ASYMPTOTIC DIMENSION AND THE DISK GRAPH II

URSULA HAMENSTÄDT

Abstract. We show that the asymptotic dimension of a hyperbolic relatively hyperbolic graph is finite provided that this holds true uniformly for the peripheral subgraphs and for the electrification. We use this to show that the asymptotic dimension of the disk graph of a handlebody of genus $g \geq 2$ is at most quadratic in the genus.

1. Introduction

A metric space (X,d) has *asymptotic dimension* $\mathrm{asdim}(X)$ at most n if for every number $R > 0$, there exists a covering of X by uniformly bounded sets such that every metric R-ball intersects at most $n+1$ of the sets in the cover. More generally, a collection of metric spaces has asdim at most n *uniformly* if for every R there are covers of each space whose elements are uniformly bounded over the whole collection.

The main goal of this article is to investigate the asymptotic dimension of a (not necessarily locally finite) hyperbolic graph $\mathcal{G}$ which is hyperbolic relative to a collection $\{H_c \mid c \in \mathcal{C}\}$ of peripheral subgraphs. This means the following. Define the $\mathcal{H}$-*electrification* $\mathcal{EG}$ of $\mathcal{G}$ to be the graph which is obtained from $\mathcal{G}$ by adding for every $c \in \mathcal{C}$ a new vertex v_c which is connected to each vertex $x \in H_c$ by an edge and which is not connected to any other vertex. We require that the graph $\mathcal{EG}$ is hyperbolic and that a property called *bounded penetration* holds true. We shall define this property in Section 2 and refer to [H16] for a detailed discussion. We show

Theorem 1. *Let $\mathcal{G}$ be a hyperbolic graph which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of peripheral subgraphs, with electrification $\mathcal{EG}$. If the collection H_c $(c \in \mathcal{C})$ has* $\mathrm{asdim}(H_c) \leq n$ *uniformly then* $\mathrm{asdim}(\mathcal{G}) \leq \mathrm{asdim}(\mathcal{EG}) + n + 1$.

Our second goal is to apply Theorem 1 to the *disk graph* of a *handlebody* of genus $g \geq 2$. Such a handlebody is a compact three-dimensional manifold H which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of g circles. Its boundary ∂H is an oriented surface of genus g.

The disk graph $\mathcal{DG}$ of H is the metric graph whose vertices are isotopy classes of properly embedded disks in H and where two such disks are connected by an edge of length one if they can be realized disjointly. Assigning to a disk its boundary then defines an embedding of the disk graph into the *curve graph* of ∂H. However, this inclusion is not a quasi-isometric embedding [MS13, H16, H11].

To describe the geometric structure of $\mathcal{DG}$ we use the next definition.

Date: February 12, 2019.

Partially supported by ERC Advanced Grant "Moduli"

AMS subject classification: 57M99.

Definition 1. A *hierarchy* of a hyperbolic metric graph $\mathcal{G}$ consists of a finite chain $\mathcal{G}_1, \dots, \mathcal{G}_k$ of hyperbolic graphs with the following properties.

(1) $\mathcal{G}_k = \mathcal{G}$.
(2) For all i, the graph $\mathcal{G}_{i+1}$ is hyperbolic relative to a family $\mathcal{H}_i$ of subgraphs, with electrification $\mathcal{G}_i$.

The graph $\mathcal{G}_1$ is called the *base* of the hierarchy, and the number k its *depth*.

Call the hierarchy *tame* if its base has finite asymptotic dimension and if moreover for each i there exists some n_i such that the family $\mathcal{H}_i$ of subgraphs of $\mathcal{G}_{i+1}$ has asdim $\leq n_i$ uniformly.

An inductive application of Theorem 1 leads to

Corollary 1. *The asymptotic dimension of a hyperbolic metric graph $\mathcal{G}$ which admits a tame hierarchy is finite.*

From [H16, H11] we deduce the following more precise version of the main result of [MS13].

Theorem 2. *The disk graph $\mathcal{DG}$ of H is hyperbolic and admits a tame hierarchy whose base is a quasi-isometrically embedded subgraph of the curve graph of ∂H. Furthermore,*

$$\operatorname{asdim}(\mathcal{DG}) \leq (3g-3)(6g-2).$$

Another geometrically defined graph which admits a tame hierarchy with base a curve graph is the graph of non-separating multicurves introduced in [H14]. Corollary 1 then yields that the asymptotic dimension of the graph of non-separating multicurves is finite as well.

Acknowledgement: This work is based upon work supported by the National Science Foundation under Grant No DMS-1440140 while the author was in residence at the Mathematical Science Research Institute in Berkeley, California, during the Fall 2016 semester.

2. Asymptotic dimension of hyperbolic relatively hyperbolic graphs

We begin with a general statement about hyperbolic relatively hyperbolic geodesic metric graphs. We mostly use the notations from [H16].

Consider a connected metric graph $\mathcal{G}$ in which a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs has been specified. Here $\mathcal{C}$ is a countable, finite or empty index set. The graph $\mathcal{G}$ is *hyperbolic relative to the family* $\mathcal{H}$ if the following properties are satisfied.

Define the *$\mathcal{H}$-electrification* $\mathcal{EG}$ of $\mathcal{G}$ to be the graph which is obtained from $\mathcal{G}$ by adding for every $c \in \mathcal{C}$ a new vertex v_c which is connected to each vertex in H_c by an edge and which is not connected to any other vertex. We require that the graph $\mathcal{EG}$ is hyperbolic in the sense of Gromov and that moreover the following *bounded penetration property* holds true.

Call a simplicial path γ in $\mathcal{EG}$ *efficient* if for every $c \in \mathcal{C}$ we have $\gamma(k) = v_c$ for at most one k. Note that if γ is an efficient simplicial path in $\mathcal{EG}$ which passes through $\gamma(k) = v_c$ for some $c \in \mathcal{C}$ then $\gamma(k-1), \gamma(k+1) \in H_c$.

We require that for every $L > 1$ there is a number $p(L) > 0$ with the following property. Let γ be an efficient L-quasi-geodesic in $\mathcal{EG}$, let $c \in \mathcal{C}$ and let $k \in \mathbb{Z}$ be such that $\gamma(k) = v_c$. If the distance in H_c between $\gamma(k-1)$ and $\gamma(k+1)$ is at least

$p(L)$ then every efficient L-quasi-geodesic γ' in $\mathcal{EG}$ with the same endpoints as γ passes through v_c. Moreover, if $k' \in \mathbb{Z}$ is such that $\gamma'(k') = v_c$ then the distance in H_c between $\gamma(k-1), \gamma'(k-1)$ and between $\gamma(k+1), \gamma'(k+1)$ is at most $p(L)$.

A family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs H_c of $\mathcal{G}$ is called *uniformly quasi-convex* if the inclusion $H_c \to \mathcal{G}$ is a quasi-isometric embedding with constant not depending on c. The following is Theorem 1 of [H16].

Theorem 2.1. *Let $\mathcal{G}$ be a metric graph which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs. If there is a number $\delta > 0$ such that each of the graphs H_c is δ-hyperbolic then $\mathcal{G}$ is hyperbolic. Moreover, the subgraphs H_c $(c \in \mathcal{C})$ are uniformly quasi-convex.*

We call a graph $\mathcal{G}$ with the properties stated in Theorem 2.1 a *hyperbolic relatively hyperbolic graph*. In the sequel we always assume that all assumptions in Theorem 2.1 are fulfilled.

We begin with collecting some easy geometric properties of a hyperbolic relatively hyperbolic graph as defined above. To this end recall that for every quasi-convex subgraph H of a hyperbolic graph $\mathcal{G}$ there is a coarsely well defined shortest distance projection $\Pi_H : \mathcal{G} \to H$, i.e. a projection which associates to a point in $\mathcal{G}$ a choice of a point in H of approximate shortest distance. Any other choice of such a point is of bounded distance, and this bound only depends on the hyperbolicity constant of $\mathcal{G}$ and the constant defining the quality of the quasi-isometric embedding $H \to \mathcal{G}$. The map Π_H is coarsely distance non-increasing, i.e. it increases distances at most by a fixed additive constant.

The following lemma shows that the subgraphs H_c of $\mathcal{G}$ fulfill the three axioms in Theorem B of [BBF15].

Lemma 2.2. *Let $\mathcal{G}$ be a hyperbolic graph which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs. We require that these subgraphs are δ-hyperbolic for some fixed number $\delta > 0$.*

(1) *There is a number $R > 0$ such that for $c \neq d \in \mathcal{C}$, $\operatorname{diam}(\Pi_{H_c}(H_d)) \leq R$.*

(2) *For $a, b, c \in \mathcal{C}$ define*

$$d_a(b, c) = \operatorname{diam}(\Pi_{H_a}(H_b) \cup \Pi_{H_a}(H_c)).$$

There exists a constant $\theta > 0$ such that for any triple of distinct elements $a, b, c \in \mathcal{C}$, at most one of the numbers

$$d_a(b, c), d_b(a, c), d_c(a, b)$$

is greater than θ.

(3) *For any $a, b \in \mathcal{C}$, the set*

$$\{c \in \mathcal{C} \mid d_c(a, b) > \theta\}$$

is finite.

Proof. To show the first property, recall from Theorem 2.1 that the subgraphs H_c of $\mathcal{G}$ are uniformly quasiconvex. By hyperbolicity of $\mathcal{G}$, this implies that there exists a number $D > 0$ such that for $c \in \mathcal{C}$, any geodesic in $\mathcal{G}$ connecting two points $x, y \in H_c$ is contained in the D-neighborhood $N_D(H_c)$ of H_c.

For $c \in \mathcal{C}$ write

$$\Pi_c = \Pi_{H_c} : \mathcal{G} \to H_c.$$

Using again hyperbolicity, we deduce the following. Let us assume that the diameter of the projection $\Pi_c(H_d)$ is large. Then there exist points $x, y \in H_d$ and a geodesic $\zeta : [a,b] \to \mathcal{G}$ connecting $\zeta(a) = x$ to $\zeta(b) = y$ which is contained both in a uniformly bounded neighborhood of H_c as well as in a uniformly bounded neighborhood of H_d. However, this violates the bounded penetration property. Namely, we can find an efficient quasi-geodesic γ in $\mathcal{E}\mathcal{G}$ connecting x to y which does not pass through v_d (but instead passes through v_c). We refer to the work [Si12] for a more detailed discussion of the various equivalent formulations of relative hyperbolicity, in particular in connection to the condition (α_1) formulated in [Si12] for the collection $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$.

To show the second property, let $R > 0$ be such that the first property is valid for this R. By hyperbolicity of $\mathcal{G}$ and uniform quasi-convexity of the subspaces $H_c \subset \mathcal{G}$ $(c \in \mathcal{C})$, there exists a number $R' > R$ such that for all $x, y \in \mathcal{G}$ with $d(\Pi_c(x), \Pi_c(y)) \geq R'$ the following two properties are satisfied.

(a) Any geodesic in $\mathcal{G}$ connecting x to y passes through a uniformly bounded neighborhood of both $\Pi_c(x)$ and $\Pi_c(y)$, say through the R''-neighborhood.
(b) Any geodesic in $\mathcal{E}\mathcal{G}$ connecting x to y passes through the special vertex v_c.

Now assume that $d_a(b,c) \geq 3R'$. Choose a point $x \in H_b$ and let $y = \Pi_c(\Pi_a(x)) \in \Pi_c(H_a)$; then $d(\Pi_a(x), \Pi_a(y)) > R'$. By the choice of $R' > R$ and by property (1) in the lemma, any geodesic connecting x to y passes through the R''-neighborhood of $\Pi_a(x)$. By hyperbolicity and uniform quasi-convexity of the subgraphs H_c, the projection $\Pi_c(x)$ is uniformly near the projection $\Pi_c(z)$ of any point z on a geodesic connecting x to a point on H_c provided that the distance between z and H_c is sufficiently large. Since furthermore the projection Π_c is coarsely distance non-increasing, the distance between $\Pi_c(x)$ and $\Pi_c(\Pi_a(x))$ is uniformly bounded. By the first part of the lemma, the same then holds true for $d_c(a,b)$. The same reasoning also shows that $d_b(a,c)$ is uniformly bounded. To summarize, the second statement in the lemma holds true for some number $\theta \geq 3R'$.

We are left with showing the third property. To this end let $x \in \Pi_a(H_b), y \in \Pi_b(H_a)$ and let γ be a geodesic connecting x to y in $\mathcal{E}\mathcal{G}$. Then γ passes through only finitely many of the special vertices v_c, say through the vertices $v_{c_1}, \dots, v_{c_u}$. However, by the above choice of the number $R' > 0$, if $c \in \mathcal{C}$ is such that $d_c(a,b) > \theta \geq 3R'$ then by property (b) above, any geodesic in $\mathcal{E}\mathcal{G}$ with endpoints x, y passes through the special vertex v_c. This shows that if $d_c(a,b) \geq \theta$ then $c \in \{c_1, \dots, c_u\}$. This completes the proof of the lemma. □

We use Lemma 2.2 to show Theorem 1 from the introduction.

Theorem 2.3. *Let $\mathcal{G}$ be a hyperbolic metric graph which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected uniformly hyperbolic subgraphs, with $\mathcal{H}$-electrification $\mathcal{E}\mathcal{G}$. If* $\operatorname{asdim}(H_c) \leq n$ *uniformly then* $\operatorname{asdim}(\mathcal{G}) \leq \operatorname{asdim}(\mathcal{E}\mathcal{G}) + n + 1$.

Proof. By Theorem B of [BBF15] and Lemma 2.2, there exists a quasi-tree of metric spaces $\mathcal{Y}$ which is built from the subgraphs H_c $(c \in \mathcal{C})$. The space $\mathcal{Y}$ is a connected geodesic metric graph, and it contains each of the graphs H_c as a subgraph. The vertices of $\mathcal{Y}$ are the vertices of the graphs H_c $(c \in \mathcal{C})$. Our goal is to construct a quasi-isometric embedding of $\mathcal{G}$ into the product of $\mathcal{E}\mathcal{G}$ with $\mathcal{Y}$.

This is sufficient for the purpose of the theorem. Namely, using the assumption on the asymptotic dimensions of the graphs H_c, Theorem B iv) of [BBF15] states that $\text{asdim}(\mathcal{Y}) \leq n+1$. Now the asymptotic dimension of the product $X \times Y$ of two metric spaces satisfies $\text{asdim}(X \times Y) \leq \text{asdim}(X) + \text{asdim}(Y)$, furthermore $\text{asdim}(X) \leq \text{asdim}(Y)$ if X admits a quasi-isometric embedding into Y (see [BD06]). Thus if $\mathcal{G}$ admits a quasi-isometric embedding into $\mathcal{EG} \times \mathcal{Y}$ equipped with the product d of the metric $d_{\mathcal{EG}}$ on $\mathcal{EG}$ and the metric $d_{\mathcal{Y}}$ of $\mathcal{Y}$ then

$$\text{asdim}(\mathcal{G}) \leq \text{asdim}(\mathcal{EG} \times \mathcal{Y}) \leq \text{asdim}(\mathcal{EG}) + n + 1.$$

The graph [BBF15] $\mathcal{Y}$ is the union of the graphs H_c ($c \in \mathcal{C}$) and a collection of additional edges of length one connecting these graphs. These edges are chosen as follows.

For all $c, e \in \mathcal{C}$, let $x_{c,e} \in \Pi_c(H_e) \subset H_c$ be a point of shortest distance in H_c to the graph H_e. Connect the point $x_{c,e}$ to the point $x_{e,c}$ by an edge if there does not exist an $a \in \mathcal{C}$ so that $d_a(c,e) \geq 2\theta$ for the threshold $\theta > 0$ from part (2) of Lemma 2.2.

Let γ be a geodesic in $\mathcal{EG}$ and let $v_{c_1}, \dots, v_{c_s}$ ($c_i \in \mathcal{C}$) be the special points on γ. For a number $R > 0$ define v_{c_i} to be *R-wide* for γ if the following holds true. Let $k_i > 0$ be such that $\gamma(k_i) = v_{c_i}$; then the distance in H_{c_i} between $\gamma(k_i - 1)$ and $\gamma(k_i + 1)$ is at least R. With this terminology, the points $x_{c,e}$ and $x_{e,c}$ are connected by an edge of length one if there exists a geodesic γ in $\mathcal{EG}$ connecting $x_{c,e}$ to $x_{e,c}$ which does not contain any θ-wide points (this is only a sufficient condition). Moreover, there exists a number $\Theta > \theta$ such that the points $x_{c,e}$ and $x_{e,c}$ are *not* connected by an edge if there exists a geodesic γ in $\mathcal{EG}$ connecting $x_{c,e}$ and $x_{e,c}$ which contains a Θ-wide point.

We now define a map $\mathcal{G} \to \mathcal{EG} \times \mathcal{Y}$ as follows. Fix a basepoint $x \in \mathcal{G}$ contained in one of the quasi-convex subspaces H_c. Associate to x the product $(x, x) \in \mathcal{EG} \times \mathcal{Y}$. For every vertex $y \in \mathcal{G}$ choose once and for all a geodesic γ_y in $\mathcal{EG}$ connecting x to y. Note that such a geodesic is efficient.

Let $v_{c_1}, \dots, v_{c_s} \in \mathcal{EG}$ be the special points traveled through by γ_y in this order ($c_i \in \mathcal{C}$). Let $k_s > 0$ be such that $v_{c_s} = \gamma_y(k_s)$ and define

$$\Psi(y) = \gamma(k_s + 1) \in H_{c_s} \subset \mathcal{Y}.$$

If γ_y does not travel through any special point then define $\Psi(y) = x \in H_c \subset \mathcal{Y}$.

We claim that the map

$$\Lambda : y \to \Lambda(y) = (y, \Psi(y)) \in \mathcal{EG} \times \mathcal{Y}$$

is a quasi-isometric embedding. To this end we show first that for any two vertices $y, z \in \mathcal{G}$ of distance one, the distance between $\Lambda(y)$ and $\Lambda(z)$ is uniformly bounded. Since $\mathcal{G}$ is a geodesic metric space, this then implies that the map Λ is coarsely Lipschitz.

Consider the geodesics γ_y, γ_z in $\mathcal{EG}$. Let $v_{c_1}, \dots, v_{c_s}$ be the special points on γ_y and let $w_{d_1}, \dots, w_{d_u}$ be those on γ_z ($c_i, d_j \in \mathcal{C}$). Let $i \leq s$ be the largest number so that $v_{c_i} \in \gamma_z$. Assume that $v_{c_i} = \gamma_y(k_i)$. By the bounded penetration property and the choice of θ, for no $j > i$ the vertex v_{c_j} is θ-wide for γ_y. If ℓ_j is such that $\gamma_z(\ell_j) = w_{d_j} = v_{c_i}$ (in fact, as γ_y, γ_z are both geodesics with the same initial point, we must have $\ell_j = k_i$, however this fact is not important for us), then using once more the bounded penetration property, the distance in $H_{c_i} = H_{d_j}$ between the exit points $q = \gamma_y(k_i + 1)$ of γ_y and $\gamma_z(\ell_j + 1)$ of γ_z is uniformly bounded.

It now suffices to show that the distance in $\mathcal{Y}$ between $\Psi(y)$ and the exit point q of γ_y in H_{c_i} is uniformly bounded. However, by definition, $\Psi(y)$ is the exit point of the intersection of γ_y with H_{c_s}. Thus if $i = s$ then $q = \Psi(y)$ and we are done. Otherwise note that by uniform quasi-convexity, the exit point q of γ_y from H_{c_i} is contained in a uniformly bounded neighborhood of $\Pi_{c_i}(H_{c_s})$. Thus by the first part of Lemma 2.2 and up to adjusting constants, the shortest distance projection of H_{c_s} into H_{c_i} is contained in the $3R' < \theta$-neighborhood of q (compare also [BBF15]).

Furthermore, as none of the vertices $v_{c_{i+1}}, \dots, v_{c_s}$ along the segment of γ_y connecting q to $\Psi(y) \in H_{c_s}$ is θ-wide, there is a segment in $\mathcal{Y}$ of length one connecting a point in H_{c_i} uniformly near q to a point in H_{c_s} uniformly near the entry point of γ_y in $H_{c_s} \subset \mathcal{Y}$. As v_{c_s} is not θ-wide along γ_y, this entry point is contained in a uniformly bounded neighborhood of $\Psi(y)$. Together this shows that indeed, $d_{\mathcal{Y}}(q, \Psi(y))$ is uniformly bounded.

The same reasoning applies to γ_z and shows that $d_{\mathcal{Y}}(\Psi(z), \gamma_z(\ell_j+1))$ is uniformly bounded. Since the distance in H_{c_i} between $\gamma_y(k_i + 1)$ and $\gamma_z(\ell_j + 1)$ is uniformly bounded, $d_{\mathcal{Y}}(\Psi(y), \Psi(z))$ is uniformly bounded. Note that this argument also shows that the map Ψ is coarsely independent on the choices of the geodesics γ_y. Replacing the geodesic γ_y in $\mathcal{EG}$ by another one with the same endpoints results in replacing $\Psi(y)$ by a point of uniformly bounded distance.

We showed so far that the map Λ is coarsely Lipschitz, and we are left with showing that there exists a constant $L > 1$ such that for all $y, z \in \mathcal{G}$, we have

$$d(\Lambda(y), \Lambda(z)) \geq d_{\mathcal{G}}(y, z)/L - L$$

where $d_{\mathcal{G}}$ is the distance in $\mathcal{G}$.

Following [H16], define the *enlargement* $\hat{\gamma}$ of a geodesic $\gamma : [0, n] \to \mathcal{EG}$ with endpoints $\gamma(0), \gamma(n) \in \mathcal{G}$ as follows. Let $0 < k_1 < \cdots < k_s < n$ be those points such that $\gamma(k_i) = v_{c_i}$ for some $c_i \in \mathcal{C}$. Then $\gamma(k_i - 1), \gamma(k_i + 1) \in H_{c_i}$. For each $i \leq s$ replace $\gamma[k_i - 1, k_i + 1]$ by a simplicial geodesic in H_{c_i} with the same endpoints.

Theorem 2.4 of [H16] shows that enlargements of geodesics in $\mathcal{EG}$ are uniform quasi-geodesics in $\mathcal{G}$. Thus it suffices to show the existence of a number $L' > 1$ with the following property. If $\hat{\gamma}$ is the enlargement of any geodesic in $\mathcal{EG}$, parametrized as a simplical edge path in $\mathcal{G}$, then

$$d(\Lambda(\hat{\gamma}(m)), \Lambda(\hat{\gamma}(n))) \geq |n - m|/L' - L'$$

for all m, n.

To this end we first show the following. Let $y \in \mathcal{G} \subset \mathcal{EG}$ and let $\hat{\gamma} : [0, R] \to \mathcal{G}$ be an enlargement of the geodesic γ connecting the basepoint $x = \hat{\gamma}(0)$ to $y = \hat{\gamma}(R)$. Then for $0 \leq u < R$ we have $d(\Lambda(\hat{\gamma}(u)), \Lambda(\hat{\gamma}(R))) \geq C_0|R - u|/ - 1/C_0$ where $C_0 < 1/\Theta$ is a universal constant.

Let $s \geq u$ be the maximum of all numbers so that $\hat{\gamma}[u, s] \in H_e$ for some $e \in \mathcal{C}$. If $\hat{\gamma}(u)$ is not contained in any of the special subspaces then put $s = u$. By construction of an enlargement, we have $\hat{\gamma}(s) = \gamma_y(t_0 + 1)$ for some t_0. Let $v_{c_1}, \dots, v_{c_j}$ ($c_i \in \mathcal{C}$) be the special points passed through by the geodesic $\gamma_y[s, R]$. Let us suppose that $v_{c_i} = \gamma_y(t_i)$ for some t_i; then $\gamma_y(t_i - 1), \gamma_y(t_i + 1) \in H_{c_i} \subset \mathcal{Y}$. With a small abuse of notation, write also $\gamma_y(t_0 - 1) = \hat{\gamma}(u)$.

Denote by $d_{H_{c_i}}$ the intrinsic path metric in H_{c_i}. By construction, the length $R - u$ of $\hat{\gamma}[u, R]$ is not larger than

$$d_{\mathcal{EG}}(\hat{\gamma}(u), \hat{\gamma}(R)) + \sum_{i \geq 0} d_{H_{c_i}}(\gamma_y(t_i - 1), \gamma_y(t_i + 1)).$$

On the other hand, by the definition of the map Ψ, by Theorem 4.13 of [BBF15] and the fact that the map Λ coarsely does not depend on the choice of the geodesics γ_y (which means that we may replace in its construction the geodesic $\gamma_{\hat\gamma(u)}$ by the restriction of γ_y), the distance in $\mathcal{EG}\times\mathcal{Y}$ between $\Lambda(\hat\gamma(u))$ and $\Lambda(\hat\gamma(R))$ is not smaller than

$$C_0\big(d_{\mathcal{EG}}(\hat\gamma(u),\hat\gamma(R))+\sum_{i\geq 0} d_{H_{c_i}}(\gamma_y(t_i-1),\gamma_y(t_i+1))\big)-1/C_0.$$

The factor C_0 arises from replacing some segments of length smaller than Θ (here Θ is as in the beginning of this proof) in the special subspaces H_e by the geodesics in $\mathcal{EG}$ with the same endpoints whose length equals two together with an application of Theorem 4.13 of [BBF15] which computes distances in $\mathcal{Y}$ up to a uniform multiplicative and additive error. Together this yields

$$d(\Lambda(\hat\gamma(u)),\Lambda(\hat\gamma(R))\geq C_0 d_{\mathcal{G}}(\hat\gamma(u),\hat\gamma(R))-1/C_0$$

which is what we wanted to show.

Now let $y,z\in\mathcal{G}$ be arbitrary vertices and let $\hat\gamma$ be an enlargement of a geodesic γ in $\mathcal{EG}$ connecting y to z. Let $\hat\gamma_y,\hat\gamma_z$ be enlargements of the geodesics γ_y,γ_z in $\mathcal{EG}$ connecting the fixed basepoint x to y,z. Assume that $\hat\gamma_y$ is parametrized on $[0,T_1]$ and $\hat\gamma_z$ is parametrized on $[0,T_2]$. By hyperbolicity of $\mathcal{G}$, there exists a point on $\hat\gamma$, say the point $\hat\gamma(u)$, which is uniformly near a point $\hat\gamma_y(s)$ on $\hat\gamma_y$ and a point $\hat\gamma_z(t)$ on $\hat\gamma_z$. As Λ is coarsely Lipschitz, the images of these three points under the map Λ are uniformly close. Furthermore, up to a uniform additive constant, the distance in $\mathcal{G}$ between y,z equals the sum of the distances between y and $\hat\gamma_y(s)$ and between z and $\hat\gamma_z(t)$. The same estimate also holds true for distances in $\mathcal{EG}$.

Using again Theorem 4.13 of [BBF15], $d_{\mathcal{Y}}(\Psi(y),\Psi(z))$ is proportional to the sum of the distances in the subgraphs H_{c_i} between entry and exit point of γ where the $c_i\in\mathcal{C}$ are those points for which v_{c_i} is wide along γ. But this implies that $d_{\mathcal{Y}}(\Psi(y),\Psi(z))$ is uniformly proportional to $d_{\mathcal{Y}}(\Psi(y),\Psi(\hat\gamma_y(s)))+d_{\mathcal{Y}}(\Psi(z),\Psi(\hat\gamma_z(t)))$. Together with the estimates from the beginning of this proof, this completes the proof of the theorem. □

Remark 2.4. It is immediate from the proof of Theorem 2.3 that upper bounds for the diameters in the sets of coverings which can be used to determine the asymptotic dimension for a hyperbolic relatively hyperbolic graph $\mathcal{G}$ are uniformly controlled by the data of the peripheral subgraphs and the bounds for the electrification $\mathcal{EG}$ of $\mathcal{G}$.

Remark 2.5. The proof of Theorem 2.3 together with Theorem B of [BBF15] also shows the following. If $\mathcal{G}$ is a hyperbolic graph as in the theorem, if $\mathcal{EG}$ is a quasi-tree and if each of the peripheral subgraphs are quasi-trees, then $\mathcal{G}$ admits an quasi-isometric embedding into the product of two quasi-trees.

3. A tame hierarchy for the disk graph

The goal of this section is to apply Theorem 1 to two geometric graphs which are related to surfaces.

Let us consider a handlebody H of genus $g\geq 2$. This is a compact 3-manifold with boundary ∂H which is a regular neighborhood of a bouquet of g circles in $\mathbb{R}^3$. A *disk* in H is a properly embedded disk $D\subset H$ whose boundary is a non-contractible curve in ∂H.

A connected essential subsurface X of ∂H is called *thick* if the following holds true.

(1) Every disk in H intersects X.
(2) If $c \subset X$ is a simple closed curve which is disjoint from all boundaries of disks which are completely contained in X then c is either contractible or homotopic into the boundary of X.

An example of a thick subsurface is the entire boundary ∂H.

The *disk graph* $\mathcal{DG}(X)$ of X is the graph whose vertices are isotopy classes of disks with boundary in X and where two such disks are connected by an edge of length one if they can be realized disjointly.

In [H16, H11], we defined two more graphs whose vertices are isotopy classes of disks with boundary in X. The *electrified disk graph* $\mathcal{EDG}(X)$ is obtained from $\mathcal{DG}(X)$ by adding an edge between any two disks with boundary in X which are disjoint from a common essential simple closed curve in X.

An *I-bundle generator* in a thick subsurface X is an essential simple closed curve $\gamma \subset X$ with the following property. There exists a compact surface F with non-empty boundary ∂F, and there is an orientation preserving embedding Ψ of the oriented I-bundle $\mathcal{J}(F)$ over F into H which maps a boundary component α of ∂F to γ and which maps the union of the I-bundle over α with the *horizontal boundary* of $\mathcal{J}(F)$ (i.e. the subset of the boundary which is disjoint from the interiors of the intervals of the I-bundle) to the complement in X of a tubular neighborhood of the boundary ∂X of X.

The *superconducting disk graph* $\mathcal{SDG}(X)$ is obtained from the electrified disk graph $\mathcal{EDG}(X)$ by adding an edge of length one between any two disks whose boundaries intersect an I-bundle generator γ in X in precisely two points.

Denote by $\mathcal{CG}(X)$ the *curve graph* of X. This is the hyperbolic geodesic metric graph whose vertices are essential simple closed curves in X and where two such vertices are connected by an edge of length one if and only if they can be realized disjointly. The following is Theorem 5.2 of [H11].

Theorem 3.1. *There is an effectively computable number $L > 1$ only depending on the genus of H such that for every thick subsurface X of ∂H, the vertex inclusion which maps a disk with boundary in X to its boundary defines an L- quasi-isometric embedding $\mathcal{SDG}(X) \to \mathcal{CG}(X)$.*

Denoting by $\chi(X)$ the Euler characteristic of X, we obtain as an immediate consequence of Theorem 3.1 the following

Corollary 3.2. *The graphs $\mathcal{SDG}(X)$ are hyperbolic, and of asymptotic dimension at most $2|\chi(X)|$, uniformly.*

Proof. A quasi-isometrically embedded geodesic metric subgraph of a hyperbolic geodesic metric graph satisfies the thin triangle condition and hence it is hyperbolic.

It is immediate from the definition that the existence of a quasi-isometric embedding $f : Y \to Z$ implies that $\text{asdim}(Y) \leq \text{asdim}(Z)$. Now the asymptotic dimension of the curve graph of an oriented surface X of finite type is at most $2|\chi(X)|$ [BB15] uniformly (see also the earlier work [BF08] for finiteness) and therefore by Theorem 3.1, the asymptotic dimension of $\mathcal{SDG}(X)$ is at most $2|\chi(X)|$ uniformly as claimed. ∎

Let again X be a thick subsurface of ∂H and let γ be an I-bundle generator in X. Denote by $\mathcal{E}(\gamma)$ the subgraph of $\mathcal{DG}(X)$ of all disks which intersect γ in precisely two points. We have

Lemma 3.3. *The graphs $\mathcal{E}(\gamma)$ satisfy* asdim $\leq |\chi(X)| + 2$ *uniformly.*

Proof. By Lemma 4.2 of [H16], the map which associates to a disk $D \in \mathcal{E}(\gamma)$ the projection of ∂D to the base surface F of the I-bundle corresponding to γ extends to a 2-quasi-isometry of $\mathcal{E}(\gamma)$ onto the *electrified arc graph* of F. This electrified arc graph is 4-quasi-isometric to the curve graph of F (see Lemma 4.1 of [H16] for a proof of this folklore result). Thus by the main result in [BB15], the asymptotic dimension of the graph $\mathcal{E}(\gamma)$ is bounded from above by $4g(F)-3+p = -2\chi(F)+1-p$ uniformly, where $g(F)$ is the genus of F and where $p \geq 1$ is the number of boundary components. The Lemma now follows from the fact that X is obtained from a two-sheeted cover of F by attaching an annulus to two boundary components and that furthermore the Euler characteristic of X is negative (see [H16]). ☐

The following summarizes Lemma 4.2, Corollary 4.3, Lemma 4.5 and Corollary 4.6 of [H16].

Theorem 3.4. *The electrified disk graph $\mathcal{EDG}(X)$ of a thick subsurface X of ∂H is hyperbolic relative to the collection of subgraphs $\mathcal{E}(\gamma)$ where γ runs through all I-bundle generators of X, with electrification $\mathcal{SDG}(X)$.*

Using Theorem 3.4, Lemma 3.3 and Corollary 3.2 we obtain

Corollary 3.5. *Let $X \subset \partial H$ be a thick subsurface of genus $g(X) \geq 0$ with p boundary components; then* $\text{asdim}(\mathcal{EG}(X)) \leq 3|\chi(X)| + 3$.

Proof. By Theorem 3.4, the graph $\mathcal{EG}(X)$ is hyperbolic and hyperbolic relative to the subgraphs $\mathcal{E}(\gamma)$, with electrification the graph $\mathcal{SDG}(X)$. By Lemma 3.3, the asymptotic dimension of each of the graphs $\mathcal{E}(\gamma)$ does not exceed $|\chi(X)| + 2$, and Corollary 3.2 shows that the asymptotic dimension of $\mathcal{SDG}(X)$ is not bigger than $2|\chi(X)|$. Thus Theorem 2.3 implies that $\text{asdim}(\mathcal{EG}(X)) \leq |3\chi(X)| + 3$ as claimed. ☐

As a consequence of Corollary 3.5 and the results in [H16] we are now ready to show

Theorem 3.6. *The asymptotic dimension of the disk graph of a handlebody of genus g is bounded from above by $(3g-3)(6g-2)$.*

Proof. By the main result of [H16] and the above estimates of asymptotic dimension, there is a sequence $\mathcal{G}_1, \dots, \mathcal{G}_{3g-3}$ of hyperbolic graphs with the following properties.

(1) $\mathcal{G}_1 = \mathcal{EG}(\partial H)$.
(2) For each $i \geq 2$, $\mathcal{G}_i$ is a hyperbolic graph which is hyperbolic relative to a family $\mathcal{H}$ of hyperbolic subgraphs. The $\mathcal{H}$-electrification of $\mathcal{G}_i$ equals the graph $\mathcal{G}_{i-1}$. The asymptotic dimension of each graph H in the family is at most $6g-3$.

The graph $\mathcal{G}_i$ is defined as follows. Its vertices are disks, and two vertices are connected by an edge of length one if either they are disjoint or if they are disjoint from an essential multicurve in ∂H with at least i components. It is shown in [H16] that for $i \geq 2$, the graph $\mathcal{G}_i$ is hyperbolic relative to a family of complete connected subgraphs $\mathcal{EDG}(i-1)$ where such a subgraph is the electrified disk graph of a thick subsurface X which is the complement in ∂H of a multicurve with $i-1$ components.

Applying Proposition 2.3 inductively $3g-3$ times (which is the maximal number of pairwise disjoint curves in ∂H) and using Corollary 3.5, we conclude that the asymptotic dimension of the disk graph is at most $(3g-3)(6g-2)$. ☐

For a surface S of finite type of genus $g \geq 2$ with $m \geq 0$ punctures define the *graph of non-separating curves* $\mathcal{NC}$ to be the complete subgraph of the curve graph of S consisting of non-separating curves.

For $m \leq 1$ this graph is quasi-isometric to the curve graph of S, but this is not true whenever $m \geq 2$. However, we showed in [H14] that this graph is hyperbolic. Furthermore, it admits a tame hierarchy with base a uniformly quasi-isometrically embedded subgraph of a curve graph and where each peripheral graph also is a uniformly quasi-isometrically embedded subgraph of some curve graph. Thus we obtain

Proposition 3.7. *For any surface S of finite type, the graph of non-separating curves on S has finite asymptotic dimension.*

In [H14] we also defined a graph of non-separating multicurves on the surface S whose vertices are k-tuples of simple closed curves on S whose complement in S is connected. We showed that for $k < g/2 + 1$ this graph is hyperbolic and admits a tame hierarchy. As a consequence, the asymptotic dimension of this graph is finite as well.

References

[BD06] G. Bell, A. Dranishnikov, *Asymptotic dimension*, Topology Appl. 155 (2008), 1265–1296.

[BF08] G. Bell, K. Fujiwara, *The asymptotic dimension of the curve graph is finite*, J. Lond. Math. Soc. 77 (2008), 33–50.

[BBF15] M. Bestvina, K. Bromberg, K. Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*. Publ. Math. Inst. Hautes Etudes Sci. 122 (2015), 1–64.

[BB15] M. Bestvina, K. Bromberg, *On the asymptotic dimension of the curve complex*, arXiv:1508.04832.

[H14] U. Hamenstädt, *Hyperbolicity of the graph of non-separating multicurves*, Alg. & Geom. Topol. 14 (2014), 1759–1778.

[H16] U. Hamenstädt, *Hyperbolic relatively hyperbolic graphs and disk graphs*, Groups, Geom. Dyn. 10 (2016), 1–41.

[H11] U. Hamenstädt, *Asymptotic dimension and the disk graph I*, arXiv:1101.1843, to appear in J. Topology

[MS13] H. Masur, S. Schleimer, *The geometry of the disk complex*, J. Amer. Math. Soc. 26 (2013), 1–62.

[Si12] A. Sisto, *On metric relative hyperbolicity*, arXiv:1210.8081.

MATH. INSTITUT DER UNIVERSITÄT BONN
ENDENICHER ALLEE 60
53115 BONN
GERMANY

e-mail: ursula@math.uni-bonn.de