ASYMPTOTIC DIMENSION AND THE DISK GRAPH

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Abstract. We show that the asymptotic dimension of a hyperbolic relatively hyperbolic graph is finite provided that this holds true uniformly for the peripheral subgraphs and for the electrification. For a handlebody H of genus $g \geq 2$ we use surgery to identify a graph whose vertices are disks and which is quasi-isometrically embedded in the curve graph of the boundary surface. We use this to show that the asymptotic dimension of the disk graph is at most quadratic in the genus and to give a new proof of hyperbolicity of the disk graph.

1. Introduction

A metric space (X, d) has *asymptotic dimension* $\mathrm{asdim}(X)$ at most n if for every number $R > 0$ there exists a covering of X by uniformly bounded sets such that every metric R-ball intersects at most $n+1$ of the sets in the cover. More generally, a collection of metric spaces as asdim at most n *uniformly* if for every R there are covers of each space whose elements are uniformly bounded over the whole collection.

The first goal of this work is to investigate the asymptotic dimension of a (not necessarily locally finite) hyperbolic graph $\mathcal{G}$ which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs, so-called *peripheral graphs*. Here $\mathcal{C}$ is a countable, finite or empty index set. Such a graph is required to have the following properties.

(1) The subgraphs H_c are uniformly *quasi-convex*, i.e. the inclusion $H_c \to \mathcal{G}$ is a quasi-isometric embedding with constant not depending on c.
(2) For $c \neq u$, the diameter of a shortest distance projection $H_u \to H_c$ is uniformly bounded, independent of c, u.
(3) Define the $\mathcal{H}$*-electrification* $\mathcal{E}\mathcal{G}$ of $\mathcal{G}$ to be the graph which is obtained from $\mathcal{G}$ by adding for every $c \in \mathcal{C}$ a new vertex v_c which is connected to each vertex $x \in \mathcal{H}_c$ by an edge and which is not connected to any other vertex. The graph $\mathcal{E}\mathcal{G}$ is hyperbolic.

We refer to [H16] and Section 2 for a discussion why this definition coincides with other notions of hyperbolic relatively hyperbolic graphs defined in the literature. We show

Theorem 1. *Let $\mathcal{G}$ be a hyperbolic graph which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of peripheral subgraphs, with electrification $\mathcal{E}\mathcal{G}$. If the collection H_c $(c \in \mathcal{C})$ has* $\mathrm{asdim}(H_c) \leq n$ *uniformly then* $\mathrm{asdim}(\mathcal{G}) \leq \mathrm{asdim}(\mathcal{E}\mathcal{G}) + n + 1$.

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The second goal of this work is to apply Theorem 1 to the *disk graph* of a *handlebody* of genus $g \geq 2$, i.e. a compact three-dimensional manifold H which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of g circles. Its boundary ∂H is an oriented surface of genus g.

The disk graph $\mathcal{DG}$ of H is the metric graph whose vertices are isotopy classes of properly embedded disks in H and where two such disks are connected by an edge of length one if they can be realized disjointly. Assigning to a disk its boundary then defines an embedding of the disk graph into the *curve graph* of ∂H. However, this inclusion is not a quasi-isometric embedding.

To describe the geometric structure of $\mathcal{DG}$ we use the next definition.

Definition 1. A *tame hierarchy* of a hyperbolic metric graph $\mathcal{G}$ consists of a finite chain $\mathcal{G}_1, \dots, \mathcal{G}_k$ of hyperbolic graphs with the following properties.

1. $\mathcal{G}_k = \mathcal{G}$.
2. For all i, the graph $\mathcal{G}_{i+1}$ is hyperbolic relative to a family $\mathcal{H}_i$ of subgraphs, with electrification $\mathcal{G}_i$.
3. For each i there exists some n_i such that the family $\mathcal{H}_i$ of graphs has asdim $\leq n_i$ uniformly.
4. $\mathcal{G}_1$ has finite asymptotic dimension.

The graph $\mathcal{G}_1$ is called the *base* of the hierarchy.

An inductive application of Proposition 2.3 shows the following

Corollary 1. *The asymptotic dimension of a hyperbolic metric graph $\mathcal{G}$ which admits a tame hierarchy is finite.*

The first part of the following result was earlier established by Masur and Schleimer [MS13].

Theorem 2. *The disk graph $\mathcal{DG}$ of H is hyperbolic and admits a tame hierarchy whose base is a quasi-isometrically embedded subgraph of the curve graph of ∂H.*

As a consequence, we obtain

Corollary 2. $\operatorname{asdim}(\mathcal{DG}) \leq (3g-3)(8g-8)$.

Another geometrically defined graphs which admits a tame hierarchy with base a curve graph is the graph of non-separating multicurves introduced in [H14].

The base of the tame hierarchy of the disk graph has an explicit description which we discuss next. We begin with introducing the following graph.

Definition 2. The *electrified disk graph* is the graph $\mathcal{EDG}$ whose vertices are isotopy classes of essential disks in H and where two vertices D_1, D_2 are connected by an edge of length one if and only if there is an essential simple closed curve on ∂H which can be realized disjointly from both $\partial D_1, \partial D_2$.

Since for any two disjoint essential simple closed curves c, d on ∂H there is a simple closed curve on ∂H which can be realized disjointly from c, d (e.g. one of the curves c, d), the electrified disk graph is obtained from the disk graph by adding some edges.

Call a simple closed curve c on ∂H *diskbusting* if c has an essential intersection with the boundary of every disk.

Define an *I-bundle generator* for H to be a diskbusting simple closed curve c on ∂H with the following property. There is a compact surface F with connected boundary ∂F, and there is a homeomorphism of the orientable I-bundle $\mathcal{I}(F)$ over F onto H which maps ∂F to c. The curve c is separating if and only the surface F is orientable. The handlebody group preserves the set of I-bundle generators.

Definition 3. The *super-conducting disk graph* is the graph $\mathcal{SDG}$ whose vertices are isotopy classes of essential disks in H and where two vertices D_1, D_2 are connected by an edge of length one if and only if one of the following two possibilities holds.

(1) There is a simple closed curve on ∂H which can be realized disjointly from both $\partial D_1, \partial D_2$.
(2) There is an I-bundle generator c for H which intersects both $\partial D_1, \partial D_2$ in precisely two points.

In particular, the superconducting disk graph is obtained from the electrified disk graph by adding some edges.

Since the distance in the curve graph $\mathcal{CG}$ of ∂H between two simple closed curves which intersect in two points does not exceed 3 [MM99], the natural vertex inclusion extends to a coarse 6-Lipschitz map $\mathcal{SDG} \to \mathcal{CG}$. We show

Theorem 3. *The natural vertex inclusion extends to a quasi-isometric embedding* $\mathcal{SDG} \to \mathcal{CG}$.

The constants for the quasi-isometric embeddings are effectively computable and bounded from above by a cubic polynomial in the genus of ∂H. The superconducting disk graph is the base of the tame hierarchy of $\mathcal{DG}$. In particular, as asymptotic dimension does not increase under quasi-isometric embedding, the base of the hierarchy for $\mathcal{DG}$ has asymptotic dimension at most $4g-4$ [BB15].

The second graph in tame hierarchy is the electrified disk graph whose asymptotic dimension is bounded from above by $8g-8$.

To use the program developed by Masur and Minsky [MM99, MM00] which led to an understanding of the geometry of the mapping class group also for handlebody groups, it is necessary to understand disk graphs of handlebodies with *spots*, i.e. handlebodies with marked points on the boundary. Here as before, the disk graph of such a spotted handlebody is the complete subgraph of the curve graph of the marked boundary whose vertex set is the set of diskbounding curves. The electrified disk graph is the graph whose vertices are disks which are connected by an edge of length one if either they are disjoint or if they are disjoint from a common essential simple closed curve.

For disk graphs of spotted handlebodies we obtain

Theorem 4. *Let H be a handlebody of genus $g \geq 2$ with $m \geq 1$ spots on the boundary.*

(1) *If $m = 1$ then the disk graph is not a quasi-convex subset of the curve graph.*
(2) *For $m \geq 2$ the map which associates to a disk its boundary is a 16-quasi-isometry $\mathcal{EDG} \to \mathcal{CG}$.*

In [H11] we show that the disk graph in handlebodies with two spots on the boundary is much more complicated and can effectively be used to understand cycles in the handlebody group which require exponential filling, thus yielding that the Dehn function of the handlebody group is exponential.

Organization: Section 2 is devoted to the proof of Theorem 1. In Section 3 we use surgery of disks to relate the distance in the superconducting disk graph of a handlebody H without spots to intersection numbers of boundary curves.

In Section 4 we give an effective estimate of the distance in the curve graph using train tracks. This together with a construction of [MM04] is used in Section 5 to show Theorem 3 and identify the Gromov boundary of $\mathcal{SDG}$. The proof of Theorem 4 is contained in Section 6 and Section 7.

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2. Asymptotic dimension of hyperbolic relatively hyperbolic graphs

We begin with a general statement about hyperbolic relatively hyperbolic geodesic metric graphs. We mostly use the notations from [H16].

Consider a connected metric graph $\mathcal{G}$ in which a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs has been specified. Here $\mathcal{C}$ is a countable, finite or empty index set. The graph $\mathcal{G}$ is *hyperbolic relative to the family* $\mathcal{H}$ if the following properties are satisfied.

Define the $\mathcal{H}$-electrification $\mathcal{EG}$ of $\mathcal{G}$ to be the graph which is obtained from $\mathcal{G}$ by adding for every $c \in \mathcal{C}$ a new vertex v_c which is connected to each vertex in H_c by an edge and which is not connected to any other vertex. We require that the graph $\mathcal{EG}$ is hyperbolic in the sense of Gromov and that moreover the following *bounded penetration property* holds true.

Call a simplicial path γ in $\mathcal{EG}$ *efficient* if for every $c \in \mathcal{C}$ we have $\gamma(k) = v_c$ for at most one k. Note that if γ is an efficient simplicial path in $\mathcal{EG}$ which passes through $\gamma(k) = v_c$ for some $c \in \mathcal{C}$ then $\gamma(k-1), \gamma(k+1) \in H_c$. We require that for every $L > 1$ there is a number $p(L) > 0$ with the following property. Let γ be an efficient L-quasi-geodesic in $\mathcal{EG}$, let $c \in \mathcal{C}$ and let $k \in \mathbb{Z}$ be such that $\gamma(k) = v_c$. If the distance between $\gamma(k-1)$ and $\gamma(k+1)$ is at least $p(L)$ then every efficient L-quasi-geodesic γ' in $\mathcal{EG}$ with the same endpoints as γ passes through v_c. Moreover, if $k' \in \mathbb{Z}$ is such that $\gamma'(k') = v_c$ then the distance in H_c between $\gamma(k-1), \gamma'(k-1)$ and between $\gamma(k+1), \gamma'(k+1)$ is at most $p(L)$.

The following is Theorem 1 of [H16].

Theorem 2.1. *Let $\mathcal{G}$ be a metric graph which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs. If there is a number $\delta > 0$ such that each of the graphs H_c is δ-hyperbolic then $\mathcal{G}$ is δ-hyperbolic. Moreover, the subgraphs $H_\gamma \mid \gamma \in \mathcal{C}\}$ are uniformly quasi-convex.*

We call a graph $\mathcal{G}$ with the properties stated in Theorem 2.1 a *hyperbolic relatively hyperbolic graph*. In the sequel we always assume that all assumptions in Theorem 2.1 are fulfilled.

We first observe that a hyperbolic relatively hyperbolic graph as defined above has the properties stated in the introduction. To this end recall that for every quasi-convex subgraph H of a hyperbolic graph $\mathcal{G}$ there is a coarsely well defined shortest distance projection $\Pi_H : \mathcal{G} \to H$, i.e. a projection which associates to a point in $\mathcal{G}$ a choice of a point in H of approximate shortest distance. Any other choice of such a point is of uniformly bounded distance. The map Π_H is coarsely distance non-increasing.

Lemma 2.2. *Let $\mathcal{G}$ be a hyperbolic graph which is hyperbolic relative to a family $\mathcal{H} = \{h_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs. Then there is a number $R > 0$ such that for $c \neq d \in \mathcal{C}$,* $\mathrm{diam}(\Pi_{H_c}(H_d)) \leq R$.

Proof. Recall from Theorem 2.1 that the subgraphs H_c of $\mathcal{G}$ are uniformly quasi-convex. This means that there exist numbers $R > 0, L > 1$ such that for $c \in \mathcal{C}$, any geodesic in H_c is an L-quasi-geodesic in the hyperbolic graph $\mathcal{G}$. Furthermore, for any two points $x, y \in H_c$, any L-quasi-geodesic in $\mathcal{G}$ connecting these two points is contained in the R-neighborhood $N_R(H_c)$ of H_c.

For $c \in \mathcal{C}$ write $\Pi_c = \Pi_{H_c}$. Using again hyperbolicity, we deduce the following. Let us assume that the diameter of the projection $\Pi_c(H_d)$ is large. Then there exists points $x, y \in H_d$ and a geodesic $\zeta : [a, b] \to H_d$ connecting $\zeta(a) = x$ to $\zeta(b) = y$ which is contained in the intersection of a uniformly bounded neighborhood of $H_c \cap H_d$. However, this violates the bounded penetration property since we can find an efficient quasi-geodesic γ in $\mathcal{E}\mathcal{G}$ passing through x and y which does not pass through v_d (but instead passes through v_c). This violates the bounded penetration property. We refer to the work [Si12] for a more detailed discussion of the various equivalent formulations of relative hyperbolicity, in particular in connection to condition (α_1) for the collection $\mathcal{H} = \{H_c \mid c\}$. □

We use this observation to show Theorem 1 from the introduction.

Theorem 2.3. *Let $\mathcal{G}$ be a hyperbolic metric graph which is hyperbolic relative to a family $\mathcal{H} = \{H_\gamma \mid \gamma \in \mathcal{C}\}$ of complete connected subgraphs, with $\mathcal{H}$-electrification $\mathcal{E}\mathcal{G}$. If* $\mathrm{asdim}(H_c) \leq n$ *uniformly then* $\mathrm{asdim}(\mathcal{G}) \leq \mathrm{asdim}(\mathcal{E}\mathcal{G}) + n + 1$.

Proof. Our goal is to construct a quasi-isometric embedding of $\mathcal{G}$ into the product of $\mathcal{E}\mathcal{G}$ with a quasi-tree $\mathcal{Y}$ of metric spaces as in Theorem B of [BBF15] whose vertices are the subgraphs H_c ($c \in \mathcal{C}$). Using the assumption on the asymptotic dimensions of the graphs H_c, Theorem B iv) of [BBF15] states that $\mathrm{asdim}(\mathcal{Y}) \leq n + 1$. Now the asymptotic dimension of the product $X \times Y$ of two metric spaces satisfies $\mathrm{asdim}(X \times Y) \leq \mathrm{asdim}(X) + \mathrm{asdim}(Y)$, furthermore $\mathrm{asdim}(X) \leq \mathrm{asdim}(Y)$ if X admits a quasi-isometric embedding into Y. Thus we conclude that

$$\mathrm{asdim}(\mathcal{G}) \leq \mathrm{asdim}(\mathcal{E}\mathcal{G} \times \mathcal{Y}) \leq m + n + 1.$$

For the construction of a quasi-tree of metric spaces from the quasiconvex subspaces $H_c \subset \mathcal{G}$ we have to verify that the axioms $(P1), (P2)$ in Theorem A of [BBF15] are fulfilled. To this end denote for $c \in \mathcal{C}$ by

$$\Pi_c : \mathcal{G} \to H_c$$

a shortest distance projection of $\mathcal{G}$ into H_c. By Lemma 2.2, there is a number $R > 0$ not depending on c such that $\mathrm{diam}(\Pi_c(H_d)) \leq R$ for all $d \neq c$.

For $a, b, c \in \mathcal{C}$ define now

$$d_a(b, c) = \mathrm{diam}(\Pi_a(H_b) \cup \Pi_a(H_c)).$$

Axiom (P1) in [BBF15].requires the existence of a constant $\theta > 0$ such that for any triple a, b, c of distinct elements in $\mathcal{C}$, at most one of the three numbers

$$d_a(b, c), d_b(a, c), d_c(a, b)$$

is greater than θ.

To show that this is indeed the case, recall that the subspaces $H_c \subset \mathcal{G}$ ($c \in \mathcal{C}$) are uniformly quasi-convex. Together with hyperbolicity of $\mathcal{G}$, this implies the existence of a number $R' > R$ with the following property. If $x, y \in \mathcal{G}$ are any two points, and if $d(\Pi_c(x), \Pi_c(y)) \geq R'$, then any geodesic in $\mathcal{G}$ connecting x to y passes through a unifomly bounded neighborhood of both $\Pi_c(x)$ and $\Pi_c(y)$, say through the R''-neighborhood.

Now assume that $d_a(b,c) \geq 3R'$. Choose points $x \in \Pi_b(H_a), y \in \Pi_c(H_a)$. By the above discussion, any geodesic connecting x to y passes through the R''-neighborhood of $\Pi_a(y)$. Since the projection Π_c is coarsely distance non-increasing, this implies that the distance between $\Pi_c(x)$ and $\Pi_c(\Pi_a(x))$ is uniformly bounded and hence the same holds true for $d_c(a,b)$. The same reasoning also shows that $d_b(a,c)$ is uniformly bounded. To summarize, axiom (P1) of [BBF15] holds true.

Axiom (P2) states that for any $a, b \in \mathcal{C}$, the set

$$\{c \in \mathcal{C} \mid d_c(a,b) > \theta\}$$

is finite. However, this follows as before. Namely, let $x \in \Pi_a(H_b), y \in \Pi_b(H_a)$ and let γ be a geodesic connecting x to y in $\mathcal{EG}$. Then γ passes through only finitely many of the special vertices v_c, say through the vertices $v_{c_1} \dots, v_{c_u}$. Now the above discussion shows that if $d_c(a,b) \geq \theta$ then $c \in \{c_1, \dots, c_u\}$. This clearly implies that axiom (P2) is fulfilled as well.

As a consequence, Theorem A of [BBF15]yields the existence of a quasi-tree of metric spaces $\mathcal{Y}$ with vertex spaces H_c. Furthermore, by assumption on the graphs H_c and by Theorem iv) of [BBF15], its asymptotic dimension is at most $n+1$.

Our goal is to show that the graph $\mathcal{G}$ admits a quasi-isometric embedding into $\mathcal{EG} \times \mathcal{Y}$. To this end we recall from [BBF15] the construction of $\mathcal{Y}$.

Namely, $\mathcal{Y}$ consists of the union of the graphs H_c $(c \in \mathcal{C})$ and a collection of additional edges of length one connecting these graphs. These edges are defined as follows. For all c, d, let $x_{c.d} \in H_c$ be a point of shortest distance in H_c to the graph H_d and choose similarly $x_{d,c}$. Connect the point $x_{c,d}$ to the point $x_{d,c}$ by an edge of length if there is no a so that $d_a(c,d) \geq \theta$. In particular, if there is a geodesic in $\mathcal{EG}$ connecting $x_{c.d}$ to $x_{d,c}$ which does not pass through a special vertex v_u for $u \neq c, d$ then the points $x_{c,d}$ and $x_{d,c}$ are connected by an edge. Let $\mathcal{Y}$ be the space constructed in this way. It is connected.

We now define a map $\mathcal{G} \to \mathcal{EG} \times \mathcal{Y}$ as follows. Fix a basepoint $x \in \mathcal{G}$ contained in one of the convex subspaces H_c. Associate to x the product $(x, x) \in \mathcal{EG} \times \mathcal{Y}$. For every vertex $y \in \mathcal{G}$ choose once and for all a geodesic γ_y connecting x to y in $\mathcal{EG}$. Note that these geodesics are efficient. Let $v_1, \dots, v_s \in \mathcal{EG}$ be the special points traveled through by γ_y in this order. For each of these special points v_i, let $z_i, u_i \in H_{v_i}$ be the two neighbors of v_i along γ_y, chosen in such a way that if $v_i = \gamma(j_i)$ then $u_i = \gamma(j_i + 1)$. Define $\Psi(y) = u_s$.

We claim that the map $\Lambda : y \to \Lambda(y) = (y, \Psi(y)) \in \mathcal{EG} \times \mathcal{Y}$ is a quasi-isometric embedding. To this end we show first that for any two points y, z of distance one, the distance between $\Lambda(y)$ and $\Lambda(z)$ is uniformly bounded.

To this end consider the geodesics γ_y, γ_z. Let $v_{c_1}, \dots, v_{c_s}$ $(c_i \in \mathcal{C})$ be the special points on γ_y and let $w_1, \dots, w_u$ be those on γ_z. For the above number $R < \theta/4$ define v_i to be *wide* for γ_y if the following holds true. Let j_i be such that $\gamma_y(j_i) = v_i$; then the distance in H_{c_i} between $\gamma_y(j_i - 1)$ and $\gamma_y(j_i + 1)$ is at least $2R$. Assume that $i \leq s$ is the largest number so that v_i is $2R$-wide. By the bounded penetration property, there is some j so that $w_j = v_i$, moreover $\gamma_z(j_i) = w_j$ and by the bounded penetration property, the distances between the exit points $q = \gamma_y(j_i+1), \gamma_z(j_i+1)$ of H_{c_i} along γ_y, γ_z is uniformly bounded.

It now suffices to show that the distance in $\mathcal{Y}$ between $\Psi(y)$ and the exit point q of γ_y in H_{c_i} is uniformly bounded. However by definition, $\Psi(y)$ is the exit point of H_{v_s}. Thus if $i = s$ then $q = \Psi(y)$ are we are done. Otherwise note that the shortest distance projection of H_{v_s} into H_{v_i} is contained in the $5R < \theta$-neighborhood of q. Namely, by uniform quasiconvexity of H_{c_i}, the exit point q for γ_y is uniformly near this projection point.

Furthermore, there is no wide point along the segment of γ_y connecting q to $\Psi(y) \in H_{c_s}$ and consequently by construction of the projection complex, there is a segment of length one connecting a point in H_{c_i} uniformly near q to a point in H_{c_s} uniformly near the entry point of γ_y in H_{c_s}. As v_s is not wide along γ_y, this entry point is contained in a uniformly bounded neighborhood of $\Psi(y)$. Together this shows that indeed, the distance in $\mathcal{Y}$ between q and $\Psi(y)$ is uniformly bounded. The same reasoning applies to γ_z and shows that the distance in $\mathcal{Y}$ between $\Psi(y)$ and $\Psi(z)$ is uniformly bounded.

As $\mathcal{G}$ is a geodesic metric graph, this implies that the map Λ is coarsely Lipschitz. We are now left with showing that there exists a constant $L > 1$ such that for all y, z, the distance between $\Lambda(y)$ and $\Lambda(z)$ is bounded from below by $d(y,z)/L - L$.

By hyperbolicity and construction, for this it suffices to show the following. Let $\gamma : [0, R] \to \mathcal{G}$ be any geodesic connecting the fixed point x to a point $y = \gamma(R)$ and let $0 \le u < R$; then $d(\Lambda(\gamma(u)), \Lambda(\gamma(R))) \ge |R - u|/L - L$. By hyperbolicity and the above discussion, it suffices furthermore to assume that $\gamma_{\gamma(u)} = \gamma[0, u]$.

Assume for the moment that $\gamma(s) = z$ is contained in one of the subspaces H_c. Let $v_{c_1}, \dots, v_{c_j}$ be the special points passed through by the geodesic $\eta = \gamma[u, R]$. If the distance between entry and exit point of η in H_{c_i} is uniformly bounded then the distance in $\mathcal{G}$ between $\gamma(u)$ and $\gamma(R)$ is uniformly equivalent to the distance in $\mathcal{EG}$.

On the other hand, if v_{c_i} is wide along η then the diameter of $\Pi_{c_i}(H_{c_{i-1}}, H_{c_{i+1}}$ is large. This implies that in $\mathcal{Y}$, the length of any path connecting these two subspaces equals the distance of the exit and entry point up to a uniform additive constant. Thus any large distance between entry and exit point gives rise to large distance in $\mathcal{Y}$, and these distances coincide up to a universal constant. This is what we wanted to show. $\square$

Remark 2.4. It is immediate from the proof of Theorem 2.3 that the bounds on coverings used in the definition of asymptotic dimension for a hyperbolic relatively hyperbolic graph $\mathcal{G}$ are uniformly controlled by the data of the peripheral subgraphs and the bounds for the electrification $\mathcal{EG}$ of $\mathcal{G}$.

Remark 2.5. The proof of Theorem 2.3 together with Theorem B of [BBF15] also shows the following. If $\mathcal{EG}$ is a quasi-tree and if each of the peripheral subgraphs are quasi-trees, then $\mathcal{G}$ admits an quasi-isometric embedding into the product of two quasi-trees.

We next give an application of this result to some geometric graphs related to surfaces.

Example 2.6. For a surface S of finite type of genus $g \ge 2$ with $m \ge 0$ punctures define the *graph of non-separating curves* $\mathcal{NC}$ to be the complete subgraph of the curve graph of S consisting of non-separating curves. We showed in [H14] that this graph is hyperbolic. Furthermore, it admits a controlled hierarchy with base a quasi-isometrically embedded subgraph of a curve graph and where each peripheral graph also is a quasi-isometrically embedded subgraph of some curve graph. By Corollary 1, the graph of non-separating curves has finite asymptotic dimension.

The main goal of the rest of this work is to construct a tame hierarchy for the disk graph of a handlebody H where the base of the hierarchy is a quasi-isometrically embedded subgraph of a curve graph. This not only gives a new proof of hyperbolicity of the disk graph but also shows that its asymptotic dimension is finite.

3. Distance and intersection

In this section we consider a handlebody H of genus $g \geq 2$ without spots on the boundary. We use surgery of disks to establish first estimates for the distance in the electrified disk graph $\mathcal{EDG}$ and in the superconducting disk graph $\mathcal{SDG}$ of H. We begin with introducing the basic surgery construction needed later on.

By a disk in the handlebody H we always mean an essential disk in H. Two disks D_1, D_2 are in *normal position* if their boundary circles intersect in the minimal number of points and if every component of $D_1 \cap D_2$ is an embedded arc in $D_1 \cap D_2$ with endpoints in $\partial D_1 \cap \partial D_2$. In the sequel we always assume that disks are in normal position; this can be achieved by modifying one of the two disks with an isotopy.

Let D be any disk and let E be a disk which is not disjoint from D. A component α of $\partial E - D$ is called an *outer arc* of ∂E relative to D if there is a component E' of $E - D$ whose boundary is composed of α and an arc $\beta \subset D$. The interior of β is contained in the interior of D. We call such a disk E' an *outer component* of $E - D$. An outer component of $E - D$ intersects ∂H in an outer arc α relative to D, and α intersects ∂D in opposite directions at its endpoints.

For every disk E which is not disjoint from D there are at least two distinct outer components E', E'' of $E - D$. There may also be components of $\partial E - D$ which leave and return to the same side of D but which are not outer arcs. An example of such a component is a subarc of ∂E which is contained in the boundary of a rectangle component of $E - D$ leaving and returning to the same side of D. The boundary of such a rectangle consists of two subarcs of ∂E with endpoints on ∂D which are homotopic relative to ∂D, and two arcs contained in D.

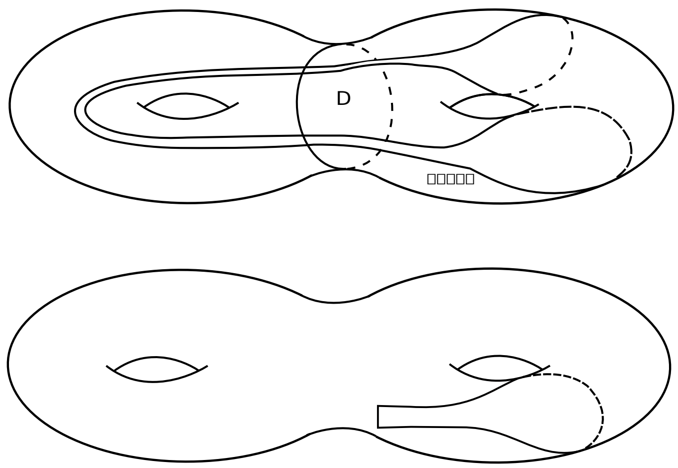
Let $E' \subset E$ be an outer component of $E - D$ whose boundary is composed of an outer arc α and a subarc $\beta = E' \cap D$ of D. The arc β decomposes the disk D into two half-disks P_1, P_2. The unions $Q_1 = E' \cup P_1$ and $Q_2 = E' \cup P_2$ are embedded disks in H which up to isotopy are disjoint and disjoint from D. For $i = 1, 2$ we say that the disk Q_i is obtained from D by *simple surgery* at the outer component E' of $E - D$ (see e.g. [S00] for this construction). Since D, E are in minimal position, the disks Q_1, Q_2 are essential.

Each disk in H can be viewed as a vertex in the disk graph $\mathcal{DG}$, the electrified disk graph $\mathcal{EDG}$ and the superconducting disk graph $\mathcal{SDG}$. We will work with all three graphs simultaneously. Denote by $d_{\mathcal{D}}$ (or $d_{\mathcal{E}}$ or $d_{\mathcal{S}}$) the distance in $\mathcal{DG}$ (or in $\mathcal{EDG}$ or in $\mathcal{SDG}$). Note that for any two disks D, E we have

$$d_{\mathcal{S}}(D, E) \leq d_{\mathcal{E}}(D, E) \leq d_{\mathcal{D}}(D, E).$$

In the sequel we always assume that all curves and multicurves on ∂H are essential. For two simple closed multicurves c, d on ∂H let $\iota(c, d)$ be the geometric

Figure A



intersection number between c, d. The following lemma [MM04] implies that the graph $\mathcal{DG}$ is connected. We provide the short proof for completeness.

Lemma 3.1. *Let $D, E \subset H$ be any two disks. Then D can be connected to a disk E' which is disjoint from E by at most $\iota(\partial D, \partial E)/2$ simple surgeries. In particular,*

$$d_{\mathcal{D}}(D, E) \leq \iota(\partial D, \partial E)/2 + 1.$$

Proof. Let D, E be two disks in normal position. Assume that D, E are not disjoint. Then there is an outer component of $E - D$. A disk D' obtained by simple surgery of D at this component is essential in ∂H. Moreover, D' is disjoint from D, i.e. we have $d_{\mathcal{D}}(D', D) = 1$, and

$$\iota(\partial E, \partial D') \leq \iota(\partial D, \partial E) - 2. \tag{1}$$

The lemma now follows by induction on $\iota(\partial D, \partial E)$. □

Lemma 3.2. *Let $D, E \subset H$ be disks. If there is an essential simple closed curve $\alpha \subset \partial H$ which intersects ∂E in at most one point and which intersects ∂D in at most $k \geq 1$ points then $d_{\mathcal{E}}(D, E) \leq \log_2 k + 3$.*

Proof. Let $D, E \subset H$ be any two disks. If D, E are disjoint then there is nothing to show, so assume that $\partial D \cap \partial E \neq \emptyset$. Let $\alpha \subset \partial H$ be a simple closed curve which intersects ∂E in at most one point and which intersects ∂D in $k \geq 0$ points. If α is disjoint from both D, E then $d_{\mathcal{E}}(D, E) \leq 1$ by definition of the electrified disk graph. Thus by perhaps exchanging D and E we may assume that $k \geq 1$. Via a small homotopy we may moreover assume that α is disjoint from $D \cap E$.

We modify D as follows. There are at least two outer components of $E - D$. Since α intersects ∂E in at most one point, one of these components, say the component E', is disjoint from α. The boundary of E' decomposes D into two subdisks P_1, P_2. Assume without loss of generality that P_1 contains fewer intersection points with α than P_2. Then P_1 intersects α in at most $k/2$ points. The disk $D' = P_1 \cup E'$ has at most $k/2$ intersection points with α, and up to isotopy, it is disjoint from D. In particular, we have $d_{\mathcal{E}}(D, D') = 1$.

Repeat this construction with D', E. After at most $\log_2 k + 1$ such steps we obtain a disk D_1 which either is disjoint from E or is disjoint from α. The distance between D and D_1 in the graph $\mathcal{EDG}$ is at most $\log_2 k + 1$.

If D_1 and E are disjoint then $d_{\mathcal{E}}(D_1, E) \leq 1$ and $d_{\mathcal{E}}(D, E) \leq \log_2 k + 2$ and we are done. Otherwise apply the above construction to D_1, E but with the roles of D_1 and E exchanged. We obtain a disk E_1 which is disjoint from both E and α, in particular it satisfies $d_{\mathcal{E}}(E_1, E) = 1$. The disks D_1, E_1 are both disjoint from α and therefore $d_{\mathcal{E}}(D_1, E_1) \leq 1$ by the definition of the electrified disk graph. Together this shows that

$$d_{\mathcal{E}}(D, E) \leq \log_2 k + 1 + d_{\mathcal{E}}(D_1, E) \leq \log_2 k + 3.$$

$\square$

A simple closed multicurve γ in ∂H is called *diskbusting* if γ intersects every disk.

Definition 3.3. An *I-bundle generator* in ∂H is a diskbusting simple closed curve $\gamma \subset \partial H$ with the following property. There is an oriented I-bundle $\mathcal{J}(F)$ over a compact surface F with connected boundary ∂F, and there is an orientation preserving homeomorphism of $\mathcal{J}(F)$ onto H which maps ∂F to γ.

We call the surface F the *base* of the I-bundle generated by γ.

If γ is a separating I-bundle generator, with base surface F, then F is orientable and we have $g = 2n$ for some $n \geq 1$. Moreover, the I-bundle $\mathcal{J}(F) = F \times [0, 1]$ is trivial. The I-bundle over every essential arc in F with endpoints in ∂F is an embedded disk in H. If γ is non-separating then the base F of the I-bundle is non-orientable.

There is an orientation reversing involution $\Phi : H \to H$ whose fixed point set intersects ∂H precisely in γ. This involution acts as a reflection in the fiber. The union of an essential embedded arc α in F with endpoints on ∂F with its image under Φ is the boundary of a disk in H (there is a small abuse of notation here- since the fixed point set of Φ intersects ∂H in a subset of the fibre over ∂F). This disk is just the I-bundle over α.

If $D, E \subset H$ are disks in normal position then each component of $D - E$ is a disk and therefore the graph dual to the cell decomposition of D whose two-cells are the components of $D - E$ is a tree. If $D - E$ only has two outer components then this tree is just a line segment. The following lemma analyzes the case that this holds true for both $D - E$ and $E - D$.

Lemma 3.4. *Let $D, E \subset H$ be disks in normal position. If $D - E$ and $E - D$ only have two outer components then one of the following two possibilities is satisfied.*

(1) $d_{\mathcal{E}}(D, E) \leq 4$.
(2) *D, E intersect some I-bundle generator γ in ∂H in precisely two points.*

Proof. Let D, E be two disks in normal position. Assume that $D - E$ and $E - D$ only have two outer components. Then each component of $D - E, E - D$ either is an outer component or a rectangle, i.e. a disk whose boundary consists of two components of $D \cap E$ and two arcs contained in $\partial D \subset \partial H$ or $\partial E \subset \partial H$, respectively. Since $d_{\mathcal{E}}(D, E) = 1$ if there is an essential simple closed curve in ∂H which is disjoint from $\partial D \cup \partial E$, we may assume without loss of generality that $\partial D \cup \partial E$ fills up ∂H. This means that $\partial H - (\partial D \cup \partial E)$ is a union of disks and peripheral annuli.

Choose tubular neighborhoods $N(D), N(E)$ of D, E in H which are homeomorphic to an interval bundle over a disk and which intersect ∂H in an embedded annulus. We may assume that the interiors $A(D), A(E)$ of these annuli are contained in the interior of ∂H. Then $\partial N(D) - A(D)$, $\partial N(E) - A(E)$ is the union of two properly embedded disjoint disks in H isotopic to D, E. We may assume that $\partial N(D) - A(D)$ is in normal position with respect to $\partial N(E) - A(E)$ and that

$$S = \partial(N(D) \cup N(E)) - (A(D) \cup A(E))$$

is a compact surface with boundary which is properly embedded in H. Since H is assumed to be oriented, the boundary $\partial(N(D) \cup N(E))$ of $N(D) \cup N(E)$ has an induced orientation which restricts to an orientation of S.

Let (P, σ) be a pair consisting of an outer component P of $D - E$ and an orientation σ of D which induces an orientation of P. The orientation σ together with the orientation of H determines an orientation of the normal bundle of D and hence σ determines a side of D in $N(D)$, say the right side. We claim that the component Q of the surface S containing the copy of P in $\partial N(D)$ to the right of D is a disk which contains precisely one other pair (P', σ') of this form, i.e. P' is an outer component of $D - E$ or of $E - D$, and σ' an orientation of P'.

Namely, by construction, each component of $\partial N(D) - (A(D) \cup N(E))$ either corresponds to an outer component of $D - E$ and the choice of a side, or it corresponds to a rectangle component of $D - E$ and a choice of a side. A component corresponding to a rectangle is glued at each of its two sides which are contained in the interior of H to a component of $\partial N(E) - (A(E) \cup N(D))$. In other words, up to homotopy, the component Q of S can be written as a chain of oriented disks beginning with P and alternating between components of $D - E$ and $E - D$ equipped with one of the two possible orientations. Since Q is embedded in H and contains P, this chain can not be a cycle and hence it has to terminate at an oriented outer component of $D - E$ or $E - D$ which is distinct from (P, σ).

To summarize, each pair (P, σ) consisting of an outer component P of $D - E$ or $E - D$ and an orientation σ of D or E determines a unique component of the oriented surface S. This component is a properly embedded disk in H which is disjoint from $D \cup E$. Each such disk corresponds to precisely two such pairs (P, σ), so there is a total of four such disks. Denote these disks by $Q_1, \dots, Q_4$. If one of these disks is essential, say if this holds true for the disk Q_i, then Q_i is an essential disk disjoint from both D, E with boundary in X and hence $d_{\mathcal{E}}(D, Q_i) \le 1, d_{\mathcal{E}}(E, Q_i) \le 1$ and we are done.

Otherwise define a *cycle* to be a subset C of $\{Q_1, \dots, Q_4\}$ of minimal cardinality so that the following holds true. Let $Q_i \in C$ and assume that Q_i contains a pair (B, ζ) consisting of an outer component B of $D - E$ (or of $E - D$) and an orientation ζ of D (or E). If Q_j is the disk containing the pair (B, ζ') where ζ' is the orientation of D (or E) distinct from ζ then $Q_j \in C$. Note that two distinct cycles are disjoint. The *length* of the cycle is the number of its components.

For each cycle C we construct a properly embedded annulus $(A(C), \partial A(C)) \subset H$ as follows. Remove from each of the disks Q_i in the cycle the subdisks which correspond to outer components of $D - E, E - D$ and glue the disks along the boundary arcs of these outer components.

To be more precise, let B be an outer component of $D-E$ corresponding to a subdisk of Q_i and let $\beta = B \cap E$. Then the complement of B in Q_i (with a small abuse of notation) contains the arc β in its boundary, and the orientation of Q_i defines an orientation of β. There is a second disk Q_j in the cycle which contains B and which induces on β the opposite orientation (Q_j is not necessarily distinct from Q_i). Glue $Q_i - B$ to $Q_j - B$ along β and note that the resulting surface is oriented. Doing this with each of the outer components of $D-E$ and $E-D$ contained in the cycle yields a properly embedded annulus $A(C) \subset H$ as claimed.

If there is a cycle C of odd length then for one of the two disks D, E, say the disk D, the cycle contains precisely one outer component (with both orientations). Since the disks Q_j are disjoint from $D \cup E$, this means that a boundary curve γ of $A(C)$ intersects the disk D in precisely one point, and it intersects the disk E in at most two points. In particular, γ is an essential curve in ∂H. Lemma 3.2 now shows that $d_{\mathcal{E}}(D,E) \leq 4$.

Similarly, if there is a cycle C of length two then there are two possibilities. The first case is that the cycle contains both an outer component of $D-E$ and an outer component of $E-D$. Then a boundary curve γ of $A(C)$ intersects each of the disks D, E in precisely one point. In particular, γ is an essential curve in ∂H and $d_{\mathcal{E}}(D,E) \leq 3$.

If C contains both outer components of say the disk D then a boundary curve γ of $A(C)$ intersects D in precisely two points, and it is disjoint from E. Let D' be a disk obtained from D by a simple surgery at an outer components of $E-D$. Then either D' is disjoint from both D, E (which is the case if D' is composed of an outer component of $E-D$ and an outer component of $D-E$) or D' intersects γ in precisely one point and is disjoint from D. As before, we conclude from Lemma 3.2 that $d_{\mathcal{E}}(D,E) \leq 4$.

We are left with the case that there is a single cycle C of length four. Let γ_1, γ_2 be the two boundary curves of $A(C)$. Then γ_1, γ_2 are simple closed curves in X which are freely homotopic in the handlebody H. We claim that γ_1, γ_2 are freely homotopic in ∂H. Namely, assume that the disks Q_i are numbered in such a way that Q_i and Q_{i+1} share one outer component of $D-E$ or $E-D$. Glue the disks $Q_1, \dots, Q_4$ successively to a single disk Q with the surgery procedure described above (namely, if P is the outer component of $D-E$ or $E-D$ contained in both Q_1, Q_2 then remove P from Q_1, Q_2 and glue Q_1 to Q_2 along the resulting boundary arc to form a disk $\hat{Q}_1$, glue $\hat{Q}_1$ to Q_3 to form a disk $\hat{Q}_2$, and glue $\hat{Q}_2$ to Q_4 to obtain the disk Q). Since by assumption none of the disks Q_i is essential, the disk Q is contractible. In particular, Q is homotopic with fixed boundary to an embedded disk in ∂H. Now the annulus $A(C)$ is obtained from Q by identifying two disjoint boundary arcs and hence $A(C)$ is homotopic into ∂H. Assume from now on that $A(C) \subset \partial H$.

By construction, each of the simple closed curves $\partial D, \partial E$ intersects $A(C)$ in precisely two arcs connecting the two boundary components of $A(C)$. These are exactly the boundary arcs of the outer components of $D-E, E-D$.

The intersection arcs $\partial D \cap A(C), \partial E \cap A(C)$ decompose $A(C)$ into four rectangles. The annulus $A(C)$ has a natural structure of an I-bundle over one of its boundary circles, with $\partial D \cap A(C), \partial E \cap A(C)$ as a union of fibres.

Let $\zeta \subset \partial D$ be a component of $\partial D - A(C)$. Then ζ is a union of boundary components of rectangles embedded in D. Two opposite sides of such a rectangle

R are contained in ∂D. There is a unique side ρ of R which is contained in ζ, and the side opposite to ρ is contained in the component ζ' of $\partial D - A(C)$ disjoint from ζ. This decomposition of D into rectangles determines for D the structure of an I-bundle over ζ. Each component of $D \cap E$ is a fibre of this I-bundle, and the two components of $\partial D \cap A(C)$ are fibres as well. Similarly, E is an I-bundle over each component ξ of $\partial E - A(C)$.

Since $\partial D \cup \partial E$ fills up ∂H, each component of $\partial H - (\partial D \cup \partial E)$ is a polygon, i.e. a disk bounded by finitely many subarcs of $\partial D, \partial E$. Such a polygon P is contained in the boundary of a component V of $H - (D \cup E)$. The boundary ∂V of V has two connected components contained in ∂H. One of these components is the polygon P, the other component P' either is a polygon component of $\partial H - (\partial D \cup \partial E)$, or it contains a boundary component of ∂H.

The complement of $P \cup P'$ in ∂V is a finite collection W of fibred rectangles. The base of such a rectangle is an edge of the boundary ∂P of the polygon P. Using again the fact that $\partial D \cup \partial E$ fills up ∂H, if P' is a polygon in ∂H then V is a 3-ball.

As a consequence, each component V of $H - (D \cup E)$ is a ball whose boundary consists of P, a finite union $\mathcal{R}$ of fibred rectangles with base ∂P and a second polygonal component P' of $X - (\partial D \cup \partial E)$. The I-bundle structure on $\mathcal{R}$ naturally extends to an I-bundle structure on V. Therefore H is an I-bundle. The involution of the I-bundle which exchanges the endpoints of the interval I preserves each component V of $H - (D \cup E)$, and it exchanges the two components of $V \cap \partial H$. This completes the proof of the lemma. □

We use Lemma 3.4 to improve Lemma 3.2 as follows.

Proposition 3.5. *Let $D, E \subset H$ be essential disks. If there is an essential simple closed curve $\alpha \subset \partial H$ which intersects $\partial D, \partial E$ in at most $k \geq 1$ points then $d_{\mathcal{S}}(D, E) \leq 2k + 4$.*

Proof. Let D, E be essential disks in normal position as in the proposition which are not disjoint.

Let α be an essential simple closed curve in ∂H which intersects both ∂D and ∂E in at most $k \geq 1$ points. We may assume that these intersection points are disjoint from $\partial D \cap \partial E$.

Let $p \geq 2$ (or $q \geq 2$) be the number of outer components of $D - E$ (or of $E - D$). If $p = 2, q = 2$ then Lemma 3.4 shows that either $d_{\mathcal{E}}(D, E) \leq 4$ or $\partial D, \partial E$ intersect some I-bundle generator γ in precisely two points, and we have $d_{\mathcal{S}}(D, E) = 1$.

Let $j \leq k, j' \leq k$ be the number of intersection points of D, E with α. If $\min\{j, j'\} \leq 1$ then $d_{\mathcal{E}}(D, E) \leq \log_2 k + 3$ by Lemma 3.2. Thus it suffices to show the following. If $\max\{p, q\} \geq 3$ and $\min\{j, j'\} \geq 2$ then there is a simple surgery transforming the pair (D, E) to a pair (D', E') with the following properties.

(1) D' is disjoint from D, E' is disjoint from E.
(2) Either $D = D'$ or $E = E'$.
(3) The total number of intersections of α with $D' \cup E'$ is strictly smaller than $j + j'$.

To this end assume without loss of generality that $q \geq 3$. If $j/2 > j'/3$ then choose an outer component E_1 of $E - D$ with at most $j'/3$ intersections with α. This is possible because $E - D$ has at least three outer components. Let D_1 be a component of $D - E_1$ which intersects α in at most $j/2$ points. Then $D_1 \cup E_1$ is a disk which is disjoint from D and has at most $j/2 + j'/3 < j$ intersections with α.

On the other hand, if $j/2 \leq j'/3$ then choose an outer component D_1 of $D - E$ with at most $j/2$ intersections with α. Let E_1 be a component of $E - D_1$ with at most $j'/2$ intersections with α and replace E by the disk $E_1 \cup D_1$ which is disjoint from E and intersects α in at most $j/2 + j'/2 < j'$ points.

This is what we wanted to show. $\square$

For easy reference we note

Corollary 3.6. *Let H be a handlebody of genus $g \geq 2$ without spots. Let $D, E \subset H$ be disks and assume that there is a simple closed curve γ on ∂H which intersects both $\partial D, \partial E$ in at most $k \geq 1$ points; then $d_{\mathcal{S}}(D, E) \leq 2k + 4$.*

Remark: The arguments in this section use the fact that every simple surgery of a disk at an outer component of another disk yields an essential disk in H. They are not valid for handlebodies with spots.

4. Distance in the curve graph

The purpose of this section is to establish an estimate for the distance in the curve graph of the boundary of the handlebody H which will be essential for a geometric description of the superconducting disk graph. The results in this section are valid for an arbitrary oriented surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$.

The idea is to use *train tracks* on S. We refer to [PH92] for all basic notions and constructions regarding train tracks.

A train track η (which may just be a simple closed curve) is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is homotopic to the identity, with $F(\eta) \subset \tau$ and such that the restriction of the differential dF of F to the tangent line of η vanishes nowhere. Write $\eta \prec \tau$ if η is carried by τ. If $\eta \prec \tau$ then the image of η under a carrying map is a *subtrack* of τ which does not depend on the choice of the carrying map. Such a subtrack is a subgraph of τ which is itself a train track. Write $\eta < \tau$ if η is a subtrack of τ.

A train track τ is called *large* [MM99] if each complementary component of τ is either simply connected or a once punctured disk. A simple closed curve η carried by τ *fills* τ if the image of η under a carrying map is all of τ. A *diagonal extension* of a large train track τ is a train track ξ which can be obtained from τ by subdividing some complementary components which are not trigons or once punctured monogons.

A *trainpath* on τ is an immersion $\rho : [k, \ell] \to \tau$ which maps every interval $[m, m+1]$ diffeomorphically onto a branch of τ. We say that ρ is *periodic* if $\rho(k) = \rho(\ell)$ and if the inward pointing tangent of ρ at $\rho(k)$ equals the outward pointing tangent of ρ at $\rho(\ell)$. Any simple closed curve carried by a train track τ defines a periodic trainpath and a *transverse measure* on τ. The space of transverse measures on τ is a cone in a finite dimensional real vector space. Each of its extreme rays is spanned by a *vertex cycle* which is a simple closed curve carried by τ. A vertex cycle defines a periodic trainpath which passes through every branch at most twice, in opposite direction [H06, Mo03].

Let η be a large train track. If $\eta \prec \tau$ then τ is large as well. In particular, if $\eta' < \eta$ is a large subtrack of η and if ξ is a diagonal extension of η', then a carrying map $F : \eta \to \tau$ induces a carrying map of η' onto a large subtrack τ' of τ, and it induces a carrying map of ξ onto a diagonal extension of τ'.

Definition 4.1. A pair $\eta \prec \tau$ of large train tracks is called *wide* if every simple closed curve which is carried by a diagonal extension of a large subtrack of η fills a diagonal extension of a large subtrack of τ.

We have

Lemma 4.2. *If $\sigma \prec \eta \prec \tau$ and if the pair $\eta \prec \tau$ is wide then $\sigma \prec \tau$ is wide.*

Proof. Let σ' be a large subtrack of σ and let ξ be a diagonal extension of σ'. Then the carrying map $\sigma \to \eta$ maps σ' onto a large subtrack η' of η, and it maps ξ to a diagonal extension ζ of η'. Similarly, η' is mapped to a large subtrack τ' of τ, and ζ is mapped to a diagonal extension ρ of τ'.

A simple closed curve α carried by ξ is carried by ζ. In particular, since $\eta \prec \tau$ is wide, α fills a large subtrack of ρ. From this the lemma follows. □

A *splitting and shifting sequence* is a finite sequence $(\tau_i)_{0\leq i\leq n}$ of train tracks so that for each i, τ_{i+1} can be obtained from τ_i by a sequence of *shifts* followed by a single *split*. We allow the split to be a collision, i.e. to reduce the number of edges. Note that τ_{i+1} is carried by τ_i for all i and the pair $\tau_{i+1} \prec \tau_i$ is *not* wide.

Recall from the introduction the definition of the curve graph $\mathcal{CG}$ of S. For an essential simple closed curve c on S let $i(c) \in \{0, \dots, n\}$ be the largest number with the following property. There is a large subtrack η of $\tau_{i(c)}$ so that c is carried by a diagonal extension ξ of η and fills ξ. If no such number exists then put $i(c) = 0$.

Define a projection $P : \mathcal{CG} \to (\tau_i)_{0\leq i\leq n}$ by

$$P(c) = \tau_{i(c)}.$$

Extend the map P to the edges of $\mathcal{CG}$ by mapping an edge to the image of one of its endpoints.

Lemma 4.3. *Let c, d be disjoint simple closed curves on S. Assume that $P(c) = \tau_i$. If $\tau_i \prec \tau_j$ is wide then $P(d) = \tau_s$ for some $s \geq j$.*

Proof. Assume that $P(c) = \tau_i \prec \tau_j$ is wide. By the definition of the map P, there is a large subtrack η of τ_i so that c fills a diagonal extension ξ of η. By Lemma 4.4 of [MM99], since d is disjoint from c, d is carried by a diagonal extension ζ of ξ. Then ζ is a diagonal extension of η.

Since $\tau_i \prec \tau_j$ is wide, d fills a diagonal extension of a large subtrack of τ_j. This implies that $P(d) = \tau_s$ for some $s \geq j$. □

Define a distance function d_g on $(\tau_i)_{0\leq i\leq n}$ as follows. For $i < j$, the *gap distance* $d_g(\tau_i, \tau_j)$ between τ_i and τ_j is the smallest number $k > 0$ so that there is a sequence $i_0 = i < i_1 < \cdots < i_k = j$ with the property that for each $p < k$, the pair $\tau_{i_{p+1}} \prec \tau_{i_p}$ is *not* wide. Note that this defines indeed a distance since for each ℓ the pair $\tau_{\ell+1} \prec \tau_\ell$ is not wide and hence $d_g(\tau_i, \tau_j) \leq j - i$. Moreover, the triangle inequality is immediate from Lemma 4.2.

The following is a consequence of Lemma 4.3. For its formulation, define a map P from a metric space X to a metric space Y to be *coarsely L-Lipschitz* for some $L > 1$ if $d(Px, Py) \leq Ld(x, y) + L$ for all $x, y \in X$.

Corollary 4.4. *The map $P : \mathcal{CG} \to ((\tau_i), d_g)$ is coarsely 2-Lipschitz.*

Define a map $\Upsilon : (\tau_i)_{0\leq i\leq n} \to \mathcal{CG}$ by associating to the train track τ_i one of its vertex cycles. We have

Lemma 4.5. *The map $\Upsilon : ((\tau_i), d_g) \to \mathcal{CG}$ is coarsely 22-Lipschitz.*

Proof. It suffices to show the following. If $\tau \prec \eta$ is not wide then the distance in $\mathcal{CG}$ between a vertex cycle α of τ and a vertex cycle β of η is at most 22.

To this end note that if α is a simple closed curve which is carried by a large train track ξ then the image of α under a carrying map is a subtrack of ξ. If this subtrack is not large then α is disjoint from an essential simple closed curve α' which can be represented by an edge-path in ξ (possibly with corners) which passes through any branch of ξ at most twice. Since a vertex cycle of ξ passes through each branch of ξ at most twice, this implies that α' intersects a vertex cycle of ξ in at most 4 points (see [H06] for details). In particular, the distance in $\mathcal{CG}$ between α and a vertex cycle of ξ is at most 6 [MM99].

On the other hand, if τ is another large train track and if ξ is a diagonal extension of a large subtrack τ' of τ then a vertex cycle of ξ intersects a vertex cycle of τ in at most 4 points. Hence the distance in $\mathcal{CG}$ between a vertex cycle of τ and a vertex cycle of ξ is at most 5. Together we deduce that the distance in $\mathcal{CG}$ between α and a vertex cycle of τ does not exceed 11.

Now by definition, if $\tau \prec \eta$ is not wide then there is a curve α which is carried by a diagonal extension ξ of a large subtrack τ' of τ and such that the following holds true. A carrying map $\tau \to \eta$ induces a carrying map of ξ onto a diagonal extension ζ of a large subtrack of η. The train track ζ carries α and so that α does not fill a large subtrack of ζ. Since a carrying map $\xi \to \zeta$ maps a large subtrack of ξ onto a large subtrack of ζ, the curve α does not fill a large subtrack of ξ.

By the above diskussion, the distance in $\mathcal{CG}$ between α and any vertex cycle of both τ and η is at most 11. This shows the lemma. □

Call a map Φ of a metric space (X, d) into a subset A of X a *coarse Lipschitz retraction* if there is a number $L > 1$ with the following properties.

(1) $d(\Phi(x), \Phi(y)) \le Ld(x, y) + L$.
(2) $d(x, \Phi(x)) \le L$ whenever $x \in A$.

We are now ready to show

Corollary 4.6. *For any splitting and shifting sequence $(\tau_i)_{0 \le i \le n}$ the map $\Upsilon \circ P$ is a coarse L-Lipschitz retraction of $\mathcal{CG}$ for a number $L > 1$ not depending on (τ_i) or on the Euler characteristic of S.*

Proof. Let $d_{\mathcal{CG}}$ be the distance in the curve graph of S. By Corollary 4.4 and Lemma 4.5 it suffices to show that $d_{\mathcal{CG}}(\alpha, \Upsilon \circ P(\alpha)) \le L$ for a universal constant $L > 1$ and every vertex cycle α of a train track τ_i from the sequence.

To this end observe that since α is a vertex cycle of τ_i, α is carried by each of the train tracks τ_j for $j \le i$, moreover α does not fill a diagonal extension of a large subtrack of τ_i. On the other hand, by definition of a wide pair, if $\tau_i \prec \tau_j$ is wide then α fills a large subtrack of τ_j. This means that $P(\alpha) = \tau_s$ for some $s \ge j$ so that the pair $\tau_i \prec \tau_{s+1}$ is *not* wide. The corollary now follows from Lemma 4.5. □

Remark: The above diskussion immediately implies that the image under Υ of a splitting and shifting sequence of train tracks is an unparametrized quasi-geodesic in $\mathcal{CG}$ for a constant not depending on the Euler characteristic of S. A non-effective version of this result was earlier established in [MM04] (see also [H06]).

5. Quasi-geodesics in the superconducting disk graph

Recall the definition of the graph $\mathcal{SDG}$. Our goal is to show that the natural map which associates to a disk its boundary defines a quasi-isometric embedding of $\mathcal{SDG}$ into the curve graph $\mathcal{CG}$ of ∂H. To simplify the notation we identify in the sequel a disk in H with its boundary circle. Thus we view the vertex set of $\mathcal{SDG}$ as a subset of the curve graph $\mathcal{CG}$.

The argument is based on the results in Section 2-3 and a construction from [MM04]. This construction uses a specific type of surgery sequences of disks which can be related to train tracks as follows.

Let H be a handlebody of genus $g \geq 2$ without spot. Let $D, E \subset H$ be two disks in normal position. Let E' be an outer component of $E - D$ and let D_1 be a disk obtained from D by simple surgery at E'.

Let α be the intersection of $\partial E'$ with ∂H. Then up to isotopy, the boundary ∂D_1 of the disk D_1 contains α as an embedded subarc. Moreover, α is disjoint from E. In particular, given an outer component E'' of $E - D_1$, there is a distinguished choice for a disk D_2 obtained from D_1 by simple surgery at E''. The disk D_2 is determined by the requirement that α is *not* a subarc of ∂D_2. Then for an outer component of $E - D_2$ there is a distinguished choice for a disk D_3 obtained from D_2 by simple surgery at an outer component of $E - D_2$ etc. We call a surgery sequence (D_i) of this form a *nested* surgery path in direction of E. Note that the boundary of each disk D_i is composed of a single subarc of ∂D and a single subarc of ∂E.

The following result is due to Masur and Minsky (this is Lemma 4.2 of [MM04] which is based on Lemma 4.1 and the proof of Theorem 1.2 in that paper).

Proposition 5.1. *Let $D, E \subset \partial H$ be any disks. Let $D = D_0, \dots, D_n$ be a nested surgery path in direction of E which connects D to a disk D_n disjoint from E. Then for each $i \leq n$ there is a train track τ_i on ∂H with a single switch such that the following holds true.*

(1) *τ_i carries ∂E and ∂E fills up τ_i.*
(2) *$\tau_{i+1} \prec \tau_i$.*
(3) *The disk D_i intersects τ_i only at the switch.*

The train tracks τ_i in the proposition are constructed as follows.

Let $\alpha = \partial D, \beta = \partial E$. Assume that the curves α, β are smooth (for a smooth structure on ∂H) and fill up ∂H. This means that the complementary components of $\alpha \cup \beta$ are all polygons or once holed polygons where in our setting, a hole is a boundary component of ∂H. Let P be a complementary polygon which has at least 6 sides. Such a polygon exists since the Euler characteristic of ∂H is negative. Its edges are subsegments of α and β. Let I be a boundary edge of P contained in α. Collapse $\alpha - I$ to a single point with a homotopy F of ∂H. This can be done in such a way that the restriction of F to β is nonsingular everywhere. The resulting graph has a single vertex. Collapsing the bigons in the graph to single arcs yields a train track τ with a single switch [MM04].

Let $b \subset \beta$ be an outer arc for $E - D$ and let $a \subset \alpha - I$ be the subarc of α which is bounded by the endpoints of b and which does not intersect the interval I. Then $a \cup b$ is the boundary of a disk D_1 obtained from D by nested surgery at b. The new train track τ_1 obtained from the above construction is obtained from $\beta \cup a$ by collapsing the arc a to a single point (we refer to [MM04] for details).

In the remainder of this section we identify a disk in H with its boundary circle on ∂H. Thus we view the vertex set of the superconducting disk graph $\mathcal{SDG}$ as a subset of the vertex set of the curve graph $\mathcal{CG}$ of ∂H. We are now ready to show

Theorem 5.2. *There is an effectively computable number $L > 1$such that the vertex inclusion defines an L- quasi-isometric embedding $\mathcal{SDG} \to \mathcal{CG}$.*

Proof. As before, let $d_{\mathcal{S}}$ be the distance in $\mathcal{SDG}$ and let $d_{\mathcal{CG}}$ be the distance in $\mathcal{CG}$. We have to show the existence of a number $c > 0$ with the following property. If D, E are any disks then

$$d_{\mathcal{S}}(D, E) \le c d_{\mathcal{CG}}(\partial D, \partial E).$$

By Proposition 5.1, there is a nested surgery path $D = D_0, \dots, D_n$ connecting the disk $D_0 = D$ to a disk D_n which is disjoint from E, and there is a sequence $(\tau_i)_{0 \le i \le n}$ of one-switch train tracks on ∂H such that $\tau_{i+1} \prec \tau_i$ for all $i < n$ and that D_i intersects τ_i only at the switch.

By Theorem 2.3.1 of [PH92], there is a splitting and shifting sequence $\tau_0 = \eta_0 \prec \cdots \prec \eta_s = \tau_n$ connecting τ_0 to τ_n and a sequence $0 = u_0 < \cdots < u_n = s$ so that $\eta_{u_q} = \tau_q$ for $0 \le q \le n$. Since the disk D_i intersects τ_i only at the switch, the boundary ∂D_i of D_i intersects a vertex cycle of τ_i in at most two points and hence the distance in the curve graph $\mathcal{CG}$ between ∂D_i and a vertex cycle of τ_i is at most three. Now the disks D_i and D_{i+1} are disjoint and consequently the distance in $\mathcal{CG}$ between a vertex cycle of τ_i and a vertex cycle of τ_{i+1} is at most 7. Thus by Corollary 4.6 and the definition of the no-gap distance, it suffices to show the existence of an effectively computable number $b > 0$ with the following property. Let $k < i$ be such that the pair $\tau_i \prec \tau_k$ is *not* wide; then $d_{\mathcal{S}}(D_i, D_k) \le b$.

Since $\tau_i \prec \tau_k$ is not wide there is a large subtrack τ_i' of τ_i, a diagonal extension ζ_i of τ_i' and a simple closed curve α carried by ζ_i with the following property. Let τ_k' be the image of τ_i' under a carrying map $\tau_i \to \tau_k$ and let ζ_k be the diagonal extension of τ_k' which is the image of ζ_i under a carrying map induced by a carrying map $\tau_i' \to \tau_k'$. Then α does not fill a large subtrack of ζ_k.

Since ζ_i is a diagonal extension of the large subtrack τ_i' of τ_i and since D_i intersects τ_i only at the switch, the intersection number between ∂D_i and ζ_i is bounded from above by a constant $\kappa \ge 2$ which does not exceed a constant multiple of the Euler characteristic of S.

For each $p \in [k, i]$, the image of τ_i' under a carrying map $\tau_i \to \tau_p$ is a large subtrack τ_p' of τ_p, and there is a diagonal extension ζ_p of τ_p' which carries α. We may assume that $\zeta_u \prec \zeta_p$ for $u \ge p$. The disk D_p intersects ζ_p in at most κ points.

For each $p \in [k, i]$ let $\beta_p \prec \zeta_p$ be the subtrack of ζ_p filled by α. Then β_p is connected and not large. The union Y_p of a thickening of β_p with the components of $\partial H - \beta_p$ which are either simply connected or once holed disks is a proper connected subsurface of ∂H for all p. The boundary of Y_p can be realized as a union of simple closed curves which are embedded in β_p (but with cusps). The carrying map $\beta_{p+1} \to \beta_p$ maps Y_{p+1} into Y_p. In particular, either the boundary of Y_{p+1} coincides up to homotopy with the boundary of Y_p or Y_{p+1} is a *proper* subsurface of Y_p. This means that there is a boundary circle of Y_{p+1} which is essential in Y_p and hence in ∂H. In other words, the subsurfaces Y_p are nested, and hence their number is bounded from above by a universal constant $h > 0$ depending linearly on the Euler characteristic of S.

Since ∂D_p intersects ζ_p in at most κ points, the number of intersections between ∂D_p and ∂Y_p is bounded from above by a universal constant $\chi > 0$ (κ times the maximal number of branches of a train track on ∂H will do, and this number is quadratic in the Euler characteristic of S). As a consequence, there are h essential simple closed curves $c_1, \dots, c_h$ in ∂H so that for every $p \in [k, i]$ there is some $r(p) \in \{1, \dots, h\}$ with

$$\iota(\partial D_p, c_{r(p)}) \leq \chi.$$

Each of the curves c_j is a fixed boundary component of one of the subsurfaces Y_p.

By reordering, assume that $r(i) = 1$. Let v_1 be the minimum of all numbers $p \in [k, i]$ such that $r(v_1) = 1$. Proposition 3.5 shows that

$$d_{\mathcal{S}}(D_i, D_{v_1}) \leq 2\chi + 8.$$

On the other hand, we have $d_{\mathcal{S}}(D_{v_1}, D_{v_1 - 1}) = 1$. Again by reordering, assume that $r(v_1 - 1) = 2$ and repeat this construction with the disks $D_{v_1-1}, \dots, D_k$ and the curve c_2. In $a \leq h$ steps we construct in this way a decreasing sequence $i \geq v_1 > \cdots > v_a = k$ such that $d_{\mathcal{S}}(D_{v_u}, D_{v_{u-1}}) \leq 2\chi + 9$ for all $u \leq a$. This implies that

$$d_{\mathcal{S}}(D_i, D_k) \leq h(2\chi + 9)$$

which is what we wanted to show. □

The *asymptotic dimension* $\mathrm{asdim}(X)$ of a metric space X is the infimum of all numbers $n \geq 1$ such that the following holds true. For any $r > 0$, there exists a covering of X of $n - 1$ families $\mathcal{U}^0 dots, \mathcal{U}^n$ of *r-disjoint* uniformly bounded subsets of X such that $\cup_i \mathcal{U}^i$ is a cover of X. Here the family $\mathcal{U}^i$ is called r-disjoint the distance between any two sets in the family $\mathcal{U}^i$ is at least r.

As an immediate consequence of Theorem 5.2 we observe

Corollary 5.3. *The graph $\mathcal{SDG}$ is hyperbolic, and its asymptotic dimension is at most $4g - 4$.*

Proof. A quasi-isometrically embedded geodesic metric subgraph of a hyperbolic geodesic metric graph satisfies the thin triangle condition and hence it is hyperbolic.

Furthermore, it is immediate from the definition that the existence of a quasi-isometric embedding $f : X \to Y$ implies that $\mathrm{asdim}(X) \leq \mathrm{asdim}(Y)$. Now the asymptotic dimension of the curve graph of a closed surface of genus g is at most $4g - 4$ [BB15] (see also the earlier work [BF08] for finiteness) and therefore the asymptotic dimension of $\mathcal{SDG}$ is at most $4g - 4$ as claimed. □

As a corollary, we obtain the statement of the theorem 3 from the introduction.

Corollary 5.4. *Let H be a handlebody of genus $g \geq 2$ without spots. Then the vertex inclusion $\mathcal{SDG} \to \mathcal{CG}$ is a quasi-isometric embedding.*

A hyperbolic geodesic metric space Y admits a *Gromov boundary.* This boundary is a topological space on which the isometry group of Y acts as a group of homeomorphisms. The remainder of this section is devoted to determine the Gromov boundary of the graph $\mathcal{SDG}$.

Let $\mathcal{L}$ be the space of all *geodesic laminations* on ∂H (for some fixed hyperbolic metric) equipped with the *coarse Hausdorff topology.* In this topology, a sequence (μ_i) converges to a lamination μ if every accumulation point of (μ_i) in the usual Hausdorff topology contains μ as a sublamination. Note that the coarse Hausdorff topology on $\mathcal{L}$ is not T_0, but its restriction to the subspace $\partial\mathcal{CG} \subset \mathcal{L}$ of all minimal geodesic laminations which fill up ∂H (i.e. which intersect every simple closed geodesic transversely) is Hausdorff. The space $\partial\mathcal{CG}$ equipped with the coarse Hausdorff topology can naturally be identified with the Gromov boundary of $\mathcal{CG}$ [K99, H06].

Let
$$\partial\mathcal{H} \subset \partial\mathcal{CG}$$
be the closed subset of all geodesic laminations which are limits in the coarse Hausdorff topology of boundaries of disks in H. The handlebody group $\mathrm{Map}(H)$ acts on $\partial\mathcal{CG}$ as a group of transformations preserving the subset $\partial\mathcal{H}$. The Gromov boundary of $\mathcal{SDG}$ can now fairly easily be determined from Corollary 5.4. To this end we first observe

Lemma 5.5. *The Gromov boundary of $\mathcal{SDG}$ is a closed* $\mathrm{Map}(H)$*-invariant subset of $\partial\mathcal{H}$.*

Proof. Since the vertex inclusion $\mathcal{SDG} \to \mathcal{CG}$ defines a quasi-isometric embedding, the Gromov boundary of $\mathcal{SDG}$ is the subset of the Gromov boundary of $\mathcal{CG}$ of all endpoints of quasi-geodesic rays in $\mathcal{CG}$ which are contained in $\mathcal{SDG}$.

By the main result of [H06] (see [K99] for an earlier account of a similar statement), a simplicial quasi-geodesic ray $\gamma : [0, \infty) \to \mathcal{CG}$ defines the endpoint lamination $\nu \in \partial\mathcal{CG}$ if and only if the curves $\gamma(i)$ converge as $i \to \infty$ in the coarse Hausdorff topology to ν. As a consequence, the Gromov boundary of $\mathcal{SDG}$ is a subset of $\partial\mathcal{H}$, and this subset is clearly $\mathrm{Map}(H)$-invariant.

We are left with showing that the Gromov boundary of $\mathcal{SDG}$ is a closed subset of $\partial\mathcal{CG}$. To this end note that by Corollary 5.4, there is a number $p > 1$ such that for every $L > 1$, any L-quasi-geodesic in $\mathcal{SDG}$ is an Lp-quasi-geodesic in $\mathcal{CG}$. Moreover, for a suitable choice of p, any vertex in $\mathcal{SDG}$ can be connected to any point in the Gromov boundary of $\mathcal{SDG}$ by a p-quasi-geodesic.

Now let (ν_i) be a sequence in the Gromov boundary of $\mathcal{SDG}$ which converges in $\partial\mathcal{CG}$ to a lamination ν. Let ∂D be the boundary of a disk and let $\gamma : [0, \infty) \to \mathcal{CG}$ be a quasi-geodesic ray issuing from $\gamma(0) = \partial D$ with endpoint ν. By hyperbolicity of $\mathcal{CG}$ and by the diskussion in the previous paragraph, there is a number $R > 0$ and for every $k \geq 0$ there is some $i(k) > 0$ such that a p-quasi-geodesic in $\mathcal{SDG} \subset \mathcal{CG}$ connecting $\gamma(0)$ to $\nu_{i(k)}$ passes through the R-neighborhood of $\gamma(k)$ in $\mathcal{CG}$. Since $k > 0$ was arbitrary, this implies that the entire quasi-geodesic ray γ is contained in the R-neighborhood of the subset $\mathcal{SDG}$ of $\mathcal{CG}$. Using once more hyperbolicity, we conclude that there is a quasi-geodesic ray in $\mathcal{CG}$ connecting $\gamma(0)$ to ν which is entirely contained in $\mathcal{SDG}$. But this just means that ν is contained in the Gromov boundary of $\mathcal{SDG}$. □

The handlebody group $\mathrm{Map}(H)$ naturally acts on the Gromov boundary of $\mathcal{SDG}$ as a group of homeomorphisms. Since $\mathrm{Map}(H)$ is a subgroup of the mapping class group $\mathrm{Mod}(\partial H)$, by naturality this action is compatible with the action of the mapping class group on the Gromov boundary of the curve graph. From Lemma 5.5 and the following observation (which is essentially contained in Theorem 1.2 of [M86]), we conclude that $\partial\mathcal{H}$ is indeed the Gromov boundary of $\mathcal{SDG}$.

Lemma 5.6. *The action of the handlebody group* $\mathrm{Map}(H)$ *on $\partial\mathcal{SDG}$ is minimal.*

Proof. Let (∂D_i) be a sequence of boundaries of disks D_i converging in the coarse Hausdorff topology to a geodesic lamination $\mu \in \partial\mathcal{H}$. For each i let E_i be a disk which is disjoint from D_i. Since the space of geodesic laminations equipped with the usual Hausdorff topology is compact, up to passing to a subsequence the sequence (∂E_i) converges in the Hausdorff topology to a geodesic lamination ν which does not intersect μ (we refer to [K99, H06] for details of this argument). Now μ is minimal and fills up ∂H and therefore the lamination ν contains μ as a sublamination. This just means that (∂E_i) converges in the coarse Hausdorff topology to μ.

Since the genus of H is at least two, for every separating disk in H we can find a disjoint non-separating disk. Thus the diskussion in the previous paragraph shows that every $\mu \in \partial\mathcal{H}$ is a limit in the coarse Hausdorff topology of a sequence of non-separating disks. However, the handlebody group acts transitively on non-separating disks. Minimality of the action of Map(H) on $\partial\mathcal{SDG}$ follows. ☐

As an immediate consequence of Lemma 5.5 and Lemma 5.6 we obtain

Corollary 5.7. *$\partial\mathcal{H}$ is the Gromov boundary of $\mathcal{SDG}$.*

The following definition is taken from the beginning of Section 3 of [H16].

Definition 5.8. A connected essential subsurface X of ∂H is called *thick* if the following conditions are satisfied.

(1) Every disk intersects X.
(2) X is filled by boundaries of disks.

By the remark after Lemma 3.1 of [H16], a thick subsurface X of ∂H is distinct from a sphere with at most four holes and from a torus with a single hole.

Our first goal is to show that the asymptotic dimension of the electrified disk graph of a thick subsurface of X is finite. To this end denote by $d_{\mathcal{CG},X}$ the distance in the curve graph $\mathcal{CG}$ of X and by $d_{\mathcal{E},X}$ the distance in the electrified disk graph of X.

In Definition 3.3 of [H16], an *I-bundle generator* in X is defined to be an essential simple closed curve $\gamma \subset X$ with the following property. There is a compact surface F with non-empty boundary ∂F, there is a boundary component α of ∂F, and there is an orientation preserving embedding Ψ of the oriented I-bundle $\mathcal{I}(F)$ over F into H which maps α to γ and which maps F_α onto the complement in X of a tubular neighborhood of the boundary ∂X of X. The surface F is called the *base* of the I-bundle generated by γ.

By what we showed so far, if X does not contain an I-bundle generator then $\mathcal{EDG}(X) = \mathcal{SDG}(X)$ and there is nothing to show. Thus assume that there is an I-bundle generator $\gamma \subset X$. Let

$$\mathcal{E}(\gamma) \subset \mathcal{EDG}(X)$$

be the complete subgraph of $\mathcal{EDG}(X)$ whose vertices are disks intersecting γ in precisely two points. Define

$$\mathcal{E} = \{\mathcal{E}(\gamma) \mid \gamma\}$$

where γ runs through all I-bundle generators in X. By definition, $\mathcal{SDG}(X)$ is 2-quasi-isometric to the $\mathcal{E}$-electrification of $\mathcal{EDG}(X)$.

Let now D be a disk with boundary $\partial D \subset X$ which intersects some I-bundle generator γ in precisely two points. Then D is an I-bundle over a simple arc $\beta \subset F$ with boundary on γ (see p.21-22 in [H16]). We call β the *projection* of ∂D to F.

By Lemma 4.2 of [H16], the map which associates to a disk $D \in \mathcal{E}(\gamma)$ the projection of ∂D to the base surface F extends to a 2-quasi-isometry of $\mathcal{E}(\gamma)$ onto the *electrified arc graph* of F which is 4-quasi-isometric to the curve graph of F (see Lemma 4.1 of [H16] for a proof of this folclore result). In particular, by [BB15] we have

Proposition 5.9. *The asymptotic dimension of the graph $\mathcal{E}(\gamma)$ is bounded from above by $4g(F)-3+p$ where $h \leq g/2$ is the genus of F and where p is the number of boundary components.*

As a corollary, we obtain

Corollary 5.10. *Let $X \subset \partial H$ be a thick subsurface of genus $g(X) \geq 0$ with p boundary components; then* $\mathrm{asdim}(\mathcal{EG}(X)) \leq 8g(X)-6+2p$.

Proof. The graph $\mathcal{EG}(X)$ is hyperbolic and hyperbolic relative to the subgraphs $\mathcal{E}(\gamma)$, with electrification the graph $\mathcal{SDG}(X)$. By Proposition 5.9, the asymptotic dimension of each of the graphs $\mathcal{E}(\gamma)$ is not smaller than $4g(F)-3+p'$ where F is the base surface of the I-bundle and p' is the number of its boundary components. Note that $4g(F)-3+p\prime \leq 4g(X)-4+p$ and therefore by Proposition 2.3, we have $\mathrm{asdim}(\mathcal{EG}(X) \leq 8g(X)-6+2p$ as claimed. □

As a consequence of Corollary 5.10 and the results in [H16] we are now ready to show

Theorem 5.11. *The asymptotic dimension of the disk graph of a handlebody of genus g is bounded from above by $(6g-6)(4g-4)$.*

Proof. By the main result of [H16] and the above estimates of asymptotic dimension, there is a sequence $\mathcal{G}_1,\dots,\mathcal{G}_{3g-3}$ of hyperbolic graphs with the following properties.

(1) $\mathcal{G}_1 = \mathcal{EG}(\partial H)$.
(2) For each $i \geq 2$, $\mathcal{G}_i$ is a hyperbolic graph which is hyperbolic relative to a family $\mathcal{H}$ of hyperbolic subgraphs. The $\mathcal{H}$-electrification of $\mathcal{G}_i$ equals the graph $\mathcal{G}_{i-1}$. The asymptotic dimension of each graph in the family is at most $2(4g-4)$.

Applying Proposition 2.3 inductively $3g-3$ times then yields that indeed, the asymptotic dimension of the disk graph is at most $((6g-6)(4g-4)$. □

6. Disks for a handlebody with a single spot

A *handlebody with spots* is a handlebody H of genus $g \geq 1$ with $m \geq 1$ marked points, called spots, on the boundary. We view the boundary ∂H of such a handlebody as a surface with m punctures.

A *disk* in a handlebody with spots is a disk D in H whose boundary ∂D is disjoint from the spots. We require that ∂D is an essential simple closed curve in ∂H, i.e. it is neither contractible nor homotopic into a spot. Two such disks are isotopic if there is an isotopy between them which is disjoint from the spots. We use the terminology introduced in Section 2 also for handlebodies with spots.

For the remainder of this section we consider a handlebody H of genus $g \geq 2$ with a single spot p. Our goal is to prove the first part of Theorem 4 from the introduction.

Let H_0 be the handlebody obtained from H by forgetting the spot and let

$$\Phi : H \to H_0$$

be the natural forgetful map. Let $\mathcal{CG}$ be the curve graph of ∂H and let $\mathcal{CG}(\partial H_0)$ be the curve graph of ∂H_0.

Lemma 6.1. *The map Φ induces a simplicial surjection*

$$\Pi : \mathcal{CG} \to \mathcal{CG}(\partial H_0)$$

which maps diskbounding curves to diskbounding curves.

Proof. Since H has a single spot, the image under the map Φ of an essential simple closed curve γ on ∂H is an essential simple closed curve $\Phi(\gamma)$ on ∂H_0. The curve γ is diskbounding if and only if this is the case for $\Phi(\gamma)$. Moreover, if γ, δ are disjoint then this holds true for $\Phi(\gamma), \Phi(\delta)$ as well. This immediately implies the lemma. ☐

Theorem 7.1 of [KLS09] shows that for every simple closed curve c in ∂H_0 the preimage $\Pi^{-1}(c)$ of c in the curve graph $\mathcal{CG}$ of ∂H is a tree T_c which can be described as follows.

Let $\mathbf{H}^2$ be the universal covering of ∂H_0 and let $\mathcal{S}(c)$ be the preimage of c in $\mathbf{H}^2$. Let T be the tree whose vertex set is the set of components of $\mathbf{H}^2 - \mathcal{S}(c)$, and where two components are connected by an edge if their closures intersect. View the spot p as the basepoint for the fundamental group of ∂H_0. We assume that p does not lie on c. Then $\pi_1(\partial H_0, p)$ acts transitively on T as a group of simplicial isometries. The tree T_c is $\pi_1(\partial H_0, p)$-equivariantly isomorphic to T (see Section 7 of [KLS09] for details).

A *section* for the projection $\Pi : \mathcal{CG} \to \mathcal{CG}(\partial H_0)$ is a map $\Lambda : \mathcal{CG}(\partial H_0) \to \mathcal{CG}$ so that $\Pi \circ \Lambda = \mathrm{Id}$. The next observation is essentially due to Harer (see [KLS09]).

Lemma 6.2. *For each $\gamma \in \mathcal{CG}$ there is an isometric section $\Lambda : \mathcal{CG}(H_0) \to \mathcal{CG}$ for the projection $\Pi : \mathcal{CG} \to \mathcal{CG}(H_0)$ passing through γ.*

Proof. Fix a hyperbolic metric on ∂H_0. Every simple closed curve on ∂H_0 can be represented by a unique simple closed geodesic. The union of these geodesics has area zero and hence there is a point $x \in \partial H_0$ which is not contained in any such geodesic. View x as the marked point in ∂H.

Define a map $\Lambda : \mathcal{CG}(\partial H_0) \to \mathcal{CG}$ as follows. To a vertex of $\mathcal{CG}(\partial H_0)$, i.e. a simple closed curve on ∂H_0, associate the isotopy class $\Lambda(\gamma)$ of γ viewed as a curve in $\partial H = (\partial H_0, x)$. Clearly $\Pi \circ \Lambda$ is the identity on vertices of $\mathcal{CG}(\partial H_0)$. If α, β are disjoint simple closed curves on ∂H_0 then their geodesic representatives are disjoint as well, and $\Lambda(\alpha), \Lambda(\beta)$ are disjoint simple closed curves in ∂H. Thus Λ extends to a simplicial section. Since Π is distance non-increasing, Λ is an isometric embedding of $\mathcal{CG}(\partial H_0)$ into $\mathcal{CG}$.

Consider the Birman exact sequence

$$0 \to \pi_1(\partial H_0, x) \to \mathrm{Mod}(\partial H) \to \mathrm{Mod}(\partial H_0) \to 0. \tag{2}$$

The action of $\pi_1(\partial H_0, x)$ on the curve graph of ∂H is fibre preserving. Indeed, the restriction of this action to a fixed fibre is just the action of $\pi_1(\partial H_0, x)$ on the geometric realization of the fibre as described in the text preceding this proof. In particular, this action is transitive on vertices in the fibres (see [KLS09] for details). Equivalently, for every simple closed curve α on ∂H and any curve $\beta \in \Pi^{-1}(\Pi(\alpha))$ there is a mapping class $\psi \in \pi_1(\partial H_0, x) < \mathrm{Mod}(\partial H)$ so that $\psi(\alpha) = \beta$.

Since the map ψ acts as a fibre preserving simplicial isometry on the curve graph of ∂H, the composition $\psi \circ \Lambda$ is a new isometric section for the map Π. This implies that for every vertex of $\mathcal{CG}$ there is an isometric section for the projection Π whose image contains that point. This is what we wanted to show. $\square$

Remark: 1) The proof of Lemma 6.2 contains some additional information. Namely, two simple closed curves γ, δ on ∂H are contained in the image of an isometric section $\mathcal{CG}(\partial H_0) \to \mathcal{CG}$ as constructed by Harer [KLS09] and used in the proof of Lemma 6.2 if the spot is not contained in a component of $\partial H - (\gamma \cup \delta)$ which is a punctured bigon, i.e. a component whose boundary consists of a single subarc of γ and a single subarc of δ.

2) In [MjS12] (see also [H05]), the authors define a *metric graph bundle* to consist of graphs V, B and a surjective simplicial map $p : V \to B$ with the following properties.

(1) For each $b \in \mathcal{V}(B)$, $F_b = p^{-1}(b)$ is a connected subgraph of V, and the inclusion maps $i : \mathcal{V}(F_b) \to V$ are uniformly metrically proper for the path metric d_b induced on F_b as measured by a non-decreasing surjective function $f : \mathbb{N} \to \mathbb{N}$.
(2) For any two adjacent vertices b_1, b_2 of $\mathcal{V}(B)$, each vertex x_1 of F_{b_1} is connected by an edge with a vertex in F_{b_2}.

If the fibres are trees then we speak about a metric tree bundle.

By Lemma 6.2, the graph projection $\Pi : \mathcal{CG} \to \mathcal{CG}(H_0)$ satisfies the second property in the definition of a metric tree bundle. However, the first property states that for every $b \in B$ the distance in V between any two points $x, y \in F_b$ is bounded from below by $f(d_b(x, y))$, and this property is violated for the map Π. As an example, let c_1, c_2 be two disjoint non-separating simple closed curves in ∂H_0, let $x \in \partial H_0 - (c_1 \cup c_2)$ and let ζ be a loop in $\pi_1(\partial H_0 - c_1, x)$ which fills $\partial H_0 - c_1$. Then the point pushing map β defined by ζ via the Birman exact sequence (2) acts as a hyperbolic isometry and hence with positive translation length on the fibre over c_2 although the distance in $\mathcal{CG}(H)$ between any two points on the orbit is at most two.

Thus we can not use Theorem 5.2 to deduce that the superconducting disk graph of H is quasi-isometrically embedded in the curve graph $\mathcal{CG}$ of ∂H. In fact we have

Proposition 6.3. *The disk graph of H is not a quasi-convex subset of the curve graph of ∂H.*

Proof. The curve graph of ∂H is hyperbolic, and pseudo-Anosov elements of the mapping class group of ∂H act as hyperbolic isometries on the curve graph. If $\gamma \in \pi_1(\partial H_0, x)$ is filling, i.e. if γ decomposes ∂H_0 into disks, then the image $\Psi(\gamma)$ of γ in $\mathrm{Mod}(\partial H)$ via the Birman exact sequence (2) is pseudo-Anosov [Kr81, KLS09].

Let φ be a diffeomorphism of ∂H_0 which fixes the basepoint x and which defines a pseudo-Anosov element of $\mathrm{Mod}(H_0)$. We require that a quasi-axis for the action of φ on $\mathcal{CG}(H_0)$ passes uniformly near the boundary of a disk and that moreover for any diskbounding simple closed curve ζ, the distance in $\mathcal{CG}(H_0)$ between $\varphi^k \zeta$ and the quasi-convex subset of diskbounding curves tends to infinity as $k \to \infty$.

Such a pseudo-Anosov element can be found as follows. Each pseudo-Anosov element fixes two filling projective measured laminations, and the set of pairs of such fixed points is dense in $\mathcal{PML} \times \mathcal{PML}$. The closure in $\mathcal{PML}$ of the set of diskbounding simple closed curves is nowhere dense in $\mathcal{PML}$, and a pseudo-Anosov element φ whose pair of fixed points is contained in the complement will do.

Since φ fixes the point x, it acts on the fundamental group $\pi_1(\partial H_0, x)$ of ∂H_0, moreover it can be viewed as an element of $\mathrm{Mod}(\partial H)$. We denote this element of $\mathrm{Mod}(\partial H)$ again by φ.

Let $\beta \subset \partial H_0 - \{x\}$ be a diskbounding curve near a quasi-axis of φ. Via the isometric section $\Lambda : \mathcal{CG}(\partial H_0) \to \mathcal{CG}$ defined by x as in Lemma 6.2, we can view β as a diskbounding curve in ∂H. Let $\gamma \in \pi_1(\partial H_0, x)$ be a filling curve. For each $k > 0$ let β_k be the image of β under point-pushing by the curve $\varphi^k(\gamma)$. Then β_k is diskbounding, moreover we have

$$\beta_k = (\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k})(\beta).$$

A quasi-axis in $\mathcal{CG}$ of the pseudo-Anosov element $\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k}$ is the image under φ^k of a quasi-axis of $\Psi(\gamma)$. By hyperbolicity of $\mathcal{CG}$, via perhaps replacing γ by a multiple we may assume that a geodesic in $\mathcal{CG}$ connecting β to β_k is close in the Hausdorff topology to the composition of three arcs. The first arc connects β to a quasi-axis ζ of $\varphi^k \circ \Psi(\gamma) \circ \varphi^{-k}$, the second arc travels along ζ and the third arc connects ζ to β_k. However, by the choice of φ, for suitable choices of k and suitable multiplicities of γ, such a curve is arbitrarily far in $\mathcal{CG}$ from the set of diskbounding curves. $\square$

7. Handlebodies with at least two spots

In this section we show the second part of Theorem 4 from the introduction. We continue to use the notations from Section 2-4.

As mentioned at the end of Section 2, the fact that surgery of disks in handlebodies with spots may produce peripheral disks causes substantial difficulty. Indeed, we showed in Section 5 that for handlebodies H with a single spot and genus $g \geq 2$, boundaries of disks do not form a quasi-convex subset of the curve graph of ∂H. To understand disk graphs in handlebodies with at least two spots we first establish a weaker analog of Lemma 3.1 and then analyze in detail disks which become peripheral after a single surgery.

As a warm-up, observe that for handlebodies H with a single spot, Lemma 3.1 is valid. Namely, for any two disks D, E with boundary in ∂H which are not disjoint and for any outer component E' of $E - D$, at least one of the disks obtained from D by surgery at E' is not peripheral. Thus the proof of Lemma 3.1 carries over without modification. Denote as before by $d_{\mathcal{D}}$ and $d_{\mathcal{E}}$ the distance in the disk graph and the electrified disk graph, respectively. We obtain

Lemma 7.1. *If ∂H contains a single spot then for any disks D, E in H we have*

$$d_{\mathcal{D}}(D, E) \leq \iota(\partial D, \partial E)/2 + 1.$$

For convenience of notation in the proof of the following lemma, we define the intersection between a peripheral curve α in ∂H and any other curve β in ∂H as $\iota(\alpha, \beta) = 0$.

Lemma 7.2. *Let H be a handlebody with $n \geq 2$ spots. Then for any two disks D, E in H we have*

$$d_{\mathcal{E}}(D, E) \leq \iota(\partial D, \partial E)/2 + 1.$$

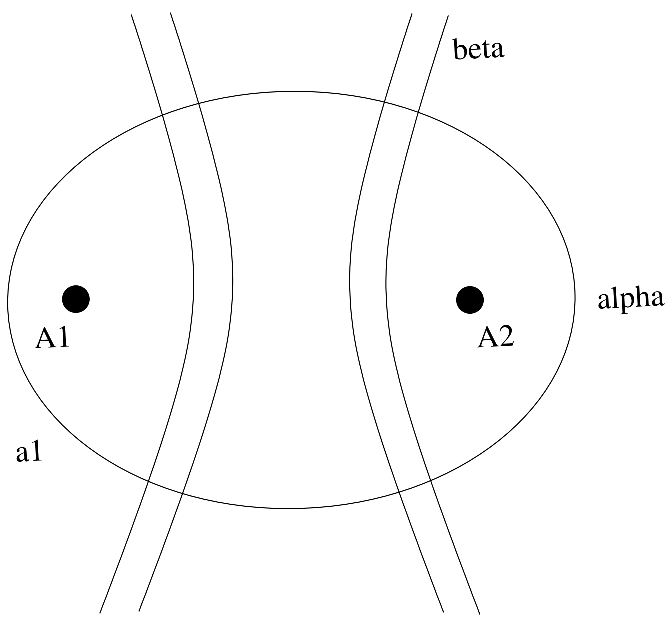
Proof. As in the proof of Lemma 3.1, we proceed by induction on $\iota(\partial D, \partial E)$. The case $\iota(\partial D, \partial E) = 0$ is immediate from the definitions. Thus assume that the claim of the lemma holds true whenever $\iota(\partial D, \partial E) \leq k - 1$ for some $k \geq 1$.

Let $\alpha = \partial D, \beta = \partial E \subset \partial H$ be diskbounding simple closed curves with $\iota(\alpha, \beta) = k$. If there is an essential simple closed curve $\gamma \subset \partial H$ disjoint from $\alpha \cup \beta$ then $d_{\mathcal{E}}(D, E) \leq 1$ by the definition of the electrified disk graph and there is nothing to show. Thus assume that $\alpha \cup \beta$ decomposes ∂H into disks and one-holed disks. Then a componenet of $\partial H - (\alpha \cup \beta)$ is a polygon or one-holed polygon with sides alternating between subarcs of α and subarcs of β. A complementary polygon has at least four sides. A *punctured bigon* is a complementary component which is a one-holed disk bounded by a single subarc of α and a single subarc of β.

Let E' be an outer component of $E - D$ (see Section 2 for the terminology). Surgery of D at E' yields two disks B_1, B_2 in H. The boundaries of these disks are simple closed curves α_1, α_2 in ∂H which are disjoint from α, moreover $\iota(\alpha_i, \beta) \leq k - 2 = \iota(\alpha, \beta) - 2$. If at least one of the disks B_1, B_2 is essential, say if this holds true for B_1, then B_1 is a disk with $d_{\mathcal{E}}(D, B_1) = 1$ and $\iota(\partial B_1, \partial E) \leq k - 2$. Thus the induction hypothesis can be applied to B_1 and E and shows the lemma.

If both disks B_1, B_2 are non-essential then $\alpha = \partial D$ bounds a twice punctured disk D_0 which is embedded in ∂H. The curve $\beta = \partial E$ decomposes D_0 into two once punctured disks A_1, A_2 and a set of rectangles.

Figure B



The once punctured disk A_i $(i = 1, 2)$ is bounded by a subarc a_i of α and a subarc b_i of β. Let p_i be the spot contained in A_i. The arc a_i is an outer arc for the disk E. For $i = 1, 2$ surger E at the outer arc a_i to a disk whose boundary β_i is obtained from β by replacing the arc b_i with the arc a_i. Then $\iota(\alpha, \beta_i) \leq \iota(\alpha, \beta) - 2$ and hence if either β_1 or β_2 is essential then the claim follows as before from the induction hypothesis.

Thus we are left with the case that both curves β_1, β_2 are peripheral. Then β bounds a disk $E_0 \subset \partial H$ punctured at p_1, p_2. The intersection $D_0 \cap E_0$ is a union of the once punctured bigons A_1, A_2 and a collection of rectangles. More precisely, if $R = E_0 - (A_1 \cup A_2)$ then the intersections of R with ∂D_0 decompose R into a chain of rectangles $R_1, \dots, R_s$ such that for even i, the rectangle R_i is contained in the twice punctured disk D_0, and for odd i the rectangle R_i is contained in $\partial H - D_0$. The number s of rectangles is odd, and $\iota(\partial D, \partial E) = \iota(\alpha, \beta) = 2s + 2$.

Assume for the moment that $s = 1$, i.e. that the rectangle R does not intersect D_0. Then R is homotopic relative to α to an embedded arc ρ in $\partial H - D_0$. Thus ∂H contains an essential simple closed curve γ which is disjoint from $D_0 \cup R$, and γ is disjoint from both D_0, E_0 and hence from D and E. This shows that $d_{\mathcal{E}}(D, E) \leq 1$ which is what we wanted to show.

If $s \geq 3$ then assume without loss of generality that the arc $a_1 \subset A_1$ is contained in the boundary of the rectangle R_1. Homotope the rectangle R_1 relative to α to a rectangle $\hat{R}_1$ such that the side ρ of $\hat{R}_1$ which is opposite to a_1 is a subarc of $a_2 \subset \alpha$. Attach to ρ a disk $G \subset A_2$ punctured at p_2. The union $A_1 \cup \hat{R}_1 \cup G$ is a disk $B_0 \subset \partial H$ punctured at p_1 and p_2. The boundary of B_0 bounds a properly embedded disk B in H. By the case $s = 1$ diskussed above, we have $d_{\mathcal{E}}(D, B) \leq 1$.

On the other hand, $\iota(\partial B, \beta) \leq 2s - 2 = \iota(\alpha, \beta) - 4$ and hence the claim follows as before from the induction hypothesis. ☐

We next establish a version of Proposition 3.5 for disks D, E in handlebodies with at least two spots on the boundary which become peripheral after closing one of the spots. To ease terminology we say that a disk $D \subset H$ *encloses two spots* p_1, p_2 in ∂H if D is homotopic with fixed boundary to a disk $D_0 \subset \partial H$ punctured at the points p_1, p_2.

Lemma 7.3. *Let D, E be two disks in H which enclose the same spots $p_1 \neq p_2 \in X$. If there is a simple closed curve $\gamma \subset \partial H$ which intersects $\partial D, \partial E$ in at most $k \geq 1$ points then $d_{\mathcal{E}}(D, E) \leq 2k + 12$.*

Proof. Let D, E be as in the lemma, with boundaries $\partial D = \alpha, \partial E = \beta$. Then α, β bound disks $D_0, E_0 \subset \partial H$ punctured at the points p_1, p_2. Up to isotopy, the intersection $D_0 \cap E_0$ is a union of two once punctured bigons A_1, A_2 and a disjoint union of rectangles. The punctured bigon A_i $(i = 1, 2)$ is bounded by a subarc of α and a subarc of β. Assume that A_i contains the spot p_i $(i = 1, 2)$.

Let $\gamma \subset \partial H$ be a simple closed curve which intersects both α, β in at most k points. If γ is not disjoint from α then $\gamma \cap D_0$ is a union of at most $k/2$ pairwise disjoint arcs. By modifying γ with an isotopy we may assume that these arcs are disjoint from the punctured disks A_1, A_2. For $i = 1, 2$ choose a compact arc $c_i \subset D_0$ which connects the punctured bigon A_i to γ and whose interior is disjoint from γ. Let moreover γ_0 be one of the two subarcs of γ which connect the two endpoints of c_1, c_2 on γ. The concatenation $c_1 \circ \gamma_0 \circ c_2^{-1}$ (read from left to right) of c_1, γ_0, c_2^{-1} is an embedded arc in ∂H connecting A_1 to A_2.

Let $C_0 \subset \partial H$ be an embedded rectangle with two opposite sides contained in the interior of $\partial A_1 \cap D_0, \partial A_2 \cap D_0$ which is a thickening of the arc $c_1 \circ \gamma_0 \circ c_2^{-1}$. The union of C_0 with $A_1 \cup A_2$ is a disk B_0 punctured at p_1, p_2 whose boundary ∂B_0 intersects γ in at most two points, one intersection point each near the endpoints of c_1, c_2, and it intersects $\partial D_0 = \alpha$ in at most $2k$ points.

Let $B \subset H$ be a properly embedded disk with boundary $\partial B = \partial B_0$. The disk B encloses the spots p_1, p_2. By Lemma 7.2, we have $d_{\mathcal{E}}(D, B) \leq k + 1$. Thus via replacing D by B we may assume that γ intersects D in at most two points.

Repeating this construction with the disk E implies that it suffices to show the following. If the simple closed curve $\gamma \subset \partial H$ intersects each of the curves $\partial D, \partial E$ in at most two points then $d_{\mathcal{E}}(D,E) \leq 10$. Note that since $\partial D, \partial E$ are separating simple closed curves, the curve γ intersects $\partial D_0 = \partial D, \partial E_0 = \partial E$ in either two or zero points.

In the case that γ is disjoint from both D_0, E_0 we are done, so assume (via possibly exchanging D_0 and E_0) that γ intersects ∂E_0 in precisely two points. If γ is not disjoint from D_0 then we may assume that $\gamma \cap D_0$ is disjoint from E_0. Moreover, in this case we may assume that each of the two curves obtained from γ by replacing $\gamma \cap D_0$ by a subarc of $\alpha = \partial D$ with the same endpoints is not peripheral. Namely, otherwise γ bounds a disk B in H and the claim follows from Lemma 7.2 applied to D, B and to B, E.

Let R be the component of $E_0 - D_0$ containing $\gamma \cap E_0$. Let E_1 be the component of $E_0 - R$ which contains the once punctured disk A_1. Note that E_1 is disjoint from γ. The intersection $E_1 \cap D_0$ is a finite union of disjoint rectangles. Let $\hat{R} \subset E_1 \cap D_0$ be the component which is closest to the once punctured disk A_2 in D_0. This means that there is an embedded arc $c_3 \subset D_0$ with one endpoint in $\hat{R}$ and the second endpoint in A_2 which intersects E_1 only at one endpoint. Let $E_2 \subset E_1$ be the union of the component of $E_1 - \hat{R}$ containing A_1 with $\hat{R}$. Note that E_2 is disjoint from γ. The union of the once punctured disk E_2, a thickening of the arc c_3 and the once punctured disk A_2 is a twice punctured disk V embedded in ∂H. Let $D_1 \subset H$ be a properly embedded disk with boundary $\partial D_1 = \partial V$. Then D_1 encloses the spots p_1, p_2. Moreover, if the rectangle component R of $E_0 - D_0$ containing the intersection of E_0 with γ is the component of $E_0 - D_0$ which contains $\partial A_2 \cap \partial D_0$ in its boundary then $\iota(\partial D_1, \partial E) = 4$.

We distinguish three cases.

Case 1: $\gamma \cap D_0 \neq \emptyset$ and $\hat{R}$ is contained in the component of $D_0 - \gamma$ containing A_1, or, equivalently, $\gamma \cap D_0$ lies between $\hat{R}$ and A_2.

Then the punctured disk V and hence D_1 is disjoint from the essential simple closed curve γ' which is obtained from γ by replacing the $\gamma \cap D_0$ by the subarc of α with the same endpoints which contains $A_2 \cap \alpha$. Since both D and D_1 are disjoint from γ', we have $d_{\mathcal{E}}(D, D_1) = 1$. Moreover, D_1 intersects γ in precisely two points. Replace D by D_1.

Case 2: $\gamma \cap D_0 = \emptyset$.

Then D_1 is disjoint from γ, and $d_{\mathcal{E}}(D, D_1) = 1$. Replace D by D_1.

Case 3: $\gamma \cap D_0 \neq \emptyset$ and $\hat{R}$ is contained in the component of $D_0 - \gamma$ containing A_2.

Then D_1 is disjoint from γ. Thus the pairs of disks D, D_1 and D_1, E satisfy the hypothesis in Case 2. Therefore this case follows from two applications of Case 2, applied to D, D_1 and D_1, E provided that we can show that under the assumption of Case 2 above, we have $d_{\mathcal{E}}(D,E) \leq 5$.

We now continue to investigate Case 1 and Case 2. Using the above notations for Case 1 and Case 2, let Q be the component of $D_0 - \hat{R}$ containing A_2. The components of $V \cap E_0$ which are different from once punctured disks are up to isotopy contained in $Q \cap E_0$. Since the subdisk $E_1 \subset E_0 - R$ is disjoint from Q this implies that the rectangle component R_1 of $E_0 - V$ which contains $\gamma \cap E_0$ has one side on the boundary of the disk component of $V \cap E_0$ which is punctured at p_1.

Reapply the above construction with the disks D_1 and E, but with the roles of the punctures p_1, p_2 exchanged. Let V' be the twice punctured disk obtained from this construction. Since the component of $E_0 - V$ containing the intersection with γ is the rectangle adjacent to the bigon component of $V \cap E_0$ punctured at p_1, the remark before Case 1 above shows that $\partial V' \cap \partial E_0$ consists of four points. In particular, if $D_2 \subset H$ is the properly embedded disk with boundary $\partial D_2 = \partial V'$ then $d_{\mathcal{E}}(D_2, E) \leq 3$ by Lemma 7.2.

An application of the above analysis to D_1 and E yields the following. If either γ is disjoint from D_1 or if γ is *not* disjoint from D_2 then $d_{\mathcal{E}}(D_1, D_2) = 1$, moreover $d_{\mathcal{E}}(D_2, E) \leq 3$ and hence $d_{\mathcal{E}}(D_1, E) \leq 4$.

To summarize, we have.

i) In Case 2 above, $d_{\mathcal{E}}(D, E) \leq 5$.
ii) In Case 1, either the pairs of disks D_1, E also satisfy Case 1 and then $d_{\mathcal{E}}(D, E) \leq 5$, or there is a disk D' which is disjoint from γ.
iii) In Case 3, there is a disk D' which is disjoint from γ.

As a consequence, either $d_{\mathcal{E}}(D, E) \leq 5$ or there is a disk D' with $d_{\mathcal{E}}(D', D) \leq 5, d_{\mathcal{E}}(D', E) \leq 5$ which is what we wanted to show. ☐

We use Lemma 7.2 and Lemma 7.3 to show the second part of Theorem 4.

Proposition 7.4. *Let H be a handlebody with $n \geq 2$ spots. Then the map $\mathcal{EDG} \to \mathcal{CG}$ which associates to a disk its boundary is a 16-quasi-isometry.*

Proof. Let $\gamma \subset \partial H$ be any simple closed curve. Let p_1, p_2 be two spots of ∂H. Then there is an embedded arc $\alpha \subset \partial H$ connecting p_1 to p_2 which intersects γ in at most one point. A thickening of α is a diskbounding simple closed curve in X which intersects γ in at most two points. Thus any simple closed curve in ∂H is at distance two in $\mathcal{CG}$ from a diskbounding simple closed curve. Moreover, by Lemma 7.2, for any disk D in H there is a disk D' which encloses the spots p_1, p_2 and such that $d_{\mathcal{E}}(D, D') \leq 2$.

As a consequence, it suffices to show the following. If the disks D, E both enclose the spots p_1, p_2 in ∂H then

$$d_{\mathcal{E}}(D, E) \leq 16 d_{\mathcal{CG}}(\partial D, \partial E)$$

where $d_{\mathcal{CG}}$ denotes the distance in the curve graph of ∂H.

Let $(\gamma_i)_{0 \leq i \leq \ell}$ be a geodesic in $\mathcal{CG}$ connecting $\partial D = \gamma_0$ to $\partial E = \gamma_\ell$. The curve γ_i is disjoint from γ_{i+1}.

Since for each $i < \ell$ the simple closed curves γ_i, γ_{i+1} are disjoint, there is a simple arc in ∂H connecting p_1 to p_2 which intersects each of the curves γ_i and γ_{i+1} in at most one point. A thickening of such an arc is a curve $\beta_{i,i+1}$ which bounds a disk $B_{i,i+1}$ enclosing p_1 and p_2. The curve $\beta_{i,i+1}$ intersects both γ_i and γ_{i+1} in at most two points.

By Lemma 7.3,

$$d_{\mathcal{E}}(B_{i-1,i}, B_{i,i+1}) \leq 16 \quad \forall i.$$

This means that D can be connected to E by a path in $\mathcal{EDG}$ whose length does not exceed $16 d_{\mathcal{CG}}(\partial D, \partial E)$. This shows the proposition. ☐

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