SIMPLICITY OF THE LYAPUNOV SPECTRUM OF FLAT COCYCLES OVER AFFINE INVARIANT MANIFOLDS

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Abstract. We show that for a non-hyperelliptic component $\mathcal{Q}$ of a stratum of quadratic differentials with at least two zeros of odd order, the Lyapunov spectrum of the Kontsevich Zorich cocycle over $\mathcal{Q}$ with respect to the invariant Lebesgue measure is simple. This is a consequence of a much more general result which applies to flat bundles over affine invariant manifolds whose monodromy is Zariski dense in $\mathrm{Sp}(2m,\mathbb{R})$ or $\mathrm{SL}(n,\mathbb{R})$.

Contents

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1. Introduction

The *mapping class group* $\mathrm{Mod}(S)$ of a closed surface S of genus $g \geq 2$ acts by precomposition of marking on the *Teichmüller space* $\mathcal{T}(S)$ of marked complex structures on S. The action is properly discontinuous, with quotient the *moduli space* $\mathcal{M}_g$ of complex structures on S.

The fiber over a Riemann surface $x \in \mathcal{M}_g$ of the *Hodge bundle*

$$\Pi : \mathcal{H} \to \mathcal{M}_g$$

equals the vector space of holomorphic one-forms (or abelian differentials) on x. The Hodge bundle is a holomorphic vector bundle (in the orbifold sense) over the complex orbifold $\mathcal{M}_g$ of complex dimension g. The complement $\mathcal{H}_+$ of its zero section decomposes into *strata* of differentials with zeros of a fixed number and fixed multiplicities. There is a natural $\mathrm{GL}^+(2,\mathbb{R})$-action on $\mathcal{H}_+$ preserving any connected component of a stratum. The subgroup $\mathrm{SL}(2,\mathbb{R})$ also preserves the sphere subbundle of *area one* abelian differentials. The action of the diagonal subgroup is called the *Teichmüller flow* Φ^t.

Period coordinates on a component $\mathcal{Q}$ of a stratum of abelian differentials with singular set $\Sigma \subset S$ are obtained by integration of a holomorphic one-form $q \in \mathcal{Q}$ over a basis of the relative homology group $H_1(S,\Sigma,\mathbb{Z})$. Thus a tangent vector of $\mathcal{Q}$ defines a point in $H_1(S,\Sigma,\mathbb{C})^*$.

By the groundbreaking work of Eskin, Mirzakhani and Mohammadi [EMM15], the orbit closures of the $\mathrm{GL}_+(2,\mathbb{R})$-action on the moduli space of abelian differentials are precisely the so-called *affine invariant manifolds*. Such manifolds $\mathcal{C}_+$ are cut out by linear equations in period coordinates. The real hypersurface $\mathcal{C} \subset \mathcal{C}_+$ of differentials of area one admits a distinguished Φ^t-invariant ergodic probability measure μ in the Lebesgue measure class.

The preimage of an affine invariant manifold $\mathcal{C}$ in the Teichmüller space of abelian differentials decomposes into connected components which are permuted by the action of the mapping class group. Choose such a component $\tilde{\mathcal{C}}$ and let $\Gamma \subset \mathrm{Mod}(S)$ be the stabilizer of $\tilde{\mathcal{C}}$; we then have $\mathcal{C} = \Gamma \backslash \tilde{\mathcal{C}}$. If $\mathcal{C}$ is a non-hyperelliptic component of a stratum of abelian differentials, then Γ is a subgroup of $\mathrm{Mod}(S)$ of finite index [CS21, H21]. In contrast, if $\mathcal{C}$ is a *Teichmüller curve* then Γ is the *Veech group* of $\mathcal{C}$ and hence it is virtually free.

Let now G be either the symplectic group $\mathrm{Sp}(2m,\mathbb{R})$ or the linear group $\mathrm{SL}(n,\mathbb{R})$ and let $\rho : \Gamma \to G$ be a homomorphism. Then ρ defines a flat bundle $\mathcal{G} \to \mathcal{C}$ with fiber G. There also is an associated flat vector bundle $\mathcal{V} \to \mathcal{C}$, defined by the standard linear action of G. Parallel transport with respect to the flat connection in $\mathcal{V}$ then defines the *monodromy* group of the flat bundle $\mathcal{V} \to \mathcal{C}$, which coincides with the group $\rho(\Gamma)$. The flat connection also determines an extension Ψ^t of the Teichmüller flow Φ^t on $\mathcal{C}$ to $\mathcal{V}$.

Let us assume that this extension fulfills the requirements for an application of the Oseledec multiplicative ergodic theorem with respect to the invariant probability measure μ on $\mathcal{C}$ in the Lebesgue measure class. Then the *Lyapunov spectrum*

$\lambda_1 < \cdots < \lambda_k$ for the extension Ψ^t of Φ^t is defined. The multiplicity of a Lyapunov exponent λ_i equals the maximal dimension of a linear subspace $V \subset \mathcal{V}_q$ for a μ-generic point $q \in \mathcal{C}$ so that for any $0 \neq X \in V$, the asymptotic growth rate $\limsup_{t\to\infty} \frac{1}{t} \log |\Psi^t X|$ for some suitable norm $| \, |$ on $\mathcal{V}$ is precisely λ_i. The Lyapunov spectrum is called *simple* if this multiplicity is one for all i.

The following is our main result.

Theorem 1. *If the image $\rho(\Gamma)$ of ρ is Zariski dense in G, then the Lyapunov spectrum of the flow Ψ^t is simple.*

There are several interesting cases to which Theorem 1 can be applied. Namely, the most natural homomorphism $\rho : \mathrm{Mod}(S) \to \mathrm{Sp}(2g, \mathbb{Z})$ is defined by the action of $\mathrm{Mod}(S)$ on the first cohomology group $H^1(S, \mathbb{Z})$ of S, equipped with the cup product. The flat vector bundle over $\mathcal{M}_g$ defined by this representation is just the Hodge bundle. The pull-back $\Pi^*\mathcal{H}$ of the Hodge bundle to the hypersurface in $\mathcal{H}$ of area one differentials is a flat vector bundle defining an extension of Φ^t which is commonly called the *Kontsevich Zorich cocycle*. The flat connection is called the *Gauss Manin* connection. The following is due to Avila and Viana [AV07b].

Corollary 1. *Let $\mathcal{Q}$ be a component of a stratum of abelian differentials. Then the Lyapunov spectrum of the Kontsevich Zorich cocycle over $\mathcal{Q}$ with respect to the invariant Lebesgue measure is simple.*

Interestingly, the corollary was established before a thorough investigation of the Zariski closure of the Kontsevich Zorich cocycle over a component of a stratum of abelian differentials was carried out (see [GR17]). Note that for general representations ρ with the property that the Zariski closure of the monodromy group of the corresponding flat bundle is a proper algebraic subgroup H of G, a Cartan subalgebra of H may not contain a regular element of the Cartan subalgebra of G and simplicity of the Lyapunov spectrum is obstructed.

The *rank* of an affine invariant manifold $\mathcal{C}$ is defined by

$$\mathrm{rk}(\mathcal{C}) = \frac{1}{2}\dim_{\mathbb{C}}(pT\mathcal{C})$$

where p is the projection of $H_1(S, \Sigma, \mathbb{C})^*$ into $H_1(S, \mathbb{C})^* = H^1(S, \mathbb{C})$ [W14]. The rank of a component of a stratum equals g.

If we denote by $\mathcal{Z}_{\mathbb{R}}$ the projection of the tangent bundle of $\mathcal{C}$ to $H^1(S, \mathbb{R}) \subset H^1(S, \mathbb{C})$ then $\mathcal{Z}_{\mathbb{R}}$ is a flat symplectic [AEM17, F16] subbundle of the restriction of $\Pi^*\mathcal{H}$ to $\mathcal{C}$. Thus $\mathcal{Z}_{\mathbb{R}}$ is invariant under the Gauss Manin connection. In Section 4 we verify that the monodromy group of $\mathcal{Z}_{\mathbb{R}}$ is Zariski dense in the symplectic group $\mathrm{Sp}(\mathcal{Z}_{\mathbb{R}}, \mathbb{R})$ of $\mathcal{Z}_{\mathbb{R}}$ and hence we obtain.

Corollary 2. *Let $\mathcal{C}$ be an affine invariant manifold; then the Lyapunov spectrum of the flat bundle $\mathcal{Z}_{\mathbb{R}} \to \mathcal{C}$ with respect to the invariant Lebesgue measure is simple.*

We can also consider the pull-back of the Hodge bundle to the moduli space $P : \mathcal{S} \to \mathcal{M}_g$ of *area one quadratic differentials* on S. As before, it decomposes into strata of differentials with the same number of zeros of the same multiplicities.

Each quadratic differential determines uniquely an orientation cover, which is a two-sheeted branched cover of S, equipped with an abelian differential. Thus components of strata of quadratic differentials define affine invariant manifolds in the moduli space of abelian differentials on the covering surface. The natural invariant measure in the Lebesgue measure class on the branched covering affine invariant manifold is the lift of the Masur Veech measure on the component.

Components of strata of quadratic differentials were classified by Lanneau [L08]. As we only consider strata of differentials without simple poles and with at least three zeros, non-hyperelliptic components of any stratum are unique [L08].

Corollary 3. *Let $\mathcal{Q}$ be a non-hyperelliptic component of a stratum of quadratic differentials on a surface of genus $g \geq 3$ with at least two zeros of odd order. Then the restriction of the Kontsevich Zorich cocycle to $\mathcal{Q}$ is Zariski dense. Thus the Lyapunov spectrum of the Kontsevich Zorich cocycle over $\mathcal{Q}$ is simple with respect to the Masur Veech measure.*

The statement of the corollary does not seem to be true for non-excecptional components of strata of quadratic differentials with a single zero. We do not know what happens for components of strata of quadratic differentials with more than one zero but all zeros of even order. Simplicity of the Lyapunov spectrum for the Kontsevich Zorich cocycle over the principal stratum of quadratic differentials was announced by Eskin and Rafi, using different methods.

Let now $h = 6g - 6$ and consider again the sphere bundle $\mathcal{S} \to \mathcal{M}_g$ of area one quadratic differentials. Denote by Λ the set of all periodic orbits for the Teichmüller flow on $\mathcal{S}$ (here we represent orbits of abelian differentials by orbits of their squares). We know that [H13]

$$\sharp\{\gamma \in \Lambda \mid \ell(\gamma) \leq R\}\frac{hR}{e^{hR}} \to 1 \quad (R \to \infty).$$

Call a subset $\mathcal{A}$ of Λ *typical* if

$$\sharp\{\gamma \in \mathcal{A} \mid \ell(\gamma) \leq R\}\frac{hR}{e^{hR}} \to 1 \quad (R \to \infty).$$

To each periodic orbit $\gamma \in \Lambda$ is associate a conjugacy class of a pseudo-Anosov mapping class which maps to a conjugacy class of an element $A(\gamma) \in \mathrm{Sp}(2g, \mathbb{R})$. As the characteristic polynomial of a symplectic matrix is invariant under conjugation, we can ask for the eigenvalues of $A(\gamma)$ without ambiguity. By abuse of notation we call the collection of these eigenvalues the spectrum of γ. Using [H23] we obtain.

Corollary 4. *The spectrum of a typical periodic orbit $\gamma \in \Lambda$ consists of $2g$ pairwise distinct real eigenvalues.*

The mapping class group $\mathrm{Mod}(S)$ also acts on the space $\mathcal{ML}$ of *measured geodesic laminations* as a group of transformations, preserving the *Thurston symplectic form.* This space is not a vector space, however via pull-back it defines a flat bundle over the bundle $\mathcal{S} \to \mathcal{M}_g$. If $\mathcal{Q} \subset \mathcal{S}$ denotes the principal stratum of area one quadratic differentials, then a measured geodesic lamnation pulls back to an absolute real cohomology class on the orientation cover of $\mathcal{Q}$. Thus with a small abuse of notation, we can ask for the Lyapunov spectrum of the Teichmüller flow on $\mathcal{Q}$ for the flat bundle with fiber $\mathcal{ML}$.

Corollary 5. *Let $\mathcal{Q} \subset \mathcal{S}$ be the principal stratum of area one quadratic differentials. Then the Lyapunov spectrum of the bundle over $\mathcal{Q}$ with fiber $\mathcal{ML}$ is simple with respect to the invariant Lebesgue measure on $\mathcal{Q}$.*

It would be interesting to relate the statement of the corollary to the embedding of Teichmüller space into the moduli space of principally polarized abelian varieties. As it stands, it seems rather curious.

Strategy of the proofs and organization of the article: Our argument has three partially independent parts.

In the first part, which is carried out in Section 3 and is based on earlier results in [H13, H23], we establish a non-uniform shadowing property for the Teichmüller flow on an affine invariant manifold $\mathcal{C}$ which is reminiscent of familiar results in hyperbolic dynamics. This is used to associate to an orbit segment beginning and ending in a suitably chosen open subset Y of $\mathcal{C}$ a pseudo-Anosov element in $\mathrm{Mod}(S)$ in such a way that concatenation of orbit segments (which is not required to be continuous) translates into multiplication of group elements. The resulting subsemigroup $\Omega(\Gamma_0)$ of $\mathrm{Mod}(S)$ consists entirely of pseudo-Anosov mapping classes.

The image of the subsemigroup $\Omega(\Gamma_0)$ of $\mathrm{Mod}(S)$ under the homomorphism ρ defines a subsemigroup of a symplectic group or the special linear group. For the standard representation $\rho : \mathrm{Mod}(S) \to \mathrm{Sp}(2g, \mathbb{Z})$ and an affine invariant manifold $\mathcal{C}$, the target group is the group $\mathrm{Sp}(\mathcal{Z}_{\mathbb{R}}, \mathbb{R})$ introduced above. The second part of our approach consists in establishing Zariski density of the monodromy of $\mathcal{Z}_{\mathbb{R}}$ over any affine invariant manifold. Its proof is contained in Section 4 and builds on results of Wright [W15] on horizontally periodic translation surfaces in affine invariant manifolds. We also use the results from Section 3 and tools from the theory of algebraic groups developed in the context of strong approximation.

Having established Zariski density, we proceed by proving an explicit local version, formulated in terms of semigroups generated by periodic orbits which start in a fixed open contractible subset of $\mathcal{C}$. This local version is then used to apply a result of Avila and Viana [AV07a] to a symbolic system encoding the Teichmüller flow on strata of abelian or quadratic differentials which was constructed in [H11]. This leads to the proof of Theorem 1 and the corollaries.

For the proof of Corollary 3 we have to verify Zariski density of the restriction of the Kontsevich Zorich cocycle to a non-hyperelliptic component of a stratum with at least two zeros of odd order. This is done by reducing the statement to Zariski density of the cocycle over a non-hyperellipitic component of a stratum of abelian differentials with a single zero, that is, to the results in Section 4.

In the introductory Section 2 we introduce the Hodge bundle and the Gauss Manin connection, and establish some basic properties of affine invariant manifolds.

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2. The geometry of affine invariant manifolds

The goal of this section is to collect some geometric and dynamical properties of affine invariant manifolds which are used throughout this article.

2.1. **The Hodge bundle.**

Let $\mathcal{M}_g$ be the moduli space of closed Riemann surfaces of genus g. This is the quotient of *Teichmüller space* $\mathcal{T}(S)$ under the action of the mapping class group $\mathrm{Mod}(S)$ and is naturally endowed with the structure of a complex orbifold.

The *Hodge bundle* $\mathcal{H} \to \mathcal{M}_g$ is a holomorphic vector bundle over $\mathcal{M}_g$ (in the orbifold sense). Its fiber over a manifold point $X \in \mathcal{M}_g$ equals the vector space of holomorphic one-forms (abelian differentials) on X. As the map which associates to a holomorphic one-form on X its real part is an isomorphism of real vector spaces, as a real vector bundle, the Hodge bundle has the following description.

The action of the mapping class group $\mathrm{Mod}(S)$ on the first real cohomology group $H^1(S,\mathbb{R})$, equipped with the symplectic structure given by the cup product, defines a homomorphism

$$\Psi : \mathrm{Mod}(S) \to \mathrm{Sp}(2g,\mathbb{Z}).$$

The Hodge bundle is then the flat orbifold vector bundle

$$\Pi : \mathcal{H} = \mathcal{T}(S) \times_{\mathrm{Mod}(S)} H^1(S,\mathbb{R}) \to \mathcal{M}_g \tag{1}$$

for the standard right action of $\mathrm{Mod}(S)$ on Teichmüller space $\mathcal{T}(S)$ by precomposition of marking, and the left action of $\mathrm{Mod}(S)$ on $H^1(S,\mathbb{R})$ via Ψ. This description determines a flat connection on $\mathcal{H}$ which is called the *Gauss Manin* connection. This connection preserves the symplectic structure on the fibers.

As the Hodge bundle $\mathcal{H}$ is a holomorphic vector bundle over the complex orbifold $\mathcal{M}_g$, it is a complex orbifold in its own right, and the same holds true for the complement $\mathcal{H}_+ \subset \mathcal{H}$ of the zero section in $\mathcal{H}$. The pull-back

$$\Pi^*\mathcal{H} \to \mathcal{H}_+$$

of $\mathcal{H}$ to $\mathcal{H}_+$ is a holomorphic vector bundle on $\mathcal{H}_+$ (in the orbifold sense). The pull-back of the Gauss-Manin connection is a flat connection on $\Pi^*\mathcal{H}$ which we call again the Gauss Manin connection.

2.2. **Strata and affine invariant manifolds.** The Hodge bundle $\mathcal{H}_+$ is naturally decomposed into *strata*, determined by the number and order of the zeros of the abelian differential. Strata need not be connected, but they have at most three connected components [KZ03]. A stratum is a complex orbifold in its own right. The closure in $\mathcal{H}_+$ of a component of a stratum equals a union of strata. The area of an abelian differential is well defined, and the locus of area one abelian differentials is a cross section for the action of the multiplicative group $(0,\infty)$ by scaling. The *Teichmüller flow* Φ^t acts on $\mathcal{H}_+$ preserving the area as well as the strata.

The fact that strata are orbifolds rather than manifolds gives rise to significant technical difficulties. As in [H13], we circumvent this difficulty by restricting all constructions to the manifold points. Concretely, let $\mathcal{Q} \subset \mathcal{H}_+$ be a component of a stratum of area one abelian differentials. Define the *good subset* $\mathcal{Q}_{\rm good}$ of $\mathcal{Q}$ to be the set of all points $q \in \mathcal{Q}$ with the following property. Let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of *marked* abelian differentials and let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q; then an element of $\mathrm{Mod}(S)$ which fixes $\tilde{q}$ acts as the identity on $\tilde{\mathcal{Q}}$ (compare [H13] for more information on this technical condition). Then $\mathcal{Q}_{\rm good}$ is precisely the subset of $\mathcal{Q}$ of manifold points. Lemma 4.5 of [H13] shows that the good subset $\mathcal{Q}_{\rm good}$ of $\mathcal{Q}$ is open, dense and Φ^t-invariant, furthermore it is invariant under scaling.

By the construction of $\mathcal{Q}_{\rm good}$, for any smooth arc $\eta : [0,a] \to \mathcal{Q}_{\rm good}$ and any choice $\tilde{q}$ of a preimage of $\eta(0)$ in the Teichmüller space $\tilde{\mathcal{H}}_+$ of marked abelian differentials, there exists a unique lift $\tilde{\eta}$ of η through $\tilde{\eta}(0) = \tilde{q}$, and this lift depends smoothly on $\tilde{\eta}$ (and $\tilde{q}$).

Definition 2.1. A closed curve $\eta : [0,a] \to \mathcal{Q}_{\rm good}$ *defines the conjugacy class of a pseudo-Anosov mapping class* $\varphi \in \mathrm{Mod}(S)$ if the following holds true. Let $\tilde{\eta} : [0,a] \to \tilde{\mathcal{Q}}$ be a lift of η to an arc in the Teichmüller space of abelian differentials. Then $\psi\tilde{\eta}(a) = \tilde{\eta}(0)$ for a unique $\psi \in \mathrm{Mod}(S)$, and we require that ψ is conjugate to φ.

As any two lifts of an arc in $\mathcal{Q}_{\rm good}$ to the Teichmüller space of marked abelian differentials are translates of each other by some element in the mapping class group, the property captured in Definition 2.1 does not depend on any choices made.

The Hodge bundle $\mathcal{H} \to \mathcal{M}_g$ is the quotient under the action of the mapping class group of the trivial bundle $\tilde{\mathcal{H}} \to \mathcal{T}(S)$ whose fiber equals the first real cohomology $H^1(S,\mathbb{R})$, equipped with the symplectic structure defined by the cup product. The mapping class group acts on the fibers of the vector bundle $\tilde{\mathcal{H}} \to \mathcal{T}(S)$ through the representation Ψ preserving the symplectic structure. The characteristic polynomial of a symplectic matrix is invariant under conjugation. Using Definition 2.1, the above discussion easily leads to the following statement (here parallel transport means parallel transport with respect to the Gauss Manin connection).

Lemma 2.2. *Let $\eta \subset \mathcal{Q}_{\rm good}$ be a closed curve which defines the conjugacy class of a pseudo-Anosov mapping class $\varphi \in \mathrm{Mod}(S)$. Then the characteristic polynomial of the holonomy map obtained by parallel transport of the bundle $\Pi^*\mathcal{H}$ along η coincides with the characteristic polynomial of the map $\Psi \circ \varphi \in \mathrm{Sp}(2g,\mathbb{Z})$.*

Proof. Since the Gauss Manin connection is flat, parallel transport along a closed based loop in $\mathcal{Q}_{\rm good}$ is invariant under homotopy with fixed basepoint in $\mathcal{Q}_{\rm good}$ and hence the holonomy along such a based loop is an invariant of its class in $\pi_1(\mathcal{Q}_{\rm good})$. Furthermore, moving the basepoint, i.e. changing the loop with a free homotopy, results in conjugation of the holonomy map.

Now the characteristic polynomial of an element $A \in {\rm Sp}(2g,\mathbb{Z})$ is invariant under conjugation and hence the characteristic polynomial of the holonomy of a loop in $\mathcal{Q}_{\rm good}$ only depends on the free homotopy class of the loop. For a loop $\eta : [0,a] \to \mathcal{Q}_{\rm good}$ which defines the conjugacy class of a pseudo-Anosov element φ, this polynomial can be computed as follows.

Choose any lift $\tilde\eta$ of η to the Teichmüller space $\tilde{\mathcal{H}}$ of area one abelian differentials. Since $\eta \subset \mathcal{Q}_{\rm good}$, such a lift only depends on η and the choice of a preimage of $\eta(0)$ in $\tilde{\mathcal{H}}$. By the definition of the Gauss Manin connection, the characteristic polynomial of the holonomy map along η is the characteristic polynomial of $\Psi \circ \zeta$ where $\zeta \in {\rm Mod}(S)$ is the unique element which maps the endpoint $\tilde\eta(a)$ of $\tilde\eta$ back to $\tilde\eta(0)$. As ζ is conjugate to φ and hence $\Psi \circ \zeta$ is conjugate to $\Psi \circ \varphi$, the lemma follows. □

Let $\mathcal{Q}_+$ be a component of a stratum of (not area normalized) abelian differentials on the surface S with fixed number and multiplicities of zeros. Throughout this article we use the notation $\mathcal{Q}_+$ if we are looking at differentials whose area may be different from one, but most of the time we consider components of strata of differentials (abelian or quadratic) of area one. Denote by $\Sigma \subset S$ the set of zeros of a differential in $\mathcal{Q}_+$.

Period coordinates for $\mathcal{Q}_+$ are defined by integration of a differential $q \in \mathcal{Q}_+$ over a basis of $H_1(S,\Sigma;\mathbb{Z})$. These coordinates take values in $H_1(S,\Sigma;\mathbb{R})^* \otimes_{\mathbb{R}} \mathbb{C}$ and induce an affine structure on $\mathcal{Q}_+$.

An abelian differential $q \in \mathcal{Q}_+$ defines an atlas on $S - \Sigma$ whose chart transitions are translations. Postcomposition of these charts with a fixed element of the group ${\rm GL}^+(2,\mathbb{R})$ defines a new such atlas and hence a new element in $\mathcal{Q}_+$. This construction defines an affine action of ${\rm GL}^+(2,\mathbb{R})$ on $\mathcal{Q}_+$. The induced action of the diagonal subgroup is just the Teichmüller flow.

An *affine invariant manifold* $\mathcal{C}_+$ in $\mathcal{Q}_+$ is the closure in $\mathcal{Q}_+$ of an orbit of the ${\rm GL}^+(2,\mathbb{R})$-action. Such an affine invariant manifold is complex affine in period coordinates [EMM15]. In particular, $\mathcal{C}_+ \subset \mathcal{Q}_+$ is a complex suborbifold. Period coordinates determine a projection

$$p : T\mathcal{C}_+ \to \Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$$

to absolute periods (see [W14] for a clear exposition). The image $p(T\mathcal{C}_+)$ is flat, i.e. it is invariant under the restriction of the Gauss Manin connection to a connection on $\Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}|_{\mathcal{C}_+}$.

By the main result of [F16], there is a *holomorphic* subbundle $\mathcal{Z}$ of $\Pi^*\mathcal{H}|_{\mathcal{C}_+}$ such that

$$p(T\mathcal{C}_+) = \mathcal{Z} \oplus \overline{\mathcal{Z}}.$$

We call $\mathcal{Z}$ the *absolute holomorphic tangent bundle* of $\mathcal{C}_+$. As a consequence, the bundle $p(T\mathcal{C}_+)$ is invariant under the complex structure on $\Pi^*\mathcal{H}\otimes_{\mathbb{R}}\mathbb{C}$ obtained by extension of scalars.

As a real vector bundle, $\mathcal{Z}$ is isomorphic to $p(T\mathcal{C}_+)\cap\Pi^*\mathcal{H}|\mathcal{C}_+$. Since $\mathcal{Z}$ is complex, the bundle $p(T\mathcal{C}_+)\cap\Pi^*\mathcal{H}\to\mathcal{C}_+$ is symplectic [AEM17].

Define the *rank* of the affine invariant manifold $\mathcal{C}_+$ as [W14]

$$\mathrm{rk}(\mathcal{C}_+)=\frac{1}{2}\dim_{\mathbb{C}} p(T\mathcal{C}_+)=\dim_{\mathbb{C}}\mathcal{Z}.$$

With this definition, components of strata are affine invariant manifolds of rank g.

3. Non-uniform hyperbolic dynamics of the Teichmüller flow

The geodesic flow Ψ^t on the unit tangent bundle T^1M of a closed negatively curved manifold M is an Anosov flow and hence has the following *strong shadowing property* [Bw73].

Fix a Riemannian metric on T^1M which induces a distance function d. There exist numbers $\epsilon>0$, $R>0$ with the following properties. Let $x_1,\dots,x_m\subset T^1M$ be an arbitrary chain of points and let $R_i>R$ $(1\leq i\leq m)$ be a sequence of sufficiently large numbers. Assume that we have $d(\Psi^{R_i}(x_i),x_{i+1})<\epsilon$ for all i, and where $x_{m+1}=x_0$. Then there exists a periodic orbit γ for Ψ^t which uniformly fellow travels the (discontinuous) concatentation of the orbit segments $\beta_i:t\to\Psi^t(x_i)$ $(0\leq t\leq R_i)$. Furthermore, the perodic orbit represents a conjugacy class in the fundamental group of M which can be reconstructed from the chain of orbit segments β_i.

A component $\mathcal{Q}$ of a stratum of area one abelian differentials is not compact, and the Teichmüller flow Φ^t acting on $\mathcal{Q}$ is not hyperbolic. However, it is non-uniformly hyperbolic in a precise quantitative sense, see [H13, H23] for more information. We shall use this non-uniform hyperbolicity to establish a non-uniform version of the shadowing property of hyperbolic geodesic flows for the restriction of the Teichmüller flow to affine invariant manifolds..

For the formulation of our main result, for an affine invariant manifold $\mathcal{C}$ of area one abelian differentials denote by $\mathcal{C}_{\mathrm{good}}\subset\mathcal{C}$ the Φ^t-invariant open dense set of good points. Call a point $q\in\mathcal{C}$ *birecurrent* if q is contained in both the α- and the ω-limit set of its orbit under Φ^t. By the Poincaré recurrence theorem, almost every point with respect to any invariant probability measure has this property.

The idea is to use non-uniform hyperbolicity of the Teichmüller flow on $\mathcal{C}$ to establish the shadowing property for orbit segments whose endpoints are contained in small contractible neighborhoods of an arbitrarily fixed finite collection $\{q_1,\dots,q_k\}\subset\mathcal{C}_{\mathrm{good}}$ of birecurrent points. The size of the neighborhoods, for example measured with respect to some choice of a Riemannian metric, depends on the points, and the minimal length of the connecting orbit segments will depend on the points as well. The following definition formalizes this concept of shadowing.

Definition 3.1. Let $\mathcal{Y} = \{Y_i \mid i \in \mathcal{I}\}$ be a non-empty finite collection of open relatively compact subsets of an affine invariant manifold $\mathcal{C}$. For some $n > 0$, an $(n, \mathcal{Y})$*-pseudo-orbit* for the Teichmüller flow Φ^t on $\mathcal{C}$ consists of a sequence of points $q_0, q_1, \dots, q_m \in \mathcal{C}$ and a sequence of numbers $t_0, \dots, t_{m-1} \in [n, \infty)$ with the following property. For every $1 \leq j \leq m$, there exists some $\kappa(j) \in \mathcal{I}$ such that $\Phi^{t_{j-1}} q_{j-1}, q_j \in Y_{\kappa(j)}$. The pseudo-orbit is called *periodic* if $q_m = q_0$.

Although we describe a pseudo-orbit by a sequence of pairs $(q_i, t_i) \in \mathcal{C} \times (0, \infty)$, we view a pseudo-orbit as a finite ordered collection of compact orbit segments such that the endpoint of the $i-1$-th segment is close to the starting point of the i-th segment.

Any periodic orbit of Φ^t in a component of a stratum is determined by the conjugacy class of a pseudo-Anosov mapping class, so that the periodic orbit is the projection of the unit tangent line of an axis of an element in this conjugacy class. We encode this information in the following notion of a characteristic curve.

Definition 3.2. Let $\mathcal{Y} = \cup_{i \in \mathcal{I}} Y_i$ be a collection of open relative compact subsets of the affine invariant submanifold $\mathcal{C}$. Assume that the closure of each Y_i is contained in an open relatively compact contractible subset V_i of $\mathcal{C}_{\rm good}$ and that the sets V_i are pairwise disjoint. Consider a periodic $(n, \mathcal{Y})$-pseudo-orbit, specified by points $q_0, q_1, \dots, q_m = q_0 \in \mathcal{C}$, numbers $t_0, \dots, t_{m-1} \in [n, \infty)$ and indices $\kappa(j) \in \mathcal{I}$. Connect $\Phi^{t_{j-1}} q_{j-1}$ to q_j by an arc α_j in $V_{\kappa(j)}$. The concatenation of the orbit segments connecting q_{j-1} to $\Phi^{t_{j-1}} q_{j-1}$ with the arcs α_j defines a closed curve η in $\mathcal{C}$ which we call a $\mathcal{V}$*-characteristic curve* of the pseudo-orbit, where $\mathcal{V} = \{V_i \mid i \in \mathcal{I}\}$.

It is immediate from this definition that a $\mathcal{V}$-characteristic curve of an $(n, \mathcal{Y})$-pseudo-orbit depends on choices, but its free homotopy class does not depend on any choices made.

The following is the main result of this section. Note that the sets Y_j are not required to satisfy any additional topological properties beyond being open and relatively compact.

Theorem 3.3. *Let $\mathcal{C}$ be an affine invariant manifold, let $q_1, \dots, q_k \in \mathcal{C}_{\rm good}$ be birecurrent points, and for each j let U_j be a neighborhood of q_j in $\mathcal{C}_{\rm good}$. Then there are open relative compact neighborhoods*

$$Y_j \subset V_j \subset U_j$$

of q_j, where V_j is contractible, and there is a number $R_0 > 0$ with the following property.

Let $\mathcal{Y} = \{Y_j \mid j\}$, let $\mathcal{V} = \{V_j \mid j\}$ and let η be a $\mathcal{V}$-characteristic curve of a periodic $(R_0, \mathcal{Y})$-pseudo-orbit, given by points $y_0, \dots, y_{m-1}, y_m = y_0$ and numbers $t_i > R_0$ such that $\Phi^{t_{i-1}} y_{i-1}, y_i \in Y_{\kappa(i)}$ for some $\kappa(i) \in \{1, \dots, m\}$. Then there is a periodic orbit $\gamma \subset \mathcal{C}_{\rm good}$ for Φ^t which passes through each of the sets $V_{\kappa(i)}$ at times close to $\sum_{s \leq i-1} t_s$ and which defines the same conjugacy class in $\mathrm{Mod}(S)$ as η.

The following is a consequence of Theorem 3.3 and Theorem C of [Ra14].

Corollary 3.4. *Under the assumption of Theorem 3.3, the periodic orbit uniformly fellow-travels its defining pseudo-orbit.*

Proof. Since the set $\{q_1,\dots,q_k\}$ is finite, its projection to the moduli space $\mathcal{M}_g$ is contained in the ϵ-thick part of moduli space for some $\epsilon>0$ depending on the set, and the same holds true for the sets $V_1,\dots,V_k$.

Now the periodic orbit γ can be decomposed into segments whose endpoints are contained in the sets V_j and hence which are close to the endpoints of the orbit segments defining the pseudo-orbit η. Furthermore, if the starting points of two such corresponding segments $\alpha\subset\gamma$ and $\beta\subset\eta$ are contained in the sets V_i and the endpoints are contained in V_j, then lifts of α,β to arcs in the Teichmüller space of abelian differentials which begin in the same lift of the set V_i have endpoints in the same lift of V_j. Thus such lifts define Teichmüller geodesic arcs with endpoints in the ϵ-thick part of Teichmüller space which are uniformly close. Theorem C of [Ra14] now states that the corresponding Teichmüller geodesics uniformly fellow travel and hence the same holds true for the characteristic curve of the pseudo-orbit and the corresponding periodic orbit. □

3.1. **Product structures and the Hodge distance.** In this subsection we introduce local product structures for affine invariant manifolds $\mathcal{C}$ consisting of area one abelian differentials and the Hodge distance on strong stable and strong unstable manifolds. We then formulate some quantitative version of non-uniform hyperbolicity of the Teichmüller flow which was established in [H13, H23].

An affine invariant manifold $\mathcal{C}_+\subset\mathcal{H}_+$ is described in period coordinates as the set of solutions of a system of linear equations [EMM15]. Here as before, we write $\mathcal{C}_+$ if we consider differentials whose area is not necessarily one. In particular, each manifold point of $\mathcal{C}_+$ has a neighborhood U which is mapped by period coordinates homeomorphically onto an open subset V of an affine subspace of $H_1(S,\Sigma;\mathbb{R})^*\otimes_{\mathbb{R}}\mathbb{C}$ where Σ is the set of zeros of the differentials in the stratum containing $\mathcal{C}_+$. This affine subspace is invariant under the complex structure induced from the complex structure on $H_1(S,\Sigma;\mathbb{R})^*\otimes_{\mathbb{R}}\mathbb{C}$ [F16].

In period coordinates, a local leaf of the *strong unstable foliation* W^{su} through a point $w\in H_1(S,\Sigma;\mathbb{R})^*\otimes_{\mathbb{R}}\mathbb{C}$ consists of all differentials whose real parts coincide with the real part of w, and the local leaf of the *strong stable foliation* W^{ss} consists of all differentials whose imaginary parts coincide with the imaginary part of w. As $\mathcal{C}_+$ is complex affine in period coordinates, we obtain

Lemma 3.5. *Let $\mathcal{C}$ be an affine invariant manifold of area one abelian differentials. Then $\mathcal{C}_{\rm good}\cap W^i$ is a smooth foliation of $\mathcal{C}_{\rm good}$ into leaves of real dimension $\dim_{\mathbb{C}}(\mathcal{C}_+)-1$ ($i=ss,su$).*

Lemma 3.5 implies that for every affine invariant manifold $\mathcal{C}$, every point $q\in\mathcal{C}_{\rm good}$ has a neighborhood *with a product structure*. We next define a set with a product structure formally. The definition we give is a bit less restrictive than other of its versions, but it is convenient for the purpose of this section.

The real and imaginary part, respectively, of a marked abelian differential ω are smooth closed one-forms on S which vanish precisely at the points in Σ. Thus their kernels define smooth one-dimensional subbundles of the tangent bundle of $S - \Sigma$ which integrate to one-dimensional oriented foliations on $S - \Sigma$. These foliations are *measured foliations* on S, that is, they are equipped with a transverse invariant measure. The transverse measure is obtained by integration of the real and imaginary part of ω, respectively, over arcs in $S - \Sigma$ which are transverse to the foliation. The foliation defined by the real part of the differential is called the *vertical foliation*, and the foliation defined by the imaginary part is called the *horizontal foliation*.

The space of *marked* equivalence classes of *projective measured foliations* $\mathcal{PMF}$ on S is equipped with a natural topology so that it is homeomorphic to a sphere of dimension $6g - 7$. Here two measured foliations are equivalent if they coincide up to Whitehead moves. As this will not be important for us, we omit a more detailed discussion. Period coordinates for the component $\mathcal{Q}$ of a stratum containing $\mathcal{C}$ show that nearby differentials in $\mathcal{Q}$ whose real parts define the same class in $H_1(S, \Sigma; \mathbb{R})^*$ determine equivalent vertical marked measured foliations on S.

Definition 3.6. Let $\mathcal{C}$ be an affine invariant manifold and let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of marked abelian differentials. A subset $\tilde{V}$ of $\tilde{\mathcal{C}}$ *admits a product structure* if there are two disjoint compact subsets D, K of the set of (marked) projective measured foliations on S, viewed as projective classes of points in $H_1(S, \Sigma; \mathbb{R})^*$ via integration of the transverse measure along arcs with endpoints in Σ, with the following properties.

(1) The sets D, K are homeomorphic to closed balls of dimension
$$m = \dim_{\mathbb{C}}(\mathcal{C}_+) - 1.$$
(2) There is a continuous map
$$\Lambda : D \times K \to \tilde{V}$$
such that for any pair $(\xi, \nu) \in D \times K$, the horizontal projective measured foliation of $\Lambda(\xi, \nu)$ equals ξ, and its vertical projective measured foliation equals ν.
(3) There is some $\epsilon > 0$ such that
$$\tilde{V} = \cup_{-\epsilon \leq t \leq \epsilon} \cup_{(\xi,\nu) \in D \times K} \Phi^t \Lambda(\xi, \nu).$$

A closed contractible set $V \subset \mathcal{C}_{\rm good}$ with dense interior *admits a product structure* if some (and hence any) component $\tilde{V}$ of V of the preimage of V in the Teichmüller space of marked abelian differentials has a product structure.

We say that an open subset U of $\mathcal{C}_{\rm good}$ *has a product structure* if its closure has a product structure in the sense of Definition 3.6. We refer to Section 3.1 of [H13] for a detailed description of this construction for strata. The requirement (1) in Definition 3.6 is made for convenience of exposition; we will occasionally talk about a set with a product structure which only has properties (2) and (3) above.

The following observation is immediate from the definition.

Lemma 3.7. *Let $U \subset \mathcal{C}_{\rm good}$ be an open or closed set with a product structure as in Definition 3.6. Then each component of the intersection of U with an orbit of the Teichmüller flow is an arc of length 2ϵ.*

Proof. Let $V \subset \mathcal{C}_{\rm good}$ be a set with a product structure, and let $\tilde V$ be a component of the preimage of V in the Teichmüller space of marked abelian differentials. As V is contained in $\mathcal{C}_{\rm good}$ and is contractible, a component of the intersection of V with an orbit of the Teichmüller flow lifts to a component of the intersection of $\tilde V$ with an orbit of the Teichmüller flow. The lemma is now immediate from the definition and the fact that the Teichmüller flow preserves the projective class of the horizontal and vertical measured foliation, respectively. □

Let $\tilde V$ be as in (3) of Definition 3.6. For each $\tilde z \in \tilde V$, the product structure determines a closed *local strong unstable manifold*

$$W^{su}_{\rm loc}(\tilde z)$$

containing $\tilde z$ which is homeomorphic to a closed ball of dimension m. This set consists of all points whose marked horizontal measured foliation coincides with the marked horizontal measured foliation of $\tilde z$, and whose marked vertical projective measured foliation is contained in the set K. Similarly we obtain a *local strong stable manifold* $W^{ss}_{\rm loc}(\tilde z)$ by exchanging the roles of the horizontal and the vertical measured foliations. The sets $W^i_{\rm loc}(\tilde z)$ $(i = ss, su)$ need not be contained in $\tilde V$, but every $\tilde y \in W^i_{\rm loc}(\tilde z)$ can be moved into $\tilde V$ with a small translate along the flow line of Φ^t through $\tilde y$. For $z \in V$ we let $W^i_{\rm loc}(z)$ be the projection to $\mathcal{C}$ of $W^i_{\rm loc}(\tilde z)$ where $\tilde z \in \tilde V$ is the preimage of z $(i = ss, su)$. Note that these sets are contained in $\mathcal{C}_{\rm good}$ by invariance of $\mathcal{C}_{\rm good}$ under the Teichmüller flow.

Example 3.8. Let $\mathcal{Q}$ be a component of a stratum of abelian or quadratic differentials. Let $q \in \mathcal{Q}_{\rm good}$ and let A^{su} be a neighborhood of q in $W^{su}_{\rm loc}(q)$. Then for a sufficiently small neighborhood A^{ss} of q in $W^{ss}_{\rm loc}(q)$ and every $z \in A^{ss}$ there exists a *holonomy homeomorphism*

$$\Xi_z : A^{su} \to \Xi_z(A^{su}) \subset W^{su}_{\rm loc}(z)$$

with $\Xi_z(q) = z$ determined by the requirement that $\Xi_z(u) \in \cup_{-\epsilon\leq t\leq\epsilon}\Phi^t W^{ss}_{\rm loc}(u)$ for some small $\epsilon > 0$ and all $u \in A^{su}$. The holonomy homeomorphisms Ξ_z are smooth and depend smoothly on z.

Define $V(A^{ss}, A^{su}) = \cup_{z\in A^{ss}}\Xi_z A^{su}$ and

$$V(A^{ss}, A^{su}, t_0) = \cup_{-t_0\leq t\leq t_0}\Phi^t V(A^{ss}, A^{su}).$$

If we choose A^i to be a sufficiently small ball neighborhood of q in $W^i_{\rm loc}(q)$ and t_0 sufficiently small, then $V(A^{ss}, A^{su}, t_0)$ is a neighborhood of q with a product structure in the sense of Definition 3.6.

The tangent bundle of the strong stable or strong unstable foliation of a component $\mathcal{Q}$ of a stratum can be equipped with the so-called *modified Hodge norm* which induces a *Hodge distance* d_H on the leaves of the foliation of a stratum of abelian differentials.

The following result is the first part of Theorem 8.12 of [ABEM12]. As before, $\mathcal{Q}$ denotes a component of a stratum of abelian differentials.

Theorem 3.9. *There exists a number $c_H > 0$ not depending on choices such that for every $q \in \mathcal{Q}$, any $q' \in W^{ss}_{\rm loc}(q)$ and all $t > 0$ we have*
$$d_H(\Phi^t q, \Phi^t q') \leq c_H d_H(q, q').$$

The following is Theorem 2 of [H23]. It quantifies the idea of non-uniform hyperbolicity of the Teichmüller flow. In its formulation, $B^i(q,r)$ denotes the ball of radius r about q for the Hodge distance on the local leaf $W^i_{\rm loc}(q)$ of the foliation W^i through q. The balls $B^i(u, r_0)$ are not required to be contained in the set U $(i = ss, su)$.

Theorem 3.10. *Let $q \in \mathcal{Q}_{\rm good}$ be a birecurrent point. Then there is a number $r_0 = r_0(q) > 0$, and there is a neighborhood U of q in $\mathcal{Q}_{\rm good}$ with the following property. Let $z \in U$ be birecurrent; then for every $a > 0$ there is a number $T(z,a) > 0$ so that for all $T > T(z,a)$, we have $\Phi^T B^{ss}(z, r_0) \subset B^{ss}(\Phi^T(z), a)$ and $\Phi^T B^{su}(z, a) \supset B^{su}(\Phi^T(z), r_0)$.*

Let us explain the similarities and differences of Theorem 3.10 with the familiar properties of an Anosov flow on a closed manifold. First, the statement is local and only applies to birecurrent points in $\mathcal{Q}_{\rm good}$. The neighborhood U of the birecurrent point q can not be made uniform in size, measured for example with respect to the distance function of a smooth Riemannian metric. The contraction times $T(z,a)$ depend on the birecurrent point $z \in U$. However, the size of the neighborhood of the point z in its local strong stable manifold does not depend on z, which is precisely what is needed to establish counting results from the mixing properties of the Masur Veech measure. By restriction, the theorem immediately carries over to affine invariant manifolds.

3.2. **Shadowing and Anosov closing.** The goal of this subsection is to prove Theorem 3.3.

Proof of Theorem 3.3. The proof is divided into three steps. In the first step, we construct the neighborhoods $Y_j \subset V_j \subset U_j$ of the points q_j and determine the number $R > 0$ whose existence is stated in the theorem. These sets have some additional properties used to obtain the dynamical control we need.

In the second step we consider the element $\varphi \in {\rm Mod}(S)$ determined by a $\mathcal{V}$-characteristic curve of a periodic $(R, \mathcal{Y})$-pseudo orbit, and we show that it is pseudo-Anosov. In particular, it determines a periodic orbit for the Teichmüller flow in the moduli space of abelian or quadratic differentials. In a thrid step We use a fixed point argument to show that this orbit is contained in $\mathcal{C}$ and has the properties stated in the proposition.

Step 1.

Using the notation from the theorem, for each $j \leq k$ choose a closed contractible neighborhood $V_j \subset U_j$ of q_j with a product structure which furthermore has the properties stated in Theorem 3.10. Recall that such a product structure is determined by a choice $\tilde{V}_j$ of a component of the preimage of V_j in the Teichmüller space of marked abelian differentials, of two closed disjoint subsets D_j, K_j of the space of projective measured foliations which are homeomorphic to closed balls of dimension $d = \dim_{\mathbb{C}}(\mathcal{C}_+) - 1$, an embedding

$$\Lambda_j : D_j \times K_j \to \tilde{V}_j$$

and a number $\epsilon_j > 0$ with the properties stated in Definition 3.6.

For $\tilde{z} \in \tilde{V}_j$ denote by $W^{ss}_{\rm loc}(\tilde{z})$ the local strong stable manifold of $\tilde{z}$ defined by $\tilde{V}_j$ as explained after Lemma 3.7 and let similarly $W^{su}_{\rm loc}(\tilde{z})$ be the local strong unstable manifold. We require that the projections into $\mathcal{C}$ of the union of all these local strong stable and strong unstable manifolds are contained in a fixed contractible subset of U_j. Note that as explained after Lemma 3.7, this is not automatic as some of these local manifolds may not be contained in V_j, but it can be achieved by making V_j smaller if necessary.

For $z \in V_j$ we denote by $W^i_{\rm loc}(z)$ the projection to $\mathcal{C}$ of the set $W^i_{\rm loc}(\tilde{z})$ where $\tilde{z}$ is the preimage of z in $\tilde{V}_j$; this does not depend on the choice of the component $\tilde{V}_j$. By perhaps decreasing the size of V_j we may assume that $W^i_{\rm loc}(\tilde{z}) \subset B^i(\tilde{z}, r_0)$ for all $\tilde{z} \in V_j$, where $r_0 > 0$ is as in Theorem 3.10.

Recall from Example 3.8 that for two points $\tilde{z}, \tilde{u} \in \tilde{V}_j$ there is a holonomy map

$$\Xi(\tilde{u}, \tilde{z}) : W^{su}_{\rm loc}(\tilde{u}) \to W^{su}_{\rm loc}(\tilde{z}).$$

For each $\tilde{v} \in W^{su}_{\rm loc}(\tilde{u})$, the point $\Xi(\tilde{u}, \tilde{z})(\tilde{v})$ is the unique point in $W^{su}_{\rm loc}(\tilde{z})$ whose marked vertical measured foliation coincides with the marked vertical measured foliation of $\tilde{v}$ (up to equivalence defined by Whitehead moves).

The holonomy maps $\Xi(\tilde{u}, \tilde{z})$ are smooth and depend smoothly on $\tilde{u}, \tilde{z}$. In particular, they are bilipschitz for the Hodge distance d_H. Furthermore, if $\tilde{z} \in W^{su}_{\rm loc}(\tilde{u})$ then $\Xi(\tilde{u}, \tilde{z}) = \mathrm{Id}$. Thus by perhaps decreasing the size of the sets V_j we may assume that the bilipschitz constants for these holonomy maps are at most 2.

Choose a compact neighborhood $Z_j \subset V_j$ of q_j with a product structure which is contained in the interior of V_j. For $z \in Z_j$ let $W^i_{{\rm loc}, Z_j}(z)$ $(i = su, ss)$ be the local strong stable and strong unstable manifold for Z_j. By continuity and compactness, there exists a number $r > 0$ such that for any $z \in Z_j$, the d_H-distance between the set $W^i_{{\rm loc}, Z_j}(z)$ and the boundary of $W^i_{\rm loc}(z)$ is at least r.

By Theorem 3.9 and Theorem 3.10 and the choice of the sets Z_j, we can find a contractible neighborhood $Y_j \subset Z_j$ of q_j with a product structure and a number $T_j > 0$ with the following property. If $z \in Y_j$ and if $T > T_j$ then

$$\begin{aligned} d_H(\Phi^T z', \Phi^T z'') &\leq \frac{r}{4} \text{ for all } z', z'' \in W^{ss}_{\rm loc}(z) \text{ and} \\ d_H(\Phi^{-T} z', \Phi^{-T} z'') &\leq \frac{r}{4} \text{ for all } z', z'' \in W^{su}_{\rm loc}(z). \end{aligned} \tag{2}$$

Namely, choose $T_j > 0$ so that the estimate (2) is satisfied for $z = q_j$ and $T = T_j$ and the constant $r/8c_H$ instead of $r/4$. Such a number exists by Theorem 3.10 and the choice of the sets V_j. By continuity, the estimate (2) with $r/4c_H$ then holds true for this number T_j and for all points z in a neighborhood Y_j of q_j which can be chosen to be contractible, with a product structure. By Theorem 3.9, the estimate (2) then holds true for all $T \geq T_j$ and for all $z \in Y_j$. Define $\mathcal{Y} = \{Y_j\}, \mathcal{V} = \{V_j\}$ and let $R = \max_j T_j$.

Step 2.

Using the notations from Step 1, let η be a $\mathcal{V}$-characteristic curve of a periodic $(R, \mathcal{Y})$-pseudo-orbit. By definition, η is determined by points $y_i \in Y_{\kappa(i)}$, numbers $t_i > R$ $(0 \leq i \leq m-1)$ and arcs in the contractible sets $V_{\kappa(i)}$. Parameterize η in such a way that for each orbit segment, the parameterization coincides with the parametrization as a flow line of the Teichmüller flow and that $\eta(\sum_{i<\ell} t_i + \ell) = y_\ell$ (i.e. the connecting arcs α_j are parametrized on a unit interval). For simplicity of notation, assume that $\eta(0) \in Y_0$. Let $T = \sum_j t_j + m > 0$ be such that $\eta(T) = \eta(0)$.

Let as before $\mathcal{Q}$ be the component of the stratum containing $\mathcal{C}$ and let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of marked abelian differentials. Let $\tilde{\mathcal{C}} \subset \tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{C}$. Let $\tilde{V}_0$ be a component of the preimage of V_0 contained in $\tilde{\mathcal{C}}$. Let $\tilde{\eta}$ be a lift of η to $\tilde{\mathcal{C}}$ which begins at $\tilde{\eta}(0) = \tilde{y}_0 \in \tilde{V}_1$. Then there is a (unique) element $\varphi \in \mathrm{Mod}(S)$ which maps the endpoint $\tilde{\eta}(T)$ of $\tilde{\eta}$ back to $\tilde{y}_0$. As any element of $\mathrm{Mod}(S)$ either stabilizes $\tilde{\mathcal{C}}$ or maps $\tilde{\mathcal{C}}$ to a disjoint component of the preimage of $\mathcal{C}$, we know that $\varphi \in \mathrm{Stab}(\tilde{\mathcal{C}})$.

By Lemma 5.1 of [H13] (and after perhaps increasing the number $R > 0$ and decreasing the sets Y_i), the mapping class φ is pseudo-Anosov (see also the bottom of p.523 of [H13] and [H23]). For completeness, we sketch the proof.

It is known that a mapping class φ is pseudo-Anosov if and only if it acts on the *curve graph* of S with positive translation length. Moreover, there exists a coarsely well defined map Υ from the Teichmüller space to the curve graph which associates to a point in Teichmüller space, viewed as a marked hyperbolic metric on S, a systole, that is, a shortest simple closed geodesic. The restriction of this map to any Teichmüller geodesic segment is a uniform unparameterized quasi-geodesic. The diameter of the image of a Teichmüller geodesic ray is infinite if this ray recurs to the thick part of Teichmüller space for arbitrarily large times.

As each of the points q_j is birecurrent, for any lift $\tilde{q}_j$ of q_j to the Teichmüller space of marked abelian differentials, the image under Υ of the Teichmüller geodesic ray whose unit cotangent line is the orbit $\{\Phi^t q_j \mid t \geq 0\}$ has infinite diameter. If $\tilde{u}$ is sufficiently close to $\tilde{q}_j$, then the Φ^t-orbits of $\tilde{u}$ and $\tilde{q}_j$ uniformly fellow travel for any a priori given time interval. This implies that up to making the sets Y_j smaller and the number $R > 0$ larger, the following holds true.

Let η be the characteristic curve of an $(R, \mathcal{Y})$-pseudo-orbit starting in Y_0 and let $\tilde{\eta}$ be the lift of η to the Teichmüller space of marked abelian differentials starting in $\tilde{V}_0$. Then $\tilde{\eta}$ projects to a path in the curve graph which consists of uniform unparameterized quasi-geodesic segments, with a priori specified lower bound on the diameter. These segments extend to segments (by extending the flow line segments in η by a large but fixed amount) which are coarsely overlapping along quasi-geodesic arcs whose diameter is bounded from below by another a priori chosen constant.

By the local to global property of such a path in the curve graph, which is a hyperbolic geodesic metric space, the lift $\tilde{\eta}$ through $\tilde{V}_0$ of the characteristic curve η maps to an unparameterized uniform quasi-geodesic in the curve graph of infinite diameter. This quasi-geodesic is invariant under the mapping class φ, and φ acts on it as a translation whose translation length is bounded from below by a positive constant not depending on the pseudo-orbit. By perhaps decreasing the size of the sets Y_j further, such a lower bound can be arbitrarily prescribed. But this just means that for sufficiently small Y_j, the mapping class φ is pseudo-Anosov.

Step 3.

Our goal is to show that the mapping class φ defines a periodic orbit γ in $\mathcal{C}$ with the properties stated in the proposition. Note that this is not implied by the fact that $\varphi \in \mathrm{Stab}(\tilde{\mathcal{C}})$. To this end we use a variation of the argument in the proof of Proposition 5.4 of [H13].

Let $\tilde{\gamma} \subset \tilde{\mathcal{H}}_+$ be the cotangent line of the axis in Teichmüller space of the pseudo-Anosov element φ. The curve $\tilde{\gamma}$ is a φ-invariant orbit of the Teichmüller flow in $\tilde{\mathcal{H}}_+$ which projects to the periodic orbit γ. The (biinfinite) lift $\tilde{\eta}$ of the characteristic curve η is contained in a uniformly bounded neighborhood of $\tilde{\gamma}$. Namely, this lift is invariant under the action of φ and hence by invariance, the Hausdorff distance (for some $\mathrm{Mod}(S)$-invariant Riemannian metric) between $\tilde{\gamma}$ and $\tilde{\eta}$ equals the Hausdorff distance between compact fundamental domains on these lifts for the action of φ and hence is finite.

The pseudo-Anosov element φ acts with north-south dynamics on the Thurston sphere $\mathcal{PMF}$ of projective measured foliations of the surface S. This means that φ has precisely two fixed points in $\mathcal{PMF}$, one is attracting, the other repelling. Furthermore, if $\tilde{u} \in \tilde{\gamma}$ is arbitrary, then the vertical projective measured foliation ν of $\tilde{u}$ equals the attracting fixed point of φ, and the horizontal projective measured foliation ξ of $\tilde{u}$ equals the repelling fixed point of φ.

Recall the definition of the sets $D_j, K_j \subset \mathcal{PMF}$ defining the set $\tilde{V}_j$, where $\tilde{V}_0$ is intersected by $\tilde{\eta}$. We claim that it suffices to verify that with the above notation, we have $\xi \in D_0, \nu \in K_0$. Namely, every flow line of the Teichmüller flow in the Teichmüller space of abelian differentials which is defined by a differential with horizontal measured foliation in D_0 and vertical measured foliation in K_0 passes through the set $\tilde{V}_0$, in particular it is entirely contained in $\tilde{\mathcal{C}}$ by invariance of $\tilde{\mathcal{C}}$ under the Teichmüller flow. Thus if $\xi \in D_0, \nu \in K_0$ then the periodic orbit γ is contained in $\mathcal{C}$, and it passes through the set V_0. As the initial point of the periodic pseudo-orbit was arbitrarily chosen among the starting points in $\mathcal{Y}$ of the orbit segments which determine the pseudo-orbit, we deduce that the periodic orbit γ passes through each of the sets $V_{\kappa(i)}$, and the crossing times fulfill the estimate stated in the proposition. Thus γ has all the properties stated in the proposition. The estimate on translation length is a consequence of the fellow traveling property for orbit segments with controlled lifts to the Teichmüller space of marked abelian differentials.

Using the argument on p.524 of [H13], we show that indeed $\nu \in K_0$. To this end we claim that

$$\Phi^{-t_0} W^{su}_{\rm loc}(\tilde\eta(t_0)) \subset W^{su}_{\rm loc}(\tilde y_0).$$

Namely, since $t_0 > R$ and since $\eta(t_0) \in Y_{\kappa(1)}$, the estimate (2) shows that the d_H-diameter of $A = \Phi^{-t_0} W^{su}_{\rm loc}(\tilde\eta(t_0))$ is at most $r/4$. On the other hand, the set A contains the point $\tilde\eta(0) = \tilde y_0 \in \tilde Y_0 \subset \tilde V_0$. As by assumption, the Hodge distance between $\tilde y_0$ and the boundary of $W^{su}_{\rm loc}(\tilde y_0)$ is at least r, we indeed have $\Phi^{-t_0} W^{su}_{\rm loc}(\tilde\eta(t_0)) \subset W^{su}_{\rm loc}(\tilde y_0)$. In particular, if we denote by $K_{\kappa(1)} \subset \mathcal{PMF}$ the closed set of all horizontal projective measured foliations for points in the component $\tilde V_{\kappa(1)}$ of the preimage of $V_{\kappa(1)}$ containing $\tilde\eta(t_0)$, then we have $K_{\kappa(1)} \subset K_0$.

The above reasoning can be iterated: For $s \geq 1$ let $K_{\kappa(s)}$ be the set of all horizontal projective measured foliations of all marked abelian differentials which are contained in the component $\tilde V_{\kappa(s)}$ of the preimage of $V_{\kappa(s)}$ containing $\tilde\eta(\sum_{j<s} t_j + s)$. We show by induction on s that for any $s \geq 1$, the set $K_{\kappa(s)}$ is entirely contained in K_0. The case $s = 1$ was discussed in the previous paragraph, so let us assume that this holds true for all $s < s_0$ for some $s_0 \geq 2$. Replacing the starting point y_0 of the periodic pseudo-orbit by y_1, we conclude from the induction hypothesis that $K_{\kappa(s_0)} \subset K_{\kappa(1)}$. However, we showed above that $K_{\kappa(1)} \subset K_0$. This yields the induction step.

To summarize, for each $t > 0$ the vertical projective measured foliation of $\tilde\eta(t)$ is contained in the compact set K_0. Now the attracting fixed point of φ is the limit as $t \to \infty$ of the vertical projective measured foliation of $\tilde\eta(t)$. Namely, the path $\tilde\eta$ is invariant under the pseudo-Anosov element φ. Since φ acts with north-south dynamics on $\mathcal{PMF}$, any non-constant orbit on $\mathcal{PMF}$ under forward iteration of φ converges to the attracting fixed point of φ. Thus this attracting fixed point of φ is indeed contained in the compact set K_0.

Reversing the direction of the flow Φ^t and replacing φ by φ^{-1}, the same argument applies to the repelling fixed point of φ and shows that this repelling fixed point is contained in D_0. In particular, the periodic orbit of Φ^t defined by φ is contained in $\mathcal{C}$, and it passes through V_0. As remarked earlier, this suffices for the proof of the proposition. $\square$

Remark 3.11. Let $\mathcal{C}$ be an affine invariant manifold, contained in a component $\mathcal{Q}$ of a stratum, and let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. If $\varphi \in \mathrm{Mod}(S)$ defines a periodic orbit of the Teichmüller flow on $\mathcal{C}$, then φ is a pseudo-Anosov mapping class which is conjugate to an element of $\mathrm{Stab}(\tilde{\mathcal{C}})$. However, it is not true that any pseudo-Anosov mapping class in $\mathrm{Stab}(\tilde{\mathcal{C}})$ determines a periodic orbit for Φ^t contained in the closure of $\mathcal{C}$. An example of this situation is the case that $\mathcal{C}$ equals a non-principal stratum of abelian differentials with at least one simple zero. In this case the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials is connected [CS21] and hence the stabilizer of this preimage equals the entire mapping class group. However, the set of periodic orbits for the Teichmüller flow contained in the closure of $\mathcal{C}$ is a proper subset of the set of all periodic orbits.

In the case of a single birecurrent point q on an affine invariant manifold $\mathcal{C}$, Proposition 3.3 predicts for every contractible neighborhood U of q a nested set of neighborhoods $Y \subset V \subset U$ of q and a number $R > 0$ with the following property. For every $y \in Y$ and $T > R$ so that $\Phi^T y \in Y$, there is a periodic orbit passing through V of period close to T which defines the same conjugacy class in $\mathrm{Mod}(S)$ as a characteristic curve of the periodic (R, Y)-pseudo-orbit (y, T).

Note also that filtering the sets $Y_i \subset V_i$ is necessary in the above argument as it is used to specify the precise location of the periodic orbit constructed from a closed pseudo-orbit.

3.3. **Semigroups defined by recurring orbits.** The goal of this subsection is to establish a parameterized version of Theorem 3.3. This is needed to associate to a periodic orbit of Φ^t on an affine invariant manifold $\mathcal{C}$ which passes through an a priori chosen subset of $\mathcal{C}$ an element of the mapping class group $\mathrm{Mod}(S)$ rather than a conjugacy class in $\mathrm{Mod}(S)$ in such a way that concatentation of orbit segments in a pseudo-orbit corresponds to multiplication of group elements. That this is possible is reminiscent of the idea that the Teichmüller flow admits a symbolic coding [AGY06, H11] by a subshift of finite type, and the characteristic property of a Markov chain is precisely that the future is independent of the past.

Let again $q \in \mathcal{C}_{\rm good}$ be a good birecurrent point. Let $U \subset \mathcal{C}_{\rm good}$ be a neighborhood of q and let $Y \subset V \subset U$ be a nested family of neighborhoods of q in $\mathcal{C}_{\rm good}$ as in Theorem 3.3. We may assume that the sets Y, V are contractible and have a product structure and that any connected component of the intersection with Y or V of an orbit segment of the Teichmüller flow is an arc of fixed length $2\delta > 0$ (this is a straightforward consequence of the construction). Put $\mathcal{V} = V$.

For $R_0 > 0$ as in Theorem 3.3 let $y \in Y$ and let $T > R_0$ be such that $\Phi^T y \in Y$. A $\mathcal{V}$-characteristic curve of this orbit segment determines uniquely a periodic orbit γ of Φ^t which intersects V in an arc of length 2δ. There may be more than one such intersection arc, but there is a unique arc determined by the requirement that the parametrized periodic orbit starting at a point in this arc uniformly fellow-travels the pseudo-orbit defined by the parameterized orbit segment $t \to \Phi^t y$ $(0 \leq t \leq T)$. Choose the midpoint of this intersection arc as a basepoint for γ and as an initial point for a unit speed parametrization of γ.

Let Γ_0 be the set of all parameterized periodic orbits of this form for points $y \in Y$ with $\Phi^T y \in Y$ $(T > R_0)$. There is a bijection between such periodic orbits and subsets of $\Phi^T V \cap V$ containing points in $\Phi^T Y \cap Y$. With some care, these subsets can be chosen to be components of $\Phi^T V \cap V$ [H13], but we will not need this somewhat technical fact in the sequel.

Fix once and for all a lift $\tilde{V}$ of the contractible set V to a component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ in the Teichmüller space of marked abelian differentials. A parametrized periodic orbit γ which starts in V lifts to a subarc of a flow line of the Teichmüller flow on $\tilde{\mathcal{C}}$ with starting point in $\tilde{V}$. The endpoint of this arc is mapped to its starting point by a pseudo-Anosov element $\Omega(\gamma) \in \mathrm{Mod}(S)$. The conjugacy class of $\Omega(\gamma)$ is uniquely determined by γ, and the element $\Omega(\gamma)$ only depends on the choice of $\tilde{V}$ (and the component of $\gamma \cap V$ as explained above). Thus a characteristic curve of a sufficiently long orbit segment beginning and ending in Y determines a pseudo-Anosov mapping class in $\mathrm{Mod}(S)$.

The following proposition is a parameterized version of shadowing as established in Theorem 3.3.

Proposition 3.12. *For $\gamma_1, \dots, \gamma_m \in \Gamma_0$, there is a point $z \in V$, and there are numbers $0 < t_1 < \cdots < t_m$ with the following properties.*

1. $\Phi^{t_i} z \in V$ *for all* i.
2. *For each $i \leq m$, a V-characteristic curve of the orbit segment $\{\Phi^t z \mid t_{i-1} \leq t \leq t_i\}$ defines the element $\Omega(\gamma_i)$ in* $\mathrm{Mod}(S)$.
3. *A V-characteristic curve of the orbit segment $\{\Phi^t z \mid 0 \leq t \leq t_m\}$ determines a parameterized periodic orbit γ for Φ^t with initial point in V, and* $\Omega(\gamma) = \Omega(\gamma_k) \circ \cdots \circ \Omega(\gamma_1)$.

Note that we can not expect that the point z is contained in the smaller set $Y \subset V$.

Proof of Proposition 3.12. The proposition is a fairly immediate consequence of Theorem 3.3 and the definitions.

Namely, recall that an orbit $\gamma \in \Gamma_0$ is constructed from a point $y \in Y$ and a number $s(\gamma, y) > R_0$ so that $\Phi^{s(\gamma,y)} y \in Y$. The orbit γ then is the unique periodic orbit determined by the characteristic curve of the pseudo-orbit $(y, s(\gamma, y))$.

Now let $\gamma_1, \dots, \gamma_m \in \Gamma_0$, and for each $i \leq m$ let (y_i, s_i) be as in the previous paragraph for γ_i. By Theorem 3.3, there exists a parameterized periodic orbit $\gamma \in \mathcal{C}$ beginning at a point $z \in V$ which passes through V at times t_i close to $\sum_{\ell < i-1} s_\ell$ and which defines the same conjugacy class in $\mathrm{Mod}(S)$ as the concatenation of the pseudo-orbits $(y_1, s_1), \dots, (y_m, s_m)$. But this just means that for each i a $\mathcal{V}$-characteristic curve of the orbit segment $\cup_{t \in [t_{i-1}, t_i]} \Phi^t z$ defines the element $\Omega(\gamma_i)$ in $\mathrm{Mod}(S)$. It is now immediate from the construction that γ can be parameterized in such a way that the properties in the proposition are fulfilled. $\square$

As a consequence, the subsemigroup

$$\langle \Omega(\Gamma_0) \rangle < \mathrm{Mod}(S)$$

generated by $\{\Omega(\gamma) \mid \gamma \in \Gamma_0\}$ consists of pseudo-Anosov elements whose corresponding periodic orbits are contained in the affine invariant manifold $\mathcal{C}$ and pass through the set V. This can be viewed as a version of Rauzy-Veech induction as used in [AV07b, AGY06] which is valid for all affine invariant manifolds, in particular for strata of quadratic differentials, or as a version of symbolic dynamics for the Teichmüller flow on affine invariant manifolds.

4. Zariski density for affine invariant manifolds

The goal of this section is to show that the mondromy of an affine invariant manifold is Zariski dense. Throughout this section we assume that $g \geq 2$, and we use the assumptions and notations from Section 2.

Let $\mathcal{Q}_+ \subset \mathcal{H}_+$ be a component of a stratum and let $\mathcal{C}_+ \subset \mathcal{Q}_+$ be an affine invariant manifold. Recall from Section 2 that the image of the projection $p : T\mathcal{C}_+ \to \Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$ to absolute periods is a flat subbundle of $\Pi^*\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}|\mathcal{C}_+$ which is invariant under both the complex structure defined by enlargement of coefficients (the tensor product) as well as the complex structure of the Hodge bundle. We denote by $2\ell \geq 2$ its complex dimension. Then $p(T\mathcal{C}_+) \cap \Pi^*\mathcal{H}|\mathcal{C}_+$ is a flat bundle $\mathcal{Z} = \mathcal{Z}_{\mathbb{R}}$ whose fibre is a symplectic subspace of the fibre of $\Pi^*\mathcal{H}$ (recall that the fibre of $\Pi^*\mathcal{H}$ can be identified with $H^1(S, \mathbb{R})$) of real dimension 2ℓ. As before, by a flat subbundle of the bundle $\Pi^*\mathcal{H}|\mathcal{C}_+$ we mean a bundle which is invariant under the restriction of the Gauss Manin connection. We call $\mathcal{Z}$ the *absolute real tangent bundle* of $\mathcal{C}_+$. The Gauss Manin connection restricts to a flat connection on $\mathcal{Z}$.

Definition 4.1. The *monodromy group* of the affine invariant manifold $\mathcal{C}_+$ of rank ℓ is the subgroup of $\mathrm{Sp}(2\ell, \mathbb{R})$ which is generated by parallel transport of the absolute real tangent bundle $\mathcal{Z}$ for the restriction of the Gauss Manin connection along loops in $\mathcal{C}_+$ based at some fixed point p.

The fact that the monodromy group is a subgroup of $\mathrm{Sp}(2\ell, \mathbb{R})$ follows from the fact that the Gauss Manin connection is symplectic. Its conjugacy class does not depend on any choices made.

A geometric description of the monodromy group of $\mathcal{C}_+$ is as follows. Observe first that the monodromy coincides with the monodromy of the restriction of the bundle $\mathcal{Z}$ to the intersection $\mathcal{C}$ of $\mathcal{C}_+$ with the moduli space of area one abelian differentials. Let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. The stabilizer $\mathrm{Stab}(\tilde{\mathcal{C}})$ of $\tilde{\mathcal{C}}$ in the mapping class group maps via the natural surjective homomorphism $\Psi : \mathrm{Mod}(S) \to \mathrm{Sp}(2g, \mathbb{Z})$ to a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$. There is a linear symplectic subspace $H \subset \mathbb{R}^{2g}$ of dimension 2ℓ which is preserved by $\Psi(\mathrm{Stab}(\tilde{\mathcal{C}}))$. The monodromy group of $\mathcal{C}$ then is the projection of $\Psi(\mathrm{Stab}(\tilde{\mathcal{C}}))$ to the group $\mathrm{Sp}(H) = \mathrm{Sp}(2\ell, \mathbb{R})$ of symplectic automorphisms of H. This description is immediate from the description of the Gauss Manin connection in Section 2.1.

Example 4.2. If $\mathcal{C}_+$ is a Teichmüller curve, then the monodromy group of $\mathcal{C}_+$ is just the Veech group of $\mathcal{C}_+$, acting on the two-dimensional symplectic subspace of $H^1(S, \mathbb{R})$ which is spanned by the real and imaginary part, respectively, of an abelian differential $\omega \in \mathcal{C}_+$. Thus this monodromy group is a lattice in $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$, in particular it is Zariski dense in $\mathrm{SL}(2, \mathbb{R})$.

Our goal is to show that the monodromy group of any affine invariant manifold is Zariski dense in $\mathrm{Sp}(2\ell, \mathbb{R})$ (using the above convention). We will make use of the fact that an abelian differential on S defines a singular euclidean metric on S with cone points of cone angle a multiple of 2π at the zeros of the differential. This singular euclidean metric is given by a family of charts, defined on the complement of the zeros of the differential, with chart transitions being translations. As it is customary in the literature, if we view an abelian differential on S as a singular euclidean metric, we refer to these data as a *translation surface*. We denote such a translation surface by X or by a pair (X,ω) if we like to specify the abelian differential ω which defines the translation structure. Note that ω can be read off from the horizontal and vertical measured foliations of the translation surface.

We begin with invoking a result of Wright [W15]. He introduced the following two deformations of a translation surface (X,ω).

The *horocycle flow* is defined as part of the $\mathrm{SL}(2,\mathbb{R})$-action,

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \subset \mathrm{SL}(2,\mathbb{R}),$$

and the *vertical stretch* is defined by

$$a_t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \subset \mathrm{GL}^+(2,\mathbb{R}).$$

For a collection $\mathcal{Y}$ of horizontal cylinders on a translation surface X (i.e. cylinders foliated by leaves of the horizontal foliation), define the *cylinder shear* $u_t^{\mathcal{Y}}(X)$ to be the translation surface obtained by applying the horocycle flow to the cylinders in $\mathcal{Y}$ but not to the rest of X. Similarly, the *cylinder stretch* $a_t^{\mathcal{Y}}(X)$ is obtained by applying the vertical stretch only to the cylinders in $\mathcal{Y}$.

A translation surface (X,ω) is called *horizontally periodic* if it is a union of horizontal cylinders. A horizontal cylinder in a translation surface defines a class in $H_1(S,\mathbb{Z})$ which is the oriented core curve of the cylinder. A cylinder family in a translation surface defines such a class if the core curves of its cylinders are all homologous. If we talk about the homology class of a cylinder family, then we implicitly require that this is the case. Note that as the core curve of a cylinder is a simple closed curve, it defines a primitive integral homology class. Hence if two cylinders in a translation surface define *collinear* homology classes, that is, classes which are scalar multiples of each other, then they define the same homology class up to sign. In fact, the signs have to match up as well. Namely, these signs are determined by integration of the cohomology class $\mathrm{Re}(\omega)$.

In the sequel, via the natural pairing

$$\langle , \rangle : H^1(S,\mathbb{R}) \times H_1(S,\mathbb{R}) \to \mathbb{R}$$

between first homology and first cohomology of S, we view a class in $H_1(S,\mathbb{R})$ as an element of $H^1(S,\mathbb{R})^*$. The following lemma is a consequence of the work of Wright [W15].

Lemma 4.3. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank ℓ. Then there exists a horizontally periodic surface $(X,\omega)\in\mathcal{C}_+$ with the following properties.*

(1) *There is a decomposition of X into $\ell+1$ collections $\mathcal{Y}_1,\dots,\mathcal{Y}_\ell,\mathcal{Y}_{\ell+1}$ of horizontal cylinder families. The family $\mathcal{Y}_{\ell+1}$ may be empty.*

(2) *The homology classes of the cylinder families* $\mathcal{Y}_i$ $(i \leq \ell)$ *span a subspace* L *of the dual* $\mathcal{Z}^*$ *of* $\mathcal{Z} = p(T\mathcal{C}_+) \cap \Pi^*\mathcal{H}|\mathcal{C}_+$ *of dimension* ℓ, *and the moduli of all of the cylinders in each of the collections* $\mathcal{Y}_i$ $(i \leq \ell)$ *are rational.*

(3) *For each* $i \leq \ell$, *the cylinder shear* $u_t^{\mathcal{Y}_i}(X)$ *remains in* $\mathcal{C}_+$.

(4) *For any contractible neighborhood* U *of* (X, ω) *in* $\mathcal{C}_+$, *the real parts*

$$\{[\mathrm{Re}(z)] \in H^1(S, \mathbb{R}) \mid z \in U\}$$

span the dual L^* *of* L.

Proof. Let $(X, \omega) \in \mathcal{C}_+$ be a translation surface with the maximal number of parallel cylinders. We may assume that these cylinders are horizontal. By Theorem 1.10 of [W15] and its proof, (X, ω) is horizontally periodic, and the core curves of the horizontal cylinders span a subspace of the dual $\mathcal{Z}^*$ of $\mathcal{Z}$ of dimension ℓ. No set of core curves of parallel cylinders on a translation surface $Y \in \mathcal{C}_+$ may span a subspace of $\mathcal{Z}^*$ of dimension greater than ℓ. As the core curves of these cylinders define integral homology classes with pairwise trivial homological intersection, there exists an isotropic (for the symplectic structure) subspace of $\mathcal{Z}^*$ of dimension $\ell = \frac{1}{2}\dim \mathcal{Z}^*$ spanned by integral points in the homology of S.

By Definition 4.6 of [W15], two cylinders in X are called $\mathcal{C}_+$*-parallel* if they are parallel at X and at every nearby $X' \in \mathcal{C}_+$. Being $\mathcal{C}_+$-parallel is an equivalence relation on the set of cylinders. Lemma 4.7 of [W15] states that two cylinders in X are $\mathcal{C}_+$-parallel if and only if their homology classes are collinear (and hence coincide).

Let $\mathcal{Z}_i$ $(i = 1, \dots, k)$ be the set of equivalence classes of horizontal cylinders in (X, ω) for this equivalence relation. By the choice of (X, ω), we have $k = \ell$, i.e. the horizontal cylinders of (X, ω) group into precisely ℓ equivalence classes. Lemma 4.11 of [W15] shows that the cylinder shear of any of the $\mathcal{C}_+$-parallel cylinder families $\mathcal{Z}_i$ remains in $\mathcal{C}_+$.

Consider one of the families $\mathcal{Z}_i$. The cylinder shear for $\mathcal{Z}_i$ remains in $\mathcal{C}_+$. Corollary 3.4 of [W15] states that if the moduli of the cylinders in this family are not all rationally dependent, then there is a proper decomposition $\mathcal{Z}_i = \mathcal{A} \cup \mathcal{B}$ so that the cylinder shears for the families $\mathcal{A}, \mathcal{B}$ remain in $\mathcal{C}_+$. Thus we can subdivide the cylinder family $\mathcal{Z}_i = \cup_j \mathcal{Z}_i^j$ where $j \geq 1$, where the moduli of the cylinders in each of the families $\mathcal{Z}_i^j$ are rationally dependent and such that for each j, the cylinder shear $u_t^{\mathcal{Z}_i^j}(X)$ remains in $\mathcal{C}_+$.

By Theorem 5.1 of [W15], for all $i \leq \ell$ the vertical stretch $a_t^{\mathcal{Z}_i}$ of the cylinder family $\mathcal{Z}_i$ is contained in $\mathcal{C}_+$. This vertical stretch changes the moduli of the cylinders in the family $\mathcal{Z}_i$ while keeping the moduli of the cylinders in the family $\mathcal{Z}_j$ fixed for all $j \neq i$. If $A_1, A_2 \subset \mathcal{Z}_i$ are cylinders with rationally dependent moduli, then the moduli of their images under the vertical stretch are rationally dependent as well. As a consequence, by successively modifying (X, ω) with a sequence of vertical stretches of the cylinder families $\mathcal{Z}_i$ $(i = 1, \dots, \ell)$ we can assure that the image surface (X', ω'), which is again horizontally periodic, has the following property. For each i, the moduli of the cylinders in the cylinder family $\mathcal{Y}_1, \dots, \mathcal{Y}_\ell$ which are the images in X' of the families $\mathcal{Z}_1^1, \dots, \mathcal{Z}_\ell^1$ are rational.

Let $\mathcal{Y}_{\ell+1} = X' - \cup_i \mathcal{Y}_i$. Then the surface (X', ω) and the cylinder families $\mathcal{Y}_i$ have the properties (1)-(3) stated in the lemma.

We are left with showing property (4). To this end choose a lift $\tilde{\omega}$ of the differential ω to the Teichmüller space of abelian differentials. This choice determines the subspace L of $H_1(S, \mathbb{R})$ spanned by the homology classes of the cylinders in the cylinder families of $\tilde{\omega}$.

The bundle $\mathcal{Z}$ is a complex subbundle of the restriction of the pull-back $\Pi^*\mathcal{H}$ of the Hodge bundle $\mathcal{H}$ to $\mathcal{C}_+$. Thus for each $q \in \mathcal{C}_+$ there is a complex structure J_q on $\mathcal{Z}$ which depends on q. This complex structure is compatible with the symplectic structure (which does not depend on q). This construction equips the flat bundle $\mathcal{Z} \to \mathcal{C}_+$ with the structure of a complex vector bundle, and this complex structure lifts to a complex structure on the pull-back bundle $\tilde{\mathcal{Z}} \to \tilde{\mathcal{C}}_+$.

Since the complex structure $J_{\tilde{\omega}}$ on $\tilde{\mathcal{Z}}_{\tilde{\omega}}$ is compatible with the symplectic structure, the Lagrangian subspace L of $\tilde{\mathcal{Z}}^*_{\tilde{\omega}}$ is totally real for $J_{\tilde{\omega}}$. Here by abuse of notation, we transfer the complex structure on $\mathcal{Z}$ to a complex structure on its dual and denote it by the same symbol. As a consequence, the real linear span of L and the linear Lagrangian subspace $J_{\tilde{\omega}}L$ of $\tilde{\mathcal{Z}}^*_{\tilde{\omega}}$ equals all of $\tilde{\mathcal{Z}}^*_{\tilde{\omega}}$.

Now the $\mathbb{C}^*$-action on $\tilde{\mathcal{C}}_+$ which associates to a pair $(q, \theta) \in \tilde{\mathcal{C}}_+ \times \mathbb{C}^*$ the abelian differential θq is holomorphic and commutes with the natural action of $\mathbb{C}^*$ on $\tilde{\mathcal{Z}}_q$: The restriction of the bundle $\tilde{\mathcal{Z}}$ to an orbit of the $\mathbb{C}^*$-action is a trivial complex vector bundle over $\mathbb{C}^*$, that is, the product of $\mathbb{C}^*$ with a complex vector space. As L is a Lagrangian subspace of $\tilde{\mathcal{Z}}^*_{\tilde{\omega}}$ which annihilates the projection of the tangent space of $W^{ss}_{\mathrm{loc}}(\tilde{\omega})$ at $\tilde{\omega}$ to absolute periods, we obtain that $J_{\tilde{\omega}}L$ is a Lagrangian subspace of $\tilde{\mathcal{Z}}^*_{\tilde{\omega}}$ which annihilates the projection of the tangent space of $W^{su}_{\mathrm{loc}}(\tilde{\omega})$ at $\tilde{\omega}$ to absolute periods. Hence the restriction of $J_{\tilde{\omega}}L$ to the projection of the tangent space of $W^{ss}_{\mathrm{loc}}(\tilde{\omega})$ at $\tilde{\omega}$ is non-degenerate. As a consequence, by the implicit function theorem, the real parts of all differentials contained in any neighborhood $\tilde{Y}$ of $\tilde{\omega}$ project to an open subset of L^*. $\square$

Remark 4.4. Algebraically primitive Teichmüller curves in the stratum of abelian differentials with a single zero on surfaces of genus 2 show that in general, the cylinder families which arise in Lemma 4.3 consist of more than one cylinder. Examples are McMullen's L-shaped billiards (Theorem 1.5 of [McM03]) which are horizontally periodic, with two homologous horizontal cylinders. A straightforward computation shows that the moduli of these cylinders are rationally dependent.

Define a *piecewise affine transformation* of a translation surface (X, ω) to be a continuous self-map $F : X \to X$ with the following property. There exists an F-invariant decomposition $X = \cup_i X_i$ into finitely many components with geodesic boundary for the singular euclidean metric, and the restriction of F to each of these components is affine. In contrast to an affine automorphism of (X, ω), we allow that F is non-trivial but that the restriction of F to some of the components X_i equals the identity. A cylinder shear of a collection $\mathcal{Y}$ of horizontal cylinders with non-empty complement is such a piecewise affine transformation. If the result of such a transformation is isometric to (X,ω) then we call the piecewise affine transformation a *piecewise affine automorphism* of (X,ω).

A *transvection* in a 2ℓ-dimensional symplectic vector space over a field K is a map $A \in \mathrm{Sp}(2\ell, K)$ which fixes a subspace of $K^{2\ell}$ of codimension one and has determinant one (see [Hl08]). Any map of the form

$$\alpha \to \alpha + \iota(\alpha,\beta)\beta$$

for some $0 \neq \beta \in K^{2\ell}$ (here as before, ι is the symplectic form) is a transvection. We call this map a *transvection by* β.

Recall that a *Dehn multitwist* along a simple closed multicurve, that is, the disjoint union $c = c_1 \cup \cdots \cup_k$ of k simple closed curves, is a map of the form $T_{c_1}^{j_1} \circ \cdots \circ T_{c_k}^{j_k}$ where for each $i \leq k$, T_{c_i} is a positive Dehn twist about c_i and $j_i \in \mathbb{Z}$.

Corollary 4.5. *Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$. Then there is a horizontally periodic surface $(X,\omega) \in \mathcal{C}_+$, and there is a free abelian group of rank ℓ of piecewise affine transformations of (X,ω) which preserves $\mathcal{C}_+$. This group of piecewise affine transformations contains a lattice Λ, that is, a subgroup isomorphic to $\mathbb{Z}^\ell$, which acts on (X,ω) as a group of Dehn-multitwists, and it acts on $H_1(S,\mathbb{R})$ as a group of transvections of rank ℓ.*

Proof. Let (X,ω) be a translation surface as in Lemma 4.3. Let $\mathcal{Y}_i$ $(i \leq \ell)$ be one of the cylinder families whose existence was shown in Lemma 4.3. The moduli of all cylinders in the family are rational. Moreover, the cylinder shear $u_t^{\mathcal{Y}_i}(X)$ for this cylinder family remains in $\mathcal{C}$.

As all the moduli of the cylinders in the cylinder families $\mathcal{Y}_i$ are rational, this cylinder shear is eventually periodic. This means that for each i there exists some number $r_i > 0$ such that for some fixed marking of the surface X, the surface $u_{r_i}^{\mathcal{Y}_i}(X)$ is the image of X by a Dehn multitwist T_i about the core curves of the cylinders in $\mathcal{Y}_i$. In particular, this multitwist defines a piecewise affine automorphism of (X,ω).

Since the core curves of the horizontal cylinders in X are pairwise disjoint, the Dehn multitwists T_i commute. Therefore these multitwists generate a free abelian group of rank ℓ of piecewise affine automorphisms of X. The multitwist T_i acts as a transvection on $H_1(S,\mathbb{R})$ by a homology class of the form $\sum_s b_i^s \zeta_i^s$ where $b_i^s \in \mathbb{Z}$ and where ζ_i^s runs through the homology classes of the waist curves of the oriented cylinders in the family $\mathcal{Y}_i$. Recall that these homology classes all coincide and are non-trivial, furthermore the coefficients b_i^s are all positive (provided that we chose the positive cylinder shear) and hence the class $\sum_s b_i^s \zeta_i^s$ is a positive multiple of the class defined by one of the cylinders in the family.

Each of the homology classes $a_i = \sum_s b_i^s \zeta_i^s$ $(i \leq \ell)$ induces a linear functional on the fibre of $pT\mathcal{C}_+ = \mathcal{Z}$ at X. The corollary now follows from the fact that by the choice of (X,ω), the rank of the subspace of $pT\mathcal{C}_+^*$ spanned by these homology classes equals ℓ. Then the subgroup of $\mathrm{Mod}(S)$ generated by the Dehn multitwists T_i $(i = 1,\dots,\ell)$ acts on $H_1(S,\mathbb{R})$ as an abelian group of transvections of rank ℓ. $\square$

Our criterion for Zariski density relies on a result of Hall [Hl08]. For its formulation, for a prime $p \geq 2$ let F_p be the field with p elements. Then $\mathrm{Sp}(2g, F_p)$ is a finite group. Therefore for every $A \in \mathrm{Sp}(2g, F_p)$ there is some $\ell \geq 1$ such that $A^\ell = A^{-1}$. As a consequence, if $G < \mathrm{Sp}(2g, F_p)$ is any subsemigroup then for all $x, y \in G$ we have $xy^{-1} \in G$ as well and hence $G < \mathrm{Sp}(2g, F_p)$ is a group.

In the formulation of the following lemma, ι denotes the symplectic form on a symplectic vector space $F_p^{2\ell}$ over F_p of rank 2ℓ.

Lemma 4.6. *Let $p \geq 3$ be an odd prime and let $G < \mathrm{Sp}(2\ell, F_p)$ be a subgroup generated by 2ℓ transvections by the elements of a set $\mathcal{E} = \{e_1, \dots, e_{2\ell}\} \subset F_p^{2\ell}$ which spans $F_p^{2\ell}$. Assume that there is no nontrivial partition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ so that $\iota(e_{i_1}, e_{i_2}) = 0$ for all $e_{i_j} \in \mathcal{E}_j$. Then $G = \mathrm{Sp}(2\ell, F_p)$.*

Proof. For each i write $A_i(x) = x + \iota(x, e_i)e_i$. Let $G < \mathrm{Sp}(2\ell, F_p)$ be the subgroup generated by the transvections $A_1, \dots, A_{2\ell}$. Since the vectors $e_1, \dots, e_{2\ell}$ span $F_p^{2\ell}$, the intersection of the invariant subspaces of the transvections A_i $(i \leq 2\ell)$ is trivial.

We claim that the standard representation of G on $F_p^{2\ell}$ is irreducible. Namely, assume to the contrary that there is an invariant proper linear subspace $W \subset F_p^{2\ell}$. Let $0 \neq w \in W$; then there is at least one i so that $\iota(w, e_i) \neq 0$. By invariance, we have $w + \iota(w, e_i)e_i \in W$ and hence $e_i \in W$ since F_p is a field.

As a consequence, W is spanned by some of the e_i, say by $e_{i_1}, \dots, e_{i_k}$, and if j is such that $\iota(e_{i_s}, e_j) \neq 0$ for some $s \leq k$ then $e_j \in W$. However, this implies that $W = F_p^{2\ell}$ by the assumption on the set $\mathcal{E} = \{e_i\}$.

To summarize, G is an irreducible subgroup of $\mathrm{Sp}(2\ell, F_p)$ generated by transvections (where irreducible means that the standard representation of G on $F_p^{2\ell}$ is irreducible). Furthermore, as p is an odd prime by assumption, the order of each of these transvections is not divisible by 2. Theorem 3.1 of [Hl08] now yields that $G = \mathrm{Sp}(2\ell, F_p)$ which is what we wanted to show. $\square$

Remark 4.7. By Proposition 6.5 of [FM12], Lemma 4.6 is not true for $p = 2$.

We use Lemma 4.6 to establish a criterion for Zariski density of a subgroup of $\mathrm{Sp}(2\ell, \mathbb{R})$ acting on a 2ℓ-dimensional symplectic subspace of $H^1(S, \mathbb{R})$. As before, we use the standard pairing $\langle , \rangle$ between homology and cohomology to view a class in $H_1(S, \mathbb{R})$ as an element of $H^1(S, \mathbb{R})^*$. A symplectic automorphism of $H_1(S, \mathbb{R})$ induces a symplectic automorphism of $H^1(S, \mathbb{R})$. Recall also that the real part $\mathrm{Re}(\tilde{q})$ and the imaginary part $\mathrm{Im}(\tilde{q})$ of a marked abelian differential $\tilde{q}$ define a cohomology class $[\mathrm{Re}(\tilde{q})], [\mathrm{Im}(\tilde{q})] \in H^1(S, \mathbb{R})$. For a symplectic subspace E of $H^1(S, \mathbb{R})$ denote by $\mathrm{Sp}(E^*)$ the group of symplectic automorphisms of its dual E^*.

By a weighted oriented simple multicurve c on S we mean a simple oriented multicurve with integral weights. Such a weighted oriented simple multicurve c defines a Dehn multitwist about the components of c, where the weight of each component determines the multiplicity of the twist about the component, and the weight together with the orientation of the multicurve determines the sign of the twist (positive or negative).

For a fixed choice of a marking of S, a weighted oriented simple multicurve defines a homology class $[c] \in H_1(S,\mathbb{Z})$ as the weighted sum of the homology classes of its components. We always assume in the sequel that this class is non-trivial. Then the Dehn multitwist about such a multicurve is a transvection in $\mathrm{Sp}(2g,\mathbb{Z})$.

The following proposition is the main technical tool towards the proof of Zariski density of the monodromy of $\mathcal{C}$.

Proposition 4.8. *Let $\mathcal{C}$ be an affine invariant manifold of rank ℓ, let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials and let $\mathcal{Z} = p(T\tilde{\mathcal{C}}_+) \cap \Pi^*\mathcal{H}|\mathcal{C}_+$.*

Let $c_1,\dots,c_\ell$ be pairwise disjoint weighted oriented simple multicurves whose (marked) homology classes $[c_i]$ span a subspace L of $\mathcal{Z}^*$ of rank ℓ. Let $U \subset \mathcal{C}$ be an open contractible set and let $\Omega(\Gamma_0) \subset \mathrm{Mod}(S)$ be the subsemigroup determined by a pair of open contractible subsets $Y \subset V$ of U and lifts $\tilde{Y} \subset \tilde{V} \subset \tilde{U}$ of U to the Teichmüller space or marked abelian differentials as in Proposition 3.12.

Assume that the cohomology classes $\{[\mathrm{Re}(\tilde{z})] \mid \tilde{z} \in \tilde{Y}\}$ span the dual L^* of L and that $\langle[\mathrm{Re}(\tilde{z})],[c_i]\rangle > 0$ for all $\tilde{z} \in \tilde{Y}$ and all i. Then the subsemigroup of $\mathrm{Sp}(\mathcal{Z}^*)$ generated by $\Psi(\Omega(\Gamma_0))$ and the transvections $\Psi(T_{c_i})$ which are the images of the Dehn multitwists T_{c_i} about the weighted oriented multicurves c_i is Zariski dense in $\mathrm{Sp}(\mathcal{Z}^*)$. If $g=\ell$ then for all but finitely many primes $p \geq 3$, this semigroup surjects onto $Sp(2g,F_p)$.

Proof. Let $\mathcal{C}$ be an affine invariant manifold of rank ℓ. Let $U \subset \mathcal{C}$ be an open contractible set with the properties stated in the proposition and let $\tilde{U}$ be a component of the preimage of U in the Teichmüller space of marked abelian differentials.

Let $Y \subset V \subset U$ be a pair of open subsets of U as in Proposition 3.3 and use these sets and a fixed component $\tilde{V} \subset \tilde{U}$ of the preimages of V to construct the subsemigroup $\Omega(\Gamma_0)$ of $\mathrm{Mod}(S)$.

Let $c_1,\dots,c_\ell$ be pairwise disjoint simple oriented weighted multicurves. With respect to some fixed marking of S, used for the choice of the lift $\tilde{U}$ of U to the Teichmüller space of marked abelian differentials, assume that the homology classes $[c_i]$ of c_i span a linear subspace L of $\mathcal{Z}^*$ of dimension ℓ. As the multicurves c_i are pairwise disjoint, this subspace is isotropic. By the assumption in the proposition, the cohomology classes $[\mathrm{Re}(\tilde{z})] \in H^1(S,\mathbb{R})$ of the real parts $\mathrm{Re}(\tilde{z})$ of the differentials $\tilde{z} \in \tilde{Y}$ span the dual L^* of L in the symplectic vector space $\mathcal{Z}$.

Any periodic orbit passing through Y defines an element of the group $\Omega(\Gamma_0)$. As periodic points for Φ^t are dense (which follows among others from Proposition 3.3 but is well known, see for example [W15]), we can choose a lift $\tilde{z} \in \tilde{Y} \subset \tilde{V}$ of a periodic point $z \in Y$ for Φ^t which defines the pseudo-Anosov mapping class $\varphi \in \Omega(\Gamma_0) < \mathrm{Mod}(S)$. By assumption, we have $\langle[\mathrm{Re}(\tilde{z})],[c_i]\rangle \neq 0$ for all i. The mapping class φ preserves the Φ^t-orbit of $\tilde{z}$.

There is a number $\kappa > 1$ such that $\varphi^*\mathrm{Re}(\tilde{z}) = \kappa^{-1}\mathrm{Re}(\tilde{z})$, moreover κ is the Perron Frobenius eigenvalue for the action of φ on $H^1(S,\mathbb{R})$. In particular, φ^*

preserves the subspace $\ker \mathrm{Re}(\tilde{z})$ and, by duality, it acts on the cone $\langle \mathrm{Re}(\tilde{z}), \cdot \rangle > 0$ as an expansion, with attracting invariant line spanned by $\mathrm{Re}(\tilde{z})^*$. This implies that as $k \to \infty$ the homology classes $[\varphi^k c_i]$ converge up to rescaling to a class $u \in H_1(S, \mathbb{R})$ whose contraction with the intersection form ι defines $\pm[\mathrm{Re}(\tilde{z})]$, viewed as a linear functional on $H_1(S, \mathbb{R})$. By this we mean that $\iota(u, a) = \langle \pm[\mathrm{Re}(\tilde{z})], a \rangle$ for all $a \in H_1(S, \mathbb{R})$. As a consequence, for all sufficiently large $n > 0$ and all $i, j \leq \ell$ we have $\iota([\varphi^n c_i], [c_j]) \neq 0$. Choose once and for all such a number n.

Let $G < \mathrm{Mod}(S)$ be the group generated by the semigroup $\Omega(\Gamma_0)$ as well as the Dehn multitwists $T_i = T_{c_i}$ $(i \leq \ell)$. Then G contains the multitwists $\varphi^n T_i \varphi^{-n} = T_{\varphi^n c_i}$ (see Fact 3.7 on p.73 of [FM12] for this equation).

Let $A_1 < \mathcal{Z}^*$ be the linear subspace of rank ℓ which is the common fixed set in $\mathcal{Z}^*$ for the transvections $\Psi(T_{c_i})$ of $\mathcal{Z}^*$ $(i = 1, \dots, \ell)$. Then A_1 is a Lagrangian subspace of the symplectic vector space $\mathcal{Z}^*$ which is spanned by the homology classes $[c_1], \cdots, [c_\ell]$. Let $A_2 \subset A_1$ be the common fixed set in $\mathcal{Z}^*$ of the transvections which are the images under the map Ψ of all multitwists $T_i, \varphi^n T_j \varphi^{-n}$. Since $\iota([\varphi^n c_i], [c_j]) \neq 0$ for all i, j by the assumption on n, the linear subspace A_2 of A_1 is of codimension $s \geq 1$. Let $i_1, \dots, i_s \subset \{1, \dots, \ell\}$ be such that the homology classes $[c_j], [\varphi^n c_{i_p}] \in H_1(S, \mathbb{Z})$ $(j \leq \ell, p \leq s)$ are independent over $\mathbb{R}$ and that the common fixed set in $\mathcal{Z}^*$ of the transvections defined by the Dehn multitwists $T_{c_i}, T_{\varphi^n c_{i_p}}$ is A_2.

Assume that the dimension of A_2 is positive. By assumption, the set of the real parts of all differentials in $\tilde{Y}$ span L^*. Thus we can find some $\tilde{y} \in \tilde{Y}$ such the linear functional $\langle [\mathrm{Re}(\tilde{y})], \cdot \rangle$ on L is non-trivial on A_2. As this condition is open, and periodic points for Φ^t are dense, we may assume that $\tilde{y}$ is the preimage of a periodic point of Y. Arguing as in the previous paragraph, we find a multitwist β in the subgroup G of $\mathrm{Mod}(S)$ generated by $\Omega(\Gamma_0)$ and the Dehn multitwists T_{c_i} so that the common fixed set in $\mathcal{Z}^*$ of the group generated by $\Psi(\beta)$ and the transvections $\Psi(T_{c_i}), \Psi(T_{\varphi^n c_{i_p}})$ has codimension at least one in A_2.

Repeat this construction. In at most ℓ steps we find integral homology classes $a_1, \dots, a_\ell, a_{\ell+1}, \dots, a_{2\ell} \in H_1(S, \mathbb{Z})$ (where for $i \leq \ell$ the class a_i is the class $[c_i]$ of the oriented weighted multicurve c_i) with the following properties.

(1) Let $W \subset H_1(S, \mathbb{R})$ be the real vector space spanned by the classes a_i. The dimension of W equals 2ℓ. Viewing W as a linear subspace of $H^1(S, \mathbb{R})^*$, its restriction to $\mathcal{Z}$ is non-degenerate. In particular, W is a symplectic subspace of $H_1(S, \mathbb{R})$.
(2) $\iota(a_j, a_i) \neq 0$ for all $i \leq \ell$, $j \geq \ell + 1$.
(3) For each j the transvection $b \to b + \iota(b, a_j) a_j$ is contained in the group generated by $\Psi(\Omega(\Gamma_0))$ and the transvections $\Psi(T_{c_i})$ $(i \leq \ell)$.

By the choice of the homology classes a_i, the $(2\ell, 2\ell)$-matrix $(\iota(a_i, a_j))$ whose (i, j)-entry is the homology intersection number $\iota(a_i, a_j)$ is integral and of maximal rank. Choose a prime $p \geq 5$ so that each of the entries of $(\iota(a_i, a_j))$ is prime to p. All but finitely many primes will do. Then the reduction mod p of the matrix $(\iota(a_i, a_j))$ is of maximal rank as well. In particular, if F_p denotes the field with p elements then the reductions mod p of the homology classes a_i span a 2ℓ-dimensional symplectic subspace W_p of $H_1(S, F_p)$.

Let $\Lambda < \mathrm{Sp}(W)$ be the subgroup of the symplectic group of W which is generated by the transvections with the elements a_i. The group Λ is defined over $\mathbb{Z}$ and hence its reduction Λ_p mod p is defined. Lemma 4.6 shows that $\Lambda_p = \mathrm{Sp}(2\ell, F_p)$. Note that property (2) above guarantees that all conditions in Lemma 4.6 are fulfilled. Since this is true for all but finitely many p, Λ is a Zariski dense subgroup of the group of symplectic automorphisms of W [Lu99]. By duality, this implies that the subgroup G of $\mathrm{Sp}(\mathcal{Z}^*)$ generated by $\Psi(T_{c_i})$ and $\Psi(\Omega(\Gamma_0))$ is Zariski dense in $\mathrm{Sp}(\mathcal{Z}^*)$.

If $g = \ell$, then any element in $\Psi(\Omega(\Gamma_0))$ as an automorphism of $\mathcal{Z}^*$ is defined over $\mathbb{Z}$ and hence $\Psi(\omega(\Gamma_0))$ can be reduced to a subset of $\mathrm{Sp}(2g, F_p)$ for any prime p. Since Γ_0 and hence $\Omega(\Gamma_0)$ is a semi-group, its reduction mod p is a subgroup of $\mathrm{Sp}(2g, F_p)$. It then follows from the above proof that this group is all of $\mathrm{Sp}(2g, F_p)$ for all but finitely many p. This completes the proof of the proposition. □

Let again $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$. Recall from Definition 4.1 the definition of the monodromy group of an affine invariant manifold $\mathcal{C}_+$ of rank ℓ. We can now summarize the discussion in this section as follows.

Corollary 4.9. *The monodromy group of any affine invariant manifold $\mathcal{C}_+$ of rank ℓ is Zariski dense in* $\mathrm{Sp}(2\ell, \mathbb{R}) = \mathrm{Sp}(\mathbb{Z}_{\mathbb{R}}, \mathbb{R})$.

Proof. Let $\mathcal{C}_+$ be an affine invariant manifold of rank $\ell \geq 1$, and let $\mathcal{C} \subset \mathcal{C}_+$ be its subset of differentials of area one.

Choose a translation surface $(X, \omega) \in \mathcal{C}$ with the properties stated in Corollary 4.5. Choose a marking of the translation surface, that is, a lift $\tilde{\omega}$ of ω to the Teichmüller space of abelian differentials. By Corollary 4.5, there exists a smooth submanifold N of the component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ which is diffeomorphic to $\mathbb{R}^\ell$ and consists of differentials with the same horizontal measured foliation as $(X, \tilde{\omega})$. Thus a neighborhood of $(X, \tilde{\omega})$ of this submanifold of $\tilde{\mathcal{C}}$ is contained in the leaf $W^{ss}_{\mathrm{loc}}(\tilde{\omega})$ of the local strong stable manifold of $(X, \tilde{\omega})$.

The submanifold N contains the orbit of $\tilde{\omega}$ under a free abelian subgroup Λ of $\mathrm{Mod}(S)$ of rank ℓ consisting of marked Dehn multitwists contained in the group of piecewise affine automorphisms of (X, ω). Denote by $[c_i]$ the homology classes of the marked weighted oriented multicurves defining these multitwists $(i = 1, \dots, \ell)$ and let L be the Lagrangian subspace of $\tilde{\mathcal{Z}}^*$ spanned by these homology classes.

Let $\tilde{U}$ be a neighborhood of $\tilde{\omega}$. As by construction, we have $\langle [\mathrm{Re}(\tilde{\omega}), [c_i] \rangle > 0$ for all i and this is an open condition, by decreasing the size of $\tilde{U}$ we may assume that that this condition is fulfilled for all $\tilde{z} \in \tilde{U}$. Via further restricting the size of $\tilde{U}$ we may assume that $\tilde{U}$ projects to a contractible subset U of $\mathcal{C}$.

Now by part (4) of Lemma 4.3, the real parts $[\mathrm{Re}(\tilde{z})]$ of the points $\tilde{z} \in \tilde{U}$ span the dual L^* of L. Thus the corollary follows from Proposition 4.8. □

5. From global to local Zariski density

Consider as before an affine invariant manifold $\mathcal{C}$. Let $\tilde{\mathcal{C}}$ be a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials; then we have $\mathcal{C} = \Lambda\backslash\tilde{\mathcal{C}}$ where Λ is the stabilizer of $\tilde{\mathcal{C}}$ in the mapping class group of S.

Let $\rho : \Lambda \to G$ be a homomorphism where for the moment, G is an arbitrary simple algebraic group defined over $\mathbb{R}$. The image $\rho(\Lambda)$ is a subgroup of G whose Zariski closure is defined. A natural example is the representation arising from the restriction of the homomorphism $\rho : \mathrm{Mod}(S) \to \mathrm{Sp}(2g, \mathbb{Z})$ defined by the restriction of the Kontsevich Zorich cocycle. Another example is the action of Λ on the symplectic subspace of $H^1(S, \mathbb{R})$ which defines the absolute real tangent bundle $\mathcal{Z}_{\mathbb{R}}$ of $\mathcal{C}_+$.

The homomorphism $\rho : \Lambda \to G$ defines a flat principal bundle $\mathcal{P} \to \mathcal{C}$ with fiber G. This fiber bundle comes equipped with a natural flat connection and hence we can talk about the monodromy of the connection.

The goal of this section is to pass from global to local information regarding the Zariski closure of $\rho(\Lambda)$. Consider a good birecurrent point $q \in \mathcal{C}_{\rm good}$. For an open contractible neighborhood $U \subset \mathcal{C}_{\rm good}$ of q let $G(U)$ be the Zariski closure of the subgroup of G generated by those periodic orbits for Φ^t which pass through U (by identifying basepoints via parallel transport in U as before). By definition, if $V \subset U$ is an open and contractible set then $G(V) \subset G(U)$.

The Zariski closure of a subgroup of G is an algebraic subgroup of G. Now any descending sequence $G \supset G_1 \supset G_2 \supset \cdots$ of properly nested algebraic subgroups of G is finite, where properly nested means that G_{i+1} is a proper algebraic subgroup of G_i. This implies that there exists an open contractible neighborhood V of q such that for any open contractible neighborhood $W \subset V$ of q, we have $G(W) = G(V)$. We call $G(V)$ the *local monodromy group of* q and denote it by $G(q)$.

We aim at comparing these local monodromy groups for different basepoints in $\mathcal{C}_{\rm good}$. To explain the dependence on basepoints, note that given a subgroup H of a given group G, conjugation by elements of H preserves H and hence this conjugation action does not cause any ambiguity towards identifying the subgroup.

Lemma 5.1. *Any two local monodromy groups $G(q), G(z)$ $(q, z \in \mathcal{C}_{\rm good})$ coincide.*

Proof. Let $q, z \in \mathcal{C}_{\rm good}$ be good birecurrent points. By symmetry, it suffices to show that the group $G(z)$ is a subgroup of $G(q)$.

To this end let U_q be any open contractible subset of $\mathcal{C}_{\rm good}$ such that $G(U_q) = G(q)$ and choose similarly an open contractible subset U_z of $\mathcal{C}_{\rm good}$ so that $G(U_z) = G(z)$.

By Theorem 3.3, we can find neighborhoods $Y_z \subset V_z \subset U_z$ of z, $Y_q \subset V_q \subset U_q$ of q and a number $n > 0$ with the following properties. The sets V_z, V_q are contractible. Write $\mathcal{Y} = \{Y_q, Y_z\}$ and let u_0, u_1, u_2, u_3 be a periodic $(n, \mathcal{Y})$-pseudo-orbit for Φ^t, with $u_0 = u_3 \in Y_z$ and $u_1, u_2 \in Y_q$. There are numbers $t_i > n$ such that $\Phi^{t_i}u_i \in Y_{\kappa(i+1)}$ where $\kappa(i+1) = q$ for $i = 0, 1$ and $\kappa(i+1) = z$ otherwise. Such a pseudo-orbit exists since the Teichmüller flow on $\mathcal{C}$ is topologically transitive.

Let $\mathcal{V} = \{V_q, V_z\}$ and let η be a $\mathcal{V}$-characteristic curve for this pseudo-orbit. By Proposition 3.3, the characteristic curve η determines a parametrized periodic orbit ν for Φ^t beginning in V_z, and this orbit passes through V_q.

Choose a component $\tilde{\mathcal{C}}$ of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials and let $\tilde{V}_z \subset \tilde{\mathcal{C}}$ be a component of the preimage of V_z. Let $\tilde{u}_0$ be the preimage of u_0 in $\tilde{V}_z$. For this fixed choice, the parameterized periodic orbit ν determines a pseudo-Anosov element $\Omega(\nu) \in \mathrm{Mod}(S)$ as follows. Let $\tilde{\eta}$ be the lift of the characteristic curve η for the pseudo-orbit beginning at $\tilde{u}_0$. Then $\Omega(\nu)$ maps the endpoint of $\tilde{\eta}$ back to its starting point.

Let $\tilde{V}_q \subset \tilde{\mathcal{C}}$ be the component of the preimage of V_q which contains $\Phi^{t_0}\tilde{u}_0$. If η' is a characteristic curve of a pseudo-orbit defined by points $u_0, u_1', u_2, u_3 = u_0$, with $u_1' \in Y_q$, and times $t_0, t_1', t_2 > n$, and if ν' is the corresponding periodic orbit, then the element $\Omega(\nu)^{-1} \circ \Omega(\nu')$ (read from right to left) of $\mathrm{Mod}(S)$ maps the endpoint of the lift beginning in $\tilde{V}_z$ of the concatentation $\eta^{-1} \circ \eta'$ (read from right to left) back to its starting point $\tilde{u}_0$. Recall that this makes sense since η, η' begin and end at the same point $u_0 \in Y_z$.

Thus $\rho(\Omega(\nu)^{-1} \circ \Omega(\nu'))$ equals the holonomy for parallel transport with respect to the flat connection of the following loop. Fix the point $u_0 \in Y_z$ as a basepoint. The $(n, \mathcal{Y})$-pseudo-orbit given by the points u_0, u_1, u_2 and the times t_0, t_1 determine the homotopy class with fixed endpoints of an arc β connecting u_0 to u_2, and there is an arc β' for the $(n, \mathcal{Y})$-pseudo-orbit given by the points u_0, u_1', u_2 and the times t_0, t_1'. These arcs are constructed in such a way that they end at u_2. The holonomy of the concatenation of β' with the inverse of β equals the element $\rho(\Omega(\nu)^{-1} \circ \Omega(\nu'))$ (again read from right to left).

Choose an open contractible neighborhood W of the distinguished orbit segment connecting u_0 to u_1. Parallel transport in W identifies the fiber $\mathcal{P}_{u_0} \sim G$ of $\mathcal{P}$ at u_0 with the fiber $\mathcal{P}_{u_1} \sim G$ of $\mathcal{P}$ at u_1. This identification maps $\rho(\Omega(\nu)^{-1} \circ \Omega(\nu'))$ to $\rho(\Omega(\xi)^{-1} \circ \Omega(\xi'))$ where $\Omega(\xi')$ is the element of $\mathrm{Mod}(S)$ constructed from $\tilde{V}_q$ and from a parametrized periodic orbit of Φ^t through V_q determined by the one-segment periodic pseudo-orbit (u_1', t_1'), and where ξ is defined by the periodic pseudo-orbit u_1, u_2, u_0, u_1 based in Y_q. As $\Omega(\xi), \Omega(\xi^| prime) \in G(q)$, this implies that indeed, $G(q) \subset G(z)$ and hence $G(z) = G(q)$ by symmetry. $\square$

Definition 5.2. An affine invariant manifold $\mathcal{C}$ of rank ℓ is *locally Zariski dense* for $\rho : \Lambda \to G$ if for every birecurrent point $q \in \mathcal{C}_{\mathrm{good}}$ we have $G(q) = G$.

We are now ready to show

Theorem 5.3. *An affine invariant manifold $\mathcal{C}$ is locally Zariski dense for ρ if and only if $\rho(\Lambda) \subset G$ is Zariski dense.*

Proof. Clearly if $\mathcal{C}$ is locally Zariski dense for ρ then ρ is Zariski dense. Thus we have to show that Zariski density implies local Zariski density.

To see that this is the case assume otherwise. Let $q \in \mathcal{C}_{\rm good}$ be a birecurrent point; then there exists a periodic orbit $\gamma \subset \mathcal{C}_{\rm good}$ whose monodromy is not contained in $G(q)$. Choose a point $z \in \gamma$; then z is birecurrent and the monodromy $\rho(\gamma)$ of γ (based at z) is contained in the local monodromy group $G(z)$ but not in $G(q)$. However by Lemma 5.1, we have $G(z) = G(q)$ and hence this is a contradiction. $\square$

Corollary 4.9 shows that the monodromy group of the absolute real tangent bundle $\mathcal{Z}_\mathbb{R}$ of $\mathcal{C}_+$ is Zariski dense in $\mathrm{Sp}(\mathcal{Z}_\mathbb{R}, \mathbb{R})$ and hence we obtain

Corollary 5.4. *The monodromy of an affine invariant manifold is locally Zariski dense.*

6. Simplicity of the Lyapunov spectrum

The goal of this section is to use Theorem 5.3 and the results of [AV07a] to complete the proof of Theorem 1 from the introduction.

Let Q be a component of a stratum of area one abelian differentials, and Φ^t the Teichmüller flow on $\mathcal{Q}$. Let $\mathcal{C} \subset Q$ be an affine invariant manifold. There exists a Φ^t-invariant ergodic Borel probability measure μ on $\mathcal{Q}$ whose support equals $\mathcal{C}$ and which is in the Lebesgue measure class of $\mathcal{C}$. The measure μ is absolutely continuous with respect to the stable and unstable foliation of $\mathcal{C}$ and ergodic (see for example [AEM17]).

Let as before $\Lambda \subset \mathrm{Mod}(S)$ be the stabilizer of a component of the preimage of $\mathcal{C}$ in the Teichmüller space of abelian differentials. Let $\rho : \Lambda \to G$ be a representation where $G = \mathrm{Sp}(2m, \mathbb{R})$ or $G = \mathrm{SL}(n, \mathbb{R})$ and assume that $\rho(\Lambda)$ is Zariski dense in G. Denote by $\mathcal{P} \to \mathcal{C}$ the flat G-principal bundle defined by ρ and let $\mathcal{V} \to \mathcal{C}$ be the induced flat vector bundle. The natural flat connection on $\mathcal{V}$ determines an extension Ψ^t of the Teichmüller flow on $\mathcal{C}$ by parallel transport.

Choose a continuous norm $|\,|$ on $\mathcal{V}$ compatible with the symplectic structure if $G = \mathrm{Sp}(2m, \mathbb{R})$. Assume that the function $q \to \log \|\Psi^1(q)\|$ is integrable with respect to μ, where $\|\,\|$ denotes the operator norm with respect to the norm $|\,|$.

Example 6.1. Using the notations from Section 4, consider as before the absolute real tangent bundle $\mathcal{Z}_\mathbb{R}$ of $\mathcal{C}_+$ which can be thought of as the projection of the tangent bundle of $\mathcal{C}$ to absolute periods. It is a flat invariant subbundle of the restriction of the Hodge bundle $\mathcal{H}$ to $\mathcal{C}$ and hence it determines an extension Ψ^t of the Teichmüller flow on $\mathcal{C}$. By Corollary 4.9, the monodromy of this bundle is Zariski dense in the corresponding symplectic group $\mathrm{Sp}(2\ell, \mathbb{R})$.

The bundle $\mathcal{Z}_\mathbb{R}$ can be equipped with the *Hodge norm* $|\,|$ which is defined as follows. A point $v \in \mathcal{Z}_\mathbb{R}$ is a real first cohomology class on the surface S, that is, $v \in H^1(S, \mathbb{R})$. Equipped with the cup product, $\mathcal{Z}_\mathbb{R}$ is a symplectic vector space. The basepoint $P(v) = q \in \mathcal{C}$ of v determines a complex structure $\Pi(q)$ on S. There exists a unique holomorphic one-form ω on S for the complex structure $\Pi(q)$ whose real part $\Re(\omega)$, as a closed one-form, defines the cohomology class v. Put $|v| = \int_S \omega$, the area of the flat metric determined by ω. We refer to [ABEM12] for a detailed account on this norm.

Now using period coordinates and the fact that the Teichmüller flow Φ^t is the restriction to the diagonal group of the natural $\mathrm{SL}(2,\mathbb{R})$ action on $\mathcal{C}$, it follows easily (and is well known) that the operator norm of Ψ^t with respect to the Hodge norm $|\,|$ on $\mathcal{Z}_{\mathbb{R}}$ satisfies

$$e^{-t} \leq \|\Psi^t(q)\| \leq e^t \text{ for all } t, q.$$

As a consequence, the function $q \to \log \|\Psi^1(q)\|$ is integrable with respect to the probability measure μ.

Let as before $U \subset \mathcal{C}_{\text{good}}$ be an open contractible subset. Since the bundle $\mathcal{V} \to \mathcal{C}$ is flat, for $x, y \in U$ the *holonomy map*

$$H_{x,y} : \mathcal{V}_x \to \mathcal{V}_y$$

defined by parallel transport along a path in the open contractible set U only depends on the choice of U. Since $|\,|$ was chosen to be a continuous norm on $\mathcal{V}$, in a flat trivialization of $\mathcal{V}$ over U, the norm of a vector $X \in \mathcal{V}_q$ depends continuously on $q \in U$. Thus we have

Lemma 6.2. *Every $q \in \mathcal{C}_{\text{good}}$ admits an open contractible neighborhood U such that for $x, y \in U$, the holonomy map $H_{x,y} : \mathcal{V}_x \to \mathcal{V}_y$ is 2-bi-Lipschitz with respect to the norm $|\,|$.*

By our assumption on $|\,|$ and ergodicity of μ, Oseledec's multiplicative theorem can be applied. It yields a Ψ^t-invariant measurable filtration $V^1 \subset V^2 \subset \cdots \subset V^k = \mathcal{V}$ of μ-measurable subbundles V^i of $\mathcal{V}$ corresponding to the *Lyapunov exponents* $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. More specifically, for μ-almost every $q \in \mathcal{C}$ and any $v \in V_q^i \setminus V_q^{i-1}$ we have

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log |\Psi^t v|.$$

The Lyapunov spectrum is called *simple* if the dimension of V^i equals i for all i.

In [AV07a], Avila and Viana established a criterion which guarantees simplicity of the Lyapunov spectrum for a certain class of cocycles over a countable Markov shift with respect to an invariant probability measure which is absolutely continuous with respect to the stable and unstable foliation and satisfies some mild extra assumptions. Our goal is to reduce Theorem 1 to [AV07a].

To reduce the study of the extension Ψ^t of the Teichmüller flow to the study of a cocycle over a countable Markov shift we use the following result from [H11] for the Teichmüller flow on a component $\mathcal{Q}$ of abelian or quadratic differentials.

Theorem 6.3 (Theorem 1 of [H11]). *There exists*

- *a topologically transitive subshift of finite type* (Ω, σ)
- *a σ-invariant dense Borel set $\mathcal{U} \subset \Omega$ containing all normal sequences*
- *a suspension (X, Θ^t) of σ over $\mathcal{U}$, given by a positive bounded continuous roof function on $\mathcal{U}$*

and a finite-to-one semi-conjugacy $\Xi : (X, \Theta^t) \to (\mathcal{Q}, \Phi^t)$ *which maps the space of* σ*-invariant Borel probability measures on* $\mathcal{U}$ *continuously onto the space of* Φ^t*-invariant probability measures on* $\mathcal{Q}$*.*

Let as before $\mathcal{Q}$ be the component of a stratum of abelian differentials containing the affine invariant manifold $\mathcal{C}$. By Theorem 6.3, there exists a σ-invariant Borel probability measure $\hat{\mu}$ on $\mathcal{U} \subset \Omega$ which is mapped by the semi-conjugacy Ξ to μ in the following sense. Let $\rho : \mathcal{U} \to (0, R]$ be the roof function which defines the suspension (X, Θ^t). Recall that this roof function is bounded. Define the measure $\tilde{\mu}$ on the suspension X by

$$d\tilde{\mu} = d\hat{\mu} \times dt.$$

If $B \subset \Omega$ is any Borel set, then we have

$$\tilde{\mu}\{(x,t) \mid x \in B, 0 \leq t < \rho(x)\} = \int_B \rho \, d\hat{\mu}.$$

As the roof function ρ is bounded, this construction defines a finite Θ^t-invariant Borel measure $\tilde{\mu}$ on (X, Θ^t) (it may not be a probability measure) which is mapped by Ξ to a positive multiple of μ. Since μ is ergodic under the action of Φ^t, the measure $\hat{\mu}$ is ergodic under the action of σ.

As Ξ is a semi-conjugacy, it maps stable and unstable manifolds for Θ^t to stable and unstable manifolds for Φ^t. Since μ is absolutely continuous with respect to the stable and unstable foliation of Φ^t, the measure $\hat{\mu}$ is absolutely continuous with respect to the stable and unstable foliation of the shift σ. However, if $\mathcal{C}$ is a proper affine invariant submanifold of $\mathcal{Q}$ then $\hat{\mu}$ is not of full support.

Via the map Ξ, the bundle $\mathcal{V}$ pulls back to a measurable flat bundle $\tilde{\mathcal{V}}$ over the intersection of the support of $\tilde{\mu}$ with the suspension of the set $\mathcal{U} \subset \Omega$. By equivariance of the semi-conjugacy Ξ with respect to the flows Θ^t and Φ^t, the pull-back of the flat connection induces an extension $\tilde{\Psi}^t$ of the flow Θ^t to a flow on $\tilde{\mathcal{V}}$ (note that this just means equivariance of $\tilde{\Psi}^t$ with respect to pull-back). The extension $\tilde{\Psi}^t$ is defined for $\tilde{\mu}$-almost every $x \in X$ and all $t \in \mathbb{R}$. Since $\Xi_* \tilde{\mu} = a\mu$ for a number $a > 0$, the Lyapunov spectrum for (Ψ^t, μ) is simple if and only if the Lyapunov spectrum for the pull-back flow $(\tilde{\Psi}^t, \tilde{\mu})$ is simple.

The measurable bundle $\tilde{\mathcal{V}}$ over X restricts to a measurable bundle $\Pi : \hat{\mathcal{V}} \to \Omega$. The flow $\tilde{\Psi}^t$ induces a transition map $\hat{F} : \hat{\mathcal{V}} \to \hat{\mathcal{V}}$ by

$$\hat{F}(x) = \tilde{\Psi}^{\rho(x)}(x).$$

Clearly $\hat{F}$ is a cocycle for σ, that is, we have

$$\hat{F}(\sigma(x)) \circ \hat{F}(x) = \tilde{\Psi}^{\rho(\sigma(x))}(\sigma(x)) \circ \tilde{\Psi}^{\rho(x)}(x)$$

(here a matrix acts on a vector by left multiplication and hence multiplication is read from right to left).

Although the bundle $\hat{\mathcal{V}}$ is only measurable and not defined outside of the support of $\hat{\mu}$, the pull-back of the norm on $\mathcal{V}$ induces a norm $|\,|$ on $\hat{\mathcal{V}}$. Furthermore, as the roof function for the suspension (X, Θ^t) is bounded, from the assumptions on $|\,|$ we deduce that all requirements for an application of Oseledec's theorem are fulfilled for $(\hat{F}, \hat{\mu})$. We have

Lemma 6.4. *The Lyapunov spectrum of* (Ψ^t, μ) *is simple if and only if the Lyapunov spectrum of* $(\hat{F}, \hat{\mu})$ *is simple.*

Proof. We only show that if the Lyapunov spectrum of $(\hat{F}, \hat{\mu})$ is simple, then the same holds true for the Lyapunov spectrum of $(\tilde{\Psi}^t, \tilde{\mu})$. Simplicity of the Lyapunov spectrum for (Ψ^t, μ) then follows by the above discussion. This is the only implication we are interested in.

Let $R > 0$ be an upper bound for the roof function ρ. Assume that the Lyapunov spectrum for $(\hat{F}, \hat{\mu})$ is simple. Let $\lambda_1 < \cdots < \lambda_k$ be the Lyapunov exponents and assume that $\min_i\{\lambda_i - \lambda_{i-1}\} = 4\epsilon > 0$. Let $x \in \Omega$ be a density point for $\hat{\mu}$ so that the Lyapunov exponents can be computed using the point x via iteration of both $\hat{F}$ and $\hat{F}^{-1}$. By ergodicity of $\hat{\mu}$ and the Birkhoff ergodic theorem, the set of all such points has full $\hat{\mu}$-measure. Applying the Oseledec multiplicative ergodic theorem to both $\hat{F}$ and $\hat{F}^{-1}$, we obtain the following.

Write $\hat{F}^N(x) = \hat{F}(\sigma^{n-1}(x)) \circ \cdots \circ \hat{F}(x)$. Then there exists a decomposition $\hat{\mathcal{V}}_x \cong \mathbb{R}^k = E_1 \oplus \cdots \oplus E_k$ and a number $N_0 > 0$ such that with respect to the norm $|\,|$, for any $N > N_0$ and any $X \in E_i$ with $|X| = 1$ we have

$$|\hat{F}^N(x)X| \in [e^{N(\lambda_i - \epsilon)}, e^{N(\lambda_i + \epsilon)}].$$

For $N > N_0$ let $T(N) > 0$ be such that $\Theta^{T(N)}x = \sigma^N x$. We know that

$$T(N)(x) = \sum_{j=0}^{N-1} \rho(\sigma^j(x)) \le RN.$$

Furthermore, as $\hat{\Psi}^{T(N)}(x) = \hat{F}^N(x)$, for $X \in E_i$ we have

$$|\hat{\Psi}^{T(N)}X| \in [e^{T(N)((N/T(N)(\lambda_i - \epsilon))}, e^{T(N)((N/T(N)(\lambda_i + \epsilon))}]. \tag{3}$$

In particular, by the choice of ϵ, the logarithms of the dilatations of the map $\hat{\Psi}^{T(N)}$ on the subspaces E_i differ by at least the factor $2\epsilon N/T(N)$, independent of $N > N_0$.

As μ is ergodic with respect to σ, by the Birkhoff ergodic theorem (and perhaps adjusting the basepoint x), as $N \to \infty$ we have $T(N)/N \to \kappa$ for $\kappa = \int \rho \, d\hat{\mu} > 0$ and hence $\frac{N}{T(N)} \to \frac{1}{\kappa}$. Inserting into (3) and taking into account the choice of ϵ yields simplicity of the Lyapunov spectrum for $(\tilde{\Psi}^t, \tilde{\mu})$. □

We next inspect the cocycle $(\hat{F}, \hat{\mu})$ over the subshift (Ω, σ), equipped with the measure $\hat{\mu}$. As for proper affine invariant manifolds the measure $\hat{\mu}$ is not of full support and, furthermore, the cocycle is only measurable, this is not possible directly. Instead we use the geometric information to construct from the affine invariant manifold $\mathcal{C}$ and the subshift (Ω, σ) a Markov shift on countably many symbols to which the work [AV07a] can be applied.

The construction of the shift requires some preparation as follows. Recall that a *cylinder set* for the subshift of finite type (Ω, σ) is a set of the form

$$[\alpha_{-i}, \ldots, \alpha_{-1}; \alpha_0; \alpha_1, \ldots, \alpha_k] = \{x \mid x_j = \alpha_j \text{ for } j = -i, \ldots, k\}.$$

Here the α_i are letters from the finite alphabet $\mathcal{A}$ defining Ω, and for any consecutive letters α_i, α_{i+1} of the word defining the cylinder, we have $A_{\alpha_i,\alpha_{i+1}} = 1$ where (A_{α_i,α_j}) is the transition matrix which determines the subshift of finite type. We call a cylinder $[\alpha_0]$ given by a single letter in position zero a *basic cylinder*. In the sequel, all cylinders will be defined a word $\alpha_{-i} \cdots \alpha_0 \cdots \alpha_k$ in the alphabet $\mathcal{A}$ with $i, k \geq 0$.

For every cylinder Y and every $x = (x_i) \in Y$ we define the *local stable manifold*

$$W^s_{\mathrm{loc},Y}(x) = \{(z_i) \in Y \mid z_i = x_i \text{ for } i \geq 0\}$$

and the *local unstable manifold* by

$$W^u_{\mathrm{loc},Y}(x) = \{(z_i) \in Y \mid z_i = x_i \text{ for } i \leq 0\}.$$

For every basic cylinder $[\alpha_0] \subset \Omega$, there exists a measurable roof function $\rho_{[\alpha_0]} : [\alpha_0] \cap \mathcal{U} \to (0, R]$ where $R > 0$ is a fixed constant. These roof functions determine the suspension (X, Θ^t). We call the set

$$\{(x,t) \in ([\alpha_0] \cap \mathcal{U}) \times [0,R] \mid 0 \leq t < \rho_{[\alpha_0]}(t)\}$$

a *box* for the suspension. A box is saturated with respect to its vertical foliation, and the finitely many boxes define a partition of X. Note however that as Ξ is not injective in general, the boxes do not define a partition of $\Xi(X)$.

From now on we only work with the subshift (Ω, σ) and we view the map Ξ as a map defined on $\mathcal{U} \subset \Omega$ (by abuse of notation). Let us suppose that $x \in \mathcal{U} \subset \Omega$ is the preimage under the map Ξ of a point in $\mathcal{C}_{\mathrm{good}}$, equivalently that $\Xi(x) = q \in \mathcal{C}_{\mathrm{good}}$. As $\mathcal{C}$ is a locally closed subset of $\mathcal{Q}$, there exists a neighborhood $U \subset \mathcal{Q}_{\mathrm{good}}$ of q in the stratum $\mathcal{Q}$ containing $\mathcal{C}$ such that $U \cap \mathcal{C}$ is connected and homeomorphic to a ball satisfying the properties in Lemma 6.2. Furthermore, we may assume that $U \cap \mathcal{C}$ has a local product structure. Choose an open contractible neighborhood $V \subset U \subset \mathcal{Q}_{\mathrm{good}}$ with the same properties as U and compact closure in U.

Since the map $\Xi : \mathcal{U} \to \mathcal{Q}$ is continuous and cylinders in Ω define a basis of the topology, there exists a cylinder $Y = [\alpha_{-i}, \dots; \alpha_0; \dots, \alpha_k]$ containing x with the property that $\Xi(Y) \subset V$. Since a local stable manifold for the Teichmüller flow on $\mathcal{C}$ is a submanifold of a local stable manifold for the Teichmüller flow on $\mathcal{Q}$, by making Y smaller if necessary we also may assume that for every $y \in Y$, the closure of $\Xi(W^u_{\mathrm{loc},Y}(y) \cap \mathcal{U})$ in $\mathcal{Q}$ intersects $\mathcal{C}$ in a subset of a contractible set of a stable manifold in $\mathcal{C}$ and similarly for $\Xi(W^s_{\mathrm{loc},Y}(y) \cap \mathcal{U})$. If these conditions are satisfied, then we call Y a *$\mathcal{C}$-good cylinder*. The above discussion yields that every $x \in \Xi^{-1}(\mathcal{C}_{\mathrm{good}})$ is contained in some $\mathcal{C}$-good cylinder.

Of course $\mathcal{C}$-good cylinders containing a given point $x \in \Xi^{-1}(\mathcal{C}_{\mathrm{good}})$ are by no means unique. Our next goal is to circumvent this problem by constructing a partition of $\Xi^{-1}(\mathcal{C}_{\mathrm{good}})$ by countably many $\mathcal{C}$-good cylinders. This is carried out in the following

Lemma 6.5. *There exists a countable collection $\mathcal{P} = \{P_i \mid i\}$ of pairwise disjoint $\mathcal{C}$-good cylinders whose intersections with $\Xi^{-1}(\mathcal{C}_{\mathrm{good}})$ define a measurable partition of $\Xi^{-1}(\mathcal{C}_{\mathrm{good}}) \subset \mathcal{U}$.*

Proof. To construct the partition we review from [H11] the basic properties of the map Ξ.

An element α_i of the finite alphabet $\mathcal{A}$ of Ω corresponds to a combinatorial type of a *large marked numbered train track* τ on the surface S. The train track τ is contained in the collection $\mathcal{LT}(\mathcal{Q})$ associated to the component $\mathcal{Q}$ and is equipped with a numbering of its branches. We refer to [H24] and the appendix for a summary of the basic properties of these train tracks.

Denote by $\mathcal{ML}$ the space of measured geodesic laminations on S, equipped with the weak*-topology. This space is naturally homeomorphic to $\mathbb{R}^{6g-6} \setminus \{0\}$. The closed subset $\mathcal{V}_0(\tau) \subset \mathcal{ML}$ of all measured geodesic laminations which are *carried* by τ and which deposit the total mass one on τ is homeomorphic to a closed cell of dimension $\dim_{\mathbb{C}}(\mathcal{Q}) - 1$ and hence contractible. Let moreover $\mathcal{V}^*(\tau)$ be the contractible cone of all measured geodesic laminations which are carried by the *dual bigon track* of τ. Equivalently, this is the cone of all measured geodesic laminations which *hit* τ *efficiently*. The appendix contains more information.

There exists a continuous function
$$\iota : \mathcal{ML} \times \mathcal{ML} \to [0, \infty),$$
the so-called *intersection form*, which is homogeneous under scaling a measured lamination with a positive real. It restricts to a function $\iota : \mathcal{V}_0(\tau) \times \mathcal{V}^*(\tau) \to [0, \infty)$. Any pair $(\mu, \nu) \in \mathcal{V}_0(\tau) \times \mathcal{V}^*(\tau)$ with $\iota(\mu, \nu) \neq 0$ determines a pair $(\mu, \hat{\nu})$ with $\iota(\mu, \hat{\nu}) = 1$ by rescaling of ν. For fixed μ, the map $(\mu, \nu) \to (\mu, \hat{\nu})$ is invariant under scaling the lamination ν with a positive real. Thus if we denote by P projectivization, then the map $(\mu, \nu) \to (\mu, \hat{\nu})$ defines an embedding of the open dense (see [H24]) subset
$$D(\tau) = \{(\mu, [\nu]) \in \mathcal{V}_0(\tau) \times P\mathcal{V}^*(\tau) \mid \iota(\mu, [\nu]) \neq 0\}$$
of $\mathcal{V}_0(\tau) \times P\mathcal{V}^*(\tau)$ into $\mathcal{V}_0(\tau) \times \mathcal{V}^*(\tau)$. In the sequel we freely identify the set $D(\tau)$ with its image under the natural embedding into $\mathcal{V}_0(\tau) \times \mathcal{V}^*(\tau)$.

It was shown in [H24] that if a pair $(\xi, \eta) \in D(\tau)$ *jointly fills up* S, that is, if for any measured lamination ν we have $\iota(\xi, \nu) + \iota(\eta, \nu) > 0$, then this pair defines an abelian differential in the closure of the component $\mathcal{Q}$. Recalling that τ corresponds to the letter $\alpha_i \in \mathcal{A}$, the set of pairs $(\xi, \eta) \in D(\tau)$ which define a point in $\mathcal{Q}$ is an open subset $U([\alpha_i])$ of $D(\tau) \subset \mathcal{V}_0(\tau) \times P\mathcal{V}^*(\tau)$. This set is an immersed suborbifold of $\mathcal{Q}$ of codimension one, and it is a transversal for the Teichmüller flow on $\mathcal{Q}$. A component of its preimage in the Teichmüller space of marked differentials is contractible. We refer to [H24] for details.

The image under the map Ξ of the basic cylinder $[\alpha_i] \subset \Omega$ corresponding to τ equals the subset of $U([\alpha_i])$ of *uniquely ergodic* pairs, which means that both measured geodesic laminations in the pair are uniquely ergodic. In other words, if we identify $[\alpha_i] \cap \mathcal{U}$ with its image under Ξ, then we can view $[\alpha_i] \cap \mathcal{U}$ as a measurable dense subset of the transversal $U([\alpha_i])$.

To summarize, for each of the finitely many elements α_i of the alphabet $\mathcal{A}$, there exists an immersion $U([\alpha_i]) \to \mathcal{Q}$ extending Ξ in the sense described in the previous paragraph. It was shown in [H11] that the finite union of the images of the sets

$U([\alpha_i])$ where α_i runs through the alphabet $\mathcal{A}$ define a global transversal Σ for the Teichmüller flow on $\mathcal{Q}$ in the following sense:

(1) Σ is an immersed suborbifold of $\mathcal{Q}$ of codimension one.
(2) There exists a number $R > 0$, and for any $q \in \mathcal{Q}$ there exists a number $t \in [0, R)$ such that $\Phi^t q \in \Sigma$.
(3) If $q \in \Sigma$, then there exists a number $t(q) > 0$ so that $\Phi^s(q) \notin \Sigma$ for all $s \in (0, t(q))$. The number $t(q)$ depends continuously on q.

Note that the number $t(q) > 0$ appearing in the third statement above is not bounded from below by a positive constant. The transversal Σ contains the image $\Xi(\mathcal{U})$ of $\mathcal{U}$ as a measurable dense subset which has full measure with respect to the disintegration of every Φ^t-invariant probability measure on $\mathcal{Q}$. We refer to [H11] for more details on this fact.

Since $\mathcal{C}$ is a closed connected Φ^t-invariant suborbifold of $\mathcal{Q}$ and hence its topology as a subspace of $\mathcal{Q}$ has a countable basis, the intersection of $\mathcal{C}$ with the transversal Σ has at most countably many connected components. Each such component is a closed subset of $\mathcal{C}$. We use this information to construct inductively a countable collection $\mathcal{P}$ of pairwise disjoint $\mathcal{C}$-good cylinders in Ω whose intersections with $\Xi^{-1}(\mathcal{C}_{\rm good})$ define a partition of $\Xi^{-1}(\mathcal{C}_{\rm good}) \subset \mathcal{U}$ as follows.

Consider the collection $\hat{\mathcal{P}}_0$ of all basic cylinders $[\alpha_i] \subset \Omega$ ($\alpha_i \in \mathcal{A}$) with the property that $U([\alpha_i])$ intersects $\mathcal{C}_{\rm good}$, equivalently that $[\alpha_i] \cap \Xi^{-1}(\mathcal{C}_{\rm good}) \neq \emptyset$. Note that $(\mathcal{C}_{\rm good} \cap \Sigma) \subset \cup\{U([\alpha_i]) \mid [\alpha_i] \in \hat{\mathcal{P}}_0\}$. If $[\alpha_i] \in \hat{\mathcal{P}}_0$ is a $\mathcal{C}$-good cylinder then we require that $[\alpha_i] \in \mathcal{P}$. Let $\mathcal{P}_0 \subset \mathcal{P}$ be the collection of all $\mathcal{C}$-good basic cylinders obtained in this way. This is a finite set.

In a second step, consider a basic cylinder $[\alpha_i] \in \hat{\mathcal{P}}_0 - \mathcal{P}_0$. This cylinder can be decomposed as a disjoint finite union of length three cylinders of the form $[\alpha_{-j}; \alpha_i; \alpha_\ell]$. Let $\hat{\mathcal{P}}_1$ be the union over all $[\alpha_i] \in \hat{\mathcal{P}}_0 - \mathcal{P}_0$ of all those subdivision cylinders which intersect $\Xi^{-1}(\mathcal{C}_{\rm good})$. As before, this is a finite set. Furthermore, it is clear from the construction that the cylinders in $\hat{\mathcal{P}}_1 \cup \mathcal{P}_0$ cover $\Xi^{-1}(\mathcal{C}_{\rm good})$. Denote by $\mathcal{P}_1$ the union of those among the cylinders from $\hat{\mathcal{P}}_1$ which are $\mathcal{C}$-good. This is a finite set. Proceed inductively by subdividing cylinders in $\hat{\mathcal{P}}_1 - \mathcal{P}_1$ into cylinders of length five etc.

Let $\mathcal{P}$ be the union of all cylinders obtained inductively in this fashion. Each cylinder in $\mathcal{P}$ is $\mathcal{C}$-good by construction. Since every point $x \in \Xi^{-1}(\mathcal{C}_{\rm good})$ is contained in some $\mathcal{C}$-good cylinder and since a subcylinder of a $\mathcal{C}$-good cylinder which intersects $\Xi^{-1}(\mathcal{C}_{\rm good})$ is $\mathcal{C}$-good by definition, the cylinders from the collection $\mathcal{P}$ define a partition of $\Xi^{-1}(\mathcal{C}_{\rm good})$. Furthermore, the number of cylinders is countable as they can be sorted by their (finite) cylinder length and the number of cylinders of uniformly bounded length is finite. This completes the proof of the lemma. □

By Lemma 6.5, a point $x \in \Xi^{-1}(\mathcal{C}_{\rm good})$ is contained in a unique $\mathcal{C}$-good cylinder P from the countable cylinder family $\mathcal{P}$. By the definition of a $\mathcal{C}$-good cylinder, for $P \in \mathcal{P}$, the set $\Delta(P) = P \cap \mathcal{U} \cap \Xi^{-1}(\mathcal{C}_{\rm good})$ is mapped by Ξ into a contractible subset of $\mathcal{C}_{\rm good}$ with the properties stated in Lemma 6.2. Furthermore, if $P \in \mathcal{P}$

and if $x \in \Delta(P)$, then there exists some $t(x) \in (0, R]$ such that $\Theta^{t(x)}(x) \in P'$ for some $P' \in \mathcal{P}$, but $\Theta^t x \notin \mathcal{P}$ for any $t \in (0, t(x))$. Equivalently, we have $\sigma(x) \in P'$. That this statement indeed holds true for all $x \in \Xi^{-1}(\mathcal{C}_{\rm good})$ rather than for points of a subset of full $\hat{\mu}$-mass is a direct consequence of the construction of the cylinder family $\mathcal{P}$ as detailed in the proof of Lemma 6.5.

Our goal is to use the cylinders $P \in \mathcal{P}$ to construct a Markov shift over the countable alphabet $\mathcal{P}$ to which the results of [AV07a] can be applied. This is done by constructing a transition matrix in the following way.

To simplify the notation, put $\Delta = \Xi^{-1}(\mathcal{C}_{\rm good})$; then $\Delta(P) = \Delta \cap P$ for all $p \in \mathcal{P}$. If $P, P' \in \mathcal{P}$ and if there exists some $y \in \Delta(P)$ with $\sigma(y) \in P'$, then we define $A_{P,P'} = 1$, and define $A_{P,P'} = 0$ otherwise. Denote by (Υ, τ) the Markov shift on the alphabet $\mathcal{P}$ defined by this transition matrix.

It follows from the construction that there exists a natural measurable embedding $\Upsilon \to \Omega$ which is equivariant with respect to the shifts τ and σ and whose image has full $\hat{\mu}$-mass. Thus the measure $\hat{\mu}$ defines a τ-invariant measure on Υ, again denoted by $\hat{\mu}$. Moreover, the extension $\hat{F}$ of the shift σ determines an extension of the shift τ, denoted again by $\hat{F}$, whose Lyapunov spectrum with respect to the measure $\hat{\mu}$ on Υ is simple if and only if the same holds true for the Lyapunov spectrum of Ψ^t with respect to μ.

We next translate the extension $\hat{F}$ of the shift τ on Υ into an iteration property of (symplectic) matrices. Namely, any measurable map $A : \Upsilon \to G$ defines a cocycle F_A over the shift τ by

$$F_A : \Upsilon \times \mathbb{R}^k \to \Upsilon \times \mathbb{R}^k, \quad F_A(x, v) = (\tau(x), A(x)v).$$

Note that this definition is just a direct translation of the cocycle into the pull-back of the flat principal bundle over $\mathcal{C}$ with fiber G.

Our next goal is to verify that there is a measurable map $L : \Upsilon \to G$ with controlled properties which induces the cocycle $\hat{F}$ over τ in this way.

Lemma 6.6. *There exists a measurable map $L : \Upsilon \to G$ with the following properties.*

1. *For any $P, P' \in \mathcal{P}$ the restriction of L to $\Delta(P) \cap \tau^{-1}(P')$ is constant.*
2. *The map L determines the cocycle $\hat{F}$ over τ.*

Proof. By the definition of a $\mathcal{C}$-good cylinder, for any $P \in \mathcal{P}$, the image under Ξ of $\Delta(P)$ is contained in an open contractible subset $U(P)$ of $\mathcal{C}_{\rm good}$ with the properties stated in Lemma 6.2.

Denote as before by $\mathcal{V}$ the flat vector bundle over $\mathcal{C}$ defined by ρ. There exists a flat trivialization of $\mathcal{V}|U(P)$ which consists of a local basis (symplectic if G is the symplectic group) whose basis elements have norm contained in the interval $[1/2, 2]$ (recall that the norm $|\,|$ is not flat). We choose a fixed such flat trivialization of $\mathcal{V}$ for each $P \in \mathcal{P}$ and call it *preferred*. In the sequel we shall also talk about the preferred trivialization of $\hat{\mathcal{V}}$ over P by abuse of notation, identifying $\hat{\mathcal{V}}|P$ with its image under Ξ.

Now let $P, P' \in \mathcal{P}$ be such that $A_{P,P'} = 1$. Then it follows from the construction that we have $P \cap \sigma^{-1}(P') \neq \emptyset$. More precisely, if we denote by $W \subset U(P)$ the closure of the intersection $\Xi(\mathcal{U} \cap P \cap \tau^{-1}(P')) \cap \mathcal{C}$, then there exists a continuous map $\tau : W \to (0, R]$ such that for all $x \in W$ we have $\Phi^{t(x)}(x) \in U(P')$. As a consequence, since the preferred trivializations of $\mathcal{V}$ on $U(P)$ and $U(P')$ are flat, the matrix $L(x) \in G$ which determines the map $x \to \Psi^{t(x)}(x)$ with respect to the preferred trivializations of $\mathcal{V}$ on $U(P)$ and $U(P')$ does not depend on $x \in W$. Note that this statement uses in a crucial way the fact that the transversal Σ for the Teichmüller flow on $\mathcal{Q}$ is an immersed codimension one suborbifold of $\mathcal{Q}$ which intersects the affine invariant manifold $\mathcal{C}$ in an immersed codimension one suborbifold, so that it makes sense to talk about continuity.

That the map L also has the second property stated in the lemma follows from its construction: Namely, the map L encodes the cocycle $\hat{F}$ with respect to preferred trivializations on the $\mathcal{C}$-good cylinders in the collection $\mathcal{P}$. □

To summarize, to complete the proof of Theorem 1 it suffices to show that the cocycle defined by the map L has simple Lyapunov spectrum with respect to the measure $\hat{\mu}$ on Υ. This is accomplished by a direct application of the main result of [AV07a]. We have to verify that the map L and the measure $\hat{\mu}$ fulfill the conditions stated in [AV07a].

From now on we shall work exclusively with the shift (Υ, τ) on the countable alphabet $\mathcal{P}$. For such a shift, cylinders $[\alpha_i, \dots; \alpha_0; \dots, \alpha_k]$ are defined as for the subshift of finite type Ω, and similarly we define cylinders of $\Upsilon^u = \mathcal{P}^{n \geq 0}$ and $\Upsilon^s = \mathcal{P}^{n \leq 0}$, corresponding to the case $i = 0$ and $k = 0$, respectively. The phase spaces Υ and Υ^u, Υ^s are endowed with the topologies generated by their cylinders. Call cylinders in $\Upsilon, \Upsilon^u, \Upsilon^s$ of the form $[\alpha_0]$ defined by a single letter of the alphabet $\mathcal{P}$ in position zero (for Υ, Υ^u) or -1 (for Υ^s) *basic*.

Let $P^u : \Upsilon \to \Upsilon^u$ and $P^s : \Upsilon \to \Upsilon^s$ be the natural projections. There are one-sided shifts $\tau^u : \Upsilon^u \to \Upsilon^u$ and $\tau^s : \Upsilon^s \to \Upsilon^s$ defined by

$$\tau^u \circ P^u = P^u \circ \tau \quad \text{ and } \tau^s \circ P^s = P^s \circ \tau^{-1}.$$

For $x = (x_n) \in \Upsilon$ put $x^u = P^u(x)$ and $x^s = P^s(x)$. Then $x \to (x^s, x^u)$ is an embedding of Υ into $\Upsilon^s \times \Upsilon^u$. The image consists of all pairs of one-sided infinite strings which are contained in Υ^u, Υ^s, that is, which fulfill the compatibility condition given by the matrix $A_{P,P'}$ which defines Υ, and whose starting letters (at position zero) coincide. Note there is a small deviation of our conventions from the ones chosen in [AV07a].

For each $(x_i) \in \Upsilon$, one can identify the *local stable set*

$$W^s_{\text{loc}}(x^u) = W^s_{\text{loc}}(x) = \{(y_n)_n \mid x_n = y_n \text{ for all } n \geq 0\}$$

with a basic cylinder in Υ^s, and the local unstable set

$$W^u_{\text{loc}}(x^s) = W^u_{\text{loc}}(x) = \{(y_n)_n \mid x_n = y_n \text{ for all } n \leq 0\}$$

with a basic cylinder in Υ^u via the projections P^s and P^u.

For $x \in \Upsilon$, $n \geq 1$ put

$$L^n(x) = L(\tau^{n-1}(x)) \circ \cdots \circ L(x)$$

(read from right to left as a product of matrices). Following [AV07a], we say that L *admits stable holonomies* if the limit

$$H_{x,y} = \lim_{n \to -\infty} L^n(y)^{-1} L^n(x)$$

exists for any pair of points x and y in the same local stable set, and depends continuously on x, y. *Unstable holonomies* are defined in the same way, with $n \to \infty$ in the same local unstable set.

Lemma 6.7. *L admits stable holonomies which are constant on local stable and unstable manifolds.*

Proof. To show that stable holonomies exist and are constant it suffices to show the following. Let $P \in \mathcal{P}$, let $x \in \Delta(P)$ and let $y \in W^s_{\mathrm{loc}}(x)$; then $L(x) = L(y)$. Namely, if this is indeed the case then constant stable holonomies follows by using this identity inductively over the forward orbit of x under τ.

However, by construction, if $x \in \Delta \cap P$ and if $y \in W^s_{\mathrm{loc}}(x)$ then we have $\tau(x), \tau(y) \in P'$ for the same cylinder P', specified by the letter in the string defining x at position 1. That $L(x) = L(y)$ now follows from Lemma 6.6.

The argument for the existence of local unstable holonomies is identical and will be omitted. $\square$

The following definition is due to Avila and Viana [AV07a]. For its formulation, let x be any periodic point for τ. A point $z \in \Upsilon$ is called a *homoclinic point* of x if $z \in W^u_{\mathrm{loc}}(x)$ and if there exists some $N > 0$ so that $\tau^N(z) \in W^s_{\mathrm{loc}}(x)$. Then we define the *transition map*

$$\psi_{x,z} : \mathbb{R}^k \to \mathbb{R}^k$$

by

$$\psi_{x,z} = H^s_{\sigma^N,x} \cdot L^N(z) \cdot H^u_{x,z}.$$

Definition 6.8. The map $L : \Upsilon \to G$ is called *simple* for τ if there exists a periodic point x of period $N > 0$ and some homoclinic point z of x such that

(1) $L^N(x)$ is diagonalizable over $\mathbb{R}$, and the absolute values of the eigenvalues are pairwise distinct.
(2) For any invariant subspaces E and V of $\mathbb{R}^k$ with $\dim(E) + \dim(V) = k$, we have $\psi_{x,z}(E) \cap V = \{0\}$.

Example 6.9. A point in the *flag variety* $\mathcal{F}$ for the symplectic group $\mathrm{Sp}(2\ell, \mathbb{R})$ is a filtration of isotropic (for the symplectic form ω) subspaces $\mathcal{E} = E_1 \subset \cdots \subset E_\ell$ such that $\dim(E_i) = i$. Two such flags $\mathcal{E}^1, \mathcal{E}^2$ are transverse if we have $E^1_\ell \cap E^2_\ell = \{0\}$. This implies in particular that the map $\eta : X \in E^1_\ell \to \omega(X, \cdot)$ is an isomorphism of E^1_ℓ onto the dual $(E^2_\ell)^*$ of E^2_ℓ.

Assume furthermore that the flags are *opposite*, which means that for each i, the annihilator of the i-dimensional linear subspace $\eta(E^1_i) \subset (E^2_\ell)^*$, which is a linear subspace of E^2_ℓ of dimension $\ell - i$, is transverse to E^2_i. Then the flag $\mathcal{E}^1$ determines a basis $v_1, \dots, v_\ell$ of E^2_ℓ up to multiplying a basis vector with a nonzero real by the following requirements.

- $\mathrm{span}\{v_1, \dots, v_i\} = E^2_i$ for all i.
- v_i is contained in the annihilator of $\eta(E^1_{\ell-i-1})$.

Note that by the above assumption, the annihilator of $\eta(E^1_{\ell-i-1})$ intersects E^2_i in a one-dimensional subspace. In the same way we obtain from $\mathcal{E}^2$ a basis of E^1_ℓ.

Let $\mathcal{E}^3$ be a third flag which is transverse to both $\mathcal{E}^1$ and $\mathcal{E}^2$. It then follows that E^3_ℓ is transverse to both E^1_ℓ and E^2_ℓ, furthermore the bases of E^1_ℓ determined by $\mathcal{E}^2, \mathcal{E}^3$ are linearly independent, that is, no two basis elements are collinear.

Now let us suppose that the opposite flags $\mathcal{E}^1, \mathcal{E}^2$ are the attracting and repelling flags, respectively, of a *proximal* element $A \in \mathrm{Sp}(2\ell, \mathbb{R})$. By proximality, the eigenvalues of A have mutually distinct absolute values, and the one-dimensional eigenspaces are the lines spanned by the basis of $\mathbb{R}^{2\ell}$ constructed from the two flags $\mathcal{E}^1$, $\mathcal{E}^2$.

Let $B \in \mathrm{Sp}(2\ell, \mathbb{R})$ be another proximal element, with attracting and repelling flags $\mathcal{E}^3, \mathcal{E}^4$, respectively, and so that any two flags $\mathcal{E}^i, \mathcal{E}^j$ are opposite. There exists a basis transformation $V : \mathbb{R}^{2\ell} \to \mathbb{R}^{2\ell}$ which transforms the basis constructed from the pair $\mathcal{E}^1, \mathcal{E}^2$ to the basis constructed from the pair $\mathcal{E}^3, \mathcal{E}^4$. Write this linear map as a matrix with respect to the basis defined by the pair $\mathcal{E}^1, \mathcal{E}^2$. As the four flags $\mathcal{E}^i$ are pairwise opposite, all the minors of this matrix are non-zero. Then for any pair of A-invariant subspaces E, V of $\mathbb{R}^{2\ell}$ with $\dim(E) + \dim(F) = 2\ell$, we have $B(E) \cap V = \{0\}$.

It was shown by Benoist (Section 3.6 of [Be97]) that any Zariski dense semigroup $\Gamma \subset \mathrm{Sp}(2\ell, \mathbb{R})$ contains proximal elements φ, ψ whose fixed point flags $\mathcal{E}^1, \mathcal{E}^2$ and $\mathcal{E}^3, \mathcal{E}^4$ are pairwise opposite.

A completely analogous discussion also applies to the group $\mathrm{SL}(n, \mathbb{R})$, where the flag variety is just the variety of full flags in $\mathbb{R}^n$. As this is well known, we omit further details.

In the following proposition, we refer to the above example for convenience.

Proposition 6.10. *Let $q \in \mathcal{C}_{\mathrm{good}}$ be a birecurrent point and let $U \subset \mathcal{C}_{\mathrm{good}}$ be a contractible neighborhood of q. If $\rho(\Lambda) \subset G$ is Zariski dense in G, then there exists a pair $(x, y) \in \Upsilon^2$ consisting of a periodic point $x \in \Upsilon$ and a homoclinic point z for x with the properties in Definition 6.8 for the extension L of the shift (Υ, τ).*

Proof. Since by assumption the subgroup $\rho(\Lambda) \subset G$ is Zariski dense, it is locally Zariski dense by Proposition 5.3. Thus any birecurrent point $q \in \mathcal{C}_{\mathrm{good}}$ has a contractible neighborhood $U \subset \mathcal{C}_{\mathrm{good}}$ with a local product structure defining a semi-group $\Gamma_0 \subset \mathrm{Sp}(2\ell, \mathbb{R})$ with the following properties.

(1) There exists a periodic point $q \in U$ for Φ^t, of period $T > 0$, such that the map $\Psi^T(q)$ is proximal, with all eigenvalues of distinct absolute values, and attracting and repelling opposite flags $\mathcal{E}^1, \mathcal{E}^2$.
(2) For any neighborhood $V \subset U$ of q, there is a periodic point $b \in V$, of period $\tau > 0$, such that the map $\Psi^\tau(b)$ is proximal, with all eigenvalues of distinct absolute values, and the attracting and repelling flags $\mathcal{E}^3, \mathcal{E}^4$ for $\Psi^\tau(y)$ are opposite to the flags $\mathcal{E}^1, \mathcal{E}^2$.

Namely, since $\rho(\Lambda) \subset G$ is Zariski dense, the same holds true for the semi-group $\rho(\Gamma_0)$. Thus we can find two elements in that semigroup by [Be97] with the above properties (see the above example for more details).

By construction of the shift space Υ and the fact that the image of the map $\Xi : X \to \mathcal{Q}$ contains all Φ^t-orbits of abelian differentials q with uniquely ergodic vertical and horizontal measured foliations, there exists a periodic point $x \in \Upsilon$ such that $\Xi(x) = \Phi^t q$ for some $t \in [0, R)$. Note that the image of the periodic point x under the map Ξ may cover the periodic orbit for q a nontrivial multiple of times, however this does not alter the properties of the orbit we are looking at. By an adjustment of U by a translate under the Teichmüller flow we may in fact assume that $\Xi(x) = q$.

Let $X = [P_0, \dots, P_k]$ (with $P_i \in \mathcal{P}$) be the fundamental cylinder for the point x so that $x = \cdots X \cdot X \cdots \in \Upsilon$. By possibly decreasing the size of the neighborhood U we may assume that there exists some $y \in X$ so that $\Xi(y) = b$. Note that this holds true by the construction of the map Ξ in spite of the fact that the shift space Υ is totally disconnected. We refer to the proof of Lemma 6.5 for more information. Then the periodic word in the alphabet $\mathcal{P}$ defining b is of the form $\cdots Z \cdot Z \cdots$ where $Z = XY$ for some nontrivial string Y.

Let $Y_s \in \mathcal{P}$ be the last letter in the word defining Y. Then $A_{Y_s, P_0} = 1$ where as before, A is the transition matrix defining the shift space (Υ, τ). As a consequence, the string $v = \cdots X \cdot XY \cdot X \cdots$ defines a word in Υ and hence it defines a homoclinic point for the periodic point x. Furthermore, since local stable and unstable holonomies are constant, the extension L of the shift τ is defined at the point v, and it has the property (2) in Definition 6.8. □

Proof of Theorem 1. By construction, the measure $\hat{\mu}$ on (Υ, τ) is absolutely continuous with respect to the stable and unstable foliation. Proposition 6.10 shows that the cocycle over (Υ, τ) fulfills the assumptions in Theorem A of [AV07a]. Thus Theorem 1 now follows from Theorem A of [AV07a]. □

7. Applications

In this section we use Theorem 1 and the results from Section 4 to prove the corollaries from the introduction.

Proof of Corollary 1. The corollary is well known. Let $\mathcal{Q}$ be a component of a stratum of abelian differentials. By Corollary 4.9, the monodromy group of $\mathcal{Q}$ is Zariski dense, that is, the Kontsevich Zorich cocycle over $\mathcal{Q}$ is Zariski dense. The corollary now follows from Theorem 1. □

Proof of Corollary 2. The corollary is immediate from Corollary 4.9 and Theorem 1. □

Proof of Corollary 3. By Theorem 1, it suffices to show that for any non-hyperelliptic component $\mathcal{Q}$ of a stratum of quadratic differentials with at least two zeros of odd order the restriction of the Kontsevich Zorich cocycle to $\mathcal{Q}$ is Zariski dense.

As the proof of this fact is a consequence of some more technical results established in [H24], we carry it out in the appendix (Proposition A.1). □

Proof of Corollary 4. Consider the Teichmüller flow Φ^t on the sphere bundle $\mathcal{S} \to \mathcal{M}_g$ of area one quadratic differentials. There is an invariant probability measure μ on $\mathcal{S}$ in the Lebesgue measure class which gives full mass to the *principal stratum*. It was shown in [H13] that this measure can be obtained with Bowen's construction: Namely, for each $R > 0$ let $\Lambda(R)$ be the set of all periodic orbits of Φ^t of length at most R, where we take the union of all orbits over all strata of abelian or quadratic differentials. This amounts to collecting all conjugacy classes of pseudo Anosov mapping classes of translation length at most R. Then

$$\mu = \lim_{R\to\infty} he^{hR} \sum_{\gamma\in\Lambda(R)} \delta_\gamma$$

where δ_γ denotes the natural invariant Lebesgue measure on the periodic orbit.

Let $\lambda_1 < \cdots < \lambda_{2g}$ be the Lyapunov spectrum of the flow Φ^t with respect to the measure μ. By Corollary 3, we know that the Lyapunov spectrum is simple and hence there exists a number $\epsilon > 0$ so that $\lambda_i - \lambda_{i-1} \geq 4\epsilon$. On the other hand, it was shown in [H23] that for every $\delta > 0$, the property $\mathcal{A}$ for a periodic orbit γ to have eigenvalues whose normalized logarithms are contained in the interval $[\lambda_i - \delta, \lambda_i + \delta]$ for all i is typical. Choosing $\delta < \epsilon$, this implies that the eigenvalues of a typical orbit are real and pairwise distinct, which is what we wanted to show. □

Proof of Corollary 5. Let $\mathcal{Q}$ be the principal stratum of area one quadratic differentials. It consists of differentials with precisely $4g - 4$ simple zeros. There is a double orientation cover for this stratum which is branched at each of the zeros. An application of the Hurwitz formula shows that the Euler characteristic of the covering surface S' equals $-8g-8$. The principal stratum $\mathcal{Q}$ lifts to an affine invariant manifold $\mathcal{C}$ in the moduli space of area one abelian differentials on S'.

Let ι be the involution of S' so that $\iota\backslash S' = S$. The cohomology $H^1(S', \mathbb{R})$ decomposes as $H^1(S', \mathbb{R}) = V^+ \oplus V^-$ where $V^\pm$ is the eigenspace with respect to the eigenvalue ± 1 for the action of ι. The absolute real tangent space of the preimage of $\mathcal{Q}$ is the space V^-, and V^+ is isomorphic to $H^1(S, \mathbb{R})$. Counting dimensions yields that the dimension of V^- equals $6g-6$, which coincides with the dimension of $\mathcal{ML}$. Furthermore, $\mathcal{ML}$ lifts to an open cone in V^-.

By Corollary 4.9, the monodromy of $\mathcal{C}$ is Zariski dense and hence the corollary follows ones more from Theorem 1. $\square$

Appendix A. Train tracks and components of strata

The goal of this appendix is to summarize some results of [H24] in the form we need. We also establish Zariski density for the Kontsevich Zorich cocycle over a non-hyperelliptic component $\mathcal{Q}$ of a stratum of quadratic differentials with at least two zeros of odd order which is used in the proof of Corollary 3.

A *train track* on a closed surface S is an embedded graph τ of class C^1 in S. All vertices are required to be trivalent. At each vertex v, called a *switch*, there is a non-trivial partition of the edges of τ incident on v corresponding to the direction of their inward pointing tangents a v. Informally (and somewhat incorrectly) we call the two sets in the partition incoming and outgoing edges, respectively. The complementary components of τ all have negative Euler characteristic. This Euler characteristic is computed by adding to the usual Euler characteristic the number $-k/2$ where k is the number of cusps of the component. Thus a disk with k cusps at the boundary has Euler characteristic $-k/2+1$.

A *geodesic lamination* μ on S is *carried* by τ if there is a *carrying map* $F : S \to S$ of class C^1 which is homotopic to the identity, with $F(\mu) \subset \tau$ and whose derivative restricted to a leaf of μ vanishes nowhere. If μ admits a transverse measure, then via a carrying map, the lamination defines a non-negative weight function on the edges, called *branches* of τ, satisfying the *switch condition*: For each switch v, the sum of the weights of the incoming branches at v equals the sum of the weights of the outgoing branches. A non-negative weight function which satisfies this system of liner equations will be called *admissible*.

Let $\mathcal{ML}$ be the space of measured geodesic laminations on S equipped with the weak*-topology. It is homeomorphic to $\mathbb{R}^{6g-6} - \{0\}$. There exists a continuous pairing

$$\iota : \mathcal{ML} \to \mathcal{ML} \to [0, \infty),$$

the so-called *intersection form*. A pair of measured geodesic laminations (μ, ν) *jointly fills up* S if for every measured lamination ζ we have $\iota(\mu, \zeta) + \iota(\nu, \zeta) > 0$. A pair of measured geodesic laminations which jointly fills up S determines uniquely a quadratic or abelien differential.

To each component $\mathcal{Q}$ of a stratum of area one abelian or quadratic differentials there is associated a distinguished collection $\mathcal{LT}(\mathcal{Q})$ of train tracks [H24]. To describe their properties, note that from a train track τ which *fills*, that is, whose complementary components are all simply connected, one can construct a second *dual bigon track* τ^*. A train track $\tau \in \mathcal{LT}(\mathcal{Q})$ fills and has the following properties.

(1) τ is *recurrent*, that is, it admits an admissible weight function which is positive on every branch.
(2) τ is *transversely recurrent*, that is, its dual bigon track admits an admissible weight function which is positive on every branch.

(3) Let $\mathcal{V}(\tau)$ and $\mathcal{V}(\tau^*)$ be the convex cone of admissible weight functions on τ and τ^*, respectively. Let

$$Z(\tau) = \{(\mu,\nu) \in \mathcal{V}(\tau) \times \mathcal{V}(\tau^*) \mid (\mu,\nu) \text{ jointly fill up } S\};$$

then each pair in $Z(\tau)$ defines a quadratic differential (or an abelian differential if τ is orientable), and the set of such differentials is the closure of an open subset of $\mathcal{Q}_+$ (not necessarily area normalized).

By the classification of Lanneau [L08], a stratum of quadratic differentials with at least two zeros on a surface of genus $g \geq 3$ has at most two connected components, and if there are two components, then one of them is *hyperelliptic*. The remainder of this appendix is devoted to the proof of the following

Proposition A.1. *Let $\mathcal{Q}$ be a non-hyperelliptic component of a stratum of area one quadratic differentials on a surface of genus $g \geq 3$ with at least two zeros of odd order. Then the monodromy of the Kontsevich Zorich cocycle over $\mathcal{Q}$ is Zariski dense.*

Proof. We begin with constructing for each non-hyperelliptic component $\mathcal{Q}$ of area one quadratic differentials with at least two zeros of odd order an explicit example of a train track $\tau \in \mathcal{LT}(\mathcal{Q})$.

Let $g \geq 3$ and consider a non-hyperelliptic component $\mathcal{E}_g$ of a stratum of *abelian differentials* with a single zero on the surface of genus g. To this component is associated a collection $\mathcal{LT}(\mathcal{E}_g)$ of large train tracks with the properties (1),(2),(3) above. Let η be such a train track. It is orientable, that is, there exists a consistent orientation of its branches, where consistent means that for each switch v the orientations of the branches incident on v match up to an orientation of a neighborhood of v. Moreover, η has a single complementary component which is an ideal polygon P with $4g-4$ sides. We refer to [H24] for more information.

Assume that the zeros of the differentials in $\mathcal{Q}$ of odd order (which is always even) are of multiplicity $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$ where $k \geq 2$ by assumption. Choose an embedded arc ζ_1 in the polygon P which connects two distinct sides of P, has endpoints in the interior of some branches of P and is tangent to the branches at its endpoints. We require that ζ cuts from P a polygon P_1 with m_1+2 sides. Since m_1 is odd, the union of η with ζ_1 is a train track which is *not* orientable.

Choose a second such arc which cuts from the polygon $P-P_1$ a polygon P_2 with m_2 sides. Subdivide in the same way the polygon $P-(P_1\cup P_2)$ into polygons whose number of sides is predicted by the zeros of differentials in $\mathcal{Q}$ and their multiplicities. As $\eta \in \mathcal{LT}(\mathcal{E}_g)$ and $\mathcal{E}_g$ is non-hyperelliptic, this construction yields a train track $\tau \in \mathcal{LT}(\mathcal{Q}')$ for a non-hyperelliptic component $\mathcal{Q}'$ of the stratum containing $\mathcal{Q}$ and hence $\tau \in \mathcal{LT}(\mathcal{Q})$ by uniqueness. Note that τ contains η as a *subtrack*, that is, it contains η as an embedded subgraph, and τ is obtained from η by adding a collection of *small* branches. We refer to [H24] for more details of this construction.

A pseudo-Anosov mapping class $\varphi \in \mathrm{Mod}(S)$ admits a train track β as a *train track expansion* if $\varphi(\beta) \prec \beta$, that is, if $\varphi(\beta)$ is *carried by* β. By definition, this means that there exists a map $F : S \to S$ of class C^1 which is homotopic to the identity and maps $\varphi(\beta) \to \beta$ in such a way that the restriction of the differential of F to $\varphi(\beta)$ vanishes nowhere. If $\beta \in \mathcal{LT}(\mathcal{Q}')$ for some component $\mathcal{Q}'$ of a stratum of abelian or quadratic differentials, then the periodic orbit of Φ^t corresponding to φ is contained in a component of a stratum in the closure of $\mathcal{Q}'$. Thus if $\mathcal{Q}' = \mathcal{E}_g$ then this orbit is in fact contained in $\mathcal{E}_g$.

If $\varphi, \psi \in \mathrm{Mod}(S)$ admit $\eta \in \mathcal{LT}(\mathcal{E}_g)$ as a train track expansion, then the same holds true for $\varphi \circ \psi$. Thus the mapping classes with this property form a semi-subgroup H_0 of $\mathrm{Mod}(S)$. It follows from [H11, H24] and Theorem 5.3 and Corollary 4.9 that the image under the homomorphism $\rho : \mathrm{Mod}(S) \to \mathrm{Sp}(2g, \mathbb{Z})$ of the semi-subgroup H_0 of $\mathrm{Mod}(S)$ is Zariski dense in $\mathrm{Sp}(2g, \mathbb{R})$.

Now note that by the construction of the train track track τ, there exists a number $k_0 > 0$ so that any $\varphi \in H$ admits a positive power φ^k for some $k \leq k_0$ with the property that φ^k also admits τ as a train track expansion. This power is characterized by the requirement that it preserves every horizontal separatrix of an abelian differential on its axis. We refer once more to [H24] for more details. As a consequence, the semi-subgroup H of H_0 of all elements which admit τ as a train track expansion maps to a Zariski dense subgroup of $\mathrm{Sp}(2g, \mathbb{R})$.

Now choose any $\varphi \in \mathrm{Mod}(S)$ which admits τ as a train track expansion and such that φ defines a periodic orbit for Φ^t contained in $\mathcal{Q}$. We refer to [H24] for the existence of such an element. Then for any $\psi \in H$, the concatentation $\psi \circ \varphi$ admits τ as a train track expansion, and the corresponding periodic orbit is contained in $\mathcal{Q}$. Since $\psi \in H$ was arbitrary and the image of H is Zariski dense in $\mathrm{Sp}(2g, \mathbb{Z})$, this implies that the monodromy of the Kontsevich Zorich cocycle over the component $\mathcal{Q}$ is Zariski dense. □

The proof of Proposition A.1 makes an essential use of the existence of zeros of odd order for differentials in $\mathcal{Q}$. It does not apply to components of strata of quadratic differentials with all zeros of even multiplicity, and we do not know how to determine the Zariski closure of the Kontsevich Zorich cocycle in that case.

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