpath with $\rho_i[0, 1] = b_i$ and $\rho_i[1, 2] \in \sigma_1$ then $\rho_i[1, m] \subset \sigma_1$ and hence once again, τ_i is not recurrent. As above, this contradicts the assumption that τ is fully recurrent. The first part of the corollary is proven. The second part follows from the same argument since a split of an orientable train track is orientable. \square

Example: 1) Figure B below shows a non-orientable recurrent train track τ of type $(4; 0)$ on a closed surface of genus two. The train track obtained from τ by a split at the large branch e and removal of the diagonal of the split track is orientable and hence τ is not fully recurrent. This corresponds to the fact established by Masur and Smillie [MS93] that every quadratic differential with a single zero and no pole on a surface of genus 2 is the square of a holomorphic one-form (see Section 4 for more information).

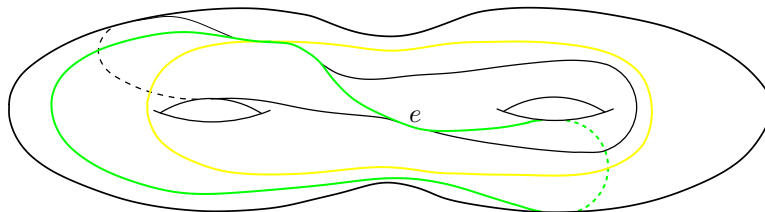


Figure B

2) To construct an orientable recurrent train track of type $(m_1, \dots, m_\ell; 0)$ which is not fully recurrent let S_1 be a surface of genus $g_1 \geq 2$ and let τ_1 be an orientable fully recurrent train track on S_1 with $\ell_1 \geq 1$ complementary components. Choose a complementary component C_1 of τ_1 in S_1 , remove from C_1 a disc D_1 and glue two copies of $S_1 - D_1$ along the boundary of D_1 to a surface S of genus $2g_1$. The two copies of τ_1 define a recurrent disconnected oriented train track τ on S which has an annulus complementary component C .

Choose a branch b_1 of τ in the boundary of C . There is a corresponding branch b_2 in the second boundary component of C . Glue a compact subarc of b_1 contained

in the interior of b_1 to a compact subarc of b_2 contained in the interior of b_2 so that the images of the two arcs under the glueing form a large branch e in the resulting train track η . The train track η is recurrent and orientable, and its complementary components are topological discs. However, by Lemma 3.5 it is not fully recurrent.

To each train track τ which fills up S one can associate a *dual bigon track* τ^* (Section 3.4 of [PH92]). There is a bijection between the complementary components of τ and those complementary components of τ^* which are not *bigons*, i.e. discs with two cusps at the boundary. This bijection maps a component C of τ which is an n -gon for some $n \geq 3$ to an n -gon component of τ^* contained in C , and it maps a once punctured monogon C to a once punctured monogon contained in C . If τ is orientable then the orientation of S and an orientation of τ induce an orientation on τ^* , i.e. τ^* is orientable.

Measured geodesic laminations which are carried by τ^* can be described as follows. A *tangential measure* on a train track τ of type $(m_1, \dots, m_\ell; -m)$ assigns to a branch b of τ a weight $\mu(b) \geq 0$ such that for every complementary k -gon of τ with consecutive sides c_1, \dots, c_k and total mass $\mu(c_i)$ (counted with multiplicities) the following holds true.

- (1) $\mu(c_i) \leq \mu(c_{i-1}) + \mu(c_{i+1})$.
- (2) $\sum_{i=j}^{k+j-1} (-1)^{i-j} \mu(c_i) \geq 0$, $j = 1, \dots, k$.

(The complementary once punctured monogons define no constraint on tangential measures). The space of all tangential measures on τ has the structure of a convex cone in a finite dimensional real vector space. By the results from Section 3.4 of [PH92], every tangential measure on τ determines a simplex of measured geodesic laminations which *hit τ efficiently*. The supports of these measured geodesic laminations are carried by the bigon track τ^* , and every measured geodesic lamination which is carried by τ^* can be obtained in this way. The dimension of this simplex equals the number of complementary components of τ with an even number of sides. The train track τ is called *transversely recurrent* if it admits a tangential measure which is positive on every branch.

In general, there are many tangential measures which correspond to a fixed measured geodesic lamination ν which hits τ efficiently. Namely, let s be a switch of τ and let a, b, c be the half-branches of τ incident on s and such that the half-branch a is large. If β is a tangential measure on τ which determines the measured geodesic lamination ν then it may be possible to drag the switch s across some of the leaves of ν and modify the tangential measure β on τ to a tangential measure $\mu \neq \beta$. Then $\beta - \mu$ is a multiple of a vector of the form $\delta_a - \delta_b - \delta_c$ where δ_w denotes the function on the branches of τ defined by $\delta_w(w) = 1$ and $\delta_w(a) = 0$ for $a \neq w$.

Definition 3.6. A train track τ of topological type $(m_1, \dots, m_\ell; -m)$ is called *fully transversely recurrent* if its dual bigon track τ^* carries a large minimal geodesic lamination $\nu \in \mathcal{LL}(m_1, \dots, m_\ell; -m)$. A train track τ of topological type $(m_1, \dots, m_\ell; -m)$ is called *large* if τ is fully recurrent and fully transversely recurrent. A large train track of type $(1, \dots, 1; -m)$ is called *complete*.

For a large train track τ let $\mathcal{V}^*(\tau) \subset \mathcal{ML}$ be the set of all measured geodesic laminations whose support is carried by τ^* . Each of these measured geodesic laminations corresponds to a tangential measure on τ . With this identification, the pairing

$$(7) \quad (\nu, \mu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau) \rightarrow \sum_b \nu(b)\mu(b)$$

is just the restriction of the intersection form on measured lamination space (Section 3.4 of [PH92]). Moreover, $\mathcal{V}^*(\tau)$ is naturally homeomorphic to a convex cone in a real vector space. The dimension of this cone coincides with the dimension of $\mathcal{V}(\tau)$.

Denote by $\mathcal{LT}(m_1, \dots, m_\ell; -m)$ the set of all isotopy classes of large train tracks on S of type $(m_1, \dots, m_\ell; -m)$.

Remark: In [MM99], Masur and Minsky define a large train track to be a train track τ whose complementary components are topological discs or once punctured monogons, without the requirement that τ is generic, transversely recurrent or recurrent. We hope that this inconsistency of terminology does not lead to any confusion.

4. STRATA

As in Section 2, for a closed oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures let $\mathcal{Q}^1(S)$ be the bundle of marked area one holomorphic quadratic differentials with a simple pole at each puncture over the Teichmüller space $\mathcal{T}(S)$ of marked complex structures on S . For a complete hyperbolic metric on S of finite area, an area one quadratic differential $q \in \mathcal{Q}^1(S)$ is determined by a pair (λ^+, λ^-) of measured geodesic laminations which jointly fill up S and such that $\iota(\lambda^+, \lambda^-) = 1$. The *vertical* measured geodesic lamination λ^+ for q corresponds to the equivalence class of the vertical measured foliation of q . The *horizontal* measured geodesic lamination λ^- for q corresponds to the equivalence class of the horizontal measured foliation of q .

A tuple (m_1, \dots, m_ℓ) of positive integers $1 \leq m_1 \leq \dots \leq m_\ell$ with $\sum_i m_i = 4g - 4 + m$ defines a *stratum* $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ in $\mathcal{Q}^1(S)$. This stratum consists of all marked area one quadratic differentials with m simple poles and ℓ zeros of order m_1, \dots, m_ℓ which are not squares of holomorphic one-forms. The stratum is a real hypersurface in a complex manifold of dimension

$$(8) \quad h = 2g - 2 + m + \ell.$$

The closure in $\mathcal{Q}^1(S)$ of a stratum is a union of components of strata. Strata are invariant under the action of the mapping class group $\text{Mod}(S)$ of S and hence they project to strata in the moduli space $\mathcal{Q}(S) = \mathcal{Q}^1(S)/\text{Mod}(S)$ of quadratic differentials on S with a simple pole at each puncture. We denote the projection of the stratum $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ by $\mathcal{Q}(m_1, \dots, m_\ell; -m)$. The strata in moduli space need not be connected, but their connected components have been identified by Laneeau [L08]. A stratum in $\mathcal{Q}(S)$ has at most two connected components.

Similarly, if $m = 0$ then we let $\mathcal{H}^1(S)$ be the bundle of marked area one holomorphic one-forms over Teichmüller space $\mathcal{T}(S)$ of S . For a tuple $k_1 \leq \dots \leq k_\ell$ of

positive integers with $\sum_i k_i = 2g - 2$, the stratum $\mathcal{H}^1(k_1, \dots, k_\ell)$ of marked area one holomorphic one-forms on S with ℓ zeros of order k_i ($i = 1, \dots, \ell$) is a real hypersurface in a complex manifold of dimension

$$(9) \quad h = 2g - 1 + \ell.$$

It projects to a stratum $\mathcal{H}(k_1, \dots, k_\ell)$ in the moduli space $\mathcal{H}(S)$ of area one holomorphic one-forms on S . Strata of holomorphic one-forms in moduli space need not be connected, but the number of connected components of a stratum is at most three [KZ03].

Recall from Section 2 the definition of the strong stable, the stable, the unstable and the strong unstable foliation W^{ss}, W^s, W^u, W^{su} of $\mathcal{Q}^1(S)$. Let $\tilde{\mathcal{Q}}$ be a component of a stratum $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ of marked quadratic differentials or of a stratum $\mathcal{H}^1(m_1/2, \dots, m_\ell/2)$ of marked abelian differentials. Using period coordinates, one sees that every $q \in \tilde{\mathcal{Q}}$ has a connected neighborhood U in $\tilde{\mathcal{Q}}$ with the following properties [V90]. For $u \in U$ let $[u^v]$ (or $[u^h]$) be the vertical (or the horizontal) projective measured geodesic lamination of u . Then $\{[u^v] \mid u \in U\}$ is homeomorphic to an open ball in \mathbb{R}^{h-1} (where $h > 0$ is as in equation (8,9)). Moreover, for $q \in U$ the set

$$\{u \in U \mid [u^v] = [q^v]\} = W_{\tilde{\mathcal{Q}}, \text{loc}}^s(q) \subset W^s(q)$$

is a smooth connected local submanifold of U of (real) dimension h which is called the *local stable manifold* of q in $\tilde{\mathcal{Q}}$ (see [V90]). Similarly we define the *local unstable manifold* $W_{\tilde{\mathcal{Q}}, \text{loc}}^u(q)$ of q in $\tilde{\mathcal{Q}}$. If two such local stable (or unstable) manifolds intersect then their union is again a local stable (or unstable) manifold. The maximal connected set containing q which is a union of intersecting local stable (or unstable) manifolds is the *stable manifold* $W_{\tilde{\mathcal{Q}}}^s(q)$ (or the *unstable manifold* $W_{\tilde{\mathcal{Q}}}^u(q)$) of q in $\tilde{\mathcal{Q}}$. Note that $W_{\tilde{\mathcal{Q}}}^i(q) \subset W^i(q)$ ($i = s, u$). A stable (or unstable) manifold is invariant under the action of the Teichmüller flow Φ^t .

Remark: There may be a component $\tilde{\mathcal{Q}}$ of a stratum and some $\tilde{q} \in \tilde{\mathcal{Q}}$ such that $W^s(\tilde{q}) \cap \tilde{\mathcal{Q}}$ has infinitely many components.

The (strong) stable and (strong) unstable manifolds define smooth foliations $W_{\tilde{\mathcal{Q}}}^s, W_{\tilde{\mathcal{Q}}}^u$ of $\tilde{\mathcal{Q}}$ which are called the *stable* and *unstable* foliations of $\tilde{\mathcal{Q}}$, respectively. Define the *strong stable* foliation $W_{\tilde{\mathcal{Q}}}^{ss}$ (or the *strong unstable* foliation $W_{\tilde{\mathcal{Q}}}^{su}$) of $\tilde{\mathcal{Q}}$ by requiring that locally the leaf $W_{\tilde{\mathcal{Q}}}^{ss}(\tilde{q})$ (or $W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$) through \tilde{q} is the subset of $W_{\tilde{\mathcal{Q}}}^s(\tilde{q})$ (or of $W_{\tilde{\mathcal{Q}}}^u(\tilde{q})$) of all marked quadratic differentials whose vertical (or horizontal) measured geodesic lamination equals the vertical (or horizontal) measured geodesic lamination of \tilde{q} . The strong stable foliation of $\tilde{\mathcal{Q}}$ is transverse to the unstable foliation of $\tilde{\mathcal{Q}}$.

The foliations $W_{\tilde{\mathcal{Q}}}^i$ ($i = ss, s, su, u$) are invariant under the action of the stabilizer $\text{Stab}(\tilde{\mathcal{Q}})$ of $\tilde{\mathcal{Q}}$ in $\text{Mod}(S)$, and they project to Φ^t -invariant singular foliations $W_{\mathcal{Q}}^i$ of $\mathcal{Q} = \tilde{\mathcal{Q}}/\text{Stab}(\tilde{\mathcal{Q}})$.

4.1. Orbifold coordinates. In this technical subsection we describe for every component \mathcal{Q} of a stratum in the moduli space of quadratic differentials and for every point $q \in \mathcal{Q}$ a basis of neighborhoods of q in \mathcal{Q} with local product structures. The material is well known to the experts but a bit difficult to find in the literature. In the course of the discussion we introduce some notations which will be used throughout.

For $\tilde{q} \in \mathcal{Q}^1(S)$ and $z \in W^s(\tilde{q})$ there is a neighborhood V of \tilde{q} in $W^{su}(\tilde{q})$ and there is a homeomorphism

$$(10) \quad \zeta_z : V \rightarrow \zeta_z(V) \subset W^{su}(z)$$

with $\zeta_z(\tilde{q}) = z$ which is determined by the requirement that $\zeta_z(u) \in W^s(u)$. We call ζ_z a *holonomy map* for the strong unstable foliation along the stable foliation.

Similarly, for $\tilde{q} \in \mathcal{Q}^1(S)$ and $z \in W^u(\tilde{q})$ there is a neighborhood Y of \tilde{q} in $W^{ss}(\tilde{q})$ and there is a homeomorphism

$$(11) \quad \theta_z : Y \rightarrow \theta_z(Y) \subset W^{ss}(z)$$

with $\theta_z(\tilde{q}) = z$ which is determined by the requirement that $\theta_z(u) \in W^u(u)$. We call θ_z a *holonomy map* for the strong stable foliation along the unstable foliation. The holonomy maps are equivariant under the action of the mapping class group and hence they project to locally defined holonomy maps in $\mathcal{Q}(S)$ which are denoted by the same symbols.

Recall from Section 2 the definition of the intrinsic path-metrics d^i on the leaves of the foliation W^i ($i = s, u$). These path metrics are invariant under the action of the mapping class group and hence they project to path metrics on the leaves of W^i in $\mathcal{Q}(S)$ which we denote by the same symbols. For $q \in \mathcal{Q}(S)$, $z \in W^i(q)$ and any preimage \tilde{q} of q in $\mathcal{Q}^1(S)$, the distance $d^i(q, z)$ is the shortest length of a path in $W^i(\tilde{q})$ connecting \tilde{q} to a preimage of z . Let moreover d^{ss}, d^{su} be the restrictions of d^s, d^u to distances on the leaves of the strong stable and strong unstable foliation of $\mathcal{Q}^1(S)$ and $\mathcal{Q}(S)$.

Let

$$\Pi : \mathcal{Q}^1(S) \rightarrow \mathcal{Q}(S)$$

be the canonical projection. For $q \in \mathcal{Q}(S)$ and $r > 0$ let

$$B^i(q, r)$$

be the closed ball of radius r about q in $W^i(q)$ ($i = ss, su, s, u$) with respect to the metric d^i . Call such a ball $B^i(q, r)$ a *metric orbifold ball* centered at q if there is a lift $\tilde{q} \in \mathcal{Q}^1(S)$ of q with the following properties.

- (1) The ball $B^i(\tilde{q}, r) \subset (W^i(\tilde{q}), d^i)$ about \tilde{q} of the same radius is contractible and precisely invariant under the stabilizer $\text{Stab}(\tilde{q})$ of \tilde{q} in $\text{Mod}(S)$.
- (2) $B^i(q, r) = B^i(\tilde{q}, r)/\text{Stab}(\tilde{q})$ which means that the restriction of the map Π to $B^i(\tilde{q}, r)$ factors through a homeomorphism $B^i(\tilde{q}, r)/\text{Stab}(\tilde{q}) \rightarrow B^i(q, r)$.

We also say that $B^i(q, r)$ is an *orbifold quotient* of $B^i(\tilde{q}, r)$. Note that every metric orbifold ball $B^i(q, r) \subset W^i(q)$ is contractible. There is also an obvious notion of an orbifold ball which is not necessarily metric.

For every point $q \in \mathcal{Q}(S)$ there is a number

$$a(q) > 0$$

such that the balls $B^i(q, a(q))$ are metric orbifold balls ($i = ss, su$) and that for any preimage \tilde{q} of q in $\mathcal{Q}^1(S)$ and any $z \in B^{ss}(\tilde{q}, a(q))$ (or $z \in B^{su}(\tilde{q}, a(q))$) the holonomy map ζ_z (or θ_z) is defined on $B^{su}(\tilde{q}, a(q))$ (or on $B^{ss}(\tilde{q}, a(q))$).

Now let

$$W_1 \subset B^{ss}(q, a(q)), W_2 \subset B^{su}(q, a(q))$$

be Borel sets and let $\tilde{W}_1 \subset B^{ss}(\tilde{q}, a(q))$, $\tilde{W}_2 \subset B^{su}(\tilde{q}, a(q))$ be the preimages of W_1, W_2 in $B^{ss}(\tilde{q}, a(q)), B^{su}(\tilde{q}, a(q))$. Then \tilde{W}_1, \tilde{W}_2 are precisely invariant under $\text{Stab}(\tilde{q})$. Define

$$V(\tilde{W}_1, \tilde{W}_2) = \cup_{z \in \tilde{W}_1} \zeta_z \tilde{W}_2 \text{ and } V(W_1, W_2) = \Pi V(\tilde{W}_1, \tilde{W}_2).$$

Note that the map $\xi : \tilde{W}_1 \times \tilde{W}_2 \rightarrow V(\tilde{W}_1, \tilde{W}_2)$ defined by $\xi(z, u) = \zeta_z(u)$ is a homeomorphism, and $V(W_1, W_2)$ is homeomorphic to the quotient of $\tilde{W}_1 \times \tilde{W}_2$ under the diagonal action of $\text{Stab}(\tilde{q})$. In particular, if W_1, W_2 are connected then $V(W_1, W_2)$ is connected as well. Similarly, define

$$Y(\tilde{W}_1, \tilde{W}_2) = \cup_{z \in \tilde{W}_2} \theta_z \tilde{W}_1 \text{ and } Y(W_1, W_2) = \Pi Y(\tilde{W}_1, \tilde{W}_2).$$

Then there is a continuous function

$$(12) \quad \sigma : V(B^{ss}(q, a(q)), B^{su}(q, a(q))) \rightarrow \mathbb{R}$$

which vanishes on $B^{ss}(q, a(q)) \cup B^{su}(q, a(q))$ and such that

$$Y(W_1, W_2) = \{\Phi^{\sigma(z)} z \mid z \in V(W_1, W_2)\}.$$

In particular, for every number $\kappa > 0$ there is a number $r(\kappa) > 0$ such that the restriction of the function σ to $V(B^{ss}(\tilde{q}, r(\kappa)), B^{su}(\tilde{q}, r(\kappa)))$ assumes values in $[-\kappa, \kappa]$.

For $t_0 > 0$ define

$$(13) \quad V(\tilde{W}_1, \tilde{W}_2, t_0) = \cup_{-t_0 \leq s \leq t_0} \Phi^s V(\tilde{W}_1, \tilde{W}_2) \\ \text{and } V(W_1, W_2, t_0) = \Pi V(\tilde{W}_1, \tilde{W}_2, t_0).$$

Then for sufficiently small t_0 , say for all $t_0 \leq t(q)$, the following properties are satisfied.

- a) $V(W_1, W_2, t_0)$ is homeomorphic to $(\tilde{W}_1 \times \tilde{W}_2) / \text{Stab}(\tilde{q}) \times [-t_0, t_0]$.
- b) Every connected component of the intersection of an orbit of Φ^t with $V(W_1, W_2, t_0)$ is an arc of length $2t_0$.

We call a set $V(W_1, W_2, t_0)$ as in (13) which satisfies the assumptions a), b) a set with a *local product structure*. Note that every point $q \in \mathcal{Q}(S)$ has a neighborhood in $\mathcal{Q}(S)$ with a local product structure, e.g. the set $V(B^{ss}(q, r), B^{su}(q, r), t)$ for $r \in (0, a(q))$ and $t \in (0, t(q))$. Moreover, the neighborhoods of q with a local product structure form a basis of neighborhoods.

The above discussion can be applied to strata as follows.

A connected component \mathcal{Q} of a stratum $\mathcal{Q}(m_1, \dots, m_\ell; -m)$ or of a stratum $\mathcal{H}(m_1/2, \dots, m_\ell/2)$ is locally closed in $\mathcal{Q}(S)$ (here we identify an abelian differential with its square). This means that for every $q \in \mathcal{Q}$ there exists an open neighborhood V of q in $\mathcal{Q}(S)$ such that $V \cap \mathcal{Q}$ is a closed subset of V .

Using period coordinates [V90], one obtains that for every point $q \in \mathcal{Q}$ there is a number $a_{\mathcal{Q}}(q) \leq a(q)$ and a number $t_{\mathcal{Q}}(q) \leq t(q)$ with the following property. For $r \leq a_{\mathcal{Q}}(q)$ let

$$B_{\mathcal{Q}}^{ss}(q, r), B_{\mathcal{Q}}^{su}(q, r)$$

be the component containing q of the intersection $B^{ss}(q, r) \cap \mathcal{Q}, B^{su}(q, r) \cap \mathcal{Q}$ (note that the intersection $B^{ss}(q, r) \cap \mathcal{Q}$ may not be closed and may have infinitely many components). Then $V(B_{\mathcal{Q}}^{ss}(q, r), B_{\mathcal{Q}}^{su}(q, r), t_{\mathcal{Q}}(q))$ is a neighborhood of q in \mathcal{Q} [V90]. We say that this neighborhood has a *local product structure*.

We say that a Borel set $Y \subset \mathcal{Q}$ has a *local product structure* if there is some $q \in Y$ and if there are Borel sets

$$W_1 \subset B_{\mathcal{Q}}^{ss}(q, a(q)), W_2 \subset B_{\mathcal{Q}}^{su}(q, a(q))$$

and a number $t_0 < t(q)$ such that $Y = V(W_1, W_2, t_0)$.

The Φ^t -invariant Borel probability measure λ on \mathcal{Q} in the Lebesgue measure class admits a natural family of conditional measures $\lambda^{ss}, \lambda^{su}$ on strong stable and strong unstable manifolds. The conditional measures λ^i are well defined up to a universal constant, and they transform under the Teichmüller geodesic flow Φ^t via

$$d\lambda^{ss} \circ \Phi^t = e^{-ht} d\lambda^{ss} \text{ and } d\lambda^{su} \circ \Phi^t = e^{ht} d\lambda^{su}.$$

Let $\mathcal{F} : \mathcal{Q}(S) \rightarrow \mathcal{Q}(S)$ be the flip $q \rightarrow \mathcal{F}(q) = -q$ and let dt be the Lebesgue measure on the flow lines of the Teichmüller flow. The conditional measures $\lambda^{ss}, \lambda^{su}$ are uniquely determined by the additional requirements that $\mathcal{F}_* \lambda^{su} = \lambda^{ss}$ and that with respect to a local product structure, λ can be written in the form

$$d\lambda = d\lambda^{ss} \times d\lambda^{su} \times dt.$$

The measures λ^u on unstable manifolds defined by $d\lambda^u = d\lambda^{su} \times dt$ are invariant under holonomy along strong stable manifolds.

To summarize, we obtain the following. The natural homeomorphism

$$\begin{aligned} \Psi : B_{\mathcal{Q}}^{ss}(q, a_{\mathcal{Q}}(q)) \times B_{\mathcal{Q}}^{su}(q, a_{\mathcal{Q}}(q)) \times [-t_{\mathcal{Q}}(q), t_{\mathcal{Q}}(q)] \\ \rightarrow V(B_{\mathcal{Q}}^{ss}(q, a_{\mathcal{Q}}(q)), B_{\mathcal{Q}}^{su}(q, a_{\mathcal{Q}}(q)), t_{\mathcal{Q}}(q)) = V \end{aligned}$$

maps the measure λ_0 on V defined by $d\lambda_0 = \Psi_* d\lambda^{ss} \times d\lambda^{su} \times dt$ to a measure of the form $e^\varphi \lambda$ where φ is a continuous function on V which vanishes on $\cup_{t \in [-t_{\mathcal{Q}}(q), t_{\mathcal{Q}}(q)]} \Phi^t B_{\mathcal{Q}}^{ss}(q, a_{\mathcal{Q}}(q))$ (see [V86]).

4.2. Train track coordinates. The goal of this subsection is to relate components of strata in $\mathcal{Q}(S)$ to large train tracks. This will be used to define product coordinates near points in the boundary of a stratum. Note that the natural product coordinates on strata are period coordinates. For a point q in the boundary of a stratum, some of the relative periods vanish and there is no canonical choice of a relative cycle near q which can be used for period coordinates in a neighborhood of q .

We chose to construct product coordinates near boundary points of a stratum using train tracks even though similar coordinates can be obtained using the usual period coordinate construction. These train track coordinates will be used in other contexts as well.

We continue to use the assumptions and notations from Section 2 and Section 3. For a large train track $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -m)$ let

$$\mathcal{V}_0(\tau) \subset \mathcal{V}(\tau)$$

be the set of all measured geodesic laminations $\nu \in \mathcal{ML}$ whose support is carried by τ and such that the total weight of the transverse measure on τ defined by ν equals one. Let

$$\mathcal{Q}(\tau) \subset \mathcal{Q}^1(S)$$

be the set of all area one marked quadratic differentials whose vertical measured geodesic lamination is contained in $\mathcal{V}_0(\tau)$ and whose horizontal measured geodesic lamination is carried by the dual bigon track τ^* of τ . By definition of a large train track, we have $\mathcal{Q}(\tau) \neq \emptyset$. The next proposition relates $\mathcal{Q}(\tau)$ to components of strata.

Proposition 4.1. (1) *For every large non-orientable train track $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -m)$ there is a component $\tilde{\mathcal{Q}}$ of the stratum $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ such that for every $\delta > 0$ the set $\{\Phi^t q \mid q \in \mathcal{Q}(\tau), t \in [-\delta, \delta]\}$ is the closure in $\mathcal{Q}^1(S)$ of an open subset of $\tilde{\mathcal{Q}}$.*

(2) *For every large orientable train track $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; 0)$ there is a component $\tilde{\mathcal{Q}}$ of the stratum $\mathcal{H}^1(m_1/2, \dots, m_\ell/2)$ such that for every $\delta > 0$ the set $\{\Phi^t q \mid q \in \mathcal{Q}(\tau), t \in [-\delta, \delta]\}$ is the closure in $\mathcal{H}^1(S)$ of an open subset of $\tilde{\mathcal{Q}}$.*

Proof. By [L83], the support ξ of the vertical measured geodesic lamination of a marked quadratic differential $z \in \mathcal{Q}^1(S)$ can be obtained from the vertical foliation of z by cutting S open along each vertical separatrix and straightening the remaining leaves with respect to the hyperbolic structure $Pz \in \mathcal{T}(S)$. In particular, up to homotopy a vertical saddle connection s of z is contained in the interior of a complementary component C of ξ which is uniquely determined by s .

Let $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -m)$. Assume first that τ is non-orientable. Let $\mu \in \mathcal{V}_0(\tau)$ be such that the support of μ is contained in $\mathcal{LL}(m_1, \dots, m_\ell; -m)$ and let $\nu \in \mathcal{V}^*(\tau)$. Then μ is non-orientable since otherwise τ inherits an orientation from μ . The measured geodesic laminations μ, ν jointly fill up S (since the support of ν is different from the support of μ and the support of μ fills up S) and hence if ν is normalized in such a way that $\iota(\mu, \nu) = 1$ then the pair (μ, ν) defines a point $q \in \mathcal{Q}(\tau)$. Our first goal is to show that $q \in \mathcal{Q}^1(m_1, \dots, m_\ell; -m)$.

The support of the geodesic lamination μ is contained in $\mathcal{LL}(m_1, \dots, m_\ell; -m)$ and therefore the orders of the zeros of the quadratic differential q are obtained from the orders m_1, \dots, m_ℓ by subdivision. There is a non-trivial subdivision, say of the form $m_i = \sum_s k_s$, if and only if there is at least one vertical saddle connection for q .

Assume to the contrary that there is a vertical saddle connection s for q . Let \tilde{q} be the lift of q to a quadratic differential on the universal covering \mathbf{H}^2 of S and let $\tilde{s} \subset \mathbf{H}^2$ be a preimage of s . Let $\tilde{\mu} \subset \mathbf{H}^2$ be the preimage of μ . As discussed in the first paragraph of this proof, the saddle connection \tilde{s} is contained in a complementary component \tilde{C} of the support of $\tilde{\mu}$. This component is an ideal polygon with finitely many sides.

A biinfinite geodesic line for the singular euclidean metric defined by \tilde{q} is a quasi-geodesic in the hyperbolic plane \mathbf{H}^2 and hence it has well defined endpoints in the ideal boundary $\partial\mathbf{H}^2$ of \mathbf{H}^2 . There are two vertical geodesic lines α_0, β_0 for \tilde{q} which contain the saddle connection \tilde{s} as a subarc and which are contained in a bounded neighborhood of a side α, β of \tilde{C} . For a fixed orientation of \tilde{s} , the geodesics α_0, β_0 are determined by the requirement that their orientation coincides with the given orientation of \tilde{s} and that moreover at every singular point x , the angle at x to the left of α_0 (or to the right of β_0) for the orientation of the geodesic and the orientation of \mathbf{H}^2 equals π .

The ideal boundary of the closed half-plane of \mathbf{H}^2 which is bounded by α (or β) and which is disjoint from the interior of \tilde{C} is a compact subarc a (or b) of $\partial\mathbf{H}^2$. The arcs a, b are disjoint (or, equivalently, the sides α, β of \tilde{C} are not adjacent). A horizontal geodesic line for \tilde{q} which intersects the interior of the saddle connection \tilde{s} is a quasi-geodesic in \mathbf{H}^2 with one endpoint in the interior of the arc a and the second endpoint in the interior of the arc b .

Now a carrying map $F : S \rightarrow S$ for μ with $F(\mu) \subset \tau$ maps the support of μ onto τ and hence it induces a bijection between the complementary components of the support of μ and the complementary components of τ . In particular, the projections of the geodesics α, β to S determine two opposite sides of the complementary component C_τ of τ corresponding to the projection of \tilde{C} to S .

On the other hand, by construction of the dual bigon track τ^* of τ (see [PH92], if $\rho : (-\infty, \infty) \rightarrow \tau^*$ is any trainpath which intersects the complementary component C_τ of τ then every component of $\rho(-\infty, \infty) \cap C_\tau$ is a compact arc with endpoints on adjacent sides of C_τ . In particular, a lift to \mathbf{H}^2 of such a trainpath is a quasi-geodesic in \mathbf{H}^2 whose endpoints meet at most one of the two arcs $a, b \subset \partial\mathbf{H}^2$. Since the support of the horizontal measured geodesic lamination ν of q is carried by τ^* by assumption, every leaf of the support of ν corresponds to a biinfinite trainpath on τ^* and hence a lift to \mathbf{H}^2 of such a leaf does not connect the arcs $a, b \subset \partial\mathbf{H}^2$. This contradicts the assumption that q has a vertical saddle connection and hence we indeed have $q \in \mathcal{Q}^1(m_1, \dots, m_\ell; -m)$.

Let $\mathcal{P}(\mu) \subset \mathcal{PML}$ be the open set of all projective measured geodesic laminations whose support is distinct from the support of μ . Then the assignment ψ which associates to a projective measured geodesic lamination $[\nu] \in \mathcal{P}(\mu)$ the area one

quadratic differential $q(\mu, [\nu])$ with vertical measured geodesic lamination μ and horizontal projective measured geodesic lamination $[\nu]$ is a homeomorphism of $\mathcal{P}(\mu)$ onto a strong stable manifold in $\mathcal{Q}^1(S)$.

The projectivization $P\mathcal{V}^*(\tau)$ of $\mathcal{V}^*(\tau)$ is homeomorphic to a ball in a real vector space of dimension $h - 1$, and this is just the dimension of a strong stable manifold in a component of $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$. Therefore by the above discussion and invariance of domain, there is a component $\tilde{\mathcal{Q}}$ of the stratum $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ such that the restriction of the map ψ to $P\mathcal{V}^*(\tau)$ is a homeomorphism of $P\mathcal{V}^*(\tau)$ onto the closure of an open subset of a strong stable manifold $W_{\tilde{\mathcal{Q}}}^{ss}(q) \subset \tilde{\mathcal{Q}}$.

Similarly, if $q \in \mathcal{Q}(\tau)$ is defined by $\mu \in \mathcal{V}_0(\tau), \nu \in \mathcal{V}^*(\tau)$ and if the support of ν is contained in $\mathcal{L}\mathcal{L}(m_1, \dots, m_\ell; -m)$ then $q \in \mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ by the above argument. Moreover, for every $[\mu] \in P\mathcal{V}(\tau)$ the pair $([\mu], \nu)$ defines a quadratic differential which is contained in a strong unstable manifold $W_{\tilde{\mathcal{Q}}}^{su}(q)$ of a component $\tilde{\mathcal{Q}}$ of the stratum $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$, and the set of these quadratic differentials equals the closure of an open subset of $W_{\tilde{\mathcal{Q}}}^{su}(q)$.

The set of quadratic differentials q with the property that the support of the vertical (or of the horizontal) measured geodesic lamination of q is minimal and of type $(m_1, \dots, m_\ell; -m)$ is dense and of full Lebesgue measure in $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ [M82, V86]. Moreover, this set is saturated for the stable (or for the unstable) foliation. Thus by the above discussion, the set of all measured geodesic laminations which are carried by τ (or τ^*) and whose support is minimal of type $(m_1, \dots, m_\ell; -m)$ is dense in $\mathcal{V}(\tau)$ (or in $\mathcal{V}^*(\tau)$). As a consequence, the set of all pairs $(\mu, \nu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau)$ with $\iota(\mu, \nu) = 1$ which correspond to a quadratic differential $q \in \mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ is dense in the set of all pairs $(\mu, \nu) \in \mathcal{V}(\tau) \times \mathcal{V}^*(\tau)$ with $\iota(\mu, \nu) = 1$. Thus the set $\mathcal{Q}(\tau)$ is contained in the closure of a component $\tilde{\mathcal{Q}}$ of the stratum $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$. Moreover, by reasons of dimension, $\{\Phi^t q \mid q \in \mathcal{Q}(\tau), t \in [-\delta, \delta]\}$ contains an open subset of this component. This shows the first part of the proposition.

Now if $\tau \in \mathcal{L}\mathcal{T}(m_1, \dots, m_\ell; -m)$ is orientable and if μ is a geodesic lamination which is carried by τ , then μ inherits an orientation from an orientation of τ . The orientation of τ together with the orientation of S determines an orientation of the dual bigon track τ^* (see [PH92]), and these two orientations determine the orientation of S . This implies that any geodesic lamination carried by τ^* admits an orientation, and if (μ, ν) jointly fill up S and if μ is carried by τ , ν is carried by τ^* then the orientations of μ, ν determine the orientation of S . As a consequence, the singular euclidean metric on S defined by the quadratic differential q of (μ, ν) is the square of a holomorphic one-form. The proposition follows. \square

If $\tilde{\mathcal{Q}}$ is a component of a stratum $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ and if the large train track $\tau \in \mathcal{L}\mathcal{T}(m_1, \dots, m_\ell; -m)$ is such that $\mathcal{Q}(\tau) \cap \tilde{\mathcal{Q}} \neq \emptyset$ then we say that τ belongs to $\tilde{\mathcal{Q}}$, and we write $\tau \in \mathcal{L}\mathcal{T}(\tilde{\mathcal{Q}})$. The next proposition is a converse to Proposition 4.1 and shows that train tracks can be used to define coordinates on strata.

- Proposition 4.2.** (1) For every $q \in \mathcal{Q}^1(m_1, \dots, m_\ell; -m)$ there is a large non-orientable train track $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -m)$ and a number $t \in \mathbb{R}$ so that $\Phi^t q$ is an interior point of $\mathcal{Q}(\tau)$.
- (2) For every $q \in \mathcal{H}^1(k_1, \dots, k_s)$ there is a large orientable train track $\tau \in \mathcal{LT}(2k_1, \dots, 2k_s; 0)$ and a number $t \in \mathbb{R}$ so that $\Phi^t q$ is an interior point of $\mathcal{Q}(\tau)$.

Proof. Fix a complete hyperbolic metric on S of finite volume. Define the *straightening* of a train track τ to be the immersed graph in S whose vertices are the switches of τ and whose edges are the geodesic arcs which are homotopic to the branches of τ with fixed endpoints.

The hyperbolic metric induces a distance function on the projectivized tangent bundle of S . As in Section 3 of [H09], we say that for some $\epsilon > 0$ a train track τ ϵ -follows a geodesic lamination μ if the tangent lines of the straightening of τ are contained in the ϵ -neighborhood of the tangent lines of μ in the projectivized tangent bundle of S and if moreover the straightening of any trainpath on τ is a piecewise geodesic whose exterior angles at the breakpoints are not bigger than ϵ . By Lemma 3.2 of [H09], for every geodesic lamination μ and every $\epsilon > 0$ there is a transversely recurrent train track which carries μ and ϵ -follows μ .

Let $q \in \mathcal{Q}^1(m_1, \dots, m_\ell; -m)$. Assume first that the support μ of the vertical measured geodesic lamination of q is large of type $(m_1, \dots, m_\ell; -m)$. This is equivalent to stating that q does not have vertical saddle connections. For $\epsilon > 0$ let τ_ϵ be a train track which carries μ and ϵ -follows μ . If $\epsilon > 0$ is sufficiently small then a carrying map $\mu \rightarrow \tau_\epsilon$ defines a bijection of the complementary components of μ onto the complementary components of τ_ϵ . The transverse measure on τ_ϵ defined by the vertical measured geodesic lamination of q is positive.

Let $\tilde{C} \subset \mathbf{H}^2$ be a complementary component of the preimage of μ in the hyperbolic plane \mathbf{H}^2 . Then \tilde{C} is an ideal polygon whose vertices decompose the ideal boundary $\partial\mathbf{H}^2$ into finitely many arcs a_1, \dots, a_k ordered counter-clockwise in consecutive order. Since q does not have vertical saddle connections, the discussion in the proof of Proposition 4.1 shows the following. Let ℓ be a leaf of the preimage in \mathbf{H}^2 of the support ν of the horizontal measured geodesic lamination of q . Then the two endpoints of ℓ in \mathbf{H}^2 either are both contained in the interior of the same arc a_i or in the interior of two adjacent arcs a_i, a_{i+1} . As a consequence, for sufficiently small ϵ the geodesic lamination ν is carried by the dual bigon track τ_ϵ^* of τ_ϵ (see the characterization of the set of measured geodesic laminations carried by τ_ϵ^* in [PH92]). Moreover, for any two adjacent subarcs a_i, a_{i+1} of $\partial\mathbf{H}^2$ cut out by \tilde{C} , the transverse measure of the set of all leaves of the preimage of ν connecting these sides is positive. Therefore for sufficiently small ϵ , the horizontal measured geodesic lamination ν of q defines an interior point of $\mathcal{V}^*(\tau_\epsilon)$.

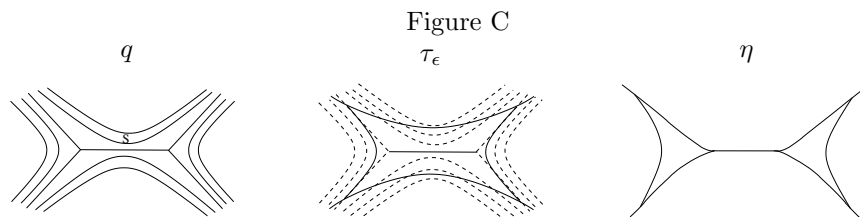
Now the set of quadratic differentials z so that the support of the horizontal measured geodesic lamination of z is large of type $(m_1, \dots, m_\ell; -m)$ is dense in the strong stable manifold $W_{\tilde{\mathcal{Q}}, \text{loc}}^{ss}(q)$ of q . The above reasoning shows that for such a quadratic differential z and for sufficiently small ϵ , the horizontal measured geodesic lamination of z is carried by τ_ϵ^* . But this just means that $\tau_\epsilon \in \mathcal{LL}(m_1, \dots, m_\ell; -m)$.

Moreover, if $r > 0$ is the total weight which the vertical measured geodesic lamination puts on τ_ϵ then $\Phi^{-\log r} q$ is an interior point of $\mathcal{Q}(\tau_\epsilon)$. Thus τ_ϵ satisfies the requirement in the proposition. Note that τ_ϵ is necessarily non-orientable.

If $q \in \mathcal{H}^1(k_1, \dots, k_s)$ is such that the support of the vertical measured geodesic lamination of q is large of type $(2k_1, \dots, 2k_s; 0)$ then the above reasoning also applies and yields an oriented large train track with the required property.

Consider next the case that the support μ of the vertical measured geodesic lamination of q fills up S but is not of type $(m_1, \dots, m_\ell; -m)$. Then q has a vertical saddle connection. The set of all vertical saddle connections of q is a finite disjoint union T of finite trees. The number of edges of this union of trees is uniformly bounded. For $\epsilon > 0$ let τ_ϵ be a train track which ϵ -follows μ and carries μ . If ϵ is sufficiently small then a carrying map $\mu \rightarrow \tau_\epsilon$ defines a bijection between the complementary components of μ and the complementary components of τ_ϵ which induces a bijection between their sides as well.

Modify τ_ϵ as follows. Up to isotopy, a vertical saddle connection s of q is contained in a complementary component C_s of τ_ϵ which corresponds to the complementary component of μ determined by s (see the proof of Proposition 4.1). Since a carrying map $\mu \rightarrow \tau_\epsilon$ determines a bijection between the sides of the complementary components of μ and the sides of the complementary components of τ_ϵ , the horizontal lines crossing through s determine two non-adjacent sides c_1, c_2 of C_s (see once more the discussion in the proof of Proposition 4.1). Choose an embedded rectangle $R_s \subset C_s$ whose boundary intersects the boundary of C_s in two opposite sides contained in the interior of the sides c_1, c_2 of C_s . Up to an isotopy we may assume that these rectangles R_s where s runs through the vertical saddle connections of q are pairwise disjoint. Collapse each of the rectangles R_s to a single segment in such a way that the two sides of R_s which are contained in τ_ϵ are identified and form a single large branch b_s as shown in Figure C. The branch b_s can be isotoped to the saddle connection s . Let η be the train track constructed in this way. Then η is of topological type $(m_1, \dots, m_\ell; -m)$.



The train track τ_ϵ can be obtained from η by splitting η at each of the large branches b_s and removing the diagonal of the split. In particular, η carries τ_ϵ and hence μ . The transverse measure on η defined by the vertical measured geodesic lamination of q is positive and consequently η is recurrent. Moreover, for sufficiently small ϵ , the horizontal measured geodesic lamination of q is carried by η^* . As above, we conclude that if $\epsilon > 0$ is sufficiently small then η is fully transversely recurrent and in fact large. There is a tangential measure on η which is defined by the

horizontal measured geodesic lamination of q and which gives positive weight to each of the branches b_s . Thus by possibly decreasing once more the size of ϵ , we can guarantee that for some $t \in \mathbb{R}$ the quadratic differential $\Phi^t q$ is an interior point of $\mathcal{Q}(\eta)$. As a consequence, η satisfies the requirements in the proposition.

If the support μ of the vertical measured geodesic lamination of q is arbitrary then we proceed in the same way. Let $\epsilon > 0$ be sufficiently small that there is a bijection between the complementary components of the train track τ_ϵ and the complementary components of the support of μ . As before, we use the horizontal measured foliation of q to construct for every vertical saddle connection s of q an embedded rectangle R_s in S whose interior is contained in a complementary component of τ_ϵ and with two opposite sides on τ_ϵ in such a way that the rectangles R_s are pairwise disjoint. Collapse each of the rectangles to a single arc. The resulting train track has the required properties.

We discuss in detail the case that the support of μ contains a simple closed curve component α . Then τ_ϵ contains α as a simple closed curve component as well. There is a vertical flat cylinder C for q foliated by smooth circles freely homotopic to α . The boundary ∂C of C is a finite union of vertical saddle connections. Some of these saddle connections may occur twice on the boundary of C (if $\mu = \alpha$ then this holds true for each of these saddle connections). Assume without loss of generality (i.e. perform a suitable isotopy) that α is a closed vertical geodesic contained in the interior of C .

For each saddle connection s in the boundary of C choose a compact arc a_s contained in the interior of s . Choose moreover a foliation \mathcal{F} of C by compact arcs with endpoints on the boundary of C which is transverse to the foliation of C by the vertical closed geodesics and such that the following holds true. If u_1, u_2 are two distinct half-leaves of \mathcal{F} with one endpoint in the arc a_s and the second endpoint on α then the endpoints on α of u_1, u_2 are distinct. In particular, each arc a_s which occurs twice in the boundary of the cylinder C determines an embedded rectangle R_s in S . Two opposite sides of R_s are disjoint subarcs of α ; we call these sides the vertical sides. Each of the other two opposite sides consists of two half-leaves of the foliation \mathcal{F} which begin at a boundary point of a_s and end in a point of α . The interior of the arc a_s is contained in the interior of R_s .

The rectangles R_s are pairwise disjoint. Therefore each of the rectangles R_s can be collapsed in S to the arc a_s . The resulting graph is a train track which carries α and contains for every saddle connection s which occurs twice in the boundary of C a large branch b_s .

If s is a saddle connection on the boundary of C which separates C from $S - C$ then the arc a_s is contained in the interior of a rectangle R_s with one side contained in α and the second side contained in the interior of a branch of the component of τ_ϵ different from α . This branch is determined by the horizontal geodesics which cross through s . As before, the rectangle R_s is collapsed to a single branch.

To summarize, the train track τ_ϵ can be modified in finitely many steps to a train track η with the required properties by collapsing for every vertical saddle

connection of q a rectangle with two sides on τ_ϵ to a single large branch. This completes the construction and finishes the proof of the proposition. \square

Remark: In the proof of Lemma 4.2, we constructed explicitly for every quadratic differential $q \in \mathcal{Q}(S)$ a train track τ_q belonging to the stratum of q . If q is a one-cylinder Strebel differential then the train track τ_q is uniquely determined by the combinatorics of its vertical saddle connections on the boundary of the cylinder. This fact in turn can be used to obtain a purely combinatorial proof of the classification results of Kontsevich-Zorich [KZ03] and of Lanneau [L08].

Let again $\tau \in \mathcal{LT}(m_1, \dots, m_\ell; -m)$. Then $\tau \in \mathcal{LT}(\tilde{\mathcal{Q}})$ for a component $\tilde{\mathcal{Q}}$ of $\mathcal{Q}^1(m_1, \dots, m_\ell; -m)$. For every $\mu \in \mathcal{V}_0(\tau)$ and every $\nu \in \mathcal{V}^*(\tau)$ so that the pair (μ, ν) jointly fills up S there is a unique $q \in \mathcal{Q}(\tau)$ with vertical measured geodesic lamination μ and horizontal measured geodesic lamination $\iota(\mu, \nu)^{-1}\nu$. Thus if $P\mathcal{V}^*(\tau)$ denotes the projectivization of the cone $\mathcal{V}^*(\tau)$ then for all $a < b$ there is a natural homeomorphism ψ from the subset of $\mathcal{V}_0(\tau) \times P\mathcal{V}^*(\tau) \times [a, b]$ corresponding to pairs $(\mu, [\nu])$ which jointly fill up S onto $C = \cup_{t \in [a, b]} \Phi^t \mathcal{Q}(\tau)$. The set C is the closure in $\mathcal{Q}^1(S)$ of an open subset of $\tilde{\mathcal{Q}}$. We say that the map ψ defines on C a *train track product structure*. If $A \subset \mathcal{V}_0(\tau), B \subset P\mathcal{V}^*(\tau)$ are Borel sets then we also say that the image of $A \times B \times [a, b]$ under the map ψ has a train track product structure. If $q \in \mathcal{Q}(\tau)$ and if C is a neighborhood of q with a train track product structure which is precisely invariant under the stabilizer of q in $\text{Mod}(S)$ then we say that the projection of C to $\mathcal{Q}(S)$ has a train track product structure.

The following proposition establishes product coordinates near boundary points of strata. For this let again \mathcal{Q} be a component of the stratum $\mathcal{Q}(m_1, \dots, m_\ell; -m)$, with closure $\overline{\mathcal{Q}}$. Let λ be the Lebesgue measure on \mathcal{Q} .

Proposition 4.3. *For every $q \in \overline{\mathcal{Q}} - \mathcal{Q}$ and every closed neighborhood A of q in $\overline{\mathcal{Q}}$ there is a closed neighborhood $K \subset A$ of q in $\overline{\mathcal{Q}}$ with the following properties.*

- (1) $K = \cup_{i=1}^k K_i$ for some $k \geq 1$, and $\lambda(K_i \cap K_j) = 0$ for $i \neq j$.
- (2) For each i , the set K_i contains q and has a train track product structure.

Proof. Our goal is to show that every point $q \in \overline{\mathcal{Q}} - \mathcal{Q}$ has a closed neighborhood W in $\overline{\mathcal{Q}}$ with the following property. Let $\tilde{\mathcal{Q}} \subset \mathcal{Q}^1(S)$ be a connected component of the preimage of \mathcal{Q} and let \tilde{q} be a lift of q contained in the closure of $\tilde{\mathcal{Q}}$. Then W lifts to a contractible neighborhood \tilde{W} of \tilde{q} in the closure of $\tilde{\mathcal{Q}}$ which is precisely invariant under $\text{Stab}(\tilde{q})$. Moreover, \tilde{W} is contained in

$$\cup_{j=1}^k \bigcup_{t \in [a_j, b_j]} \Phi^t \mathcal{Q}(\eta_j)$$

for some $a_j < b_j$ where $\eta_j \in \mathcal{LT}(\tilde{\mathcal{Q}})$ and where \tilde{q} is contained in the boundary of $\Phi^{-s_j} \mathcal{Q}(\eta_j)$ for some $s_j \in (a_j, b_j)$ ($j = 1, \dots, k$). For $i \neq j$ we have

$$(14) \quad \lambda \left(\bigcup_{t \in [a_i, b_i]} \Phi^t \mathcal{Q}(\eta_i) \cap \bigcup_{t \in [a_j, b_j]} \Phi^t \mathcal{Q}(\eta_j) \right) = 0.$$

For this assume that $q \in \mathcal{Q}(n_1, \dots, n_s; -m)$ for some $s < \ell$. Assume moreover for the moment that q does not have vertical saddle connections.

Let $(q_i) \subset \mathcal{Q}$ be a sequence converging to q . Since the subset of \mathcal{Q} of quadratic differentials without vertical saddle connection is dense in \mathcal{Q} , we may assume that for each i , q_i does not have a vertical saddle connection. Let $\tilde{q}_i \in \tilde{\mathcal{Q}}$ be a preimage of q_i such that $\tilde{q}_i \rightarrow \tilde{q}$. For each i the support μ_i of the vertical measured geodesic lamination of \tilde{q}_i is large of type $(m_1, \dots, m_\ell; -m)$.

We claim that up to passing to a subsequence, the geodesic laminations μ_i converge in the Hausdorff topology to a large geodesic lamination ξ of topological type $(m_1, \dots, m_\ell; -m)$. The lamination ξ then contains the support ν of the vertical measured geodesic lamination of \tilde{q} as a sublamination. Since q does not have vertical saddle connections, ν fills up S and ξ can be obtained from ν by adding finitely many isolated leaves. These isolated leaves subdivide some of the complementary components of ν . The number of such limit laminations is uniformly bounded.

To see that this claim indeed holds true it is enough to assume that $s = \ell - 1$ and that $n_u = m_j + m_p$ for some $j < p \leq \ell$ and some $u \leq s$ [MZ08]- the purpose of this assumption for our argument is to simplify the notations. Then for each sufficiently large i the quadratic differential \tilde{q}_i has a saddle connection s_i connecting a zero x_1^i of order m_j to a zero x_2^i of order m_p whose length (measured in the singular euclidean metric defined by \tilde{q}_i) tends to zero as $i \rightarrow \infty$. More precisely, the saddle connections s_i converge to a zero x_0 of \tilde{q} of order $n_u \geq 2$. The length of any other saddle connection of \tilde{q}_i is bounded from below by a universal positive constant.

Since \tilde{q}_i does not have vertical saddle connections, locally near x_1^i the interior of the saddle connection s_i is contained in the interior of an euclidean sector based at x_1^i of angle π bounded by two vertical separatrices α_1^i, α_2^i of \tilde{q}_i which issue from x_1^i . The union $\alpha_i = \alpha_1^i \cup \alpha_2^i$ is a smooth vertical geodesic line passing through x_1^i , i.e. a geodesic which is a limit in the compact open topology of geodesic segments not passing through a singular point. There are two vertical separatrices β_1^i, β_2^i issuing from x_2^i so that the sum of the angles at x_1^i, x_2^i of the (local) strip bounded by $\alpha_1^i, s_i, \beta_1^i$ equals π and that the same holds true for the angle sum of the (local) strip bounded by $\alpha_2^i, s_i, \beta_2^i$. The vertical length of s_i is positive. The union $\beta_i = \beta_1^i \cup \beta_2^i$ is a smooth vertical geodesic line passing through x_2^i .

Equip S with the marked hyperbolic metric $P\tilde{q} \in \mathcal{T}(S)$. For each i lift the singular euclidean metric on S defined by \tilde{q}_i to a $\pi_1(S)$ -invariant singular euclidean metric on the universal covering \mathbf{H}^2 of S . Let \tilde{s}_i be a lift of the saddle connection s_i . Since \tilde{s}_i is not vertical, the leaves of the vertical foliation of \tilde{q}_i which pass through \tilde{s}_i define a strip of positive transverse measure in \mathbf{H}^2 . This strip is bounded by the two lifts $\tilde{\alpha}_i, \tilde{\beta}_i$ of the smooth vertical geodesics α_i, β_i which pass through the endpoints of \tilde{s}_i . As $i \rightarrow \infty$, up to normalization and by perhaps passing to a subsequence, the vertical geodesics $\tilde{\alpha}_i, \tilde{\beta}_i$ converge in the compact open topology to vertical geodesics $\tilde{\alpha}, \tilde{\beta}$ for the singular euclidean metric defined by \tilde{q} which pass through a preimage \tilde{x}_0 of the zero x_0 of \tilde{q} of order $n_u = m_j + m_p \geq 2$. By construction, the geodesics $\tilde{\alpha}, \tilde{\beta}$ coincide in a neighborhood of \tilde{x}_0 . Since there are no vertical saddle connections for \tilde{q} , we necessarily have $\tilde{\alpha} = \tilde{\beta}$. Let $\tilde{\gamma} \subset \mathbf{H}^2$ be the hyperbolic geodesic with the same endpoints as $\tilde{\alpha}$ in the ideal boundary of \mathbf{H}^2 .

(see [L83] and the proof of Proposition 4.1). The projection of $\tilde{\gamma}$ to S subdivides the complementary component of ν containing x_0 into two ideal polygons with $m_j + 2$ and $m_p + 2$ sides, respectively. The union of ν with this geodesic is a large geodesic lamination ξ of type $(m_1, \dots, m_\ell; -m)$. This lamination is the limit in the Hausdorff topology of the laminations μ_i .

Let $\xi_1, \dots, \xi_k \in \mathcal{LL}(m_1, \dots, m_\ell; -m)$ be the (finitely many) large geodesic laminations obtained in this way. Each of the laminations ξ_s contains ν as a sublamination, and it is determined by a decomposition of a complementary $n_u + 2$ -gon of ν into an ideal $m_j + 2$ -gon and an ideal $m_p + 2$ -gon. The set ξ_1, \dots, ξ_k is invariant under the action of $\text{Stab}(\tilde{q})$. For sufficiently small $\epsilon > 0$, a train track η_j which carries ξ_j and ϵ -follows ξ_j (for the hyperbolic metric $P\tilde{q}$) is a simple extension of a train track τ which carries ν and ϵ -follows ν . The added branch is a diagonal of the complementary $m_j + m_p + 2$ -gon of τ defined by the zero x_0 of \tilde{q} of order $m_j + m_p$. It decomposes this component into an $m_j + 2$ -gon and an $m_p + 2$ -gon in a combinatorial pattern determined by ξ_j . The vertical measured geodesic lamination ν of \tilde{q} defines a transverse measure on η_j which gives full mass to the subtrack τ and hence it is contained in the boundary of the cone $\mathcal{V}(\eta_j)$. We also may assume that the horizontal measured geodesic lamination of \tilde{q} is carried by the dual bigon track η_j^* (compare the proof of Lemma 4.2) and that the set η_1, \dots, η_k is invariant under the action of $\text{Stab}(\tilde{q})$.

Since the set of geodesic laminations carried by a train track is open and closed in the Hausdorff topology [H09], for each j the train track η_j carries a minimal large geodesic lamination of type $(m_1, \dots, m_\ell; -m)$ (namely, the support of the vertical measured geodesic lamination of a quadratic differential $\tilde{q}_i \in \tilde{\mathcal{Q}}$ sufficiently close to \tilde{q} from the sequence which determines η_j) and hence it follows as in the proof of Proposition 4.1 that $\eta_j \in \mathcal{LT}(\tilde{\mathcal{Q}})$. Moreover, if $s_j \in \mathbb{R}$ is such that $\Phi^{s_j} \tilde{q} \in \mathcal{Q}(\eta_j)$ then for every $\epsilon > 0$ the set $\cup_j \cup_{t \in [-s_j - \epsilon, -s_j + \epsilon]} \Phi^t \mathcal{Q}(\eta_j)$ is a closed neighborhood of \tilde{q} in the closure of $\tilde{\mathcal{Q}}$.

Now if $i \neq j$ then $\mathcal{V}(\eta_i) \cap \mathcal{V}(\eta_j) = \mathcal{V}(\tau)$ and hence this intersection is contained in an affine subspace of codimension one. Since the measure class of the conditional measures λ^u of λ coincides with the Lebesgue measure class defined by the linear coordinates for the cone $\mathcal{V}(\eta_j)$, the equation (14) holds true.

As a consequence, for suitable numbers $a_j < b_j$, the set

$$\cup_j \cup_{t \in [a_j, b_j]} \Phi^t \mathcal{Q}(\eta_j)$$

is a $\text{Stab}(\tilde{q})$ -invariant closed neighborhood of \tilde{q} in the closure of $\tilde{\mathcal{Q}}$. In other words, there is a $\text{Stab}(\tilde{q})$ -invariant finite collection of closed sets with train track product structures which cover a neighborhood of \tilde{q} in the closure of $\tilde{\mathcal{Q}}$ and contain \tilde{q} in their boundary. This completes the proof of the proposition in the case that the support of the vertical measured geodesic lamination of \tilde{q} is large of type $(n_1, \dots, n_s; -m)$.

If the support of the vertical measured geodesic lamination of \tilde{q} is not large of type $(n_1, \dots, n_s; -m)$ then we argue in the same way. In this case \tilde{q} has a vertical saddle connection whose horizontal length is positive. Consider the action of the group $SO(2)$ on the space of quadratic differentials by rotation. There is a sequence $\theta_j \in (0, \pi/2)$ with $\theta_j \rightarrow 0$ such that the quadratic differential $e^{i\theta_j} \tilde{q}$ does

not have any vertical or horizontal saddle connection. Then the supports of the horizontal and the vertical measured geodesic laminations of $e^{i\theta_j} \tilde{q}$ are large of type $(n_1, \dots, n_s; -m)$.

Let $\tau \in \mathcal{LT}(n_1, \dots, n_s; -m)$ be a train track as in Lemma 4.2 so that for some $\sigma > 0$, $\Phi^\sigma \tilde{q}$ is an interior point of $\mathcal{Q}(\tau)$. For sufficiently small θ , say whenever $0 < |\theta| < \epsilon$, we have $e^{i\theta} \tilde{q} \in \cup_{s \in [\sigma-b, \sigma+b]} \Phi^s \mathcal{Q}(\tau)$ where $b > 0$ is a fixed number. If $\theta \in (-\epsilon, \epsilon)$ is such that $e^{i\theta} \tilde{q}$ does not have any vertical saddle connection then the argument in the beginning of this proof shows that up to passing to a subsequence, for sufficiently large j the vertical measured geodesic lamination of $e^{i\theta} \tilde{q}_j$ is carried by a simple extension of τ which is large of type $(m_1, \dots, m_\ell; -m)$. As before, there are only finitely many such simple extensions, and these simple extensions define train track coordinates on a neighborhood of \tilde{q} in the closure of $\tilde{\mathcal{Q}}$ as before. From this the proposition follows. \square

As an immediate consequence, we obtain the following. Let $q \in \overline{\mathcal{Q}} - \mathcal{Q}$ and let $K = \cup_{i=1}^k K_i$ be as in Proposition 4.3. Then for each $i \leq k$ there is an open subset $U_i \subset K_i$ of a strong unstable submanifold of \mathcal{Q} whose closure A_i contains q . The set

$$(15) \quad W_{\mathcal{Q}, \text{loc}}^{su}(q) = \cup_i A_i.$$

is a compact subset of $W^{su}(q)$ which contains the intersection with $W^{su}(q)$ of every sufficiently small neighborhood of q in $\overline{\mathcal{Q}}$. Moreover, $\lambda^{su}(A_i \cap A_j) = 0$ for $i \neq j$.

5. ABSOLUTE CONTINUITY

Let again \mathcal{Q} be a connected component of a stratum in $\mathcal{Q}(S)$. Then \mathcal{Q} is invariant under the Teichmüller flow Φ^t . For a periodic orbit $\gamma \subset \mathcal{Q}$ for Φ^t , the Lebesgue measure supported in γ is a Φ^t -invariant Borel measure $\sigma(\gamma)$ on \mathcal{Q} whose total mass equals the prime period $\ell(\gamma)$ of γ . If we denote for $R > 0$ by $\Gamma(R)$ the set of all periodic orbits for Φ^t of period at most R which are contained in \mathcal{Q} then we obtain a finite Φ^t -invariant Borel measure μ_R on \mathcal{Q} by defining

$$(16) \quad \mu_R = e^{-hR} \sum_{\gamma \in \Gamma(R)} \sigma(\gamma).$$

Let μ be any weak limit of the measures μ_R as $R \rightarrow \infty$. Then μ is a Φ^t -invariant Borel measure on $\mathcal{Q}(S)$ supported in the closure $\overline{\mathcal{Q}}$ of \mathcal{Q} (which may a priori be zero or locally infinite). The purpose of this section is to show

Proposition 5.1. *The measure μ on $\overline{\mathcal{Q}}$ satisfies $\mu \leq \lambda$.*

This means that $\mu(U) \leq \lambda(U)$ for every open relative compact subset U of $\overline{\mathcal{Q}}$. In particular, the measure μ is finite and absolutely continuous with respect to the Lebesgue measure, and it gives full mass to \mathcal{Q} .

A point $q \in \mathcal{Q}$ is called *forward recurrent* (or *backward recurrent*) if it is contained in its own ω -limit set (or in its own α -limit set) under the action of Φ^t . A point $q \in \mathcal{Q}$ is *recurrent* if it is forward and backward recurrent. The set $\mathcal{R} \subset \mathcal{Q}$ of

recurrent points is a Φ^t -invariant Borel subset of \mathcal{Q} . It follows from the work of Masur [M82] that a forward recurrent point $q \in \mathcal{Q}(S)$ has uniquely ergodic vertical and horizontal measured geodesic laminations whose supports fill up S . As a consequence, the preimage $\tilde{\mathcal{R}}$ of \mathcal{R} in $\mathcal{Q}^1(S)$ is contained in the set $\tilde{\mathcal{A}}$ defined in (5) of Section 2.

Using the notations from Section 2, there is a number $p > 1$ such that for every $q \in \mathcal{Q}^1(S)$ the map $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t q)$ is an unparametrized p -quasi-geodesic in the curve graph $\mathcal{C}(S)$. If q is a lift of a recurrent point in $\mathcal{Q}(S)$ then this unparametrized quasi-geodesic is of infinite diameter.

Recall from (3) of Section 2 the definition of the distances δ_x ($x \in \mathcal{T}(S)$) on $\partial\mathcal{C}(S)$ and of the sets $D(q, r) \subset \partial\mathcal{C}(S)$ ($q \in \tilde{\mathcal{A}}, r > 0$). The following lemma is a version of Lemma 2.1 of [H10b].

Lemma 5.2. *There are numbers $\alpha_0 > 0, \beta > 0, b > 0$ with the following property. Let $q \in \tilde{\mathcal{R}}$ and for $s > 0$ write $\sigma(s) = d(\Upsilon_{\mathcal{T}}(Pq), \Upsilon_{\mathcal{T}}(P\Phi^s q))$; then*

$$\beta e^{-b\sigma(s)} \delta_{P\Phi^s q} \leq \delta_{Pq} \leq \beta^{-1} e^{-b\sigma(s)} \delta_{P\Phi^s q} \text{ on } D(\Phi^s q, \alpha_0).$$

The map $F : \tilde{\mathcal{A}} \rightarrow \partial\mathcal{C}(S)$ defined in Section 2 is equivariant under the action of the mapping class group on $\tilde{\mathcal{A}} \subset \mathcal{Q}^1(S)$ and on $\partial\mathcal{C}(S)$. In particular, for $q \in \tilde{\mathcal{A}}$ and $r > 0$ the set $D(q, r) \subset \partial\mathcal{C}(S)$ is invariant under $\text{Stab}(q)$, and the same holds true for $F^{-1}D(q, r)$.

Let $\tilde{\mathcal{Q}} \subset \mathcal{Q}^1(S)$ be a component of the preimage of \mathcal{Q} and let $\text{Stab}(\tilde{\mathcal{Q}}) < \text{Mod}(S)$ be the stabilizer of $\tilde{\mathcal{Q}}$ in $\text{Mod}(S)$. The Φ^t -invariant Borel probability measure λ on \mathcal{Q} in the Lebesgue measure class lifts to a $\text{Stab}(\tilde{\mathcal{Q}})$ -invariant locally finite measure on $\tilde{\mathcal{Q}}$ which we denote again by λ . The conditional measures $\lambda^{ss}, \lambda^{su}$ of λ on the leaves of the strong stable and strong unstable foliation of \mathcal{Q} lift to a family of conditional measures on the leaves of the strong stable and strong unstable foliation $W_{\tilde{\mathcal{Q}}}^{ss}, W_{\tilde{\mathcal{Q}}}^{su}$ of $\tilde{\mathcal{Q}}$, respectively, which we denote again by $\lambda^{ss}, \lambda^{su}$ (see the discussion in Section 4).

Lemma 5.3. *For every $\tilde{q} \in \tilde{\mathcal{Q}} \cap \tilde{\mathcal{R}}$ and for all compact neighborhoods $W_1 \subset W_2$ of \tilde{q} in $W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$ there are compact neighborhoods $K \subset C \subset W_1$ of \tilde{q} in $W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$ with the following properties.*

- (1) K, C are precisely invariant under $\text{Stab}(\tilde{q})$.
- (2) There are numbers $0 < r_1 < r_2 < \alpha_0/2$ such that

$$K = \overline{W_1 \cap F^{-1}D(\tilde{q}, r_1)}, \quad C = \overline{W_1 \cap F^{-1}D(\tilde{q}, r_2)}.$$

- (3) $\lambda^{su}(K)(1 + \epsilon) \geq \lambda^{su}(C)$.
- (4) If $z \in K \cap \tilde{\mathcal{A}}$ and if $Z \subset \overline{F^{-1}D(z, (r_2 - r_1)/2)} \cap W_2$ then $Z \subset C$.

Proof. Let $q \in \mathcal{Q}$ be a recurrent point and let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q . Let $W_1 \subset W_2 \subset W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$ be compact neighborhoods of \tilde{q} and let $r > 0$ be such that $B_{\tilde{\mathcal{Q}}}^{su}(\tilde{q}, 2r) \subset W_1 \subset W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$ is precisely invariant under $\text{Stab}(\tilde{q})$ and projects to a metric orbifold ball in $W_{\mathcal{Q}}^{su}(q)$.

By Lemma 2.2, the map $F : \tilde{\mathcal{A}} \rightarrow \partial\mathcal{C}(S)$ is continuous and closed, and the sets $F(B^{su}(\tilde{q}, \nu) \cap \tilde{\mathcal{A}})$ ($\nu > 0$) form a neighborhood basis of $F\tilde{q}$ in $\partial\mathcal{C}(S)$. Thus there is a number $u_0 > 0$ such that

$$D(\tilde{q}, u_0) \cap F(W_2 \cap \tilde{\mathcal{A}}) \subset F(B_{\tilde{\mathcal{Q}}}^{su}(\tilde{q}, r) \cap \tilde{\mathcal{A}}).$$

For $u \leq u_0$ let $K_u \subset W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$ be the closure of the set

$$F^{-1}(D(\tilde{q}, u)) \cap W_2.$$

Then K_u is a closed neighborhood of \tilde{q} in $W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$ which is precisely invariant under $\text{Stab}(\tilde{q})$. Moreover, $K_t \subset K_u$ for $t < u$, and Lemma 2.2 shows that $\bigcap_{u>0} K_u = \{\tilde{q}\}$. Since the conditional measure λ^{su} on $W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$ is Borel regular, for every $\epsilon > 0$ there are numbers $r_1 < r_2 < u_0$ so that

$$\lambda^{su}(K_{r_1}) \geq \lambda^{su}(K_{r_2})(1 + \epsilon)^{-1}.$$

For these number $r_1 < r_2$, all requirements in the lemma hold true. This shows the lemma. \square

Remark: Since $\tilde{\mathcal{A}}$ is dense in $\mathcal{Q}^1(S)$ and the map $F : \tilde{\mathcal{A}} \rightarrow \partial\mathcal{C}(S)$ is continuous and closed, the sets $K \subset C \subset W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q})$ have dense interior. Moreover, we may assume that their boundaries have vanishing Lebesgue measures.

Let again $\tilde{\mathcal{Q}} \subset \mathcal{Q}^1(S)$ be a component of the preimage of \mathcal{Q} . For $q \in \mathcal{Q}$ let \tilde{q} be a preimage of q in $\tilde{\mathcal{Q}}$ and let $|\text{Stab}(q)|$ be the cardinality of the quotient of $\text{Stab}(\tilde{q})$ by the normal subgroup of all elements of $\text{Stab}(\tilde{q})$ which fix $\tilde{\mathcal{Q}}$ pointwise (for example, the hyperelliptic involution of a closed surface of genus 2 acts trivially on the entire bundle $\mathcal{Q}^1(S)$). We note

Lemma 5.4. *The set $\mathcal{S} = \{q \in \mathcal{Q} \mid |\text{Stab}(q)| = 1\}$ is an open dense Φ^t -invariant submanifold of \mathcal{Q} .*

Proof. The mapping class group preserves the Teichmüller metric on $\mathcal{T}(S)$ and hence an element $h \in \text{Mod}(S)$ which stabilizes a quadratic differential $\tilde{q} \in \mathcal{Q}^1(S)$ fixes pointwise the Teichmüller geodesic with initial cotangent \tilde{q} . Therefore the set \mathcal{S} is Φ^t -invariant, moreover it is clearly open. Since the Teichmüller flow on \mathcal{Q} has dense orbits, either \mathcal{S} is empty or dense. However, $\text{Mod}(S)$ acts properly discontinuously on $\mathcal{T}(S)$ and consequently the first possibility is ruled out by the fact that the conjugacy class of an element of $\text{Mod}(S)$ which fixes an entire component of the preimage of \mathcal{Q} does not contribute towards $|\text{Stab}(q)|$. \square

For a control of the measure μ we use a variant of an argument of Margulis [Mar04]. Namely, for numbers $R_1 < R_2$ let $\Gamma(R_1, R_2)$ be the set of all periodic orbits of Φ^t which are contained in \mathcal{Q} , with prime periods in the interval (R_1, R_2) . For an open or closed subset V of $\overline{\mathcal{Q}}$ and numbers $R_1 < R_2$ define

$$H(V, R_1, R_2) = \sum_{\gamma \in \Gamma(R_1, R_2)} \int_{\gamma} \chi(V)$$

where $\chi(V)$ is the characteristic function of V .

To obtain control on the quantities $H(V, R_1, R_2)$ we use a tool from [ABEM10]. Namely, every leaf $W^{ss}(q)$ of the strong stable foliation of $\mathcal{Q}(S)$ can be equipped with the *Hodge distance* d_H (or, rather, the modified Hodge distance, [ABEM10]). This Hodge distance is defined by a norm on the tangent space of $W^{ss}(q)$ (with a suitable interpretation). In particular, closed d_H -balls of sufficiently small finite radius are compact, and balls about a given point q define a neighborhood basis of q in $W^{ss}(q)$. We also obtain a Hodge distance on the leaves of the strong unstable foliation as the image under the flip \mathcal{F} of the Hodge distance on the leaves of the strong stable foliation. These Hodge distances restrict to Hodge distances on the leaves of the foliations $W_{\mathcal{Q}}^{ss}, W_{\mathcal{Q}}^{su}$ which we denote by the same symbol d_H .

The following result is Theorem 8.12 of [ABEM10],

Theorem 5.5. *There is a number $c_H > 0$ such that*

$$(17) \quad d_H(\Phi^t q, \Phi^t q') \leq c_H d_H(q, q').$$

for all $q \in \mathcal{Q}(S), q' \in W^{ss}(q)$ and all $t \geq 0$.

The next lemma provides some first volume control for the measure μ .

Lemma 5.6. *For every recurrent point $q \in \mathcal{Q}$ with $|\text{Stab}(\tilde{q})| = 1$, for every neighborhood V of q in \mathcal{Q} and for every $\epsilon > 0$ there is a number $t_0 > 0$ and there is an open neighborhood $U \subset V$ of q such that*

$$\limsup_{R \rightarrow \infty} H(U, R - t_0, R + t_0) e^{-hR} \leq 2t_0 \lambda(U)(1 + \epsilon).$$

Proof. We use the strategy of the proof of Lemma 6.1 of [Mar04]. The idea is to find for every recurrent point $q \in \mathcal{Q}$ with $|\text{Stab}(\tilde{q})| = 1$, for every neighborhood V of q in \mathcal{Q} and for every $\epsilon \in (0, 1)$ some number $t_0 > 0$ and closed neighborhoods $Z_1 \subset Z_2 \subset Z_3 \subset V_0 \subset V$ of q in \mathcal{Q} with dense interior such that for all sufficiently large $R > 0$ the following properties hold.

- (1) V_0 is connected and has a local product structure.
- (2) $\lambda(Z_3) \leq \lambda(Z_1)(1 + \epsilon)$.
- (3) Let $z \in Z_1$ and assume that $\Phi^\tau z = z$ for some $\tau \in (R - t_0, R + t_0)$. Let \hat{E} be the component containing z of the intersection $\Phi^\tau V_0 \cap V_0$ and let $E = \hat{E} \cap \Phi^\tau Z_2 \cap Z_3$. Then

$$\lambda(E) \in [e^{-hR} \lambda(Z_1)/(1 + \epsilon), e^{-hR} \lambda(Z_1)(1 + \epsilon)],$$

and the length of the connected orbit subsegment of $(\cup_{t \in \mathbb{R}} \Phi^t z) \cap Z_1$ containing z equals $2t_0$.

- (4) There is at most one periodic orbit for Φ^t of prime period $\sigma \in (R - t_0, R + t_0)$ which intersects E , and the intersection of this orbit with E is connected.

The construction is as follows.

Let $q \in \mathcal{Q}$ be recurrent with $|\text{Stab}(\tilde{q})| = 1$ and let V be a neighborhood of q in \mathcal{Q} . Using the notations from Subsection 4.1, for $\epsilon > 0$ there are numbers $a_0 < a_{\mathcal{Q}}(q), t_0 < \min\{t_{\mathcal{Q}}(q)/4(1 + \epsilon), \log(1 + \epsilon)/h\}$ such that

$$V_0 = V(B_{\mathcal{Q}}^{ss}(q, a_0), B_{\mathcal{Q}}^{su}(q, a_0), t_0(1 + \epsilon)) \subset V$$

is a set with a local product structure.

Let $\tilde{q} \in \mathcal{Q}^1(S)$ be a preimage of q . By construction (see the discussion in Section 4.1), the set

$$\tilde{V}_0 = V(B_{\mathcal{Q}}^{ss}(\tilde{q}, a_0), B_{\mathcal{Q}}^{su}(\tilde{q}, a_0), t_0(1 + \epsilon))$$

is precisely invariant under $\text{Stab}(\tilde{q})$. In particular, since $|\text{Stab}(\tilde{q})| = 1$, the set \tilde{V}_0 is mapped homeomorphically onto V_0 by the projection $\tilde{\mathcal{Q}} \rightarrow \mathcal{Q}$.

Since periodic orbits for Φ^t are in bijection with conjugacy classes of pseudo-Anosov elements of $\text{Mod}(S)$, up to making a_0 smaller we may assume that the following holds true. For every $r > 8t_0$, every component of the intersection $\Phi^r V_0 \cap V_0$ is intersected by at most one periodic orbit for the Teichmüller flow with prime period contained in the interval $[r - 2t_0, r + 2t_0]$, and if such an orbit exists then its intersection with $\Phi^r V_0 \cap V_0$ is connected.

As in (11) of Section 4, for $z \in V_0$ let $\theta_z : B_{\mathcal{Q}}^{ss}(q, a_0) \rightarrow W_{\mathcal{Q}, \text{loc}}^{ss}(z)$ be defined by the requirement that $\theta_z(u) \in W_{\mathcal{Q}, \text{loc}}^u(u)$ for all u . Similarly, as in (10) of Section 4, let $\zeta_z : B_{\mathcal{Q}}^{su}(q, a_0) \rightarrow W_{\mathcal{Q}, \text{loc}}^{su}(z)$ be defined by $\zeta_z(u) \in W_{\mathcal{Q}, \text{loc}}^s(u)$. We claim that for sufficiently small $a_1 < a_0$ and for every

$$z \in V_1 = V(B_{\mathcal{Q}}^{ss}(q, a_1), B_{\mathcal{Q}}^{su}(q, a_1), t_0)$$

the following holds true.

- a) The Jacobian of the embedding $\theta_z : B_{\mathcal{Q}}^{ss}(q, a_1) \rightarrow W_{\mathcal{Q}, \text{loc}}^{ss}(z)$ and of the embedding $\zeta_z : B_{\mathcal{Q}}^{su}(q, a_1) \rightarrow W_{\mathcal{Q}, \text{loc}}^{su}(z)$ with respect to the measures λ^{ss} and λ^{su} , respectively, is contained in the interval $[(1 + \epsilon)^{-1}, 1 + \epsilon]$.
- b) The restriction to V_1 of the function σ defined in (12) takes values in the interval $[-(\log(1 + \epsilon)/h), (\log(1 + \epsilon)/h)]$.
- c) If $z \in V(B_{\mathcal{Q}}^{ss}(q, a_1), B_{\mathcal{Q}}^{su}(q, a_1))$ and if $t > 8t_0$ is such that

$$\Phi^t z \in V(B_{\mathcal{Q}}^{ss}(q, a_1), B_{\mathcal{Q}}^{su}(q, a_1))$$

then $\Phi^t(V_1 \cap W_{\mathcal{Q}, \text{loc}}^s(z)) \subset V_0$ and $\Phi^{-t}(V_1 \cap W_{\mathcal{Q}, \text{loc}}^u(z)) \subset V_0$.

Here and in the sequel, for $z \in V_1$ we denote by $V_1 \cap W_{\mathcal{Q}, \text{loc}}^s(z)$ the connected component containing z of the intersection $V_1 \cap W_{\mathcal{Q}}^s(z)$.

To verify the claim, note first that property b) can be fulfilled since σ is continuous and Φ^t -invariant and equals one at q . Property a) is fulfilled for sufficiently small a_1 since the measures λ^s (or λ^u) are invariant under holonomy along the strong unstable (or the strong stable) foliation and since $d\lambda^s = d\lambda^{ss} \times dt$ and $d\lambda^u = d\lambda^{su} \times dt$ and hence Jacobians of the maps θ_z, ζ_z are controlled by the function σ .

By Property b) above and by Theorem 5.5, the last property is fulfilled if we choose $a_1 > 0$ small enough so that for some $r > 0$ and every $z \in V_1$ the following is satisfied. For every $u \in V_1$ the diameter of $\theta_u(B_{\mathcal{Q}}^{ss}(q, a_1))$ with respect to the Hodge distance does not exceed r , and the Hodge distance between $\theta_u(B_{\mathcal{Q}}^{ss}(q, a_1))$ and the boundary of $\theta_u(B_{\mathcal{Q}}^{ss}(q, a_0))$ is not smaller than $c_H r$.

Since $h \geq 1$, Property b) implies the following. For all closed sets $A^i \subset B_{\mathcal{Q}}^i(q, a_1)$ ($i = ss, su$) and for every $z \in V(B_{\mathcal{Q}}^{ss}(q, a_1), B_{\mathcal{Q}}^{su}(q, a_1))$ we have

$$(18) \quad \begin{aligned} V(A^{ss}, A^{su}, t_0(1 + \epsilon)^{-1}) &\subset V(\theta_z(A^{ss}), \zeta_z(A^{su}), t_0) \\ &\subset V(A^{ss}, A^{su}, t_0(1 + \epsilon)). \end{aligned}$$

Moreover, we have

$$(19) \quad \lambda(V(\theta_z(A^{ss}), \zeta_z(A^{su}), t_0))/2t_0\lambda^{ss}(A^{ss})\lambda^{su}(A^{su}) \in [(1 + \epsilon)^{-4}, (1 + \epsilon)^4].$$

By the estimate (4) in Section 2, there is a number $\kappa > 0$ such that for any two points $u, x \in \mathcal{T}(S)$ with $d_{\mathcal{T}}(u, x) \leq 1$ the distances δ_u, δ_x on $\partial\mathcal{C}(S)$ are e^κ -bilipschitz equivalent.

Let $\tilde{\mathcal{Q}}$ be a component of the preimage of \mathcal{Q} in $\mathcal{Q}^1(S)$. Let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q . Choose closed neighborhoods $K^{ss} \subset C^{ss} \subset B_{\tilde{\mathcal{Q}}}^{ss}(\tilde{q}, a_1) \subset B_{\tilde{\mathcal{Q}}}^{ss}(\tilde{q}, a_0)$ of \tilde{q} whose images under the flip \mathcal{F} satisfy the properties in Lemma 5.3 for some numbers $0 < r_1 < r_2 < \alpha_0/2e^\kappa$ where $\alpha_0 > 0$ is as in Lemma 5.2. Choose also closed neighborhoods $\tilde{K}^{su} \subset \tilde{C}^{su} \subset B_{\tilde{\mathcal{Q}}}^{su}(\tilde{q}, a_1) \subset B_{\tilde{\mathcal{Q}}}^{su}(\tilde{q}, a_0)$ of \tilde{q} with the properties in Lemma 5.3 for some numbers $0 < \tilde{r}_1 < \tilde{r}_2 < \alpha_0/2\kappa$. By the choice of the set V_0 , for any two points $u, z \in V(C^{ss}, \tilde{C}^{su}, t_0(1 + \epsilon))$ the distances δ_{P_u} and δ_{P_z} are e^κ -bilipschitz equivalent. As a consequence, for all $u \in V(C^{ss}, \tilde{C}^{su}, t_0(1 + \epsilon))$ the δ_{P_u} -diameter of $F(\mathcal{F}C^{ss} \cap \mathcal{A})$ and $F(\tilde{C}^{su} \cap \mathcal{A})$ does not exceed $\alpha_0/2$. Let

$$\rho_0 \in (0, \min\{(r_2 - r_1)/2, (\tilde{r}_2 - \tilde{r}_1)/2\}).$$

By assumption, q is recurrent and hence by Lemma 5.2, applied to both \tilde{q} and $-\tilde{q} = \mathcal{F}(\tilde{q})$, there is a number $R_0 > 0$ so that for every $R \geq R_0$ and for every $z \in B_{\tilde{\mathcal{Q}}}^{su}(\tilde{q}, a_1)$ with $d_{\mathcal{T}}(P\Phi^{Rz}, P\Phi^R\tilde{q}) \leq 1$ we have

$$(20) \quad \begin{aligned} \delta_{P\Phi^{Rz}} &\leq \rho_0\delta_{Pz}/\alpha_0 \text{ on } F(\mathcal{F}C^{ss} \cap \tilde{\mathcal{A}}) \text{ and} \\ \delta_{P\Phi^{Rz}} &\geq \alpha_0\delta_{Pz}/\rho_0 \text{ on } D(\Phi^R\tilde{q}, \alpha_0). \end{aligned}$$

Moreover, there is a mapping class $h \in \text{Stab}(\tilde{\mathcal{Q}})$ and a number $R_1 > R_0$ such that $\Phi^{R_1}\tilde{q}$ is an interior point of $hV(K^{ss}, \tilde{K}^{su})$.

By equivariance under the action of the mapping class group, for every $u \in hV(C^{ss}, \tilde{C}^{su})$ the δ_{P_u} -diameter of $F(hV(C^{ss}, \tilde{C}^{su}) \cap \tilde{\mathcal{A}})$ is smaller than $\alpha_0/2$. In particular, the $\delta_{P_{\Phi^{R_1}\tilde{q}}}$ -diameter of $F(h\tilde{C}^{su} \cap \tilde{\mathcal{A}})$ is smaller than $\alpha_0/2$. The second part of inequality (20) then implies that the $\delta_{P_{\tilde{q}}}$ -diameter of $F(h\tilde{C}^{su} \cap \tilde{\mathcal{A}})$ does not exceed ρ_0 . Thus by Property c) above, by the choice of ρ_0 and by Lemma 5.3, we have

$$F(h\tilde{C}^{su} \cap \tilde{\mathcal{A}}) \subset F(\tilde{C}^{su} \cap \tilde{\mathcal{A}}).$$

Define

$$\begin{aligned} K^{su} &= \overline{\{x \in W_{\tilde{\mathcal{Q}}, \text{loc}}^{su}(\tilde{q}) \cap \tilde{\mathcal{A}} \mid F(x) \in F(h\tilde{K}^{su} \cap \tilde{\mathcal{A}})\}} \text{ and} \\ C^{su} &= \overline{\{x \in W_{\tilde{\mathcal{Q}}, \text{loc}}^{su}(\tilde{q}) \cap \tilde{\mathcal{A}} \mid F(x) \in F(h\tilde{C}^{su} \cap \tilde{\mathcal{A}})\}}. \end{aligned}$$

Then \tilde{q} is an interior point of K^{su} (as a subset of $W_{\mathcal{Q},\text{loc}}^{su}(\tilde{q})$), and K^{su}, C^{su} are precisely invariant under $\text{Stab}(\tilde{q})$ (since a non-trivial element of $\text{Stab}(\tilde{q})$ fixes $\tilde{\mathcal{Q}}$ pointwise).

The conditional measures λ^{su} are invariant under holonomy along the strong stable foliation and transform under the Teichmüller flow by $\lambda^{su} \circ \Phi^t = e^{ht} \lambda^{su}$. Moreover, $\lambda^{su}(\tilde{K}^{su}) \geq \lambda^{su}(\tilde{C}^{su})(1+\epsilon)^{-1}$ and hence properties a) and b) above and the definition of the function σ imply that

$$\lambda^{su}(K^{su}) \geq \lambda^{su}(C^{su})(1+\epsilon)^{-3}.$$

Define

$$\tilde{Z}_1 = V(K^{ss}, K^{su}, t_0), \tilde{Z}_2 = V(K^{ss}, C^{su}, t_0), \tilde{Z}_3 = V(C^{ss}, C^{su}, t_0(1+\epsilon))$$

and let Z_i be the projection of \tilde{Z}_i to \mathcal{Q} . Note that we have $Z_1 \subset Z_2 \subset Z_3$ and

$$\lambda(Z_1) \geq \lambda(Z_3)(1+\epsilon)^{-8}$$

by the choice of K^{ss}, C^{ss} , by the estimate in a) above, by invariance of λ under the flow Φ^t (which implies that $\lambda(\tilde{Z}_3) \leq \lambda V(C^{ss}, C^{su}, t_0)(1+\epsilon)^2$) and by the fact that \tilde{Z}_i is mapped homeomorphically onto Z_i for $i = 1, 2, 3$. Moreover, each of the sets Z_i is closed with dense interior.

Let $R > R_1 + t_0$ and let $z \in Z_1$ be a periodic point for Φ^t of period $r \in [R - t_0, R + t_0]$. Since every orbit of Φ^t which intersects Z_1 also intersects $V(K^{ss}, K^{su})$ we may assume that $z \in V(K^{ss}, K^{su})$. Let \hat{E} be the component containing z of the intersection $\Phi^r V_0 \cap V_0$ and let

$$E = \hat{E} \cap \Phi^r Z_2 \cap Z_3 \subset Z_3.$$

We claim that

$$(21) \quad \lambda(E) \in [e^{-hr} \lambda(Z_1)(1+\epsilon)^{-10}, e^{-hr} \lambda(Z_1)(1+\epsilon)^{11}].$$

To see that this is indeed the case, let $\tilde{z} \in \tilde{Z}_1$ be a lift of z . By the choice of the set C^{su} and by the first part of the estimate (20), the $\delta_{P\Phi^R\tilde{z}}$ -diameter of the set $F(\mathcal{F}\Phi^R C^{ss} \cap \tilde{\mathcal{A}})$ does not exceed ρ_0 . In particular, since $z \in Z_1$ and Property c) above holds true, we have

$$(22) \quad \Phi^r(W_{\mathcal{Q},\text{loc}}^s(z) \cap Z_2) \subset E$$

and similarly

$$(23) \quad \Phi^{-r}(W_{\mathcal{Q},\text{loc}}^u(z) \cap Z_1) \subset E.$$

Let $D \subset C^{ss}$ be such that

$$\theta_z D = \Phi^r(W_{\mathcal{Q},\text{loc}}^s(z) \cap Z_2) \cap \theta_z C^{ss}.$$

Then by the estimate (18) and by (23), we have

$$Q_1 = V(\theta_z(D), \zeta_z(K^{su}), t_0(1+\epsilon)^{-1}) \subset E \subset V(\theta_z(D), \zeta_z(C^{su}), t_0(1+\epsilon)) = Q_2.$$

Now by the estimate (19) and the fact that Φ^r preserves the stable foliation and contracts the measures λ^s by the factor e^{-hr} , we conclude that

$$\lambda(Q_1) \geq e^{-hr} \lambda^{ss}(K^{ss}) \lambda^{su}(K^{su}) / 2t_0(1+\epsilon)^6$$

and similarly

$$\lambda(Q_2) \leq e^{-hr} \lambda^{ss}(K^{ss}) \lambda^{su}(C^{su}) (1 + \epsilon)^6 / 2t_0.$$

Together with the estimate (19) this implies the estimate (21).

Since $r \in [R - t_0, R + t_0]$ and $e^{ht_0} \leq 1 + \epsilon$ we conclude that the measure of the set E is at least

$$e^{-hR} \lambda(Z_1) (1 + \epsilon)^{-12}.$$

Moreover, the Lebesgue measure of the component containing z of the orbit segment $\{\Phi^t z \mid -t_0 < t < t_0\} \cap Z_2$ equals $2t_0$.

On the other hand, if $z \neq z' \in Z_0$ are periodic points of prime periods $r, s \in [R - t_0, R + t_0]$ then by our choice of V_0 the components containing z, z' of the intersection $\Phi^r V_0 \cap V_0$ are disjoint. Thus there are at most

$$\lambda(\Phi^R Z_2 \cap Z_3) e^{hR} (1 + \epsilon)^{12} / \lambda(Z_1)$$

such intersection arcs which are subarcs of periodic orbits of prime period in $[R - t_0, R + t_0]$. However, since the Lebesgue measure λ is mixing for the Teichmüller flow [M82, V86], for sufficiently large R we have

$$\lambda(\Phi^R Z_2 \cap Z_3) \leq \lambda(Z_2) \lambda(Z_3) (1 + \epsilon) \leq \lambda(Z_1)^2 (1 + \epsilon)^{17}.$$

From this we deduce that

$$H(Z_1, R - t_0, R + t_0) e^{-hR} \leq 2t_0 \lambda(Z_1) (1 + \epsilon)^{29}$$

for all sufficiently large $R > 0$. This shows the lemma. \square

Now we are ready for the proof of Proposition 5.1.

Proof of Proposition 5.1. Let μ be a weak limit of the measures μ_R as $R \rightarrow \infty$. Then μ is a (a priori locally infinite) Φ^t -invariant Borel measure supported in the closure $\overline{\mathcal{Q}}$ of \mathcal{Q} . This measure is moreover invariant under the flip $\mathcal{F} : q \rightarrow -q$.

By Lemma 5.6 it suffices to show the following. Let $A \subset \overline{\mathcal{Q}}$ be a closed Φ^t -invariant set of vanishing Lebesgue measure. Then for all $\epsilon > 0$, every $q \in A$ has a neighborhood U in $\overline{\mathcal{Q}}$ such that $\mu(A \cap U) < \epsilon$.

First let $q \in A \cap \mathcal{Q}$. Choose compact balls $B^i \subset C^i \subset W_{\mathcal{Q}, \text{loc}}^i(q)$ about q for the Hodge distance of radius $r_1 > 0, r_2 > 2c_H r_1 > 0$ ($i = ss, su$) and numbers $t_0 > 0, \delta > 0$ such that $V_3 = V(C^{ss}, C^{su}, t_0(1 + \delta))$ is a set with a local product structure. In particular, for every preimage \tilde{q} of q in $\mathcal{Q}^1(S)$ the component of the preimage of V_3 containing \tilde{q} is precisely invariant under $\text{Stab}(\tilde{q})$. Then

$$V_0 = V(B^{ss}, B^{su}, t_0(1 - \delta)) \subset V(C^{ss}, C^{su}, t_0(1 + \delta)) = V_3$$

are closed neighborhoods of q in \mathcal{Q} . Let moreover

$$V_1 = V(B^{ss}, B^{su}, t_0) \subset V_2 = V(B^{ss}, C^{su}, t_0).$$

We may assume that for one (and hence every) component \tilde{V}_3 of the preimage of V_3 in $\mathcal{Q}^1(S)$ the diameter of the projection $P\tilde{V}_3$ of \tilde{V}_3 to $\mathcal{T}(S)$ does not exceed one. As in the proof of Lemma 5.6 we require that moreover the following holds true.

(*) If $z \in V(B^{ss}, B^{su})$ and if $t > 8t_0$ is such that $\Phi^t z \in V(B^{ss}, B^{su})$ then $\Phi^t(V_1 \cap W_{\mathcal{Q}, \text{loc}}^s(z)) \subset V_3$ and moreover $\Phi^{-t}(V_0 \cap W_{\mathcal{Q}, \text{loc}}^u(z)) \subset V_2$.

That this requirement can be met follows from Theorem 5.5 and the discussion in the proof of Lemma 5.6.

If $q \in \overline{\mathcal{Q}} - \mathcal{Q}$ then we choose closed neighborhoods

$$V_0 = \cup_i V(B_i^{ss}, B_i^{su}, t_0(1 - \delta)) \subset V_3 = \cup_i V(C_i^{ss}, C_i^{su}, t_0(1 + \delta))$$

of q in $\overline{\mathcal{Q}}$ as in Proposition 4.3 such that $\cup_i B_i^j$ and $\cup_i C_i^j$ are the intersections with $W_{\mathcal{Q}, \text{loc}}^j(q)$ of closed balls for the Hodge norm. We require that property (*) above holds true (with a slight abuse of notation).

Let $u \in V_1$ and let $r > 0$ be such that $\Phi^r u = u$. Let Y be the connected component containing u of the intersection $V_3 \cap \Phi^r(V_2)$. By the property (*), we have $Y \supset \Phi^r(V_1 \cap W_{\mathcal{Q}, \text{loc}}^s(u))$. Moreover, the connected component containing u of the intersection $V_3 \cap \Phi^r(V_2 \cap W_{\mathcal{Q}, \text{loc}}^u(u))$ contains the component containing u of the intersection $W_{\mathcal{Q}, \text{loc}}^u(u) \cap V_0$. Thus as in the proof of Lemma 5.6, we observe that for any point $u \in V_0$ and every $r > 0$ such that $\Phi^r u = u$ the Lebesgue measure of the intersection $\Phi^r V_2 \cap V_3$ is bounded from below by $e^{-hr} \chi$ where $\chi > 0$ is a fixed constant which only depends on V_1, V_2, V_3 . Moreover, the number of periodic points $z \in V_1$ of period $s \in [r - t_0, r + t_0]$ such that the intersection components $\Phi^r V_2 \cap V_3, \Phi^s V_2 \cap V_3$ containing u, z are *not* disjoint is bounded from above by the cardinality of $\text{Stab}(\tilde{q})$ where \tilde{q} is a preimage of q in $\mathcal{Q}^1(S)$.

For $q, z \in \overline{\mathcal{Q}}$ and $t > 0$ write $q \approx_t z$ if there are lifts \tilde{q}, \tilde{z} of q, z to $\mathcal{Q}^1(S)$ such that

$$d(P\Phi^s \tilde{q}, P\Phi^s \tilde{z}) < 1 \text{ for } 0 \leq s \leq t.$$

Write moreover $q \sim_u z$ if there are lifts \tilde{q}, \tilde{z} of q, z to $\mathcal{Q}^1(S)$ such that

$$d(\tilde{q}, \tilde{z}) < 1, d(P\Phi^u \tilde{q}, P\Phi^u \tilde{z}) < 1.$$

Note that if $y \approx_t z$ then also $y \sim_t z$. For a subset D of $\overline{\mathcal{Q}}$ define

$$(24) \quad \begin{aligned} U_t(D) &= \{z \mid z \approx_t y \text{ for some } y \in D\} \text{ and} \\ Y_u(D) &= \{z \mid z \sim_u y \text{ for some } y \in D\}. \end{aligned}$$

Then $U_t(D)$ and $Y_u(D)$ are open neighborhoods of D .

For $j > 0$ define

$$(25) \quad Z_j = U_j(A \cap V_1) \cap V_1 \text{ and}$$

$$(26) \quad W_{j,k} = Y_k(Z_j) \cap V_1.$$

Then for all $k > 0, j > 0$ each $j > 0$, Z_j is an open neighborhood of $A \cap V_1$ in V_1 , and $W_{j,k}$ is an open neighborhood of Z_j in $A \cap V_1$. Moreover, we have $Z_j \supset Z_{j+1}$ for all j and $\cap_j Z_j \supset A \cap V_1$. If $z \in \cap_j Z_j - A$ then there is some $y \in A$ and there are lifts \tilde{z}, \tilde{y} of z, y to $\mathcal{Q}^1(S)$ such that $d(P\Phi^t(\tilde{z}), P\Phi^t(\tilde{y})) \leq 1$ for all $t \geq 0$. However, up to removing from $\cap_j Z_j$ a set of vanishing Lebesgue measure, this implies that $z \in W_{\mathcal{Q}, \text{loc}}^{ss}(y)$ [M82, V86]. But $\lambda(A) = 0$ and therefore $\lambda(\cap_j Z_j) = \lambda(A \cap V_1) = 0$ by absolute continuity. Since λ is Borel regular, the Lebesgue measures of the sets Z_j tend to zero as $j \rightarrow \infty$.

Similarly, we infer that $\lambda(Z_j) = \limsup_{k \rightarrow \infty} \lambda(W_{j,k})$. Thus for every $\kappa > 0$ there are numbers $j_0 = j_0(\kappa) > 0$ and $k_0 = k_0(\kappa) > j_0$ such that we have $\lambda(W_{j,k}) < \kappa$ for all $j \geq j_0, k \geq k_0$.

Now let $R > k_0 + 2\epsilon$ and let $w \in V_1 \cap Z_{j_0}$ be a periodic point for Φ^t of prime period $r \in [R - \epsilon, R + \epsilon]$. Let Z be the component of $\Phi^r V_2 \cap V_3$ containing w . Then every point in Z is contained in $W_{j_0, R}$. By Lemma 5.6 and its proof, the Lebesgue measure of this intersection component is bounded from below by χe^{-hR} where $\chi > 0$ is as above. Moreover, the number of periodic points $u \neq z$ for which these intersection components are not disjoint is uniformly bounded. In particular, there is a number $\beta > 0$ not depending on R, j_0 such that the number of these intersection components is bounded from above by βe^{hR} times the Lebesgue measure of $W_{j_0, R}$, i.e. by $e^{hR} \beta \kappa$. This implies that we have $\mu(Z_{j_0}) \leq \beta \kappa / 2t_0$. Since $\kappa > 0$ was arbitrary we conclude that $\mu(A \cap V_1) = 0$. Proposition 5.1 follows.

6. PROOF OF THE THEOREM

In this section we complete the proof of the theorem from the introduction. We continue to use the assumptions and notations from Sections 2-5.

As before, let $\mathcal{Q} \subset \mathcal{Q}(S)$ be a component of a stratum, equipped with the Φ^t -invariant Lebesgue measure λ . Let $\mathcal{S} \subset \mathcal{Q}$ be the open dense Φ^t -invariant subset of full Lebesgue measure of all points q with $|\text{Stab}(q)| = 1$. Then \mathcal{S} is a manifold.

Let $q \in \mathcal{S}$ and let $U \subset \mathcal{S}$ be an open relative compact contractible neighborhood of q . For $n > 0$ define a *periodic* (U, n) -*pseudo-orbit* for the Teichmüller flow Φ^t on \mathcal{Q} to consist of a point $x \in U$ and a number $t \in [n, \infty)$ such that $\Phi^t x \in U$. We denote such a periodic pseudo-orbit by (x, t) . A periodic (U, n) -pseudo-orbit (x, t) determines up to homotopy a closed curve beginning and ending at x which we call a *characteristic curve* (compare Section 4 of [H10b]). This characteristic curve is the concatenation of the orbit segment $\{\Phi^s x \mid 0 \leq s \leq t\}$ with a smooth arc in U which is parametrized on $[0, 1]$ and connects the endpoint $\Phi^t x$ of the orbit segment with the starting point x .

Recall from Section 5 the definition of a recurrent point for the Teichmüller flow on \mathcal{Q} . Lemma 4.4 of [H10b] shows

Lemma 6.1. *There is a number $L > 0$ and for every recurrent point $q \in \mathcal{S}$ there is an open relative compact contractible neighborhood V of q in \mathcal{S} and there is a number $n_0 > 0$ depending on V with the following property. Let (x, t_0) be a periodic (V, n_0) -pseudo-orbit and let γ be a lift to $\mathcal{Q}^1(S)$ of a characteristic curve of the pseudo-orbit. Then the curve $t \rightarrow \Upsilon_{\mathcal{T}}(P\gamma(t))$ is an infinite unparametrized L -quasi-geodesic in $\mathcal{C}(S)$.*

Remark: Lemma 4.4 of [H10b] is formulated for $\mathcal{Q}(S)$ rather than for a component of a stratum. However, the statement and its proof immediately carry over to the result formulated in Lemma 6.1.

For a point $q \in \mathcal{Q}$ choose a preimage $\tilde{q} \in \mathcal{Q}^1(S)$ of q and for $t > 0$ define

$$\beta(q, t) = d(\Upsilon_{\mathcal{T}}(P\tilde{q}), \Upsilon_{\mathcal{T}}(P\Phi^t \tilde{q})).$$

Note that $\beta(q, t)$ depends on the choice of the map $\Upsilon_{\mathcal{T}}$ (and on the choice of the lift \tilde{q}). However, by Lemma 3.3 of [H10a], there is a *continuous* function $\tilde{\beta} :$

$\mathcal{Q} \times [0, \infty) \rightarrow \mathbb{R}$ and a number $a > 0$ such that $|\tilde{\beta}(q, t) - \beta(q, t)| \leq a$ for all (q, t) . In particular, the values $\liminf_{t \rightarrow \infty} \frac{1}{t} \beta(q, t)$ and $\limsup_{t \rightarrow \infty} \frac{1}{t} \beta(q, t)$ are independent of any choices made and coincide with the corresponding values for $\tilde{\beta}$. We use this observation to show

Lemma 6.2. *There is a number $c > 0$ such that for λ -almost every $q \in \mathcal{Q}$ we have*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \beta(q, t) = c.$$

Proof. It suffices to show the lemma for the continuous function $\tilde{\beta}$.

By the choice of $a > 0$ and by the triangle inequality, we have

$$\tilde{\beta}(q, s+t) \leq \tilde{\beta}(q, s) + \tilde{\beta}(\Phi^s q, t) + 3a$$

for all $q \in \mathcal{Q}, s, t \in \mathbb{R}$. Therefore the subadditive ergodic theorem shows that for λ -almost all $q \in \mathcal{Q}$ the limit $\lim_{t \rightarrow \infty} \frac{1}{t} \tilde{\beta}(q, t)$ exists and is independent of q . We are left with showing that this limit is positive.

By Lemma 2.4 of [H10a], there is a number $r > 0$ such that for every $z \in \mathcal{Q}^1(S)$ and all $t \geq s \geq 0$ we have

$$d(\Upsilon_{\mathcal{T}}(Pz), \Upsilon_{\mathcal{T}}(P\Phi^t z)) \geq d(\Upsilon_{\mathcal{T}}(Pz), \Upsilon_{\mathcal{T}}(P\Phi^s z)) - r.$$

Let $q \in \mathcal{Q}$ be a periodic point for Φ^t . Then there is a number $b > 0$ such that for every lift \tilde{q} of q to $\mathcal{Q}^1(S)$ the map $t \rightarrow \Upsilon_{\mathcal{T}}(P\Phi^t \tilde{q})$ is a biinfinite b -quasi-geodesic in $\mathcal{C}(S)$ [H10a]. Thus by inequality (2) and continuity of Φ^t we can find an open neighborhood $U \subset \mathcal{Q}$ of q and a number $T > 0$ such that

$$\tilde{\beta}(u, T) \geq 3r + 3a \text{ for all } u \in U.$$

Now if $z \in \mathcal{Q}$ and if $n > k > 0$ are such that the cardinality of the set of all numbers $i \leq n$ with $\Phi^{Ti} z \in U$ is not smaller than k then $\tilde{\beta}(z, nT) \geq kr$.

The measure λ is Φ^T -invariant and ergodic, and $\lambda(U) > 0$. Thus by the Birkhoff ergodic theorem, the proportion of time a typical orbit for the map Φ^T spends in U is positive. The lemma follows. \square

The next proposition is the main remaining step in the proof of the theorem from the introduction.

Proposition 6.3. *For every recurrent point $q \in \mathcal{S}$, for every neighborhood V of q in \mathcal{S} and for every $\epsilon > 0$ there is an open neighborhood $U \subset V$ of q in \mathcal{S} and a number $t_0 > 0$ such that*

$$\liminf_{R \rightarrow \infty} H(U, R - t_0 - \epsilon, R + t_0 + \epsilon) e^{-hR} \geq 2t_0 \lambda(U)(1 - \epsilon).$$

Proof. Let $q \in \mathcal{S}$ be recurrent and let V be an open neighborhood of q which satisfies the conclusion of Lemma 6.1 for some $n_0 > 0$. Let $\epsilon > 0$. With the notations from Section 4, let $a_0 < a_{\mathcal{Q}}(q), t_0 < \min\{t_{\mathcal{Q}}(q), \log(1+\epsilon)/2h, \epsilon/4\}$ be such that $V_0 = V(B_{\mathcal{Q}}^{ss}(q, a_0), B_{\mathcal{Q}}^{su}(q, a_0), t_0) \subset V$. Choose a number $a_1 < a_0$ which is

sufficiently small that for every $z \in V_1 = V(B_{\mathcal{Q}}^{ss}(q, a_1), B_{\mathcal{Q}}^{su}(q, a_1), t_0)$ the Jacobian at z of the homeomorphism

$$V(B_{\mathcal{Q}}^{ss}(q, a_1), B_{\mathcal{Q}}^{su}(q, a_1), t_0) \rightarrow B_{\mathcal{Q}}^{ss}(q, a_1) \times B_{\mathcal{Q}}^{su}(q, a_1) \times [-t_0, t_0]$$

with respect to the measures λ and $\lambda^{ss} \times \lambda^{su} \times dt$ is contained in the interval $[(1 + \epsilon)^{-1}, (1 + \epsilon)]$. We may assume that any two points in a component \tilde{V}_1 of the preimage of V_1 can be connected in \tilde{V}_1 by a smooth curve whose projection to $\mathcal{T}(S)$ is of length at most $\epsilon/2$.

Let $\alpha_0 > 0$ be as in Lemma 5.2. Let \tilde{q} be a lift of q to a component $\tilde{\mathcal{Q}}$ of the preimage of \mathcal{Q} in $\mathcal{Q}^1(S)$. Recall from Section 2 the definition of the map $F : \tilde{\mathcal{A}} \rightarrow \partial\mathcal{C}(S)$. Since q is recurrent, the horizontal and the vertical measured geodesic laminations of \tilde{q} are uniquely ergodic [M82]. Let

$$Z_1 \subset Z_2 \subset Z_3 \subset V_1$$

be neighborhoods of q as in the proof of Lemma 5.6 and let $\tilde{Z}_1 \subset \tilde{Z}_2 \subset \tilde{Z}_3 \subset \tilde{V}_1$ be components of lifts of $Z_1 \subset Z_2 \subset Z_3 \subset V_1$ to $\tilde{\mathcal{Q}}$ which contain \tilde{q} . These sets have the following property.

- (1) There are closed sets $K^i \subset C^i \subset W_{\mathcal{Q}}^i(q)$ with dense interior ($i = ss, su$) and there are numbers $t_0 > 0, \delta > 0$ such $Z_1 = V(K^{ss}, K^{su}, t_0), Z_2 = V(K^{ss}, C^{su}, t_0), Z_3 = V(C^{ss}, C^{su}, t_0(1 + \delta))$.
- (2) For every $u \in \tilde{Z}_3$ the δ_{P_u} -diameter of $F(\tilde{Z}_3 \cap \tilde{\mathcal{A}})$ and of $F(\mathcal{F}\tilde{Z}_3 \cap \tilde{\mathcal{A}})$ is not bigger than α_0 .
- (3) $\lambda(Z_3) \leq \lambda(Z_1)(1 + \epsilon)$.
- (4) There is a number $\rho > 0$ with the following property. If $z \in \tilde{Z}_1$ and if $C \subset B_{\mathcal{Q}}^{su}(z, a_1)$ (or $C \subset B_{\mathcal{Q}}^{ss}(z, a_1)$) is an open neighborhood of z such that the δ_{P_z} -diameter of $F(C \cap \tilde{\mathcal{A}})$ (or of $F(\mathcal{F}(C) \cap \tilde{\mathcal{A}})$) is not bigger than ρ then $C \subset \tilde{Z}_3$ and the Φ^t -orbit of every point of C intersects \tilde{Z}_3 in an arc of length $2t_0$.

Let $\Pi : \tilde{\mathcal{Q}} \rightarrow \mathcal{Q}$ be the canonical projection. By Lemma 6.2 and Lemma 5.2, there is a number $T > 0$ and there is a Borel subset $Z_0 \subset Z_1 \cap \Pi(\tilde{\mathcal{A}})$ with

$$\lambda(Z_0) > \lambda(Z_1)/(1 + \epsilon)$$

such that for every $z \in \tilde{Z}_0 = \tilde{Z}_1 \cap \Pi^{-1}(Z_0)$ and every $t \geq T$ we have

$$\delta_{P_z} \leq \rho \delta_{P_{\Phi^t z}} / e^{\kappa} \text{ on } D(\Phi^t z, \alpha_0)$$

where $\kappa > 0$ is as in the estimate (4). We may assume that $Z_0 = V(A_0, K^{su}, t_0)$ for some Borel set $A_0 \subset K^{ss}$. In particular, we conclude as in the proof of Lemma 5.6 (see the estimate (21) that (with some a-priori adjustment of the constant ϵ) the following holds true. Let $z \in Z_0$ and let $t \geq T$ be such that $\Phi^t z \in Z_1$. Let \hat{E} be the connected component containing $\Phi^t z$ of the intersection $\Phi^t V_1 \cap V_1$. Then the Lebesgue measure of the intersection $\Phi^t Z_2 \cap Z_3 \cap \hat{E}$ is not bigger than

$$e^{-ht} \lambda(Z_1)(1 + \epsilon)^3 \leq e^{-ht} \lambda(Z_0)(1 + \epsilon)^4.$$

On the other hand, since the Lebesgue measure is mixing, for sufficiently large $t > T$ we have

$$\lambda(\Phi^t Z_0 \cap Z_0) \geq \lambda(Z_0)^2 / (1 + \epsilon).$$

Together this implies that the number of such intersection components is at least

$$e^{ht}\lambda(Z_0)/(1+\epsilon)^5.$$

Next we claim that for sufficiently large $n \geq T$ and for a point $z \in Z_0$ with $\Phi^n z \in Z_1$ there is a periodic orbit for the flow Φ^t which intersects Z_3 in an arc of length at least $2t_0$ and whose period is contained in the interval $[n - \epsilon, n + \epsilon]$. To this end let $n_1 > \max\{n_0, T\}$; then the conclusion of Lemma 6.1 is satisfied for every periodic (Z_1, n_1) -pseudo-orbit beginning at a point $z \in Z_0 \subset V$.

Let $u \in Z_0$ be such that $\Phi^n u \in Z_1$ for some $n > n_1$. Let γ be a characteristic curve of the periodic (Z_1, n_1) -pseudo-orbit (u, n) which we obtain by connecting $\Phi^n u \in Z_0$ with $u \in Z_0$ by a smooth arc contained in Z_1 . Up to replacing n by $R = n + \tau$ for some $\tau \in [-2t_0, 2t_0] \subset [\epsilon/2, \epsilon/2]$ we may assume that $u \in V(K^{ss}, K^{su})$, $\Phi^R u \in V(K^{ss}, K^{su})$.

Let $\tilde{\gamma}$ be a lift of γ to \tilde{Q} with starting point $\tilde{\gamma}(0) \in \tilde{Z}_0$. Then $\tilde{\gamma}$ is invariant under a mapping class $g \in \text{Mod}(S)$ whose conjugacy class defines the homotopy class of γ in \mathcal{S} . A fundamental domain for the action of g on $\tilde{\gamma}$ projects to a smooth arc in $\mathcal{T}(S)$ of length at most $R + \epsilon/2 < n + \epsilon$.

By Lemma 6.1 and the choice of Z_0, R the curve $t \rightarrow \Upsilon_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an unparametrized L -quasi-geodesic in $\mathcal{C}(S)$ of infinite diameter. Up to perhaps a uniformly bounded modification, this quasi-geodesic is invariant under the mapping class $g \in \text{Mod}(S)$, and g acts on the quasi-geodesic $\Upsilon_{\mathcal{T}}(P\tilde{\gamma})$ as a translation. As a consequence, g acts on $\mathcal{C}(S)$ with unbounded orbits and hence it is pseudo-Anosov. By invariance of $\tilde{\gamma}$ under g , the attracting fixed point of g is just the endpoint of $\Upsilon_{\mathcal{T}}(P\tilde{\gamma})$ in $\partial\mathcal{C}(S)$.

Since g is pseudo-Anosov, there is a closed orbit ζ for Φ^t on $\mathcal{Q}(S)$ which is the projection of a g -invariant flow line $\tilde{\zeta}$ for Φ^t in $\mathcal{Q}^1(S)$. The length of the orbit is at most $R + \epsilon$. The image under the map $\Upsilon_{\mathcal{T}}P$ of the orbit $\tilde{\zeta}$ in $\mathcal{Q}^1(S)$ is an unparametrized p -quasi-geodesic in $\mathcal{C}(S)$ which connects the two fixed points for the action of g on $\partial\mathcal{C}(S)$.

Assume that the characteristic curve γ is parametrized on $[0, R+1]$ with $\gamma(0) = u$. As in the proof of Theorem 4.3 of [H10b], we claim that for every $i > 0$ we have

$$\delta_{P\tilde{\gamma}(0)}(F(\tilde{\gamma}(0)), F(\tilde{\gamma}(iR+i))) < \rho$$

(note that this makes sense since the points $\tilde{\gamma}(iR+i)$ are lifts of recurrent points in $\mathcal{Q}(S)$ by assumption). To see this we proceed by induction on i . The case $i = 1$ follows from the definition and from (4) above, so assume that the claim is known for all $j \leq i - 1$ and some $i \geq 0$. By equivariance under the action of the mapping class group we have

$$(27) \quad \delta_{P\tilde{\gamma}(R+1)}(F\tilde{\gamma}(R), F\tilde{\gamma}(R+1)) \leq e^\kappa \rho,$$

moreover the distances $\delta_{P\tilde{\gamma}(R)}, \delta_{P\tilde{\gamma}(R+1)}$ are e^κ -bilipschitz equivalent.

Now $F(\tilde{\gamma}(jR+j)) \in B_{\tilde{\gamma}(R+1)}(F(\tilde{\gamma}(R+1)), \rho)$ for all $j \in \{1, \dots, i\}$ by the induction hypothesis and therefore

$$(28) \quad \delta_{P\tilde{\gamma}(R)}(F(\tilde{\gamma}(R)), F(\tilde{\gamma}(jR+j))) \leq 2e^\kappa \rho.$$

On the other hand, by the choice of ρ and the choice of R and the fact that $\tilde{\gamma}(0) \in \tilde{Z}_0$ we obtain that

$$(29) \quad \delta_{P\tilde{\gamma}(R)}(F\tilde{\gamma}(R), F\tilde{\gamma}(jR+j)) \geq \delta_{P\tilde{\gamma}(0)}(F\tilde{\gamma}(0), F\tilde{\gamma}(jR+j))/2e^\kappa.$$

Together this implies the above claim.

As a consequence, the attracting fixed point ξ for the action of the pseudo-Anosov element g on $\partial\mathcal{C}(S)$ is contained in the ball $D(\tilde{\gamma}(0), \rho)$, moreover it is contained in the closure of the set $F(W_{\tilde{\mathcal{Q}}}^{su}(\tilde{q}) \cap \tilde{\mathcal{A}}) \subset F(\tilde{\mathcal{A}} \cap \tilde{\mathcal{Q}})$. The same argument also shows that the repelling fixed point of g is contained in the intersection of $D(-\tilde{\gamma}(0), \rho)$ with the closure of $F(\mathcal{F}W_{\tilde{\mathcal{Q}}}^{ss}(\tilde{q}) \cap \tilde{\mathcal{A}}) \subset F(\tilde{\mathcal{A}} \cap \tilde{\mathcal{Q}})$. Since the map F is closed we conclude that the axis of g is contained in the closure of $\tilde{\mathcal{Q}}$. Since $\tilde{\gamma}(0) \in Z_1$, by property 4) above, this axis passes through the lift \tilde{Z}_3 of Z_3 containing \tilde{q} . In other words, the projection of this axis to $\overline{\mathcal{Q}}$ passes through Z_3 , and, in particular, it is contained in \mathcal{Q} . Moreover, it intersects the component of $\Phi^R Z_1 \cap Z_3$ which contains $\Phi^R u$. As a consequence, the length of the axis is contained in $[R - \epsilon/2, R + \epsilon/2] \subset [n - \epsilon, n + \epsilon]$.

To summarize, there is an injective assignment which associates to every $R > n_0$ and to every connected component of the intersection $\Phi^R Z_1 \cap Z_1$ for $R > n_0 > T$ which contains points in $\Phi^R Z_0 \cap Z_0$ a subarc of length $2t_0$ of the intersection with Z_3 of a periodic orbit for Φ^t whose period is contained in $[n - \epsilon, n + \epsilon]$. Together with the above discussion, this completes the proof of the proposition. \square

We use Proposition 6.3 to complete the proof of our theorem from the introduction.

Theorem 6.4. *The Lebesgue measure on every stratum \mathcal{Q} is obtained from Bowen's construction.*

Proof. By Proposition 5.1 and Proposition 6.3, it suffices to show the following. Let $q \in \mathcal{Q}$ be birecurrent and let $\epsilon > 0$. For $R > 0$ let $\Gamma(R)$ be the set of all periodic orbits of Φ^t in \mathcal{Q} of period at most R . Then there is a compact neighborhood K of q in \mathcal{Q} and there is a number $n > 0$ such that for every $N > n$ the measure

$$\mu_N = e^{-hN} \sum_{\gamma \in \Gamma(R)} \delta(\gamma)$$

assigns the mass

$$\mu_N(K) \in [(1 - \epsilon)\lambda(K), (1 + \epsilon)\lambda(K)]$$

to K . However, this holds true by Proposition 5.1 and Proposition 6.3. This completes the proof of the theorem. \square

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