INVARIANT MEASURES FOR THE TEICHMÜLLER FLOW

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To the memory of Martine Babillot

Abstract. Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g-3+m \geq 2$. The Teichmüller flow Φ^t acts on the moduli space $\mathcal{Q}(S)$ of area one holomorphic quadratic differentials leaving invariant a collection of so-called strata. Each such stratum is defined by a sequence $(k_1, \dots, k_\ell)$ of positive integers with $\sum_i k_i = 4g-4+m$. We show that the Φ^t-invariant Lebesgue measure on each connected component $\mathcal{Q}$ of a stratum defined by a sequence of odd integers is the unique measure of maximal entropy. We also construct an uncountable family of Φ^t-invariant probability measures on $\mathcal{Q}$ containing the usual Lebesgue measure which are mixing, absolutely continuous with respect to the stable and unstable foliation and exponentially recurrent to a compact set.

1. Introduction

Let S be an oriented surface of finite type, i.e. S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g-3+m \geq 2$, i.e. that S is not a sphere with at most 4 punctures or a torus with at most 1 puncture. We then call the surface S *nonexceptional*. Since the Euler characteristic of S is negative, the *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all hyperbolic metrics on S under the action of the group of diffeomorphisms of S which are isotopic to the identity. The fibre bundle $\mathcal{Q}^1(S)$ over $\mathcal{T}(S)$ of all *holomorphic quadratic differentials* of area one can naturally be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the *Teichmüller metric*. If the surface S has punctures, i.e. if $m > 0$, then we mean by a holomorphic quadratic differential on S a meromorphic quadratic differential on the closed Riemann surface obtained from S by filling in the punctures which has a simple pole at each of the punctures.

The *moduli space of area one quadratic differentials* is the quotient $\mathcal{Q}(S)$ of $\mathcal{Q}^1$ under the natural action of the *mapping class group* $\mathcal{M}(S)$ of all isotopy classes of orientation preserving self-homeomorphisms of S. The space $\mathcal{Q}(S)$ can be partitioned into so-called *strata*. Namely, let $1 \leq k_1 \leq \dots \leq k_\ell$ $(\ell \geq 1)$ be a sequence of positive integers with $\sum_i k_i = 4g-4+m$. The stratum $\mathcal{Q}(k_1, \dots, k_\ell)$ defined by the ℓ-tuple $(k_1, \dots, k_\ell)$ is the moduli space of pairs (C, φ) where C is a Riemann surface of genus g with m punctures and where φ is a holomorphic quadratic differential on C with ℓ zeros of order k_i. We call a stratum *odd* if each of the

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integers k_i defining the stratum is odd. A stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ is a real hypersurface in a complex algebraic orbifold of complex dimension $2g+\ell+m-2$ which is invariant under the Teichmüller flow Φ^t. Masur and Smillie [MS93] showed that the stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ is non-empty unless $\ell=2, k_1=1, k_2=3$ and S is a closed surface of genus 2. The strata need not be connected, however they have at most three connected components [L06, KZ03]. The open stratum $\mathcal{Q}(1,\dots,1)$ is always connected. The closure in $\mathcal{Q}(S)$ of a stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ is a union of strata $\mathcal{Q}(n_1,\dots,n_s)$ where $s\leq \ell$.

The *Teichmüller flow* Φ^t acts on $\mathcal{Q}(S)$ preserving the strata. For a component $\mathcal{Q}$ of a stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ let $\mathcal{M}_{\rm inv}(\mathcal{Q})$ the set of all Φ^t-invariant Borel probability measures on $\mathcal{Q}$. Denote by h_ν the *entropy* of a measure $\nu\in\mathcal{M}_{\rm inv}(\mathcal{Q})$. Masur and Veech [M82, V86] showed that there is a natural measure $\lambda\in\mathcal{M}_{\rm inv}(\mathcal{Q})$ in the Lebesgue measure class. This measure is ergodic, mixing [M82, V86] and exponentially recurrent to a compact set [A06]. Morover, its entropy h_λ coincides with the complex dimension $2g+m+\ell-2$ of the complex orbifold defining the stratum (note that we use a normalization for the Teichmüller flow which is different from the one used by Masur and Veech). In particular, the entropy of the Lebesgue measure on the open connected stratum $\mathcal{Q}(1,\dots,1)$ equals $6g-6+2m$.

Define

$$h_{\rm top}(\mathcal{Q})=\sup\{h_\nu \mid \nu\in\mathcal{M}_{\rm inv}(\mathcal{Q})\}.$$

A *measure of maximal entropy* for the component $\mathcal{Q}$ is a measure $\mu\in\mathcal{M}_{\rm inv}(\mathcal{Q})$ such that $h_\mu=h_{\rm top}(\mathcal{Q})$. A priori, such a measure need not exist. In [H07b] we showed that the *topological* entropy $h_{\rm top}(K)$ of the restriction of Φ^t to any compact invariant set $K\subset\mathcal{Q}(S)$ does not exceed $6g-6+2m$, and

$$6g-6+2m=\sup\{h_{\rm top}(K)\mid K\subset\mathcal{Q}(S) \text{ compact } \Phi^t-\text{invariant}\}.$$

In particular, $6g-6+2m$ is the supremum of the entropies of all Φ^t-invariant Borel probability measures on $\mathcal{Q}(S)$ which are supported in a compact subset of $\mathcal{Q}(S)$.

If the surface S is closed, i.e. if $m=0$, then the moduli space $\mathcal{A}(S)$ of squares of *abelian* differentials is a subspace of $\mathcal{Q}(S)$. This moduli space is a union of strata contained in the strata $\mathcal{Q}(k_1,\dots,k_\ell)$ for which all the numbers k_i are even. In this case much more is known. Avila, Gouëzel and Yoccoz [AGY06] established that the Lebesgue measure λ_0 on each component of a stratum is exponentially mixing, i.e. exponential decay of correlations for Hölder observables holds. Recently Bufetov and Gurevich [BG07] showed that the Φ^t-invariant probability measure in the Lebesgue measure class is the unique measure of maximal entropy for the component. Both results rely on the fact that the Teichmüller flow on a stratum of abelian differentials can be represented as a suspension over a Markov shift with countably many symbols, with a roof function of bounded variation which is moreover bounded from below by a positive constant. Such a structural result is not available at the moment for quadratic differentials (see however [BL07]).

In this note we establish an analog of the result of Bufetov and Gurevich for the Teichmüller flow on odd strata of the moduli space of quadratic differentials. We show.

Theorem 1. *For every component of an odd stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ the Φ^t-invariant probability measure in the Lebesgue measure class is the unique measure of maximal entropy.*

We conjecture that Theorem 1 holds true for every component of any stratum of $\mathcal{Q}(S)$.

For the proof of the above theorem we construct for every component $\mathcal{Q}$ of an odd stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ a subshift of finite type (Ω,T) and a Borel suspension X over (Ω,T) which admits a finite-to-one semi-conjugacy Ξ into the Teichmüller flow on $\mathcal{Q}$. Here a Borel suspension is a suspension with a bounded continuous roof function defined on a shift invariant Borel subset of (Ω,T). We show that every Φ^t-invariant Borel probability measure on $\mathcal{Q}$ is the image under Ξ of a shift invariant Borel measure on our subshift of finite type. Moreover, every Gibbs equilibrium state for a Hölder continuous function on (Ω,T) induces via the map Ξ a Φ^t-invariant Borel probability measure on $\mathcal{Q}$ which we call a *Bernoulli measure*.

To describe the properties of these Bernoulli measures, recall that every area one quadratic differential $q\in\mathcal{Q}^1(S)$ determines a pair (q_h,q_v) of *measured geodesic laminations*. The measured geodesic lamination q_h is called *horizontal* and q_v is called *vertical*. For every $t\in\mathbb{R}$ the pair $(e^tq_h,e^{-t}q_v)$ corresponds to the quadratic differential Φ^tq. For a quadratic differential $q\in\mathcal{Q}^1(S)$ define the *unstable manifold* $W^u(q)\subset\mathcal{Q}^1(S)$ to be the set of all quadratic differentials whose vertical measured geodesic lamination is a multiple of the vertical measured geodesic lamination for q. Then $W^u(q)$ is a submanifold of $\mathcal{Q}^1(S)$ which projects homeomorphically onto $\mathcal{T}(S)$ [HM79]. Similarly, define the *strong stable manifold* $W^{ss}(q)$ to be the set of all quadratic differentials whose horizontal measured geodesic lamination coincides with the horizontal measured geodesic lamination of q. The sets $W^u(q)$ (or $W^{ss}(q)$) $(q\in\mathcal{Q}^1(S))$ define a foliation of $\mathcal{Q}^1(S)$ which is invariant under the mapping class group and hence projects to a singular foliation on $\mathcal{Q}(S)$ which we call the *unstable foliation* (or the *strong stable foliation*). These foliations intersect each of the strata $\mathcal{Q}(k_1,\dots,k_\ell)$ in a foliation which we call again the unstable (or strong stable) foliation of the stratum.

We summarize the properties of Bernoulli measures on a component $\mathcal{Q}$ of the odd stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ as follows.

Theorem 2. *There is an uncountable family of Φ^t-invariant probability measures on $\mathcal{Q}$ including the Lebesgue measure which are mixing, absolutely continuous with respect to the strong stable and the unstable foliation and which moreover are exponentially recurrent to a compact set.*

The organization of the paper is as follows. In Section 2 we review some properties of train tracks and geodesic laminations needed in the sequel. In Section 3 we use train tracks to construct for every connected component $\mathcal{Q}$ of an odd stratum a special subshift of finite type (Ω,T). In Section 4 we define a bounded roof function ρ for our subshift of finite type and obtain a semi-conjugacy of the suspension of (Ω,T) with roof function ρ into the Teichmüller flow. This is used in Section 5 to show Theorem 3. In Section 6 we use the results of the earlier sections to show that the Lebesgue measure on $\mathcal{Q}$ is a measure of maximal entropy. Uniqueness of such a measure is established in Section 7.

2. Train tracks and geodesic laminations

In this section we summarize some results and constructions from [T79, PH92, H06b] which will be used throughout the paper (compare also [Mo03]).

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and where $3g - 3 + m \geq 2$. A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination λ is called *minimal* if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*. Every geodesic lamination λ consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of λ either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components [CEG87].

A geodesic lamination λ on S is said to *fill up* S if its complementary regions are all topological discs or once punctured monogons (note that the definition of filling laminations by other authors require only that the complementary regions of λ are topological discs or once punctured topological discs). A *maximal* geodesic lamination is a geodesic lamination whose complementary regions are all ideal triangles or once punctured monogons.

Definition 2.1. A geodesic lamination λ is called *web* if λ fills up S and if moreover λ can be approximated in the *Hausdorff topology* by simple closed geodesics. A web maximal geodesic lamination is called *complete*.

Since every minimal geodesic lamination can be approximated in the Hausdorff topology by simple closed geodesics [CEG87], a minimal geodesic lamination which fills up S is web.

Let λ be a web geodesic lamination. Then every complementary component of λ is a hyperbolic surface P with geodesic boundary. The Euler characteristic of P is defined to be half the Euler characteristic of the double of P glued along its boundary geodesics. With this convention, the Euler characteristic of an ideal triangle is $-1/2$. Assigning to each complementary region of λ without puncture twice the absolute value of its Euler characteristic determines a sequence of positive numbers $1 \leq k_1 \leq \cdots \leq k_\ell$ with $\sum_i k_i = 4g - 4 + m$. In this way the set of all geodesic networks on S is partitioned into disjoint subsets $\mathcal{L}(k_1, \ldots, k_\ell)$. A web geodesic lamination contained in $\mathcal{L}(k_1, \ldots, k_\ell)$ is called *of topological type* $(k_1, \ldots, k_\ell)$. A complete geodesic lamination is of topological type $(1, \ldots, 1)$.

We call a web geodesic lamination $\lambda \in \mathcal{L}(k_1, \ldots, k_\ell)$ *special* if λ contains a single minimal component which is a simple closed curve. Note that by definition, the set of all special laminations $\lambda \in \mathcal{L}(k_1, \ldots, k_\ell)$ is invariant under the action of the mapping class group and decomposes into finitely many orbits for this action.

Special geodesic laminations $\lambda \in \mathcal{L}(1,\dots,1)$ can be constructed as follows. Choose a simple closed geodesic c on S, a point $x \in c$ and a triangulation T of S with a single vertex x which contains $c - x$ as an edge. Let φ be a Dehn twist about c. For each $k > 0$ let T_k be the triangulation of S with the single vertex x obtained from $\varphi^k T$ by replacing the edges by geodesic arcs in the same homotopy class. As $k \to \infty$, the graphs T_k converge in the Hausdorff topology to a special lamination (see [T79]).

A *measured geodesic lamination* is a geodesic lamination λ together with a translation invariant transverse measure. Such a measure assigns a positive weight to each compact arc in S with endpoints in the complementary regions of λ which intersects λ nontrivially and transversely. The geodesic lamination λ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. The space $\mathcal{ML}$ of all measured geodesic laminations on S equipped with the weak*-topology is homeomorphic to $S^{6g-7+2m} \times (0,\infty)$. Its projectivization is the space $\mathcal{PML}$ of all *projective measured geodesic laminations*. There is a continuous symmetric pairing $i : \mathcal{ML} \times \mathcal{ML} \to [0,\infty)$, the so-called *intersection form*, which extends the geometric intersection number between simple closed curves. The measured geodesic lamination $\mu \in \mathcal{ML}$ *fills up* S if its support fills up S. This support is then necessarily connected and hence minimal, and it defines a point in one of the sets $\mathcal{L}(k_1,\dots,k_\ell)$ for some tuple $(k_1,\dots,k_\ell)$. The projectivization of a measured geodesic lamination which fills up S is also said to fill up S. We call $\mu \in \mathcal{ML}$ *strongly uniquely ergodic* if the support of μ fills up S and admits a unique transverse measure up to scale.

A *large generic train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Every switch is trivalent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. In particular, the branches which are incident on a fixed switch are divided into "incoming" and "outgoing" branches according to their inward pointing tangent at the switch. The complementary regions of the train track are all polygons, i.e. discs with at least three cusps at the boundary, or once punctured monogons, i.e. once punctured discs with one cusp at the boundary. We always identify train tracks which are isotopic (see [PH92] for a comprehensive account on train tracks). If all the complementary regions of τ are trigons or once punctured monogons then τ is called *maximal*.

A large generic train track or a geodesic lamination σ is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is homotopic to the identity and maps σ into τ in such a way that the restriction of the differential of F to the tangent space of σ vanishes nowhere; note that this makes sense since a train track has a tangent line everywhere. We call the restriction of F to σ a *carrying map* for σ. Write $\sigma \prec \tau$ if the train track or the geodesic lamination σ is carried by the train track τ.

A *transverse measure* on a large generic train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: For every switch s of τ, the sum of the weights over all incoming branches at s is required to coincide with the sum of the weights over all outgoing branches at s. The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. We call such a transverse measure *positive*. The space $\mathcal{V}(\tau)$ of all transverse measures on τ has the structure of an euclidean cone. Via a carrying map, a measured geodesic lamination carried by τ defines a transverse measure on τ, and every transverse measure arises in this way [PH92]. Thus $\mathcal{V}(\tau)$ can naturally be identified with a subset of $\mathcal{ML}$ which is invariant under scaling.

Following [P88] we define a *tangential measure* μ for a large generic train track τ to be an assignment of a nonnegative weight $\mu(b)$ to every branch b of τ such that for every complementary polygon with consecutive sides $c_1, \dots, c_s$ the following is satisfied.

(1) $\mu(c_i) \leq \mu(c_{i-1}) + \mu(c_{i+1})$.
(2) $\sum_{i=j}^{s+j-1} (-1)^{i-j} \mu(c_i) \geq 0$ for $j = 1, \dots, s$.

We refer to [P88] for a precise definition which takes into account that sides of complementary polygons may have self-intersections. For a complementary triangle these two requirements just mean that the mass of one of the sides does not exceed the sum of the masses of the other two. Note however that Penner calls such a tangential measure a metric. The space $\mathcal{V}^*(\tau)$ of all tangential measures on τ has the structure of an euclidean cone. The large generic train track τ is called *transversely recurrent* if it admits a tangential measure μ which is positive on every branch [P88]. We call such a tangential measure *positive*.

Definition 2.2. A large generic train track τ is called *web* if it is recurrent and transversely recurrent. A web maximal train track is called *complete*.

The set of large train tracks decomposes into *topological types* according to the shapes of the complementary polygons. This topological type is given by a sequence of positive integers $k_1 \leq \dots \leq k_\ell$ ($\ell \geq 1$) with $\sum_i k_i = 4g - 4 + m$ where ℓ is the number of complementary regions and where the complementary regions are $k_i + 2$-gons or once punctured monogons. Denote by $\mathcal{T}(k_1, \dots, k_\ell)$ the set of all *web* train tracks whose combinatorial type is given by the ℓ-tuple $(k_1, \dots, k_\ell)$. The sets $\mathcal{T}(k_1, \dots, k_\ell)$ define a partition of the collection of all web train tracks which is invariant under the natural action of the mapping class group. We call a web train train $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$ *odd* if k_i is odd for all i.

Define a *thickening* of a full geodesic lamination $\lambda \in \mathcal{L}(k_1, \dots, k_\ell)$ to be an ϵ-neighborhood $N_\epsilon(\lambda)$ of λ in the surface S (equipped with a marked hyperbolic metric for which λ is geodesic) where $\epsilon > 0$ is small enough that every complementary component of λ contains a complementary component of $N_\epsilon(\lambda)$. If $\epsilon > 0$ is sufficiently small then $N_\epsilon(\lambda)$ can be foliated by compact arcs which are transverse to λ. The singular leaves of the foliation are leaves through the corners of $S - N_\epsilon(\lambda)$. Collapsing the leaves of this foliation to a single point yields a transversely recurrent train track τ whose complementary regions are in one-to-one correspondence with the complementary regions of λ and which carries λ. For a suitable choice of ϵ, the train track is moreover generic [PH92]. We call τ a *collapse of a λ-thickening.*

A transversely recurrent train track η which carries a web geodesic lamination λ also carries every simple closed curve which is sufficiently close to λ in the Hausdorff topology. Thus if a carrying map $\lambda \to \tau$ is surjective then since λ can be approximated in the Hausdorff topology by simple closed curves, τ carries a simple closed curve c with the property that a carrying map $c \to \tau$ is surjective. In particular, the counting measure on τ defined by c is a *positive* transverse measure on τ. Therefore τ is full. This shows the following.

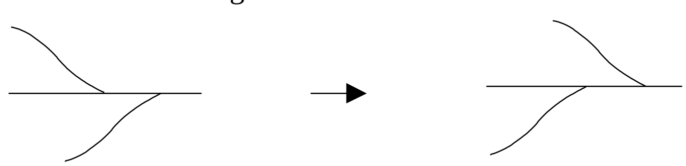
Lemma 2.3. (1) *For every $\lambda \in \mathcal{L}(k_1, \dots, k_\ell)$ a collapse of a λ-thickening is a web train track $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$ which carries λ.*
(2) *A transversely recurrent large train track τ of topological type $(k_1, \dots, k_\ell)$ is web if and only if it carries a web geodesic lamination $\lambda \in \mathcal{L}(k_1, \dots, k_\ell)$.*

Proof. Let τ be a large train track of topological type $(k_1, \dots, k_\ell)$. If τ carries a geodesic network $\lambda \in \mathcal{L}(k_1, \dots, k_\ell)$ then a carrying map $\lambda \to \tau$ is necessarily surjective and hence $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$ is web by our above consideration. Thus all what remains to be shown is that a web train track $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$ carries a web geodesic lamination $\lambda \in \mathcal{L}(k_1, \dots, k_\ell)$. However, this follows as in [PH92]. Namely, τ is recurrent and hence the dimension of the space of all transverse measures carried by τ equals $2g + \ell + m - 2$. Since this space is a cone in the PL-manifold of all measured geodesic laminations and since the set of all minimal geodesic laminations which are carried by τ and which are not contained in $\mathcal{L}(k_1, \dots, k_\ell)$ is contained in a countable union of affine subspaces of smaller dimension (see [PH92]), the train track τ carries a minimal lamination $\nu \in \mathcal{L}(k_1, \dots, k_\ell)$. This shows the lemma. □

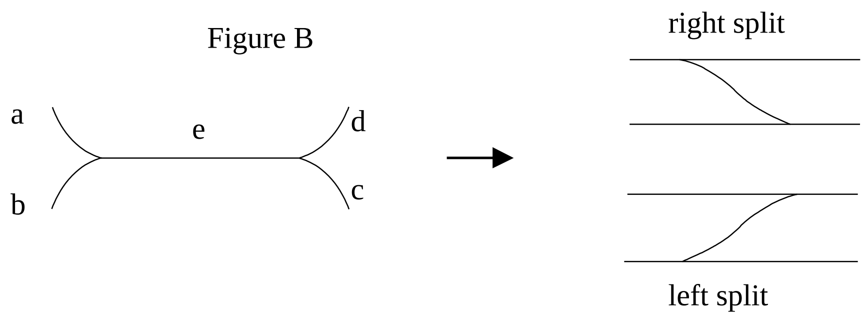
A half-branch $\hat{b}$ in a web train track τ incident on a switch v of τ is called *large* if every arc of class C^1 which is embedded in τ and contains v in its interior passes through $\hat{b}$. A half-branch which is not large is called *small*. A branch b in a web train track τ is called *large* if each of its two half-branches is large; in this case b is necessarily incident on two distinct switches, and it is large at both of them. A branch b is called *small* if each of its two half-branches is small. A branch is called *mixed* if one of its half-branches is large and the other half-branch is small (for all this, see [PH92] p.118).

There are two simple ways to modify a web train track τ to another web train track. First, we can *shift* τ along a mixed branch to a train track τ' as shown in Figure A below. If τ is web then the same is true for τ'. Moreover, a train track or a lamination is carried by τ if and only if it is carried by τ' (see [PH92] p.119). In particular, the shift τ' of τ is carried by τ. Note that there is a natural bijection of the set of branches of τ onto the set of branches of τ'.

Figure A



Second, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure B. Note that a right split at e is uniquely determined by the orientation of S and does not depend on the orientation of e. Using the labels in the figure, in the case of a right split we call the branches a and c *winners* of the split, and the branches b, d are *losers* of the split. If we perform a left split, then the branches b, d are winners of the split, and the branches a, c are losers of the split. The split τ' of a train track τ is carried by τ, and there is a natural choice of a carrying map which maps the switches of τ' to the switches of τ. This carrying map then induces a bijection of the set of branches of τ onto the set of branches of τ' which maps the branch e to the diagonal e' of the split. The split of a full train track is large, transversely recurrent and generic, but it may not be recurrent. However, for every full train track $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$ and every full geodesic lamination $\lambda \in \mathcal{L}(k_1, \dots, k_\ell)$ carried by τ there is a unique choice of a right or left split of τ such that the split track τ' carries λ, and then $\tau' \in \mathcal{T}(k_1, \dots, k_\ell)$, in particular τ' is full.



If $\nu \in \mathcal{L}(k_1, \dots, k_\ell)$ is a special lamination, then the collapse of a thickening of ν is a train track $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$ which contains the minimal component of ν as a simple closed embedded curve and such that up to shift, every switch of τ is contained in c. We call such a train track *special.* Note that a special train track τ which is a collapse of a thickening of ν carries ν. Moreover, ν is determined by τ. We have.

Lemma 2.4. *Let $\nu \in \mathcal{L}(k_1, \dots, k_\ell)$ be minimal and let τ be the collapse of a thickening of ν. Then there is a special train track η which carries ν and is carried by τ.*

Proof. Let $\nu \in \mathcal{L}(k_1, \dots, k_\ell)$ be minimal and let $\epsilon > 0$ be sufficiently small that the ϵ-neighborhood $N_\epsilon(\nu)$ is foliated into compact arcs, so-called *ties*, which are transverse to ν and intersect ν transversely and can be collapsed to a full train track τ which carries ν. Let $\alpha : [0, \infty) \to N_\epsilon(\nu)$ be a half-leaf of ν which is not a boundary leaf. We assume that $\alpha(0)$ is contained in a tie a which is collapsed to an interior point of a branch of τ. Orient a in such a way that the orientation of α at $\alpha(0)$ together with the orientation of a defines the orientation of S. Then α cuts through a from left to right at $\alpha(0)$.

Let $T > 0$ be the smallest positive number such that $\alpha(T) \in a$. Such a number exists since α is dense in ν. If α cuts through a from left to right at $\alpha(T)$ then connect $\alpha(T)$ to $\alpha(0)$ by the subarc of a with endpoints $\alpha(T), \alpha(0)$. The resulting simple closed curve c is homotopically nontrivial and carried by τ. In the case that α cuts through a from right to left there is a smallest number $\hat{T} > T$ such that $\alpha(\hat{T})$ is contained in the interior of the subarc of a bounded by $\alpha(0)$ and $\alpha(T)$ since α is not a boundary leaf of ν by assumption. If α cuts through a from left to right at $\alpha(\hat{T})$ then connect $\alpha(\hat{T})$ to $\alpha(0)$ by a subarc of a. Note as before that the resulting simple closed curve c is carried by τ. Otherwise connect $\alpha(\hat{T})$ to $\alpha(T)$ by a subarc of a and observe that the resulting curve c is carried by τ.

By possibly replacing α by $\alpha[T,\infty)$ and by renaming the parameters of the endpoints of the subarc of α defining the simple closed curve c we may assume that our curve c is constructed from the subarc $\alpha[0,T]$ of α and a subarc a_0 of a and that α cuts through a from left to right at $\alpha(0)$ and $\alpha(T)$. Via rounding off the corners of c we may assume that c is smooth and contains a_0. We successively construct a special train track η carried by τ by attaching branches to c at switches contained in a_0 as follows. With respect to the orientation of a, define a switch $p \in a_0$ to be a *right* switch if the half-branch incident on p and not contained in c lies to the right of a. Call the switch $p \in a_0$ a *left* switch otherwise. A switch is called *outgoing* if an oriented subarc of a passing through the switch travels through a large half-branch incident on the switch before entering a small half-branch. Call the switch *incoming* otherwise.

Let $T_1 > T$ be the smallest number such that $\alpha(T_1)$ is contained in a_0. If the simple closed curve which is the concatenation of $\alpha[T,T_1]$ with a subarc of a_0 is not freely homotopic to c then we proceed as follows. Let $\gamma \subset N_\epsilon(\nu)$ be an embedded arc which is freely homotopic to $\alpha[T,T_1]$ relative to a_0 and is transverse to the ties. If α cuts through a_0 from left to right at $\alpha(T_1)$ then we choose γ in such a way that its intitial point $\gamma(T)$ lies behind the endpoint $\gamma(T_1)$ along the oriented arc a_0. Connect γ to c at $\gamma(T)$ and $\gamma(T_1)$ with two switches so that a right switch is always outgoing and a left switch is incoming. If the arc is freely homotopic to c then use the same consideration for a subarc of α issuing from $\alpha(T_1)$. After finitely many steps we obtain a special train track η which carries ν, which is contained in $N_\epsilon(\nu)$ and is transverse to the ties. Thus η is carried by τ. This completes the proof of our lemma. □

Equip the space of geodesic laminations with the Hausdorff topology. With this topology, the space of complete geodesic laminations is compact and totally disconnected. For a sequence $(k_1,\dots,k_\ell) \neq (1,\dots,1)$ as above, the space $\mathcal{L}(k_1,\dots,k_\ell)$ is locally compact but non-compact. Namely, by Lemma 2.3, every lamination $\nu \in \mathcal{L}(k_1,\dots,k_\ell)$ is carried by a full train track $\tau \in \mathcal{T}(k_1,\dots,k_\ell)$. Moreover, the space of all laminations $\nu \in \mathcal{L}(k_1,\dots,k_\ell)$ carried by τ is both open and closed in the Hausdorff topology and hence compact [H06b].

Let $\mathcal{Q}$ be a connected component of a stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ defined by the sequence $(k_1,\dots,k_\ell)$. Let $\mathcal{M}_{\rm inv}(\mathcal{Q})$ be the space of all Φ^t-invariant Borel probability measures on $\mathcal{Q}$. The next easy lemma gives some useful information on $\mathcal{M}_{\rm inv}(\mathcal{Q})$. For its formulation, we say that a measured geodesic lamination is contained in $\mathcal{L}(k_1,\dots,k_\ell)$ if this is the case for its support. Note that the support of such a lamination is necessarily minimal.

Lemma 2.5. *Every $\nu \in \mathcal{M}_{\rm inv}(\mathcal{Q})$ gives full measure to the set of quadratic differentials $q \in \mathcal{Q}$ whose horizontal and vertical measured geodesic laminations are contained in $\mathcal{L}(k_1, \dots, k_\ell)$.*

Proof. Since every measure $\nu \in \mathcal{M}_{\rm inv}(\mathcal{Q}$ is a generalized convex combination of ergodic invariant measures its is enough to show the lemma for ergodic measures $\nu \in \mathcal{M}_{\rm inv}(\mathcal{Q}(S))$. Note that such a measure gives full mass to the set of quadratic differentials whose orbits under the Teichmüller flow return to a fixed compact subset of $\mathcal{Q}(S)$ for arbitrarily large positive and negative times and hence it gives full measure to the set of quadratic differentials with strongly uniquely ergodic horizontal and vertical measured geodesic lamination [M82].

Now assume that $q \in \mathcal{Q}$ has strongly uniquely ergodic vertical measured geodesic lamination ν but that $\nu \notin \mathcal{L}(k_1, \dots, k_\ell)$. Then q admits at least one vertical *saddle connection*, i.e. a vertical geodesic segment for the singular euclidean metric defined by $\mathcal{Q}$ connecting two singular points. This saddle connection has finite length. Now the length of a vertical saddle connection is exponentially decreasing under the action of the Teichmüller flow. On the other hand, since ν is an ergodic invariant probability measure on $\mathcal{Q}$, if ν gives full mass to a set of differentials addmitting a vertical saddle connection, then there is some $r > 0$ such that the ν-mass of the set A of points with a vertical saddle connection of minimal length contained in the interval (e^{-r}, e^r) is bigger than 3/4. Now $\Phi^{2r} A \cap A = \emptyset$ which contradicts invariance of ν. This shows the lemma. □

By the results of Masur and Veech [M82, V86], every component $\mathcal{Q}$ of $\mathcal{Q}(k_1, \dots, k_\ell)$ admits a Φ^t-invariant ergodic probability measure in the Lebesgue measure class. The Φ^t-orbit of a typical point q for this measure is dense in $\mathcal{Q}$, and the horizontal measured geodesic lamination of q is minimal and contained in $\mathcal{L}(k_1, \dots, k_\ell)$. We use this fact to show.

Lemma 2.6. *Let $\mathcal{Q}$ be a connected component of a stratum $\mathcal{Q}(k_1, \dots, k_\ell)$ and let $\tilde{\mathcal{Q}}$ be the lift of $\mathcal{Q}$ to $\mathcal{Q}^1(S)$. Then there is a special train track $\eta \in \mathcal{T}(k_1, \dots, k_\ell)$ with the following properties.*

1. *For every $q \in \tilde{\mathcal{Q}}$ without horizontal saddle connection, the horizontal geodesic lamination of q is carried by a train track in the $\mathcal{M}(S)$-orbit of η.*
2. *Let ζ be the special geodesic lamination defined by η. Then for every $q \in \tilde{\mathcal{Q}}$ without horizontal saddle connection, the horizontal geodesic lamination of q is contained in thc closure of the $\mathcal{M}(S)$-orbit of ζ.*

Proof. By the results of Masur and Veech [M82, V86] discussed above, there is some $q \in \mathcal{Q}$ whose orbit under Φ^t is dense in $\mathcal{Q}$ and such that the following holds. Let $\tilde{q}$ be a lift of q to the preimage $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathcal{Q}^1(S)$. Then the horizontal measured geodesic lamination ν of $\tilde{q}$ is minimal with support in $\mathcal{L}(k_1, \dots, k_\ell)$. The orbit of ν under the action of the mapping class group is dense in the closure $\mathcal{A} \subset \mathcal{L}(k_1, \dots, k_\ell)$ of all horizontal measured geodesic laminations of all quadratic differentials $z \in \tilde{\mathcal{Q}}$ without horizontal saddle connection. Or, equivalently, for every $z \in \tilde{\mathcal{Q}}$ with minimal horizontal geodesic lamination of support $\zeta \in \mathcal{L}(k_1, \dots, k_\ell)$ there is a sequence of mapping classes $g_i \in \mathcal{M}(S)$ such that $g_i \nu \to \zeta$.

By Lemma 2.4, there is a sequence of special train tracks $\tau_i \in \mathcal{T}(k_1,\dots,k_\ell)$ $(i \geq 0)$ with $\nu \prec \tau_i \prec \tau_{i-1}$ such that the special geodesic laminations defined by τ_i converge to ν in the Hausdorff topology. Since there are only finitely many $\mathcal{M}(S)$-orbits of special train tracks under the action of the mapping class group we may assume that this sequence of train tracks is contained in a single $\mathcal{M}(S)$-orbit. Together with the above consideration, the second part of the lemma follows. In particular, there is a special geodesic lamination ξ and there is a sequence $h_j \subset \mathcal{M}(S)$ such that $h_j\xi \to \nu$ in the Hausdorff topology.

Now the space of web geodesic lamination $\nu \in \mathcal{L}(k_1,\dots,k_\ell)$ carried by a train track $\tau \in \mathcal{T}(k_1,\dots,k_\ell)$ is open and closed in $\mathcal{L}(k_1,\dots,k_\ell)$. Using the above notations, this implies that $g_i^{-1}\zeta$ is carried by τ_0 for sufficiently large i. This shows the first part of the lemma. □

For a web train track $\tau \in \mathcal{T}(k_1,\dots,k_\ell)$ the interior $\mathrm{int}\mathcal{V}(\tau)$ of the convex cone $\mathcal{V}(\tau)$ (or the interior $\mathrm{int}\mathcal{V}^*(\tau)$ of $\mathcal{V}^*(\tau)$) consists of the positive transverse measures (or of the positive tangential measures). Now let ξ be a measured geodesic lamination which hits τ efficiently. Then every branch of τ is transverse to ξ and hence the transverse measure of ξ defines a weight function on τ. This weight function is a tangential measure on τ. We say that the weight function is determined by τ.

The following result of Penner [P88] relates web odd train tracks τ equipped with pairs $(\mu,\nu) \in \mathrm{int}\mathcal{V}(\tau) \times \mathrm{int}\mathcal{V}^*(\tau)$ of positive transverse and tangential measures, respectively, to quadratic differentials. For its formulation, call a web train track $\tau \in \mathcal{T}(k_1,\dots,k_\ell)$ *odd* if each k_i is odd, i.e. if none of the complementary polygons of τ has an even number of sides.

Lemma 2.7. *Let $\tau \in \mathcal{T}(k_1,\dots,k_\ell)$ be a full train track and let $(\mu,\nu) \in \mathrm{int}\mathcal{V}(\tau) \times \mathrm{int}\mathcal{V}^*(\tau)$ with $\sum_b \mu(b)\nu(b) = 1$. Then there is a quadratic differential $q(\mu,\nu) \in \mathcal{Q}^1(k_1,\dots,k_\ell)$ with horizontal measured geodesic lamination μ and whose vertical measured geodesic lamination defines ν. If τ is odd then $q(\mu,\nu)$ is uniquely determined by the pair (μ,ν). The set*

$$q(\tau) = \{q(\mu,\nu) \mid (\mu,\nu) \in \mathrm{int}\mathcal{V}(\tau) \times \mathrm{int}\mathcal{V}^*(\tau)\} \subset \mathcal{Q}(k_1,\dots,k_\ell)$$

of all such quadratic differetials is connected and has non-empty interior.

Proof. Let $(\mu,\nu) \in \mathrm{int}\mathcal{V}(\tau) \times \mathrm{int}\mathcal{V}^*(\tau)$ be such that $\sum_b \mu(b)\nu(b) = 1$. Let N be a small closed neighborhood of τ in S. Assume that N is foliated into compact arcs in such a way that τ is obtained from N by collapsing each such arc to a point.

The singular leaves of the foliation which are collapsed to the switches of τ decompose N into a collection of rectangles which are in one-to-one correpondence with the branches of τ. We equip the rectangle corresponding to the branch b of τ with an euclidean metric so that the height of the rectangle equals $\mu(b)$ and that its length is $\nu(b)$. This defines an euclidean metric of area one N.

The complementary regions of N in S can be collapsed in such a way that the euclidean metric on N induces a singular euclidean metric on S with one standard k_i+2-pronged singularity in each complementary region of τ which is a k_i+2-gon. In every once punctured monogon the metric has a cone singularity of cone angle

π. If all the complementary polygons of τ have an odd number of sides then such a collapse is uniquely determined by (μ,ν) [P88]. The resulting singular euclidean euclidean metric defines a quadratic differential in the stratum $\mathcal{Q}(k_1,\dots,k_\ell)$. For each polygon with an even number of sides there is a one-parameter family of possible collapses which all yield a quadratic differential with the required properties [P88]. The set $q(\tau)$ of all such quadratic differentials obtained in this way from μ,ν and as μ,ν vary is clearly connected. A dimension count shows that $q(\tau)$ also has non-empty interior. □

There is also a converse to Lemma 2.7.

Lemma 2.8. *Let $q \in \mathcal{Q}^1(S)$ be a quadratic differential with ℓ zeros of order $k_1,\dots,k_\ell$ and without any horizontal or vertical saddle connection. Then there is a train track $\tau \in \mathcal{T}(k_1,\dots,k_\ell)$ such that $q \in q(\tau)$.*

Proof. Let τ be a collapse of a thickening of the horizontal measured geodesic lamination ν of a quadratic differential $q \in \mathcal{Q}(k_1,\dots,k_\ell)$ without horizontal or vertical saddle connection. Then $\tau \in \mathcal{T}(k_1,\dots,k_\ell)$ and ν is carried by τ. Moreover, a carrying map $\nu \to \tau$ is surjective. If the neighborhood of ν collapsed to τ is sufficiently small then the vertical measured geodesic lamination of q hits τ efficiently and hence defines a tangential measure on τ. It is now immediate from the proof of Lemma 2.7 that $q \in q(\tau)$. □

3. A symbolic system

In this section we use web train track to construct a subshift of finite type which we use in the following sections to study the Teichmüller geodesic flow.

Define a *numbered web train track* to be a train track τ together with a numbering of the branches of τ. Since a mapping class which preserves a web train track τ as well as each of its branches is the identity (compare the proof of Lemma 3.3 of [H06b]), the mapping class group $\mathcal{M}(S)$ acts *freely* on the set $\mathcal{NT}(k_1,\dots,k_\ell)$ of all isotopy classes of numbered web train track on S of topological type $(k_1,\dots,k_\ell)$.

Define a *numbered combinatorial type* to be an orbit of a numbered web train track under the action of the mapping class group. The set $\mathcal{E}_0$ of numbered combinatorial types equals the quotient of the set of all numbered web train tracks under the action of the mapping class group. If the numbered combinatorial type defined by a numbered web train track τ is contained in a subset $\mathcal{E}$ of $\mathcal{E}_0$, then we say that τ is *contained* in $\mathcal{E}$ and we write $\tau \in \mathcal{E}$. The partition of the set of all numbered web train tracks into the sets $\mathcal{NT}(k_1,\dots,k_\ell)$ projects to a partition of $\mathcal{E}_0$ into sets $\mathcal{E}(k_1,\dots,k_\ell)$.

If the web train track τ' can be obtained from a web train track τ by a single split, then a numbering of the branches of τ naturally induces a numbering of the branches of τ' and therefore such a numbering defines a *numbered split*. Define a *full split* of a numbered web train track τ to be a numbered web train track τ' with the property that τ' can be obtained from τ by splitting τ at each large branch precisely once. A *numbered splitting sequence* is a (finite or infinite) sequence (τ_i)

of numbered web train tracks such that for each i, the numbered web train tracks τ_{i+1} can be obtained from τ_i by a single numbered split. Similarly, a *full numbered splitting sequence* is a sequence (τ_i) of numbered web train tracks such that for each i, the numbered web train track τ_{i+1} can be obtained from τ_i by a full numbered split.

We say that a numbered combinatorial type $c \in \mathcal{E}_0$ is *splittable* to a numbered combinatorial type c' if there is a numbered web train track $\tau \in c$ which can be connected to a numbered web train track $\tau' \in c'$ by a *full* numbered splitting sequence.

As in Section 2 we call a web train track $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$ *special* if it is a collapsed thickening of a special geodesic lamination $\lambda \in \mathcal{L}(k_1, \dots, k_\ell)$. Define a *twist connector* in a train track η to be an embedded simple closed curve of class C^1 in η consisting of a large branch and a small branch. We can shift a special train track τ to a train track τ' which contains a single twist connector, and this twist connector contains the unique large branch of τ'. We call τ' *special* as well.

Call two web train track τ, τ' *shift equivalent* if τ' can be obtained from τ by a sequence of shifts. This clearly defines an equivalence relation on the set of all web train tracks on S. We have.

Lemma 3.1. *For every connected component $\mathcal{Q}$ of a stratum $\mathcal{Q}(k_1, \dots, k_\ell)$ there is a set $\mathcal{E} = \mathcal{E}(\mathcal{Q}) \subset \mathcal{E}(k_1, \dots, k_\ell)$ of numbered combinatorial types with the following properties.*

1. *For all $x, x' \in \mathcal{E}$, x is splittable to x'.*
2. *If τ is contained in $\mathcal{E}$ and if (τ_i) is any full numbered splitting sequence issuing from $\tau_0 = \tau$ then τ_i is contained in $\mathcal{E}$ for all $i \geq 0$.*
3. *For every $q \in \mathcal{Q}$ without horizontal and vertical saddle connections and every lift $\tilde{q}$ of q to $\mathcal{Q}^1(S)$ there is some $\tau \in \mathcal{E}$ such that $\tilde{q} \in q(\tau)$.*

Proof. Let for the moment $\eta \in \mathcal{T}(k_1, \dots, k_\ell)$ be an unnumbered special web train track. We choose η in such a way that it contains a unique twist connector. This twist connector contains the unique large branch e of η, and it defines an embedded simple closed curve c in η. Let σ be a web train track which can be obtained from η by a sequence of shifts. Then σ contains c as an embedded curve consisting of $k+2$ branches for some $k \geq 0$, and k of these branches are mixed. The branch e_0 in σ corresponding to the branch e in η is large. The train track σ_1 obtained from σ by a split at e_0 with a branch $h \subset c$ as a winner is obtained from σ by a $1/(k+2)$-*Dehn twist* about the simple closed curve c. This means that σ_1 contains c as an embedded simple closed curve consisting of $k+2$ branches, with a single large branch e_1 and k mixed branches. Repeat this splitting procedure with the train track σ_1 and the large branch e_1. After $k+2$ steps we obtain a train track σ_{k+2} which is obtained from σ by a single (left or right) Dehn twist about c. In particular, the train track σ_{k+2} and hence σ_1 is a web train track. Note that each web train track σ_i has a unique large branch and hence the above splitting sequence is full.

Let $\mathcal{Q}$ be a connected component of a stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ and let $\tilde{\mathcal{Q}}$ be the lift of $\mathcal{Q}$ to $\mathcal{Q}^1(S)$. Let $\mathcal{A}\subset\mathcal{L}(k_1,\dots,k_\ell)$ be the closure in the locally compact space $\mathcal{L}(k_1,\dots,k_\ell)$ of the set of all geodesic networks $\zeta\in\mathcal{L}(k_1,\dots,k_\ell)$ which are horizontal for some $\tilde{q}\in\tilde{\mathcal{Q}}$ without horizontal saddle connection. Then $\mathcal{A}$ is a closed subset of $\mathcal{L}(k_1,\dots,k_\ell)$ which is invariant under the action of the mapping class group. By Lemma 2.6 there is a special geodesic lamination $\nu\in\mathcal{L}(k_1,\dots,k_\ell)$ with the property that the closure of its orbit under the action of the mapping class group $\mathcal{M}(S)$ contains $\mathcal{A}$. Let $\eta\in\mathcal{T}(k_1,\dots,k_\ell)$ be a collapse of a thickening of ν. By Lemma 2.6 we may assume that $q(\eta)$ contains an open subset U of $\tilde{\mathcal{Q}}$. The set U is invariant under the Teichmüller flow and hence it projects to an open Φ^t-invariant subset of $\mathcal{Q}$.

Let μ be the web geodesic lamination determined by a quadratic differential $q\in U$ without horizontal or vertical saddle connections. By Lemma 2.6 there is a sequence $g_i\subset\mathcal{M}(S)$ such that the laminations $g_i\nu$ are carried by η and such that $g_i\nu\to\mu$. Then for each i, η carries a collapsed thickening η_i of $g_i\nu$. Via replacing η_i by a shift equivalent train track we may assume that $\eta_i=\tilde{g}_i\eta$ for some $\tilde{g}_i\in\mathcal{M}(S)$. By the results of [H06b], the train track η can be connected to a shift of η_i by a splitting sequence.

Let $\mathcal{Z}$ be the set of all train tracks which can be obtained from a train track which is shift equivalent to η by a sequence of partial Dehn twists as described in the first paragraph of this proof. For $\sigma\in\mathcal{Z}$ let $\mathcal{A}(\sigma)$ be the collection of all complete train tracks ξ which are shift equivalent to σ and with the following additional property. There is an element $g\in\mathcal{M}(S)$ such that σ can be connected to $g\xi$ with a full splitting sequence. By our above observation, the set $\mathcal{A}(\sigma)$ is not empty. Since $\mathcal{Z}$ is finite we can choose $\sigma\in\mathcal{Z}$ in such a way that $\mathcal{A}(\sigma)$ has the smallest cardinality.

Let $\xi\in\mathcal{A}(\sigma)$. By definition, ξ is shift equivalent to σ and we have $\mathcal{A}(\xi)\subset\mathcal{A}(\sigma)$. Therefore since σ was chosen in such a way that the cardinality of $\mathcal{A}(\sigma)$ is minimal we conclude that in fact equality holds. But this just means that for all $\xi,\xi'\in\mathcal{A}(\sigma)$ there is a nontrivial full splitting sequence connecting ξ to a train track $g\xi'$ for some $g\in\mathcal{M}(S)$.

Now let $\tau\in\mathcal{NT}(k_1,\dots,k_\ell)$ be a numbered train network obtained from σ by a numbering of the branches of ξ. By our above observation, τ can be connected with a nontrivial full splitting sequence to a numbered train track ζ which can be obtained from a train track in the $\mathcal{M}(S)$-orbit of τ by a permutation of the numbering. The permutations of the numbering of τ obtained in this way clearly form a group and hence τ can be connected to a train track in its own orbit under the mapping class group by a full numbered splitting sequence.

Define $\mathcal{E}\subset\mathcal{E}(k_1,\dots,k_\ell)$ to be the set of all numbered combinatorial types of all complete train tracks which can be obtained from τ by a full numbered splitting sequence. By the first part of Lemma 2.6, for every $aq\in\tilde{\mathcal{Q}}$ whose horizontal foliation does not have any saddle connection there is some $\eta\in\mathcal{E}$ such that η carries the horizontal geodesic lamination μ of q. Let (η_i) be a full numbered splitting sequence issuing from η which consists of train networks carrying μ. Since $\cap\mathcal{V}(\eta_i)$ equals the set of all measured geodesic laminations supported in μ, for sufficiently large i the vertical measured geodesic lamination of q hits η_i efficiently. Thus the set $\mathcal{E}$ has property (3) required in the lemma.

We claim that $\mathcal{E}$ has properties (1) and (2) in the lemma as well. Namely, if $(\tau_i)_{0\leq i\leq s}$ is any finite full numbered splitting sequence issuing from $\tau_0=\tau$ then τ_i carries a minimal geodesic lamination $\zeta\in\mathcal{L}(k_1,\dots,k_\ell)$ and this lamination is the horizontal measured geodesic lamination of a quadratic differential $q\in q(\tau)\cap\tilde{\mathcal{Q}}$. Now by our choice of the special geodesic lamination ν the train track τ_s carries a geodesic lamination in the $\mathcal{M}(S)$-orbit of ν and hence it carries a special train track in the $\mathcal{M}(S)$-orbit of τ. By our above consideration, this just means that $\tau_s\in\mathcal{E}$. In other words, properties (1) and (2) in the lemma hold true for $\mathcal{E}$. Moreover, every special train track $\sigma\in\mathcal{T}(k_1,\dots,k_\ell)$ which is carried by $\tau\in\mathcal{E}$ is shift equivalent to a web train track σ' contained in a numbered class from $\mathcal{E}$ for some numbering of the branches of σ. ☐

Let $p>0$ be the cardinality of a set $\mathcal{E}\subset\mathcal{E}(k_1,\dots,k_\ell)$ as in Lemma 3.1 and number the p elements of $\mathcal{E}$ in an arbitrary order. We identify each element of $\mathcal{E}$ with its number. Define $a_{ij}=1$ if the numbered combinatorial type i can be split with a single full numbered split to the numbered combinatorial type j and define $a_{ij}=0$ otherwise. The matrix $A=(a_{ij})$ defines a *subshift of finite type.* Its phase space is the set of biinfinite sequences $\Omega\subset\prod_{i=-\infty}^{\infty}\{1,\dots,p\}$ with the property that $(x_i)\in\Omega$ if and only if $a_{x_ix_{i+1}}=1$ for all i. Every biinfinite full numbered splitting sequence $(\tau_i)\subset\mathcal{NT}(k_1,\dots,k_\ell)$ contained in $\mathcal{E}$ defines a point in Ω. Vice versa, since the action of $\mathcal{M}(S)$ on the set of numbered large train tracks is free, a point in Ω determines an $\mathcal{M}(S)$-orbit of biinfinite full numbered splitting sequences. We say that such a full numbered splitting sequence *realizes* (x_i).

The shift map $T:\Omega\to\Omega, T(x_i)=(x_{i+1})$ acts on Ω. For $m>0$ write $A^m=(a_{ij}^{(m)})$; the shift T is *topologically transitive* if for all i,j there is some $m>0$ such that $a_{ij}^{(m)}>0$. Namely, if we define a finite sequence $(x_i)_{0\leq i\leq m}$ of points $x_i\in\{1,\dots,p\}$ to be *admissible* if $a_{x_ix_{i+1}}=1$ for all i then $a_{ij}^{(m)}$ equals the number of all admissible sequences of length m connecting i to j [Mn87]. The following observation is immediate from the definitions.

Lemma 3.2. *The shift (Ω,T) is topologically transitive.*

Proof. Let $i,j\in\{1,\dots,p\}$ be arbitrary. By Lemma 3.1, there is a nontrivial finite full numbered splitting sequence $\{\tau_i\}_{0\leq i\leq n}\subset\mathcal{NT}(k_1,\dots,k_\ell)$ connecting a train track τ_0 of numbered combinatorial type $i\in\mathcal{E}$ to a train track τ_n of numbered combinatorial type j. This splitting sequence then defines an admissible sequence $(x_i)_{0\leq i\leq m}\subset\mathcal{E}$ connecting i to j. ☐

4. Symbolic dynamics for the Teichmüller flow

In this section we relate the subshift of finite type (Ω,T) constructed in Section 3 to the Teichmüller flow. As in Section 2, for $\tau\in\mathcal{NT}(k_1,\dots,k_\ell)$ denote by $\mathcal{V}(\tau)$ the convex cone of all transverse measures on τ. Recall that $\mathcal{V}(\tau)$ coincides with the space of all measured geodesic laminations which are carried by τ. Define moreover

$\mathcal{V}_0(\tau) \subset \mathcal{V}(\tau)$ to be the subset of all measured geodesic laminations μ which are carried by τ and such that the total weight disposed by μ on τ equals one. Then $\mathcal{V}_0(\tau)$ can naturally be identified with the projectivization of $\mathcal{V}(\tau)$, i.e. with the space $\mathcal{PML}(\tau)$ of all projective measured geodesic laminations which are carried by τ. Note that $\mathcal{PML}(\tau)$ is a *compact* subset of the compact space $\mathcal{PML}$ of all projective measured geodesic laminations on S.

If $(\tau_i)_{0\leq i} \subset \mathcal{NT}(k_1,\dots,k_\ell)$ is any full numbered splitting sequence then $\emptyset \neq \mathcal{PML}(\tau_{i+1}) \subset \mathcal{PML}(\tau_i)$ and hence $\cap_i \mathcal{PML}(\tau_i)$ is a non-empty compact subset of $\mathcal{PML}$. By the results of Mosher [Mo03], this set consists of all projective measured geodesic laminations which are supported in a single geodesic lamination ν. The lamination ν is a union of minimal components, but it may be disconnected. If the lamination ν has a minimal component which fills up S and which supports a single transverse measure up to scale then $\cap_i \mathcal{PML}(\tau_i)$ consists of a unique point which is the projective class of a strongly uniquely ergodic measured geodesic lamination. In this case we call (τ_i) *uniquely ergodic*. Using the notations from Section 3, we call the sequence $(x_i) \in \Omega$ *uniquely ergodic* if one (and hence every) full numbered splitting sequence which realizes (x_i) is uniquely ergodic with support in $\mathcal{L}(k_1,\dots,k_\ell)$. This implies in particular that for every i the transverse measure on τ_i defined bz (x_i) is positive on every branch of τ.

Let $\mathcal{U} \subset \Omega$ be the set of all uniquely ergodic sequences. We define a function $\rho : \mathcal{U} \to \mathbb{R}$ as follows. For $(x_i) \in \mathcal{U}$ choose a full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(k_1,\dots,k_\ell)$ which realizes (x_i). By the definition of a strongly uniquely ergodic sequence there is a distinguished uniquely ergodic measured geodesic lamination $\mu \in \mathcal{V}_0(\tau_0)$ which is carried by each of the train tracks τ_i. Define $\rho(x_i) \in \mathbb{R}$ by the requirement that the total weight that the measured geodesic lamination $e^{\rho(x_i)}\mu$ disposes on τ_1 equals one. By equivariance under the action of the mapping class group, the number $\rho(x_i) \in \mathbb{R}$ only depends on the sequence $(x_i) \in \mathcal{U}$. In other words, ρ is a function defined on $\mathcal{U}$. We have.

Lemma 4.1. *The function $\rho : \mathcal{U} \to \mathbb{R}$ is continuous.*

Proof. Let $(x_i) \in \mathcal{U}$ and let $\epsilon > 0$. By the definition of the topology on our shift space it suffices to show that there is some $j \geq 0$ such that

$$|\rho(y_i) - \rho(x_i)| \leq 2\epsilon$$

whenever $(y_i) \in \mathcal{U}$ is such that $x_i = y_i$ for $0 \leq i \leq j$. For this let (τ_i) be a full numbered splitting sequence which realizes (x_i). Then (τ_i) defines a measured geodesic lamination λ which is carried by τ_1 and such that the sum of the weights which are disposed by λ on the branches of τ_1 equals one. By definition, $\rho(x_i)$ equals the logarithm of the total weight disposed by λ on τ_0. Denote by $[\lambda]$ the projective class of λ.

Let $p > 0$ be the number of branches of a train track in $\mathcal{T}(k_1,\dots,k_\ell)$. Recall from [PH92] that this number only depends on the sequence $(k_1,\dots,k_\ell)$. Then the set $\mathcal{V}_0(\tau_1)$ of all transverse measures on τ_1 can be identified with a compact convex subset of $\mathbb{R}^p$. The topology on $\mathcal{V}_0(\tau_1)$ obtained from this identification coincides with the topology on $\mathcal{V}_0(\tau_1)$ viewed as closed subset of the set of all projective measured geodesic laminations equipped with the weak* topology [T79]. Thus there is a neighborhood V of $[\lambda]$ in the space $\mathcal{PML}$ with the following property. Every $\nu \in \mathcal{V}_0(\tau_1)$ whose projective class $[\nu]$ is contained in V defines a transverse measure on τ_0 whose total weight is contained in the interval $(e^{\rho(x_i)-\epsilon}, e^{\rho(x_i)+\epsilon})$.

On the other hand, for every $j > 0$ the set $\mathcal{PML}(\tau_j)$ of all projective measured geodesic laminations which are carried by τ_j is a compact subset of $\mathcal{PML}$ containing $[\lambda]$, and we have $\mathcal{PML}(\tau_j) \subset \mathcal{PML}(\tau_i)$ for $j \geq i$ and $\cap_j \mathcal{PML}(\tau_j) = [\lambda]$. As a consequence, there is some $j_0 > 0$ such that $\mathcal{PML}(\tau_{j_0}) \subset V$. By the definition of ρ, this implies that the value of ρ on the intersection with $\mathcal{U}$ of the cylinder $\{(y_i) \mid y_j = x_j \text{ for } 0 \leq j \leq j_0\}$ is contained in the interval $(\rho(x_i) - \epsilon, \rho(x_i) + \epsilon)$. This shows the lemma. □

The next lemma gives additional information on the function ρ. For its formulation, let again $p > 0$ be the number of branches of a train track $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$.

Lemma 4.2. *The function ρ maps $\mathcal{U}$ to $(0, p\log 2]$.*

Proof. We have to show that $0 \leq \rho(x_i) \leq p\log 2$ for every $(x_i) \in \mathcal{U}$. For this choose a full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(k_1, \dots, k_\ell)$ which realizes (x_i). Using the above notations, let λ be the strongly uniquely ergodic measured geodesic lamination defined by (τ_i). Then λ is carried by each of the train tracks τ_i, and the total weight of the transverse measure μ on τ_0 defined by λ equals one.

Let e be a large branch of τ_0 and let τ' be the full train track which is obtained from τ_0 by a single split at e and which is splittable to τ_1. Let e' be the branch in τ' which is the *diagonal* of the split of τ_0 at e. This means that e' is the branch in τ' which is the image of e under the natural bijection Λ of the branches of τ_0 onto the branches of τ'. Let μ' be the transverse measure on τ' defined by the measured geodesic lamination λ. If b, d are the two losing branches of the split of τ_0 at e then $\mu(e) = \mu'(e') + \mu'(\Lambda(b)) + \mu'(\Lambda(d))$. Moreover, we have $\mu(a) = \mu'(\Lambda(a))$ for every branch $a \neq e$ of τ_0 and the corresponding branch of τ'. Thus the total weight disposed by μ' on τ' is contained in the interval $[1/2, 1]$. Since τ_1 can be obtained from τ by a splitting sequence whose length is bounded from above by the number p of branches of τ_0, this immediately implies that ρ is nonnegative and bounded from above by $p\log 2$.

On the other hand, since the measure μ is positive on every branch of τ the same consideration also shows that $\rho > 0$ on $\mathcal{U}$. Note however that ρ is not bounded from below by a positive constant. □

Define a measured geodesic lamination λ on S to be *recurrent* if for some (and hence every, see [M80]) quadratic differential $q \in \mathcal{Q}^1(S)$ with horizontal measured geodesic lamination λ the projection to $\mathcal{Q}(S)$ of the flow line $t \to \Phi^t q$ intersects a fixed compact set K for arbitrarily large t. By the results of Masur [M82], a recurrent measured geodesic lamination is strongly uniquely ergodic. The projective class of a recurrent measured geodesic lamination is called recurrent as well.

Call a biinfinite sequence $(x_j) \in \Omega$ *normal* if every finite admissible sequence occurs in (x_j) infinitely often in forward and backward direction. Following [H06c] we call a finite admissible sequence $(y_i)_{0\leq i\leq \ell}\subset \mathcal{E}$ *tight* if for one (and hence every) full numbered splitting sequence $(\tau_i)_{0\leq i\leq\ell}$ realizing (y_i) the natural carrying map $\tau_\ell\to\tau_0$ maps every branch b of τ_ℓ *onto* τ_0. By the definition of $\mathcal{E}$, by Lemma 3.1 and by the considerations in Section 5 of [H06c], tight finite admissible sequences exist. We use tight admissible sequences to show.

Lemma 4.3. *Let $(x_i)\in\Omega$ be normal and let $(\tau_i)\subset\mathcal{NT}(k_1,\dots,k_\ell)$ be a full numbered splitting sequence which realizes (x_i); then $\cap_i\mathcal{PML}(\tau_i)$ consists of a single recurrent projective measured geodesic lamination with support in $\mathcal{L}(k_1,\dots,k_\ell)$.*

Proof. Let $(y_i)_{0\leq i\leq\ell}$ be a tight admissible sequence and let $(\tau_i)_{0\leq i\leq\ell}$ be a full numbered splitting sequence which realizes $(y_i)_{0\leq i\leq\ell}$. Let p be the number of branches of a full train track $\tau\in\mathcal{T}(k_1,\dots,k_\ell)$ on S. Let $\lambda\in\mathcal{V}_0(\tau_\ell)$ be a measured geodesic lamination which is carried by τ_ℓ and such that the total weight disposed by λ on the branches of τ_ℓ equals one. Then by linearity of the carrying map and by the definition of a tight sequence, the *minimal* weight disposed by λ on any branch of τ_0 is not smaller than $1/p$. Since the λ-weight of a *large* branch of τ_0 equals the sum of the weights of two distinct branches of τ_0 and since τ_0 has at least one large branch, the total weight disposed by λ on τ_0 is at least $\frac{p+1}{p}$. In particular, if $(x_i)\in\mathcal{U}$ is such that $x_i=y_i$ for $0\leq i\leq k$ then we have $\sum_{j=0}^{k-1}\rho(T^j(x_i))\geq\log\frac{p+1}{p}$.

By Lemma 5.2 of [H06c], if $(\tau_i)_{0\leq i\leq n}$ is any finite numbered splitting sequence such that for some $\ell<n$ both sequences $(\tau_i)_{0\leq i\leq\ell}$ and $(\tau_i)_{\ell\leq i\leq n}$ are tight then there is a compact subset K of $\mathcal{Q}(S)$ depending on the sequence with the following property. Let (μ,ν) be a pair of measured geodesic laminations with $i(\mu,\nu)=1$ and such that μ is carried by τ_n and that ν hits τ_0 efficiently. Assume that μ is normalized in such a way that the total weight disposed by μ on the branches of τ_0 equals one. Let $q\in\mathcal{Q}^1(S)$ be the quadratic differential defined by (μ,ν); then q projects into K. Moreover by our above consideration, we have $\sum_{\ell=0}^{n-1}\rho(T^\ell(x_j))\geq 2\log\frac{p+1}{p}$ for every $(x_j)\in\mathcal{U}$ which is realized by a full numbered splitting sequence (η_j) with $\eta_i=\tau_i$ for $0\leq i\leq n$.

Now let $(x_i)\in\Omega$ be normal and let (τ_i) be a biinfinite full splitting sequence which realizes (x_i). For each i let $\mathcal{L}(\tau_i)$ be the set of all *full* geodesic laminations $\lambda\in\mathcal{L}(k_1,\dots,k_\ell)$ which are carried by τ_i. Then $\mathcal{L}(\tau_i)$ is a *compact* subset of the space $\mathcal{L}(k_1,\dots,k_\ell)$. Moreover we have $\mathcal{L}(\tau_{i+1})\subset\mathcal{L}(\tau_i)$ and $\cap_i\mathcal{L}(\tau_i)$ consists of a unique point λ_0 (see [H06b, Mo03]). Let μ be *any* measured geodesic lamination which is supported in λ_0 and such that the total weight disposed by μ on the branches of τ_0 equals one. Choose a quadratic differential $q\in\mathcal{Q}^1(S)$ with horizontal measured geodesic lamination μ and whose vertical measured geodesic lamination hits τ_0 efficiently. Since $(x_i)\in\Omega$ is normal, a finite admissible sequence $(y_i)_{0\leq i\leq n}$ which is realized by a full numbered splitting sequence $(\tau_i)_{0\leq i\leq n}$ as above occurs in $(x_i)_{0\leq i}$ infinitely often. Our above consideration then shows that the projection to $\mathcal{Q}(S)$ of the orbit of the Teichmüller flow defined by q intersects the compact subset K of $\mathcal{Q}(S)$ for arbitrarily large times. But this just means that μ is recurrent, in particular μ is strongly uniquely ergodic (compare also [K85]). As a consequence, we have $\cap_i\mathcal{V}(\tau_i)=(0,\infty)\mu$ and $(x_i)\in\mathcal{U}$.

To show that the support of μ is necessarily contained in $\mathcal{L}(k_1, \dots, k_\ell)$ we assume otherwise. Following [PH92], there is then some $j \geq 0$ such that the measure disposed by μ on τ_j is not positive. This means that the subtrack σ_j of τ_j consisting of all branches of positive μ-weight does not coincide with τ_j. Let b be a branch of $\tau_j - \sigma_j$. Since the sequence is normal, there is number $n > j$ such that *every* branch of τ_n is mapped *onto* τ_j under a natural carrying map. This means that the image of a branch in τ_n of positive μ-mass contains b. Since a covering map induces a linear map on transverse measures, we conclude that the transverse measure on τ_j defined by μ is positive on b which is a contradiction. This completes the proof of the lemma. □

By Lemma 4.3, the set of normal points in Ω is contained in the set $\mathcal{U}$ of uniquely ergodic points. Since normal points are dense in Ω, the same is true for uniquely ergodic points.

As in Section 2, for $\tau \in \mathcal{NT}(k_1, \dots, k_\ell)$ let $\mathcal{V}^*(\tau)$ be the cone of all tangential measures on τ. If $\tau' \in \mathcal{NT}(k_1, \dots, k_\ell)$ is obtained from $\tau \in \mathcal{NT}(k_1, \dots, k_\ell)$ by a single split at a large branch e and if C is the matrix which describes the transformation $\mathcal{V}(\tau') \to \mathcal{V}(\tau)$ then there is a natural transformation $\mathcal{V}^*(\tau) \to \mathcal{V}^*(\tau')$ given by the transposed matrix C^t. Denote by $\mathcal{PH}(\tau)$ the space of all projective tangential measures on τ. As in the proof of Lemma 4.3 we observe.

Lemma 4.4. *Let $(x_i) \in \Omega$ be normal and let $(\tau_i) \subset \mathcal{NT}(k_1, \dots, k_\ell)$ be a full numbered splitting sequence which realizes (x_i); then $\cap_i \mathcal{PH}(\tau_i)$ consists of a single positive projective tangential measure.*

We call the sequence $(x_i) \in \Omega$ *doubly uniquely ergodic* if (x_i) is uniquely ergodic as defined above and if moreover for one (and hence every) full numbered splitting sequence $(\tau_i) \in \mathcal{NT}(k_1, \dots, k_\ell)$ which realizes (x_i) the intersection $\cap_{i<0} \mathcal{PH}(\tau_i)$ consists of a unique projective tangential measure which is positive on every branch of τ_0. In particular, the supports of the two measured geodesic laminations λ, ν defined by (x_i) are contained in $\mathcal{L}(k_1, \dots, k_\ell)$. By Lemma 4.3 and Lemma 4.4, every normal sequence is doubly uniquely ergodic and hence the Borel set $\mathcal{DU} \subset \Omega$ of all doubly uniquely ergodic sequences $(x_i) \in \Omega$ is dense. Moreover, if each of the numbers k_i is odd then for each such sequence $(x_i) \in \mathcal{DU}$ and every numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(k_1, \dots, k_\ell)$ which realizes (x_i) there is a unique quadratic differential $\Xi(\tau_i) \in \mathcal{Q}^1(k_1, \dots, k_\ell)$ with the following properties.

(1) The horizontal measured geodesic lamination λ^+ of q is carried by each of the train tracks τ_i and the total mass disposed by λ^+ on the large branches of τ_0 equals one.
(2) The vertical measured geodesic lamination λ^- of q hits each of the train tracks τ_i efficiently.
(3) q is determined as in Lemma 2.7 by λ^+ and the tangential measure on τ defined by λ^-.

If $s \leq \ell$ is the cardinality of the even numbers among the numbers k_i then there is a set of quadratic differentials with the above properties which is homeomorphic to a cube $[0, 1]^s$.

By equivariance under the action of the mapping class group, every sequence $(x_i) \in \mathcal{DU}$ determines a quadratic differential $\Xi(x_i) \in \mathcal{Q}(k_1, \dots, k_\ell)$. This quadratic differential is contained in the subset $\mathcal{UQ} \subset \mathcal{Q}$ of all area one quadratic differentials in $\mathcal{Q}$ whose horizontal measured geodesic laminations is strongly uniquely ergodic. Note that $\mathcal{UQ}$ is a Φ^t-invariant Borel subset of $\mathcal{Q}$.

Our next goal is to show that the map Ξ is finite-to-one. For this denote for any web train track τ on S by $\mathcal{Q}(\tau) \subset \mathcal{Q}^1(S)$ the set of all quadratic differentials with horizontal measured geodesic lamination q_h contained in $\mathcal{V}_0(\tau)$ and measured geodesic lamination q_v which hits τ efficiently. Define a quadratic differential $q \in \mathcal{Q}^1(S)$ to be *strongly uniquely ergodic* if its horizontal and vertical measured geodesic laminations are strongly uniquely ergodic. Note that this definition does not refer to the stratum of $\mathcal{Q}^1(S)$ containing q. Denote by $\mathcal{TT} = \mathcal{T}(1, \dots, 1)$ the set of all complete train tracks on S. We have.

Lemma 4.5. *Let $q \in \mathcal{Q}^1(S)$ be strongly uniquely ergodic. Then there is a neighborhood V of q in $\mathcal{Q}^1(S)$ and there are finitely many train tracks $\tau_1, \dots, \tau_n \in \mathcal{TT}$ with the following property. If $\eta \in \mathcal{TT}$ is such that $\Phi^t z \in \mathcal{Q}(\eta)$ for some $z \in V$ and some $t \in [0, p \log 2]$ the $\eta \in \{\tau_1, \dots, \tau_n\}$.*

Proof. A *framing* (or *marking* in the terminology of Masur and Minsky [MM99]) consists of a pants decomposition P for S together with a system of simple closed *spanning curves*. For each curve $\gamma \in P$ there is a unique spanning curve which is contained in $S - (P - \gamma)$ and which intersects γ in the minimal number of points. The spanning curves may intersect, but we require that the number of intersection points between any two such curves is bounded from above by a universal constant.

By equivariance under the action of the mapping class group there is a number $\chi_0 > 0$ and for every framing F of S there is a complete finite volume hyperbolic metric $h \in \mathcal{T}(S)$ such that the h-length of each curve from our framing F is at most χ_0. We call such a hyperbolic metric *short for F*. By standard hyperbolic trigonometry, there is a number $\epsilon > 0$ such that every hyperbolic metric which is short for some framing F of S is contained in the set $\mathcal{T}(S)_\epsilon$ of all hyperbolic metrics whose *systole*, i.e. the shortest length of a closed geodesic, is at least ϵ. Moreover, the diameter in the Teichmüller space $\mathcal{T}(S)$ of the set of all hyperbolic metrics which are short for a fixed framing F is bounded from above by a universal constant.

The mapping class group $\mathcal{M}(S)$ naturally acts on the collection of all framings of S. By equivariance and the fact that the number of orbits for the action of $\mathcal{M}(S)$ on the set $\mathcal{TT}$ of all *complete* train tracks on S is *finite*, there is a number $k > 0$ and for every complete train track $\tau \in \mathcal{TT}$ there is a framing F of S which consists of simple closed curves carried by τ and such that the total mass of the counting measures on τ defined by these curves does not exceed k (compare the discussion in [MM99]). We call such a framing *short for τ*.

Define a map $\Lambda : \mathcal{TT} \to \mathcal{T}(S)$ by associating to a complete train track τ a hyperbolic metric $\Lambda(\tau) \in \mathcal{T}(S)$ which is short for a short framing for τ. Since the intersection number $i(c, c')$ between any two closed curves c, c' which are carried by some $\tau \in \mathcal{TT}$ and which define counting measures on τ of total mass at most k

is bounded from above by a universal constant (see Corollary 2.3 of [H06a]), there is a number $\chi_1 > 0$ only depending on the topological type of S such that if Λ' is another choice of such a map then we have $d(\Lambda(\tau), \Lambda'(\tau)) \leq \chi_1$ for every $\tau \in \mathcal{TT}$. In particular, the map Λ is *coarsely $\mathcal{M}(S)$-equivariant*: For every $\tau \in \mathcal{TT}$ and every $g \in \mathcal{M}(S)$ we have $d(\Lambda(g\tau), g\Lambda(\tau)) \leq \chi_1$.

Let $q \in \mathcal{Q}^1(S)$ be strongly uniquely ergodic. It suffices to show that there is a neighborhood V of q in $\mathcal{Q}^1(S)$ and a number $R > 0$ with the following property. If $z \in V$, if $t \in [0, p\log 2]$ and $\eta \in \mathcal{TT}$ are such that $\Phi^t z \in \mathcal{Q}(\eta)$ then $d(\Lambda(\eta), Pq) \leq R$ where $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ is the canonical projection. Namely, by coarse equivariance under the action of the mapping class group and cocompactness, for every $x \in \mathcal{T}(S)$ and every $R > 0$ there are only finitely many complete train tracks $\eta \in \mathcal{TT}$ with $d(\Lambda(\eta), x) \leq R$.

To show that this indeed holds true we follow the reasoning in Section 4 and Section 5 of [H07a]. Namely, up to increasing our above constant $\chi_0 > 0$, we may assume that for every quadratic differential $z \in \mathcal{Q}^1(S)$ there is a simple closed curve on S whose *q-length*, i.e. the length with respect to the singular euclidean metric defined by q, is at most χ_0. Recall that the *curve graph* $\mathcal{C}(S)$ of S is the metric graph whose vertices are the essential simple closed curves on S and where two such vertices are connected by an edge of length one if and only if they can be realized disjointly (see [MM99]). Define a map $\Upsilon_{\mathcal{Q}} : \mathcal{Q}^1(S) \to \mathcal{C}(S)$ by associating to a quadratic differential q a simple closed curve $\Upsilon_{\mathcal{Q}}(q)$ of q-length at most χ_0. Then there is a number $L > 0$ such that the image under $\Upsilon_{\mathcal{Q}}$ of every flow line of the Teichmüller flow is an *unparametrized L-quasi-geodesic*: For every $z \in \mathcal{Q}^1(S)$ there is an increasing homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that the curve $t \to \Upsilon_{\mathcal{Q}}(\Phi^{\varphi(t)} z)$ is a L-quasi-geodesic in $\mathcal{C}(S)$ (see [MM99] and als [H07a]).

A *vertex cycle* for a complete train track τ is a simple closed curve carried by τ whose counting measure defines an extreme point for the space of all transverse measures on τ. The distance in $\mathcal{C}(S)$ between every vertex cycle of τ and a curve of $\Lambda(\tau)$-length at most χ_0 is bounded from above by a universal constant $b > 0$. Moreover, for every $q \in \mathcal{Q}(\tau)$, the distance in $\mathcal{C}(S)$ between a vertex cycle for τ and the curve $\Upsilon_{\mathcal{Q}}(q)$ is at most b.

Let $P : \mathcal{Q}^1(S) \to \mathcal{T}(S)$ be the canonical projection. By Lemma 4.2 of [H07a], there is a number $\ell > 0$ and for every $\epsilon > 0$ there is a number $m(\epsilon) > 0$ with the following property. Let $\sigma, \tau \in \mathcal{TT}$ and assume that σ is carried by τ and that the distance in $\mathcal{C}(S)$ between any vertex cycle of σ and any vertex cycle of τ is at least ℓ. Let $q \in \mathcal{Q}(\tau)$ be a quadratic differential whose horizontal measured geodesic lamination q_h is carried by σ. If the total mass of the transverse measure on σ defined by q_h is not smaller than ϵ, then $d(\Lambda(\tau), Pq) \leq m(\epsilon)$.

Now let $q \in \mathcal{Q}^1(S)$ be strongly uniquely ergodic. Then $d(\Upsilon_{\mathcal{Q}}(\Phi^t q), \Upsilon_{\mathcal{Q}}(q)) \to \infty$ $(t \to \infty)$ (compare [MM99, H06a, H07a]) and therefore there is a small neighborhood V of q in $\mathcal{Q}^1(S)$ and there is a number $T > 0$ such that

$$d(\Upsilon_{\mathcal{Q}}(\Phi^t z), \Upsilon_{\mathcal{Q}}(\Phi^s z)) \geq \ell + 2b + 2p\log 2$$

for all $t \geq T$, all $s \in [0, p\log 2]$ and for all $z \in V$. Let $z \in V$, let $s \in [0, a]$ and let $\tau \in \mathcal{TT}$ be such that $\Phi^s z \in \mathcal{Q}(\tau)$. If σ is a complete train track with $\Phi^t q \in \mathcal{Q}(\sigma)$

for some $t \in [T, T + p \log 2]$ and if σ can be obtained from τ by a full splitting sequence then the distance in $\mathcal{C}(S)$ between a vertex cycle of τ and a vertex cycle of σ is at least ℓ. Thus σ, τ satisfy the hypothesis in Lemma 4.2 of [H07a] as stated above with a number $\epsilon \geq e^{-T-p\log 2}$. This implies that the distance between Pz and $\Lambda(\tau)$ is uniformly bounded and shows the lemma. □

Recall that the function ρ is bounded from above by $p \log 2$. A *suspension* for the shift T on the subspace $\mathcal{DU} \subset \Omega$ of all doubly uniquely ergodic sequences in the phase space Ω with continuous roof function $\rho : \mathcal{DU} \to [0, p \log 2]$ is the space $X = \{(x_i) \times [0, \rho(x_i)] \mid (x_i) \in \mathcal{DU}\}/\sim$ where the equivalence relation $\sim$ identifies the point $((x_i), \rho(x_i))$ with the point $(T(x_i), 0)$. Note that $\sim$ is a closed equivalence relation since the function ρ is continuous. There is a natural flow Θ^t on X defined by $\Theta^t(x, s) = (T^j x, \tilde{s})$ (for $t > 0$) where $j \geq 0$ is such that $0 \leq \tilde{s} = s - \sum_{i=0}^{j-1} \rho(T^i x) < \rho(T^j x)$. A *semi-conjugacy* of (X, Θ^t) into a flow space (Y, Φ^t) is a continuous map $\Xi : X \to Y$ such that $\Phi^t \Xi(x) = \Xi(\Theta^t x)$ for all $x \in X$ and all $t \in \mathbb{R}$. We call a semi-conjugacy Ξ *finite-to-one* if the number of preimages of any point is finite. As an immediate consequence of Lemma 4.2 and Lemma 4.5 we obtain.

Corollary 4.6. *There is a continuous finite-to-one semi-conjugacy Ξ of the suspension X for the shift T on $\mathcal{DU}$ with roof function ρ into the Teichmüller geodesic flow. The image of Ξ equals the Φ^t-invariant subset $\mathcal{UQ} \subset \mathcal{Q}$.*

Proof. Every web train track $\eta \in \mathcal{T}(k_1, \dots, k_\ell)$ is a subtrack of a complete train track, and the number of complete train tracks τ containing η as a subtrack is uniformly bounded. Therefore Lemma 4.5 shows that for every $(x_i) \in \mathcal{DU}$ the cardinality of $\Xi^{-1}(\Xi(x_i))$ is finite. The map Ξ is clearly a semi-conjugacy. Thus we are left with showing that Ξ is continuous and that its image equals the set $\mathcal{UQ}$

Continuity of Ξ follows as in the proof of Lemma 4.1. To show that the image of Ξ is all of $\mathcal{UQ}$ let $q \in \mathcal{Q}$ be a quadratic differential with horizontal and vertical laminations $\lambda^+, \lambda^- \in \mathcal{L}(k_1, \dots, k_\ell)$. Then we have $i(\lambda^+, \lambda^-) = 1$, and every leaf of λ^+ intersects every leaf of λ^- transversely. By Lemma 2.7 there is a web train track $\tau \in \mathcal{T}(k_1, \dots, k_\ell)$ which carries λ^+ and which hits λ^- efficiently. The proof of Lemma 3.1 shows that we may assume that τ is contained in the set $\mathcal{E}$ defining our subshift of finite type. By construction, this means that there is an infinite full numbered splitting sequence $(\tau_i) \subset \mathcal{NT}(k_1, \dots, k_\ell)$ such that the intersection $\cap_i \mathcal{PM}(\tau_i)$ consists of a unique point which is just the class of λ^+ and that the intersection $\cap_i \mathcal{EM}(\tau_i)$ consists of a unique point which is the class of λ^-. This shows that our map Ξ maps $\mathcal{DU}$ onto $\mathcal{UQ}$. □

5. Bernoulli measures for the Teichmüller flow

Masur and Veech [M82, V82, V86] constructed a probability measure in the Lebesgue measure class on each connected component of a stratum in the moduli space $\mathcal{Q}(S)$ of area one quadratic differentials. This measure is invariant, ergodic and mixing under the Teichmüller flow Φ^t, and it gives full measure to the set of quadratic differentials whose horizontal and vertical measured geodesic laminations are strongly uniquely ergodic. Moreover, the measure is absolutely continuous with respect to the strong stable and the unstable foliation.

In this section we use the results from Section 4 to construct an uncountable family of Φ^t-invariant Borel probability measures on a component $\mathcal{Q}$ of an odd stratum $\mathcal{Q}(k_1, \dots, k_\ell)$ including the Lebesgue measure. These measures are ergodic and mixing, they are exponentially recurrent to a fixed compact set and absolutely continuous with respect to the strong stable and the unstable foliation. This completes the proof of Theorem 3 from the introduction.

Recall from Section 4 the construction of the semi-conjugacy Ξ from the suspension (X, Θ^t) of our subshift of finite type (Ω, T) with roof function ρ onto the Φ^t-invariant set $\mathcal{UQ} \subset \mathcal{Q}$. Since the roof function ρ on $\mathcal{DU} \subset \Omega$ is uniformly bounded and essentially positive, every T-invariant Borel probability measure ν on Ω which gives full mass to $\mathcal{DU}$ induces a finite invariant measure $\tilde{\nu}$ for the ρ-suspension flow whose total mass is $\int \rho d\nu < \infty$. The measure $\tilde{\nu}$ is defined by $d\tilde{\nu} = d\nu \times dt$ where dt is the Lebesgue measure on the flow lines of our flow. The image of $\tilde{\nu}$ under the semi-conjugacy Ξ is a finite Φ^t-invariant Borel measure on $\mathcal{Q}$ which we may normalize to have total mass one. Thus if $\mathcal{M}_T(\Omega)$ denotes the space of all T-invariant Borel probability measures on Ω which give full measure to $\mathcal{DU}$ then Ξ induces a map Ξ_* from $\mathcal{M}_T(\Omega)$ into the space $\mathcal{M}_{\rm inv}(\mathcal{Q})$ of Φ^t-invariant probability measures on $\mathcal{Q}$. We equip both spaces with the weak*-topology. We have.

Lemma 5.1. *The map Ξ_* is continuous.*

Proof. Since Ω is a compact metrizable space, the space of all probability measures on Ω equipped with the weak*-topology is compact. Thus we only have to show that whenever $\mu_i \to \mu$ in $\mathcal{M}_T(\Omega)$. then $\Xi_*(\mu_i) \to \Xi_*(\mu)$.

Now by Lemma 4.2 the function ρ is continuous on $\mathcal{DU}$, bounded and essentially positive and hence if $\mu_i \to \mu$ in $\mathcal{M}_T(\Omega)$ then $\int \rho d\mu_i \to \int \rho d\mu > 0$. In particular, we have $\tilde{\mu}_i(X) \to \tilde{\mu}(X)$ where $\tilde{\mu}_i, \tilde{\mu}$ are the finite Borel measures on the suspension space X for ρ defined by the measures μ_i, μ. Therefore $\Xi_*(\mu_i) \to \Xi_*(\mu)$ if and only if for every continuous function f on $\mathcal{Q}(S)$ with compact support we have $\int f \circ \Xi d\tilde{\mu}_i \to \int f \circ \Xi d\tilde{\mu}$. However, since Ξ is continuous this is immediate. □

The next observation is our most important technical tool.

Lemma 5.2. *The map $\Xi_* : \mathcal{M}_T(\Omega) \to \mathcal{M}_{\rm inv}(\mathcal{Q})$ is surjective.*

Proof. It suffices to show that every ergodic Φ^t-invariant Borel probability measure on $\mathcal{Q}$ is contained in the image of Ξ_*.

Thus let ν be an ergodic Φ^t-invariant Borel probability measure on $\mathcal{Q}$. By the Birkhoff ergodic theorem, there is a density point $q \in \mathcal{Q}$ for ν such that the Borel probability measures

$$\nu_T = \frac{1}{T} \int_0^T \delta_{\Phi^t q} dt$$

converge weakly to ν as $T \to \infty$ where δ_x denotes the Dirac mass at x. By Lemma 2.5, we have $q \in \mathcal{UQ}$. Hence up to possibly replacing q by $\Phi^t q$ for some $t \in \mathbb{R}$ there is some $(x_i) \in \mathcal{DU}$ with $\Xi(x_i) = q$.

By Lemma 4.5, for every density point z for ν there is a neighborhood V of z in $\mathcal{UQ}$ such that the preimage of V under the map Ξ is a *finite* union of Borel subsets $W_1, \dots, W_k$ of our suspension flow space X. Since $\nu_T|V \to \nu|V$ weakly as $T \to \infty$ and since the map Ξ is equivariant with respect to the suspension flow and the Teichmüller flow we conclude that the restriction to $\cup_i W_i$ of the Borel probability measures

$$\tilde{\nu}_T = \frac{1}{T}\int_0^T \delta_{\Theta^t q} dt$$

converge weakly to a measure on $\cup_i W_i$ which projects to the measure ν on V. Since q was an arbitrary density point for ν we conclude that the measures $\tilde{\nu}_T$ converge weakly to a Θ^t-invariant Borel probability measure on Ω whose image under the map Ξ is just ν. Thus Ξ_* is surjective. □

Let $k > 0$ be the cardinality of the set $\mathcal{E}$ of numbered combinatorial types as in Lemma 3.1 and let $C = (c_{ij})$ be any (k, k)-matrix with non-negative entries $c_{ij} \geq 0$ such that $c_{ij} > 0$ if and only if $a_{ij} > 0$ (where the matrix $A = (a_{ij})$ is defined in the paragraph after the proof of Lemma 3.1). We require that (c_{ij}) is *stochastic*, i.e. that $\sum_j p_{ji} = 1$ for all i. Since for all i, j there is some n such that $c_{ij}^{(n)} > 0$, by the Perron Frobenius theorem there is a nonnegative *probability vector* $c = (c_1, \dots, c_k)$ with $c_i \geq 0, \sum_i c_i = 1$ and such that $Cc = c$ by our normalization. Then the probability vector c together with the stochastic matrix C defines a Markov probability measure ν for the subshift (Ω, T) of finite type which is invariant under the shift and gives full measure to the set of normal points. The ν-mass of a cylinder $\{x_0 = i\}$ $(i \leq p)$ equals c_i, and the ν-mass of $\{x_0 = i, x_1 = j\}$ is $c_{ji}c_i$. Note that the identity $\sum_i c_{ji}c_i = c_j$ for all j is equivalent to invariance of ν under the shift. If our shift (Ω, T) is topologically mixing, then the shift (Ω, T, ν) is equivalent to a Bernoulli shift (see p.79 of [Mn87]). In particular, it is ergodic and strongly mixing, with exponential decay of correlations, and it gives full measure to the subset $\mathcal{R} \subset \Omega$ of normal points. We call a shift invariant probability measure on Ω of this form a *Bernoulli measure*.

The support of every Bernoulli measure μ on Ω is all of Ω. Moreover, μ gives full mass to the normal points and hence to the full points in Ω. Thus the push-forward $\Xi_*\mu$ is a finite Φ^t-invariant ergodic measure on $\mathcal{Q}$. We call its normalization to a probability measure on $\mathcal{Q}$ a *Bernoulli measure* for the Teichmüller geodesic flow Φ^t. Since the strong stable and the unstable manifolds for the suspension flow (X, Θ^t) are mapped by Ξ to strong stable and unstable manifolds for the Teichmüller flow, a Bernoulli measure for Φ^t is absolutely continuous with respect to the strong stable and the unstable foliation.

Proposition 5.3. *A Bernoulli measure for the Teichmüller flow is mixing.*

Proof. Let $\mathcal{UE}(S) \subset \mathcal{PML}$ be the Borel set of all strongly uniquely ergodic projective measured geodesic laminations on S. Denote by Δ the diagonal in $\mathcal{PML} \times \mathcal{PML} \supset \mathcal{UE}(S) \times \mathcal{UE}(S)$. The set $\mathcal{V}$ of quadratic differentials in $\mathcal{Q}^1(S)$ with strongly uniquely ergodic horizontal and vertical measured geodesic lamination is (non-canonically) homeomorphic to $(\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta)\times\mathbb{R}$ by choosing a Borel map which associates to a pair of transverse projective measured geodesic laminations in $(\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta)\times\{0\}$ a point on the Teichmüller geodesic determined by the pair and extending this map in such a way that it commutes with the natural $\mathbb{R}$-actions. With this identification, the Teichmüller geodesic flow acts on $(\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta)\times\mathbb{R}$ via $\Phi^t(\lambda,\nu,s)=(\lambda,\nu,t+s)$. In particular, there is a natural orbit space projection $\Pi:\mathcal{V}\to\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta$ which is equivariant with respect to the natural action of the mapping class group on $\mathcal{V}$ and the diagonal action of $\mathcal{M}(S)$ on $\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta$.

Let μ be a Bernoulli measure for the Teichmüller geodesic flow on the component $\mathcal{Q}$ of the odd stratum $\mathcal{Q}(k_1,\dots,k_\ell)$. Then μ lifts to a Φ^t-invariant $\mathcal{M}(S)$-invariant Radon measure on the preimage $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ to $\mathcal{Q}^1(S)$, and this Radon measure disintegrates to a Radon measure $\hat\mu$ on $\mathcal{PML}\times\mathcal{PML}-\Delta$ which is invariant under the diagonal action of $\mathcal{M}(S)$ and which gives full mass to the invariant subset $\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta$. Since μ is ergodic under the Teichmüller geodesic flow, the measure $\hat\mu$ is ergodic under the action of $\mathcal{M}(S)$. Assume to the contrary that the measure μ is not mixing. Then there is a continuous function φ on $\mathcal{Q}$ with compact support and $\int\varphi d\mu=0$ so that $\varphi\circ\Phi^t$ does not converge weakly to zero as $t\to\infty$. We follow [B02] and conclude that there is a non-constant function ψ which is the almost sure limit of Cesaro averages of φ of the form $\frac{1}{k}\sum_{j=1}^k\varphi_{n_j}$, $\frac{1}{k}\sum_{j=1}^k\varphi_{-n_j}$, i.e. for positive and negative times. Replacing ψ by the function $v\to\int_0^\epsilon\psi(\Phi^s v)ds/\epsilon$ for sufficiently small ϵ guarantees that there is a subset $E_0\subset\mathcal{UE}(S)\times\mathcal{UE}(S)-\Delta$ of full $\hat\mu$-measure so that the lift $\tilde\psi$ to $\tilde{\mathcal{Q}}$ of the resulting function is well defined and continuous on the lines in $\Pi^{-1}(E_0)$. The periods of the function $\tilde\psi$ define a measurable $\mathcal{M}(S)$-invariant map of E_0 into the set of closed subgroups of $\mathbb{R}$ which is constant by ergodicity. Since $\tilde\psi\not\equiv 0$ by assumption, the set $E_1\subset E_0$ where this group is of the form $a\mathbb{Z}$ for some $a\geq 0$ has full measure.

The Radon measure $\hat\mu$ on $\mathcal{PML}\times\mathcal{PML}-\Delta$ is absolutely continuous with respect to the product foliation of $\mathcal{PML}\times\mathcal{PML}$. This means that $\hat\mu$ is locally of the form $d\hat\mu=fd\hat\mu^+\times d\hat\mu^-$ where $\hat\mu^\pm$ is a measure on the space of projective measured geodesic laminations whose measure class is invariant under the action of $\mathcal{M}(S)$ and where f is a positive measurable function. Since the action of $\mathcal{M}(S)$ on $\mathcal{PML}$ is minimal, the measures μ^+,μ^- are of full support. Now the horizontal measured geodesic lamination of a typical point for the measure μ on $\mathcal{Q}$ is strongly uniquely ergodic and hence if v,w are typical points for μ on the same strong stable manifold in $\mathcal{Q}$ then $\lim_{t\to\infty}d(\Phi^t v,\Phi^t w)=0$ for a suitable choice of a distance function d on $\mathcal{Q}(S)$ [M80]. From continuity of the function φ and absolute continuity of μ with respect to the strong stable and the unstable manifolds we deduce that $\psi(v)=\psi(w)$. Similarly we argue for the strong unstable manifolds (see the nice exposition of this argument in [B02]). As a consequence, there is an $\mathcal{M}(S)$-invariant subset $E_2\subset E_1$ of full $\hat\mu$-mass such that for every $\zeta\in E_2$ and every $q\in\Pi^{-1}(\zeta)$ the function $\tilde\psi$ is constant almost everywhere along the strong stable manifold $W^{ss}(q)$ and along the strong unstable manifold $W^{su}(q)$.

As in [B02] we define

$$E^+ = \{x \in \mathcal{UE}(S) \mid (x, y') \in E_2 \text{ for } \hat{\mu}^- \text{ a.e. } y'\},$$
$$E^- = \{y \in \mathcal{UE}(S) \mid (x', y) \in E_2 \text{ for } \hat{\mu}^+ \text{ a.e. } x'\}.$$

By absolute continuity, the set $E^+ \times E^-$ has full measure with respect to $\hat{\mu}$.

Choose a density point $(u^+, u^-) \in E^+ \times E^-$ for $\hat{\mu}$ and let V^+, V^- be small disjoint neighborhoods of u^+, u^- in $\mathcal{PML}$ with the property that for every $x \in V^+$ and every $y \in V^-$ the projective measured geodesic laminations x, y together fill up S. Such neighborhoods are well known to exist. Then every pair $(x, y) \in V^+ \times V^-$ determines a unique Teichmüller geodesic. Define a *cross ratio* function $\sigma : V^+ \times V^- \to \mathbb{R}$ as follows. Choose a quadratic differential $q \in \tilde{\mathcal{Q}}$ whose horizontal measured geodesic lamination λ^+ is in the class u^+ and whose vertical measured geodesic lamination λ^- is in the class u^-. For $y \in V^-$, the pair (u^+, y) defines a Teichmüller geodesic whose corresponding one-parameter family of quadratic differentials intersects $W^{ss}(q)$ in a unique point $\alpha_1(x, y)$. Similarly, the family of quadratic differentials defined by the pair (x, y) intersects $W^{su}(\alpha_1(x, y))$ in a unique point $\alpha_2(x, y)$, the family of quadratic differentials defined by the pair (x, u^-) intersects $W^{ss}(\alpha_2(x, y))$ in a unique point $\alpha_3(x, y)$, and finally the family of quadratic differentials defined by (u^+, u^-) intersect $W^{su}(\alpha_3(x, y))$ in a unique point $\Phi^{\sigma(x,y)}(q)$. The value $\sigma(x, y) \in \mathbb{R}$ does not depend on the choice of the quadratic differential q defining the pair (u^+, u^-). The resulting function $\sigma : V^+ \times V^- \to \mathbb{R}$ is continuous and satisfies $\sigma(u^+, u^-) = 0$.

Recall that there is a natural action of the group $SL(2, \mathbb{R})$ on $\tilde{\mathcal{Q}}$ where the diagonal subgroup of $SL(2, \mathbb{R})$ acts as the Teichmüller geodesic flow. Each $SL(2, \mathbb{R})$-orbit can naturally and equivariantly be identified with the unit tangent bundle of the hyperbolic plane together with its usual action by isometries. In particular, for every $z \in \tilde{\mathcal{Q}}$ the orbit through z of the unipotent subgroup of $SL(2, \mathbb{R})$ of all upper triangular matrices of trace two is an embedded line in $W^{ss}(z)$, and the orbit through z of the unipotent subgroup of all lower triangular matrices of trace two is contained in $W^{su}(z)$. Thus the $SL(2, \mathbb{R})$-orbit through the above quadratic differential q defines two embedded line segments $I^+ \subset V^+, I^- \subset V^-$ containing u^+, u^- in their interior so that the restriction of the function σ to $I^+ \times I^-$ coincides with the restriction of the usual dynamical cross ratio on the space $S^1 \times S^1 - \Delta$ of oriented geodesics in the hyperbolic plane. In particular, the restriction of the function σ to $I^+ \times I^-$ is not constant in no neighborhood of (u^+, u^-). On the other hand, if we write $U^+ = V^+ \cap E^+, U^- = V^- \cap E^-$ then U^+, U^- are dense in V^+, V^-. Since our function σ is continuous and not constant in any neighborhood of (u^+, u^-) we conclude that for every $\epsilon > 0$ there are points $x \in U^+, y \in U^-$ such that $\sigma(x, y) \in (0, \epsilon)$. However, by our choice of E^+, E^-, the function $\tilde{\psi}$ is constant along the manifolds $W^{ss}(q), W^{su}(\alpha_1(x, y)), W^{ss}(\alpha_2(x, y))$ and $W^{su}(\alpha_3(x, y))$ and therefore $\tilde{\psi}(\Phi^{\sigma(x,y)}q) = \tilde{\psi}(q)$. In other words, the function $\tilde{\psi}$ has arbitrarily small periods which is a contradiction to our assumption that the set of periods of $\tilde{\psi}$ equals $a\mathbb{Z}$ for some $a \geq 0$. This completes the proof of our proposition. □

We also obtain a control of the return time to a suitably chosen compact subset of $\mathcal{Q}(S)$ for a flow line of the Teichmüller geodesic flow Φ^t on the component $\mathcal{Q}$

of $\mathcal{Q}(k_1,\dots,k_\ell)$ which is typical for a Bernoulli measure μ on $\mathcal{Q}$. The following observation is a version of Theorem 2.15 of [AGY06] for all Bernoulli measures (see also [A06]).

Lemma 5.4. *There is a compact subset K of $\mathcal{Q}(S)$ and for every Bernoulli measure μ for the Teichmüller flow Φ^t on $\mathcal{Q}$ there is a number $\epsilon=\epsilon(\mu)>0$ and a constant $C>0$ such that*

$$\mu\{q \mid \text{for all}\, s\in[0,t], \Phi^s q\not\in K\}\leq Ce^{-\epsilon t}.$$

Proof. Let $(y_i)_{0\leq i\leq k}$ be an admissible sequence with the additional property that there is some $\ell<k$ such that the sequences $(y_i)_{0\leq i\leq \ell}$ and $(y_i)_{\ell\leq i\leq k}$ are both tight (see the paragraph before Lemma 4.3 for the definition of a tight admissible sequence). By Lemma 5.2 of [H06c] there is a compact subset K of $\mathcal{Q}(S)$ such that for every full numbered splitting and shifting sequence $(\tau_i)_{0\leq i\leq k}$ which realizes $(y_i)_{0\leq i\leq k}$ the following holds. Let $\lambda\in\mathcal{ML}$ be a measured geodesic lamination which is carried by τ_k and which defines a transverse measure on τ_0 for which the maximal mass disposed on a large branch of τ_0 equals one. Let $\nu\in\mathcal{ML}$ be a measured geodesic lamination which hits τ_0 efficiently and such that $i(\lambda,\nu)=1$. Then the quadratic differential $q(\lambda,\nu)$ with horizontal measured geodesic lamination λ and vertical measured geodesic lamination ν is contained in the lift of K to $\mathcal{Q}^1(S)$.

Let μ be any Bernoulli measure on the shift space (Ω,T). Then μ is exponentially mixing and hence there are constants $c_0>0,\epsilon>0$ such that $\mu\{(x_i)\in\Omega\mid (y_i)_{0\leq i\leq k}\not\subset(x_j)_{0\leq j\leq m}\}\leq c_0e^{-\epsilon m}$. The corollary now follows from this observation and the fact that the roof function ρ on $\mathcal{UQ}$ which defines our Borel suspension is uniformly bounded. □

We conclude this section with a description of the $\mathcal{M}(S)$-invariant Φ^t-invariant measure λ in the Lebesgue measure class in the preimage in $\mathcal{Q}^1(S)$ of the component $\mathcal{Q}$ of an odd stratum $\mathcal{Q}(k_1,\dots,k_\ell)$ which is invariant under the natural action of the group $SL(2,\mathbb{R})$ and which projects to a probability measure on $\mathcal{Q}$. This measure induces a $\mathcal{M}(S)$-invariant Radon measure $\hat\lambda$ on $\mathcal{PML}\times\mathcal{PML}-\Delta$.

Using the notations from Section 3, let $\mathcal{E}$ be the collection of all numbered combinatorial types as in Lemma 3.1. For every $j\in\mathcal{E}$ choose a numbered complete train track $\tau(j)$ of combinatorial type j. Let $W^+(j)\subset\mathcal{PML}$ be the space of all projective measured geodesic laminations which are carried by $\tau(j)$ and let $W^-(j)$ be the space of all projective measured geodesic laminations which hit $\tau(j)$ efficiently. Note that $W^+(j),W^-(j)$ are closed disjoint subsets of $\mathcal{PML}$ with dense interior.

Recall that the train tracks $\sigma^1,\dots,\sigma^\ell$ which can be obtained from $\tau(j)$ by a full split are numbered. Then $W^+(j)=\cup_{i=1}^\ell W_i$ where $W^i\subset W^+(j)$ is the closed set of all projective measured geodesic laminations which are carried by σ^i $(i=1,\dots,\ell)$. For $i\neq j$ the intersection $W^i\cap W^j$ is contained in a hyperplane of $\mathcal{PML}$ (with respect to the natural piecewise linear structure which defines the Lebesgue measure class) and hence it has vanishing Lebesgue measure. If $k(i)\in\mathcal{E}$ is the combinatorial type of the numbered train track σ^i then define

$$p_{k(i)j}=\lambda_0(W^i\times W^-(j))/\lambda_0(W^+(j)\times W^-(j))$$

and define $p_{ju} = 0$ for $u \notin \{k(i) \mid i\}$. Note that we have $p_{sj} < 1$ for all s and $\sum_s p_{sj} = 1$. By invariance of the measure $\hat{\lambda}$ under the action of the mapping class group, the probabilities $p_{jk(i)}$ only depend on the combinatorial type $j \in \mathcal{E}$ and the choice of a representative in the shift equivalence class of j.

By invariance under the action of the mapping class group, the shift invariant measure μ on the subshift (Ω, T) of finite type with alphabet $\mathcal{E}$ and transition matrix $P = (p_{ij})$ is contained in the Lebesgue measure class. By ergodicity, it coincides with the standard Lebesgue measure λ on $\mathcal{Q}$. We use this observation to obtain a new (and simpler) proof of the main theorem of [A06].

Corollary 5.5. *Let λ be the Φ^t-invariant Borel probability measure on the component $\mathcal{Q}$ of the odd stratum $\mathcal{Q}(k_1, \dots, k_\ell)$ in the Lebesgue measure class. There is a compact subset K of $\mathcal{Q}(S)$ and a number $\epsilon > 0$ such that $\lambda\{q \mid \Phi^s q \notin K$ for every $s \in [0,t]\} \leq e^{-\epsilon t}/\epsilon$.*

Proof. The corollary is immediate from the above description and Lemma 5.4. □

6. A measure of maximal entropy

As in Section 5 denote by $\mathcal{M}_{\rm inv}(\mathcal{Q})$ the space of all Φ^t-invariant Borel probability measures on a connected component $\mathcal{Q}$ of a stratum $\mathcal{Q}(k_1, \dots, k_\ell)$. For every $\nu \in \mathcal{M}_{\rm inv}(\mathcal{Q})$ let h_ν be the *entropy* of ν and define

$$h_{\rm top} = \sup\{h_\nu \mid \nu \in \mathcal{M}_{\rm inv}(\mathcal{Q})\}.$$

A *measure of maximal entropy* is by definition a measure $\nu \in \mathcal{M}_{\rm inv}(\mathcal{Q})$ such that $h_\nu = h_{\rm top}$. The goal of this section is to show that the Lebesgue measure λ on $\mathcal{Q}$ is a measure of maximal entropy. Recall that the entropy of the Lebesgue measure on the open stratum $\mathcal{Q}(1, \dots, 1)$ (which is connected) equals $6g - 6 + 2m$.

We resume the assumptions and notations from Section 5. Recall in particular the definition of the subshift of finite type (Ω, T) and the continuous roof function ρ on the Borel subset $\mathcal{DU}$ of Ω. Let again $\mathcal{M}_T(\Omega)$ be the space of all T-invariant Borel probability measures on Ω which give full measure to the set $\mathcal{DU}$ of doubly uniquely ergodic sequences. Denote by h_ν the entropy of a measure $\nu \in \mathcal{M}_T(\Omega)$. Define the *pressure* ${\rm pr}(f)$ of a continuous function f on $\mathcal{DU} \subset \Omega$ by

$${\rm pr}(f) = \sup\{h_\nu - \int f d\nu \mid \nu \in \mathcal{M}_T(\Omega)\}.$$

Our first goal is to show that the pressure of the function $h\rho$ vanishes where $h = h_\lambda$ is the entropy of the Lebesgue measure λ. Note that this makes sense since the topological entropy of a subshift of finite type is finite and hence the entropy of any T-invariant measure on Ω is uniformly bounded from above. Moreover, the function ρ is nonnegative and uniformly bounded. We need the following preparation.

Lemma 6.1. *For $a > \sup \rho$ there is a sequence of Hölder continuous non-negative functions $\rho_i : \Omega \to [0, a]$ $(i > 0)$ on Ω such that $\rho_i \geq \rho$ for all i and that $\rho_i \to \rho$ uniformly on compact subsets of $\mathcal{DU}$.*

Proof. Let $a > \sup \rho$ and let $\mathcal{A} = \{\varphi_s \mid s \in I\}$ be the family of all continuous functions $\varphi_s : \Omega \to [0, a]$ which satisfy $\varphi_s \geq \rho$. Note that the family $\mathcal{A}$ is partially ordered by $\varphi_s \geq \varphi_t$ if $\varphi_s(x) \geq \varphi_t(x)$ for all x. Moreover, if $\varphi_s, \varphi_t \in \mathcal{A}$ then the function $\psi(x) = \min\{\varphi_s(x), \varphi_t(x)\}$ is a continuous majorant of ρ and hence it is contained in $\mathcal{A}$.

Define $\varphi_0 = \inf\{\varphi_s \mid s \in I\}$. Then φ_0 is the infimum of a family of continuous functions and hence φ_0 is lower semi-continuous. Moreover we have $\varphi_0 \geq \rho$. We claim that $\varphi_0(x) = \rho(x)$ for every $x \in \Omega$. For this assume otherwise. Then there is some $x \in \mathcal{DU}$ such that $\varphi_0(x) > \rho(x)$. Since ρ is continuous and φ_0 is lower semi-continuous there is a neighborhood U of x in Ω and a number $\epsilon > 0$ such that $\varphi_0 \geq \rho(x) + 2\epsilon \geq \rho(x) + \epsilon \geq \rho$ on $U \cap \mathcal{DU}$. Choose a continuous function $f : U \to [\rho(x) + \epsilon, a]$ which equals $\rho(x) + \epsilon$ at x and which equals a on $\Omega - U$. Then for each s the function $\min\{\varphi_s, f\}$ is contained in $\mathcal{A}$ and assumes the value $\rho(x) + \epsilon < \varphi_0(x)$ at x which is a contradiction.

Now since Ω is a compact metrizable space, there is a countable set $J \subset I$ such that $\varphi_0 = \inf\{\varphi_s \mid s \in J\}$ (see [D84]). Define a countable sequence $\psi_i \subset \mathcal{A}$ by $\psi_i = \min\{\varphi_1, \dots, \varphi_i\}$ where indices are taken with respect to an enumeration of J. Then the sequence (ψ_i) is decreasing and satisfies $\psi_i \to \rho$.

Define $\tilde{\psi}_i = \psi_i + 1/i$. Since Hölder functions on Ω are dense in the space of all continuous functions, there is a sequence $\rho_i : \Omega \to [0, a+1]$ of Hölder functions such that for each i we have $\|\rho_i - \tilde{\psi}_i\| < 1/i$. Since $a > \sup \rho$ was arbitrarily chosen, we obtain with this construction a sequence ρ_i with the required properties. □

We need the following simple pressure computation.

Lemma 6.2. $0 \leq \mathrm{pr}(h\rho) \leq \liminf_{i\to\infty} \mathrm{pr}(h\rho_i)$.

Proof. Let $\epsilon > 0$ and let $\mu \in \mathcal{M}_T(\Omega)$ be such that

$$h_\mu - \int h\rho d\mu \geq \mathrm{pr}(h\rho) - \epsilon.$$

Since the functions ρ_i, ρ are nonnegative and bounded and satisfy $\rho_i \to \rho$ pointwise on $\mathcal{DU}$ there is some $i_0 \geq 0$ such that $\int h\rho_i d\mu \leq \int h\rho d\mu + \epsilon$ for all $i \geq i_0$. But this just means that $\mathrm{pr}(h\rho_i) \geq h_\mu - \int h\rho_i d\mu \geq \mathrm{pr}(h\rho) - 2\epsilon$ for all $i \geq i_0$. Now $\epsilon > 0$ was arbitrary from which we obtain that $\mathrm{pr}(h\rho) \leq \liminf_{i\to\infty} \mathrm{pr}(h\rho_i)$.

To show the second part of the lemma, note by Lemma 5.2 that there is a T-invariant Borel probability measure μ on Ω with $\Xi_*\mu = \lambda$ where λ is the Φ^t-invariant Lebesgue measure on $\mathcal{Q}(S)$. Since λ is ergodic under the action of the Teichmüller flow, we may assume that the measure μ is ergodic under the shift T. Thus by Abramov's formula, the entropy $h_{\tilde{\mu}}$ of the (normalized) invariant measure $\tilde{\mu}$ for the suspension flow (X, Θ^t) defined by μ equals $h_\mu / \int \rho d\mu$. On the other hand, since Ξ is a semi-conjugacy, the entropy $h = h_\lambda$ of λ does not exceed the entropy of $\tilde{\mu}$ and hence we have $h_\mu / \int \rho d\mu \geq h$. From this the second part of the lemma is immediate. □

Since each of the functions ρ_i on Ω is Hölder continuous, there is a unique *Gibbs equilibrium state* for $h\rho_i$. By definition, this is the unique T-invariant Borel probability measures μ_i on Ω such that

$$h_{\mu_i} - \int h\rho_i d\mu_i = \sup\{h_\nu - \int h\rho_i d\nu \mid \nu\} = \mathrm{pr}(h\rho_i)$$

where here ν runs through *all* T-invariant Borel probability measure on Ω. The measures μ_i gives full mass to the normal points and is *absolutely continuous* with respect to the stable and unstable foliation. More precisely, there is a family μ_i^+ of conditional measures on stable manifolds for the shift space Ω such that

$$\frac{d\mu_i^+ \circ T}{d\mu_i^+}(x_i) = (h\rho_i + \mathrm{pr}(h\rho_i))(x_i).$$

Let $\tilde{\mu}_i$ be the Θ^t-invariant probability measure for the suspension flow on X defined by μ_i. The image under the semi-conjugacy Ξ of the measure $\tilde{\mu}_i$ is a Φ^t-invariant Borel probability measure $\Xi_*(\mu_i)$ on the component $\mathcal{Q}$ of $\mathcal{Q}(k_1, \dots, k_\ell)$. We use Proposition 3.3 of [H07a] to show.

Lemma 6.3. *For every i we have $h_{\mu_i}/\int \rho d\mu_i \leq h$.*

Proof. Since for each i the measure μ_i is a Gibbs equilibrium state for the Hölder function ρ_i the measure μ_i is absolutely continuous with respect to the stable and unstable foliation, with conditional measures μ_i^+ on unstable manifolds. Moreover, by Abramov's formula, if $\tilde{\mu}_i$ denotes the invariant measure for the suspension flow (X, Θ^t) then for $\tilde{\mu}_i$-almost every $x \in X$ we have

$$h_{\mu_i}/\int \rho d\mu_i = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{d}{dt}\mu_i^+(\Theta^t x)|_{t=0} dt.$$

Since our semi-conjugacy Ξ is locally finite-to-one there is a piecewise continuous section $\sigma : \Xi(\mathcal{DU}) \to X$. Choose an open subset U of $\mathcal{Q}(S)$ such that the restriction of this section to $U \cap \Xi(\mathcal{DU})$ is continuous. Then the restriction of μ_i to the image of U defines a measure ν on U which is absolutely continuous with respect to the stable and the unstable foliation.

Let q be a density point for ν. Then q is the image of a normal sequence $(x_i) \in \mathcal{DU}$ and hence q is recurrent. Moreover, the orbit of (x_i) under the shift T comes arbitrarily close to (x_i) arbitrarily often. By Proposition 3.3 of [H07a], there is a neighborhood V of q in $\mathcal{Q}(S)$ and a neighborhood D of q in $W^u(q)$ with the following property. If ν^+ is the family of conditional measures of ν on unstable manifolds then there is a small constant $c > 0$ such that whenever $\Phi^t q \in V$ for some $t > 0$ then the ν^+-mass of $\Phi^{-t}D$ is contained in the interval

$$[e^{\int_0^t h\rho_i + \mathrm{pr}(h\rho_i) ds}/c, ce^{\int_0^t h\rho_i + \mathrm{pr}(h\rho_i) ds}].$$

Moreover the sets $\{\Phi^{-t}D \mid \Phi^t q \in V\}$ form a Vitali relation for ν^+ as well as for a family λ^u of conditionals for the Φ^t-invariant Lebesgue measure λ on $\mathcal{Q}$. Since the conditionals λ^u of the Lebesgue measure on unstable manifolds transform via $\lambda^u \circ \Phi^t = e^{ht}\lambda^u$, if $\int h\rho_i + \mathrm{pr}(h\rho_i) d\mu_i/\int \rho d\mu_i = h_{\mu_i}/\int \rho d\mu_i \geq h$ then we obtain a contradiction. This shows the lemma. $\square$

As an immediate corollary we obtain.

Corollary 6.4. $\mathrm{pr}(h\rho) = 0$.

Proof. Since $\rho_i \geq \rho$ and μ_i is a Gibbs equilibrium state for ρ_i, for every $i > 0$ we have $\mathrm{pr}(h\rho_i) \leq h_{\mu_i} - \int h\rho d\mu_i$. Thus $\mathrm{pr}(h\rho_i) \leq 0$ for all i by Lemma 6.3 and therefore $\mathrm{pr}(h\rho) = 0$ by Lemma 6.2. □

The above corollary immediately implies the first part of Theorem 1 from the introduction.

Proposition 6.5. *The Lebesgue measure on a component $\mathcal{Q}$ of an odd stratum $\mathcal{Q}(k_1, \dots, k_\ell)$ is a measure of maximal entropy.*

Proof. Let $\epsilon > 0$ and let μ be a Φ^t-invariant Borel probability measure on the component $\mathcal{Q}$ of an odd stratum $\mathcal{Q}(k_1, \dots, k_\ell)$ with $h_\mu \geq h_{\mathrm{top}} - \epsilon$. We may assume that μ is ergodic. By Lemma 5.2, there is a measure ν on Ω such that $\Xi_*\nu = \mu$. If h_ν is the entropy of ν then $h_\mu \leq h_\nu / \int \rho d\nu$ by Abramov's formula and therefore $h_\mu \leq h$ by Corollary 6.4. This shows the proposition. □

7. Uniqueness

In this section we show that a measure of maximal entropy for the Teichmüller flow Φ^t on a connected component $\mathcal{Q}$ of an odd stratum $\mathcal{Q}(k_1, \dots, k_\ell)$ is unique. Following [BG07], we use the results of Section 5 and construct a topological Markov shift on a countable set S of symbols, given by a transition matrix $A = (a_{ij})_{S \times S}$. The phase space of this shift is the space

$$\Sigma = \{(y_i) \in S^{\mathbb{Z}} \mid a_{y_i y_{i+1}} = 1 \text{ for all } i\}.$$

We find a positive roof function $\varphi : \Sigma \to (0, \infty)$ of bounded variation and only depending on the future such that the suspension of the shift $\sigma : \Sigma \to \Sigma$ with roof function φ admits a semi-conjugacy into Φ^t. From this and the results of Buzzi and Sarig [BS03], uniqueness of a measure of maximal entropy easily follows.

The subset $\mathcal{MQ}(S)$ of $\mathcal{Q}$ of all quadratic differentials with horizontal measured geodesic lamination whose support is contained in $\mathcal{L}(k_1, \dots, k_\ell)$ is a Φ^t-invariant Borel subset $\mathcal{Q}$. By Lemma 2.5 this set has full mass for every Φ^t-invariant Borel probability measure on $\mathcal{Q}$.

We say that a simple closed curve c on S *fills* a web train track τ if c is carried by τ and if the transverse measure on τ defined by c is positive on every branch. A *vertex cycle* of a web train track τ is a measured geodesic lamination which is carried by τ and which defines an extreme point for the compact convex space of projective measured geodesic laminations carried by τ. Masur and Minsky [MM99] observed that a vertex cycle is always defined by a simple closed curve which is carried by τ. Denote by $|\mu|$ the total weight of a transverse measure μ on a complete train track τ. Using the assumptions and notations from the Sections 2-6 we observe.

Lemma 7.1. *Let $\tau_0 \in \mathcal{T}(k_1,\dots,k_\ell)$ be a web train track, let $\zeta \in \mathcal{V}_0(\tau_0)$ be a uniquely ergodic measured geodesic lamination whose support is contained in $\mathcal{L}(k_1,\dots,k_\ell)$ and let $(\tau_i) \subset \mathcal{T}(k_1,\dots,k_\ell)$ be a full splitting sequence with $\cap_i \mathcal{V}(\tau_i) = (0,\infty)\zeta$. Then there is some $k > 0$ with the following properties.*

(1) *Every vertex cycle of τ_k fills τ_0.*
(2) *Let $\mu,\nu \in \mathcal{V}_0(\tau_0)$ be normalized transverse measures on τ_0 defined by measured geodesic laminations which are carried by τ_k. Then*
$$1/\sqrt{2} \leq \mu(b)/\nu(b) \leq \sqrt{2}$$
for every branch b of τ_0.
(3) *There is a constant $\delta > 0$ with the following property. Let $\mu,\nu \in \mathcal{V}(\tau_k)$ be positive on every branch and let $a_0 = \min\{\mu(b)/\nu(b)\}$ where b runs through the branches of τ_k. Then the measures μ_0,ν_0 on τ_0 induced by μ,ν satisfy*
$$\min\{\mu(b)/\nu(b) \mid b\} \geq a_0 + \delta(|\mu|/|\nu| - a_0).$$

Proof. Let $\tau_0 \in \mathcal{T}(k_1,\dots,k_\ell)$ be a web train track and let $\zeta \in \mathcal{V}_0(\tau_0)$ be a uniquely ergodic measured geodesic lamination whose support is contained in $\mathcal{L}(k_1,\dots,k_\ell)$. Let (τ_i) be a full splitting sequence such that $\cap_i \mathcal{V}(\tau_i) = (0,\infty)\zeta$. For every $i > 0$ let c_i be a vertex cycle for τ_i. Since ζ is uniquely ergodic, the normalized transverse measures $\mu_i \in \mathcal{V}_0(\tau_0)$ defined by c_i converge in $\mathcal{V}_0(\tau_0)$ to ζ. Now the support of ζ is contained in $\mathcal{L}(k_1,\dots,k_\ell)$ and hence ζ gives positive weight to every branch of τ_0. Thus for all sufficiently large i, the same holds true for the measures μ_i. Since the number of vertex cycles of a web train track τ on S is bounded from above by a universal constant, we can find a number $i_0 > 0$ such that for every $i \geq i_0$, every vertex cycle of τ_i fills τ_0. This shows the first part of the lemma.

To show the second part, using the above notations, recall that a sequence $(\mu_i) \subset \mathcal{V}_0(\tau_0)$ defined by normalized images of vertex cycles on τ_i under a carrying map $\tau_i \to \tau_0$ converges weakly to ζ. Since ζ is positive on every branch of τ_0, there is a number $i_1 \geq i_0$ such that $\zeta(b)/\mu_i(b) \in [1/2^{1/4}, 2^{1/4}]$ for every branch b of τ_0, every $i \geq i_1$ and every transverse measure $\mu_i \in \mathcal{V}_0(\tau_0)$ defined by a vertex cycle of τ_i. Now every transverse measure on a full train track can be written as a linear combination of vertex cycles with nonnegative coefficients, moreover a carrying map is linear on transverse measures. From this the second part of the lemma follows.

To show the third part of the lemma, let $i_0 > 0$ be as above, let $k \geq i_0$, let $\xi,\chi \in \mathcal{V}_0(\tau_k)$ be normalized vertex cycles on τ_k and let ξ_0,χ_0 be the transverse measures on τ_0 defined by ξ,χ via a carrying map. By the choice of k, the measures ξ_0,χ_0 on τ_0 are positive and hence there is a number $\kappa = \kappa(\xi,\chi) > 0$ such that $\xi_0 = \kappa\chi_0 + \alpha$ where α is a transverse measure on τ_0. Since the number of vertex cycles for τ_k is finite, the minimum δ of the numbers $\kappa(\xi,\chi)$ for all pairs of normalized vertex cycles for τ_k is positive. Now every measure $\mu \in \mathcal{V}_0(\tau_k)$ is a convex combination of vertex cycles and hence this means the following. If $\mu,\nu \in \mathcal{V}_0(\tau_k)$ and is μ_0,ν_0 are the transverse measures on τ_0 defined by μ,ν then $\mu_0 = \delta\nu_0 + \alpha$ where α is a transverse measure on τ_0.

Now let μ,ν be any *positive* transverse measures on τ_k. If $a_0 = \min\{\mu(b)/\nu(b) \mid b\}$ then $\mu - a_0\nu$ is a transverse measure on τ_k of total weight $|\mu - a_0\nu| = |\mu| - a_0|\nu|$.

Let μ_0, ν_0 be the transverse measures on τ_0 defined by μ, ν. By our choice of δ we have

$$\mu_0 - a_0\nu_0 = \delta(|\mu| - a_0|\nu|)\nu_0/|\nu| + \alpha$$

for a transverse measure α on τ_0 and hence

$$\mu_0 = (a_0 + \delta(|\mu|/|\nu| - a_0))\nu_0 + \alpha.$$

This immediately implies the third part of the lemma. □

We call a finite full splitting sequence $(\tau_i)_{0\leq i\leq k}$ *weakly tight* if the train track τ_k satisfies the conclusion of Lemma 7.1.

We need the following simple observation.

Lemma 7.2. *Let $(\tau_i)_{0\leq i\leq k} \subset \mathcal{T}(k_1, \dots, k_\ell)$ be any full splitting sequence, let $a_0 > 0$ and let μ, ν be transverse measures for τ_k with $\mu(b) \leq a_0\nu(b)$ for every branch b of τ_k. Then the transverse measures μ_0, ν_0 on τ_0 defined by μ, ν satisfy $\mu_0(h) \leq a_0\nu_0(h)$ for every branch h of τ_0.*

Proof. The lemma follows immediately from the fact that the natural map $\mathcal{V}(\tau_k) \to \mathcal{V}(\tau_0)$ can be represented by a linear map with nonnegative entries from the finite dimensional vector space of weight functions on the branches of τ_k to the vector space of weight functions on the branches of τ_0. □

Let μ be an ergodic Φ^t-invariant probability measure on $\mathcal{Q}$. Then the horizontal measured geodesic lamination of every density point $q \in \mathcal{Q}(k_1, \dots, k_\ell)$ for μ is contained in $\mathcal{L}(k_1, \dots, k_\ell)$. Let $p > 0$ be as in Lemma 4.2. By Lemma 4.5 there are finitely many web train tracks $\tau_1, \dots, \tau_n \in \mathcal{T}(k_1, \dots, k_\ell)$ such that $\Phi^t q \in \mathcal{Q}(\tau_i)$ for some $t \in [0, p\log 2]$. Since the horizontal measured geodesic lamination ζ of q is strongly uniquely ergodic, for every $\epsilon > 0$ there is a number $k > 0$ such that the following holds. Let $i \leq n$ and let $(\sigma_j^i)_{0\leq j\leq k}$ be a full numbered splitting sequence of length k issuing from $\sigma_0^i = \tau_i$ with the property that σ_k^i carries ζ. Then the sequence $(\sigma_j^i)_{0\leq j\leq k}$ is weakly tight. Since q is a density point for μ we may assume that the set V of all quadratic differentials $z \in V$ such that $\Phi^t z \in \mathcal{Q}(\tau_i)$ for some i and for some $t \in [0, p\log 2]$ is a neighborhood of q in $\mathcal{Q}$ of positive μ-mass.

Recall from Section 3 the definition of an admissible sequence. Define S to be the set of all finite admissible sequences $(x_i)_{0\leq i\leq s}$ with the following additional properties.

(1) $s \geq 2k$ and the sequences $(x_j)_{0\leq j\leq k}$ and $(x_j)_{s-k\leq j\leq s}$ are realized by one of the full splitting sequences $(\sigma_j^i)_{0\leq j\leq k}$.
(2) There is no number $t \in [2k, s)$ such that the sequence $(x_j)_{t-s\leq j\leq t}$ is realized by one of the full splitting sequences $(\sigma_j^i$. Property (1) above.

Note that S is a countable set of states. Define a transition matrix $A = (a_{ij})_{S\times S}$ by requiring that $a_{ij} = 1$ if and only if the sequence $(x_p)_{0\leq p\leq s}$ representing the symbol i and the sequence $(y_t)_{0\leq t\leq u}$ representing the symbol j satisfy $y_t = x_{s-k+t}$ for every $t \in \{0, \dots, k\}$. Let Σ be the set of all biinfinite sequences $(y_i) \subset S^{\mathbb{Z}}$ with $a_{y_iy_{i+1}} = 1$ for all i, equipped with the (binifinite) shift $\sigma : \Sigma \to \Sigma$. There is a natural continuous injective map $\Pi : \Sigma \to \Omega$ whose image contains the set of all normal sequences and is contained in the set $\mathcal{DU}$ of all uniquely ergodic sequences. Moreover by the choice of S and Σ, if $\tilde\lambda, \tilde\mu$ are shift invariant probability measures on Ω whose images under the map Ξ_* are the Lebesgue measure λ and the measure μ, respectively, then $\tilde\lambda, \tilde\mu$ give full mass to $\Pi(\Sigma)$. Since Π is injective, the measures $\tilde\mu, \tilde\lambda$ define shift invariant probability measures on Σ.

Define a roof function φ on Σ by associating to an infinite sequence $(y_i) \in \Sigma$ with $\Pi(y_i) = (x_j) \in \Omega$ and $y_0 = (x_i)_{0\leq i\leq s}$ the value $\sum_{i=0}^{s-k} \rho(T^i x_j)$. It follows from Lemma 4.2 that φ is positive and unbounded.

Define the n-th variation of φ by

$$\mathrm{var}_n(\varphi) = \sup\{\varphi(y) - \varphi(z) \mid y_i = z_i \text{ for } i = 0, \ldots, n-1\}.$$

We have.

Lemma 7.3. $\sum_{n\geq 2} \mathrm{var}_n(\varphi) < \infty$.

Proof. Let $n \geq 2$ and let $(y_i), (z_i) \in \Sigma$ be such that $y_i = z_i$ for $i = 0, \ldots, n-1$. By definition and by Lemma 4.2, there is a finite full numbered splitting sequence $(\tau_i)_{0\leq i\leq u}$, there are numbers $k \leq \ell_1 - k < \ell_1 < \cdots < \ell_{n-1} = u$ and there are two uniquely ergodic measured geodesic laminations $\mu, \nu \in \mathcal{V}_0(\tau_0)$ with support in $\mathcal{L}(k_1, \ldots, k_\ell)$ such that the following holds.

(1) The measured geodesic lamination μ, ν projects to the horizontal measured geodesic laminations of $\Xi\Pi(y_i), \Xi\Pi(z_i)$.
(2) μ, ν are carried by τ_u.
(3) The sequence $(\tau_i)_{0\leq i\leq k}$ and each of the sequences $(\tau_i)_{\ell_j - k\leq i\leq \ell_j}$ is weakly tight.
(4) $\varphi(y_i) = \sum_{j=0}^{\ell_1-k} \rho(T^j\Pi(y_i))$ and $\varphi(z_i) = \sum_{j=0}^{\ell_1-k} \rho(T^j\Pi(z_i))$.

We show by induction on n that there is a number $\kappa \in (0,1)$ not depending on our sequences such that

$$1 - \kappa^{n-1} \leq \min\{\mu(b)/\nu(b) \mid b\} \leq \max\{\mu(b)/\nu(b) \mid b\} \leq \frac{1}{1-\kappa^{n-1}}.$$

For this note first that the claim holds true for $n = 2$. Namely, since μ, ν are carried by τ_{ℓ_1} they are carried by τ_{ℓ_1-k} as well. Thus the normalized measures $\tilde\mu, \tilde\nu \in \mathcal{V}_0(\tau_{\ell_1-k})$ are defined. By Lemma 7.1 and by the definition of the sequence $(\tau_j)_{\ell_1-k\leq j\leq k}$, these measures satisfy $\tilde\mu(b)/\tilde\nu(b) \in [1/\sqrt{2}, \sqrt{2}]$ for every branch b of τ_{ℓ_1-k}. Now let $c > 0$ be such that the transverse measure μ_0 on τ_0 defined by the measured geodesic lamination $c\tilde\mu$ is normalized. By Lemma 7.2, the transverse measure $c\tilde\nu$ on τ_0 satisfies $\mu_0(b)/c\tilde\nu(b) \in [1/\sqrt{2}, \sqrt{2}]$ for every branch b of τ_0. Since μ_0 is normalized, the total weight χ of the transverse measure $c\tilde\nu$ on τ_0 is contained in $[1/\sqrt{2}, \sqrt{2}]$. For $\nu_0 = c\tilde\nu/\chi \in \mathcal{V}_0(\tau_0)$ we then have $\mu_0(b)/\nu_0(b) \in [1/2, 2]$ for every branch b of τ_0. This shows our claim for $n = 2$ with the constant $\kappa = 1/2$.

Now let $\delta \leq \frac{1}{2}$ be as in the third part of Lemma 7.1 and assume that the claim holds true for $\kappa = 1 - \delta$ and all $m \in [2, n-1]$. Assume that $(y_i), (z_i) \in \Sigma$ are such that $y_i = z_i$ for $0 \leq i \leq n-1$. Using the above notations, let $\mu_1, \nu_1 \in \mathcal{V}_0(\tau_{\ell_1-k})$ be normalized multiples of μ,ν. By our induction hypothesis, applied to the sequences $\sigma(y_i),\sigma(z_i)$, we have $1-\kappa^{n-2}\leq \mu_1(b)/\nu_1(b)=1+\kappa^{n-2}$ for every branch b of τ_{ℓ_1-k}. Lemma 7.1 and Lemma 7.2 then show that the transverse measures $\tilde\mu_1,\tilde\nu_1$ on τ_0 which are the images of the measures μ_1,ν_1 under a carrying map $\tau_{\ell_1-k}\to\tau_0$ satisfy

$$\tilde\mu_1(b)/\tilde\nu_1(b)\geq 1-\kappa^{n-2}+\delta(\kappa^{n-2})\geq 1-\kappa^{n-1}$$

for every branch b of τ_0 as required. Then the same holds true for the normalized measures.

By definition, our above estimate implies that $|\varphi(y_i)-\varphi(z_i)|\leq -\log(1-\kappa^{n-1})$ if $y_i=z_i$ for $0\leq i\leq n-1$. Now $\frac{-\log(1-t)}{t}\to 1$ $(t\to 0)$ from which the lemma follows. □

As before, let μ be an ergodic Borel probability measure of maximal entropy for the Teichmüller flow on $\mathcal{Q}$ with the additional property that q is a density point for μ. We observed above that both μ and the Lebesgue measure λ induce a shift invariant Borel probability measure on the space Σ constructed above. Let Σ^+ be the one-sided shift defined by the alphabet S and the transition matric $A=(a_{ij})$ and let $\pi:\Sigma\to\Sigma^+$ be the natural projection. By construction, the roof function φ on Σ only depends on the future, i.e. we have $\varphi(y_i)=\varphi(z_i)$ whenever $y_i=z_i$ for all $i\geq 0$. Thus φ induces a positive roof function on Σ^+. By Lemma 7.3, this roof function is of bounded variation. As in [BG07], it now follows from the work of Buzzi and Sarig [BS03] that the measures μ,λ are *equilibrium states* for the function φ on Σ^+. However, by Theorem 1.1 of [BS03] there can be only one such equilibrium state. In other words, we have $\lambda=\mu$.

As a consequence, the Lebesgue measure λ is the only Φ^t-invariant Borel probability measure of maximal entropy on the component $\mathcal{Q}$ of the odd stratum $\mathcal{Q}(k_1,\dots,k_\ell)$. This completes the proof of Theorem 1 from the introduction.

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