ERGODICITY OF THE ABSOLUTE PERIOD FOLIATION

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Abstract. We show that the absolute period foliation of the principal stratum of abelian differentials on a surface of genus $g \geq 3$ is ergodic.

1. Introduction

The moduli space of area one abelian differentials on a surface of genus $g \geq 2$ naturally decomposes into *strata* of differentials with prescribed numbers and multiplicties of zeros. Period coordinates on such a stratum $\mathcal{Q}$ are defined by evaluation of an abelian differential on a basis for homology of the surface relative to the zeros of the differential. These coordinates define on $\mathcal{Q}$ the structure of a real analytic orbifold. If $\mathcal{Q}$ is a stratum of differentials with more than one zero then it admits a natural foliation whose leaves consist of differentials with (locally) fixed absolute periods. This foliation is smooth and has been analyzed in [McM13, McM14, MinW14]; it is called the *absolute period foliation*.

A smooth foliation of an orbifold $\mathcal{Q}$ is called *ergodic* if any Borel subset of $\mathcal{Q}$ which is saturated for the foliation either has full or vanishing Lebesgue measure. In [McM14], tools from homogeneous dynamics are used to show that the absolute period foliation of the principal stratum in $g = 2, 3$ is ergodic. Here the principal stratum is the stratum of all differentials with only simple zeros. Calsamiglia, Deroin and Francaviglia [CDF15] completely classified the closures of the leaves of the absolute period foliation of the principal stratum. As a consequence, they obtain ergodicity of the absolute period foliation of the principal stratum in every genus.

Our main goal is to give a simple proof of the latter fact.

Theorem. *The absolute period foliation of the principal stratum is ergodic in every genus $g \geq 2$.*

We do not know whether the absolute period foliation of a componente of a stratum which is not principal is ergodic. Ergodicity seems likely in the presence of a dense leaf. The existence of a dense leaf for a specific component of the stratum of differentials with two zeros of order $g-1$ (namely, the component containing the so-called Arnoux-Yoccoz surface) was established in [HW15].

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2. The absolute period foliation

In this section we establish some properties of the absolute period foliation on arbitrary components of strata of area one abelian differentials with at least two zeros.

Thus let $\mathcal{Q}$ be a component of a stratum of differentials with $k \geq 2$ zeros in the moduli space of area one abelian differentials on a surface of genus $g \geq 2$. Then $\mathcal{Q}$ is a hypersurface in a component $\mathcal{Q}_{\mathbb{R}_+}$ of a stratum of nontrivial abelian differentials without area constraint. The absolute periods of an abelian differential $\omega \in \mathcal{Q}_{\mathbb{R}_+}$, obtained by integrating the differential over a basis of absolute homology, define a local submersion of orbifolds

$$\mathcal{Q}_{\mathbb{R}_+} \supset U \to H^1(X,\mathbb{C})/\mathrm{Aut}(X)$$

whose fibres are the leaves of the *absolute period foliation* $\mathcal{AP}(\mathcal{Q}_{\mathbb{R}_+})$ of $\mathcal{Q}_{\mathbb{R}_+}$. The local submersion of orbifolds commutes with the natural action of the multiplicative group $\mathbb{R}_+$ by rescaling and hence the absolute period foliation of $\mathcal{Q}_{\mathbb{R}_+}$ restricts to a foliation $\mathcal{AP}(\mathcal{Q})$ of the hypersurface $\mathcal{Q}$. This foliation is transverse to the fibres of the canonical projection $\pi : \mathcal{Q} \to \mathcal{M}_g$ (here $\mathcal{M}_g$ denotes the moduli space of Riemann surfaces of genus g).

The leaf $\mathcal{AP}(\omega)$ of $\mathcal{AP}(\mathcal{Q})$ through ω is locally a flat submanifold of $\mathcal{Q}$ which can explicitly be described [MinW14, McM13].

Assume for the moment that $\mathcal{Q}$ is the principal stratum. Denote by $Z(\omega)$ the zero set of $\omega \in \mathcal{Q}$. The cardinality of $Z(\omega)$ equals $2g-2$. At each zero $p \in Z(\omega)$ there is an infinitesimal deformation of ω called the *Schiffer variation* [McM13] which is defined as follows.

Let X be the Riemann surface underlying ω. Choose a complex coordinate z for X near p so that in this coordinate, ω can be written as $\omega = (z/2)dz$. Such a coordinate is unique up to multiplication with -1. Choose a vertical arc $A_t = i[-2u, 2u]$ in this coordinate where $t = u^2$. Slit X open along A_t and fold each of the two resulting arcs so that z is identified with $-z$ (see p.1235 of [McM13]).

The result is a new Riemann surface X_t with a distinguished arc B_t and a natural holomorphic map $f_t : X - A_t \to X - B_t$. The one-form ω_t with $f_t^*\omega_t = \omega$ is globally defined, and it only depends on the parameter t and on the choice of the zero p of ω. The Schiffer variation of X is

$$\mathrm{Sch}(\omega, p) = dX_t/dt|_{t=0}.$$

It will be useful to have a geometric description of the deformation of the one-forms ω_t defining the Schiffer variation. Namely, there are four horizontal separatrices at p for the flat metric defined by ω. In a complex coordinate z near p so that $\omega = (z/2)dz$, the horizontal separatrices are the four rays contained in the real or the imaginary axis. The restriction of ω to these rays defines an orientation on the rays. This orientation is determined by the requirement that a nontrivial tangent vector X for one of these rays defines the positive orientation if and only if $\omega(X) > 0$. Note that this makes sense since the restriction of ω to these rays is a real valued one-form. With respect to this orientation, the two rays contained in the real axis are outgoing from p, while the rays contained in the imaginary axis are incoming. The Schiffer variation slides the singular point backwards along the incoming rays in the imaginary axis. Thus if one of the two separatrices in the imaginary axis is a saddle connection for ω, then the flat length of the corresponding saddle connection for ω_t is decreasing with t.

If ω has a zero of order $n \geq 2$ at p then the Schiffer variation at p is defined as follows (see p.1235 of [McM13]). Choose a coordinate z near p so that $\omega = z^n dz$ in this coordinate. This choice of coordinate is unique up to multiplication with $e^{\ell 2\pi i/n+1}$ for some $\ell \leq n$. There are $n+1$ horizontal separatrices at p for the flat metric defined by ω whose orientations point towards p. For small $u > 0$ cut the surface S open along the initial subsegments of length $2u$ of these $n+1$ horizontal segments. The result is a $2n+2$-gon which we refold as in the case of a simple zero.

Now let C be a smooth simple loop enclosing the zero $p \in Z(\omega)$. Then the Schiffer variation at p is the real part $\delta(C, -1/\omega)$ of a complex twist deformation $\delta(C, v)$ of X about C where v is a holomorphic vector field along C.

Namely, there is a vector $\mathrm{tw}(C)$ tangent to $\mathcal{AP}(\mathcal{Q})$ at ω defined by

$$\langle \mathrm{tw}(C), E\rangle = C \cdot E$$

where $E \in H_1(X, Z(\omega))$ and where $\cdot$ is the natural intersection pairing

$$H_1(X - Z(\omega)) \times H_1(X, Z(\omega)) \to \mathbb{R}.$$

The tangent space at ω to the absolute period foliation is generated by the transformations $\mathrm{tw}(C_p)$, $p \in Z(\omega)$, subject to the relation $\sum \mathrm{tw}(C_{p_i}) = 0$ (see p.1236 of [McM13]). The leaves of the absolute period foliation $\mathcal{AP}(\mathcal{Q})$ are complex suborbifolds of $\mathcal{Q}$ (this is viewed as a local statement). If $\mathcal{Q}$ is the principal stratum then the leaves of $\mathcal{AP}(\mathcal{Q})$ have complex dimension $2g-3$. We refer to [McM13] for details and an explanation of these notations.

Note that the tangent bundle of $\mathcal{AP}(\mathcal{Q})$ is naturally equipped with a complex structure J as well as with a real structure. The subbundle of $T\mathcal{AP}(\mathcal{Q})$ spanned by the twist deformations corresponding to the Schiffer variations is a maximal real subbundle for this real structure. By abuse of notation, we call a twist deformation corresponding to a Schiffer variation again a Schiffer variation, i.e. we view Schiffer variations as locally defined paths in the leaves of the absolute period foliation.

Let $\hat{\mathcal{Q}}$ be a finite normal cover of the stratum $\mathcal{Q}$ such that there is a consistent numbering of the zeros of $q \in \hat{\mathcal{Q}}$ varying continuously with q. To construct such a cover observe that an arbitrary numbering of the zeros of $q \in \mathcal{Q}$ extends locally continuously to a numbering of the zeros of any nearby differential. Thus such a numbering can continuously be extended along any loop in $\mathcal{Q}$. In this way one obtains a homomorphism from the fundamental group of $\mathcal{Q}$ into the symmetric group in k variables whose kernel defines the desired cover $\hat{\mathcal{Q}}$ of $\mathcal{Q}$.

The absolute period foliation of $\mathcal{Q}$ lifts to a foliation $\mathcal{AP}(\hat{\mathcal{Q}})$ of $\hat{\mathcal{Q}}$. With the above convention, the real subspace of the tangent space of $\mathcal{AP}(\hat{\mathcal{Q}})$ can be identified with the vector space of all k-tuples $\mathfrak{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ of *real* numbers with $\sum_i a_i = 0$ in the following way. At a zero $p \in Z(\omega)$, the Schiffer variation with weight parameter $a \in \mathbb{R}$ at p is just the deformation $s \to \omega_{as}$ of ω where the differential ω_{as} on the right hand side is obtained in the above way by a slit along an arc A_{sa} of length $4\sqrt{as}$. In this way a real weight at the zero p of ω determines a deformation of ω. The vector $\mathfrak{a} \in \mathbb{R}^k$ defines a smooth vector field $X_\mathfrak{a}$ on $\hat{\mathcal{Q}}$ by requiring that for each $\omega \in \hat{\mathcal{Q}}$, the value of $X_\mathfrak{a}$ at ω is the tangent at ω of the deformation for the weight parameters $(a_1, \dots, a_k)$ at the numbered zeros of ω. Thus $X_\mathfrak{a}$ is tangent to the absolute period foliation.

An abelian differential ω on a Riemann surface X defines a flat metric on X and two singular foliations. The tangent bundle of the *horizontal foliation* consists precisely of all tangent vectors on X on which the evaluation of ω is real.

Example 2.1. Let ω_1, ω_2 be two abelian differentials on two closed surfaces S_1, S_2 of genus g_1, g_2. Assume that the area of ω_i is a_i for some $a_i > 0$ with $a_1 + a_2 = 1$. Cut a small horizontal slit into S_1, S_2 of the same length. The differentials ω_i define a decomposition of these slits into two oriented arcs of the same length with the same endpoints. Glue S_1 to S_2 by a crosswise isometric orientation preserving identification of these oriented arcs. The result is an area one abelian differential ω on a surface of genus $g_1 + g_2$ with two singular points p_1, p_2 (the images of the endpoints of the slits) which are connected by two homologous horizontal saddle connections of the same length. Assume that the orientation defined by ω of these saddle connections points from p_1 to p_2. The deformation induced by the Schiffer variation with weight parameters $(-1, 1)$ at the pair (p_1, p_2) decreases the length of the slit and limits in a surface with nodes. This surface with nodes consists of the surfaces S_1, S_2 attached at one point, equipped with an abelian differential which maps to the differentials ω_1, ω_2 by the marked point forgetful map.

The *Teichmüller flow* Φ^t on $\mathcal{Q}$ is defined by $\Re\Phi^t\omega = e^{t/2}\Re\omega$ and $\Im\Phi^t\omega = e^{-t/2}\omega$ where we view $\Re\omega, \Im\omega$ as real relative cohomology classes. This flow lifts to a smooth flow on $\hat{\mathcal{Q}}$ denoted by the same symbol. Its derivative acts on the tangent bundle of $\hat{\mathcal{Q}}$. We have

Lemma 2.2. $d\Phi^t X_\mathfrak{a} = e^t X_\mathfrak{a}$.

Proof. Let $\omega \in \hat{\mathcal{Q}}$ and let $\mathcal{F}$ be the horizontal foliation of ω. The Teichmüller flow expands the horizontal foliation $\mathcal{F}$ of ω with the expansion rate $e^{t/2}$. Thus if A_s is an arc of length $4\sqrt{s}$ in the imaginary axis for the preferred coordinate near the zero p of ω (recall that this arc is horizontal for the flat metric defined by ω) then the image $\Phi^t A_s$ of A_s in $\Phi^t\omega$ is an arc of length $4e^{t/2}\sqrt{s} = 4\sqrt{e^t s}$ in the imaginary axis of a preferred coordinate. Thus the push-forward by Φ^t of the deformation of ω defined by the vector field $X_\mathfrak{a}$ is the deformation of $\Phi^t\omega$ defined by the vector field $X_{e^t\mathfrak{a}} = e^t X_\mathfrak{a}$. $\square$

The vector field $X_\mathfrak{a}$ defines a flow $\Lambda^t_\mathfrak{a}$ on $\hat{\mathcal{Q}}$. This flow is incomplete as a horizontal saddle connection may give rise to a finite flow line limiting on a lower dimensional stratum (see p.1235 of [McM13]). There may also be limit points on surfaces with nodes as described in Example 2.1. However, if q does not have any horizontal saddle connection then the flow line of $\Lambda^t_{\mathfrak{a}}$ through q is defined for all times [MinW14].

The component $\mathcal{Q}$ is equipped with the *Masur-Veech measure* λ. This measure is a Φ^t-invariant Borel probability measure contained in the Lebesgue measure class which is constructed as follows. Period coordinates on $\mathcal{Q}_{\mathbb{R}_+}$ define on $\mathcal{Q}_{\mathbb{R}_+}$ the structure of a complex orbifold. The standard Lebesgue measure in these coordinates does not depend on choices as transition maps are volume preserving and hence gives rise to a Lebesgue measure on $\mathcal{Q}_{\mathbb{R}_+}$. Up to normalization, the Masur-Veech measure associates to a set $U \subset \mathcal{Q}$ the measure of the cone $\cup_{0<t<1} tU$ (here as before, $\mathbb{R}_+$ acts on abelian differentials by scaling).

The Masur-Veech measure lifts to a Φ^t-invariant finite Borel measure on $\hat{\mathcal{Q}}$ which we denote again by λ. As almost every point with respect to the Masur-Veech measure λ is a differential without horizontal saddle connection, $\Lambda^t_{\mathfrak{a}}$ defines a flow on a subset of $\hat{\mathcal{Q}}$ of full Lebesgue measure.

For $\mathfrak{a} \neq \mathfrak{b}$ the flows $\Lambda^t_{\mathfrak{a}}$ and $\Lambda^s_{\mathfrak{b}}$ commute and hence these flows fit together to a (local) action of the group $\mathbb{R}^{k-1}$ on $\hat{\mathcal{Q}}$. This action is smooth, and its local orbits naturally develop to a smooth foliation of $\hat{\mathcal{Q}}$ called the *real REL foliation* [MinW14]. This foliation is a subfoliation of the absolute period foliation. It can be more conceptually defined using period coordinates on the component $\mathcal{Q}_{\mathbb{R}_+}$ of a stratum in the moduli space of all abelian differentials, but as we do not need this description, we refer to [MinW14] for details.

A leaf of the *local strong unstable foliation* of $\hat{\mathcal{Q}}$ consists of abelian differentials with the same horizontal foliation (up to Whitehead moves).

Lemma 2.3. *The real REL foliation is a subfoliation of the strong unstable foliation of $\hat{\mathcal{Q}}$ which is invariant under the action of the Teichmüller flow and under holonomy along the strong stable foliation.*

Proof. By construction, for every $\mathfrak{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ with $\sum_i a_i = 0$, the local flow of the vector field $X_{\mathfrak{a}}$ preserves the horizontal measured foliation of an abelian differential up to Whitehead moves. Now the strong unstable foliation of $\hat{\mathcal{Q}}$ is defined as follows. Its leaf through ω locally consists of all area one abelian differentials whose horizontal measured foliation coincides up to Whitehead moves with the horizontal measured foliation of ω. Thus the vector fields $X_{\mathfrak{a}}$ are tangent to the strong unstable foliation of $\hat{\mathcal{Q}}$. As a consequence, the real REL foliation is a subfoliation of the strong unstable foliation, and it is smooth. We refer to [McM14] for a detailed analysis of this foliation in the case $g = 2$.

Together with Lemma 2.2, this implies invariance under the action of the Teichmüller flow. Invariance under holonomy along the strong stable foliation follows from the fact that a leaf of the real REL foliation can be characterized as the set of all abelian differentials in the stratum with fixed horizontal foliation and the property that the vertical foliations all define the same absolute cohomology class (see [MinW14]). The lemma follows. $\square$

Remark 2.4. It is easy to see that the flows $\Lambda^t_{\mathfrak{a}}$ preserve the Masur-Veech measure λ of the stratum. It is an interesting question whether any of these flows is ergodic. Our proof of ergodicity of the absolute period foliation does not give any information to this end.

The above discussion shows that the absolute period foliation has an affine structure (see [MinW14] and p.1236 of [McM13] for more details and compare also [LNW15]).

Recall that there is a natural circle action on $\mathcal{Q}$ and on $\hat{\mathcal{Q}}$. To a point e^{is} on the unit circle $S^1 \subset \mathbb{C}^*$ and an abelian differential q we associate the differential $e^{is}q$. For $\mathfrak{a} \in \mathbb{R}^k$ with zero mean, for $e^{is} \in S^1$ and for $\omega \in \hat{\mathcal{Q}}$ let

$$\Lambda^t_{e^{is}\mathfrak{a}}(\omega) = e^{-is}\Lambda^t_{\mathfrak{a}}(e^{is}\omega).$$

Then $(t,\omega) \to \Lambda^t_{e^{is}\mathfrak{a}}\omega$ defines a flow on $\hat{\mathcal{Q}}$ which preserves the absolute period foliation.

Following [EMZ03] we define the *principal boundary* of the component $\mathcal{Q}$ of a stratum as follows. Let $\omega \in \mathcal{Q}$ and assume that ω has a horizontal saddle connection and that the set of horizontal saddle connections of ω is a forest, i.e. it does not have cycles. Let p_1, p_2 be the endpoints of such a saddle connection α, chosen such that α points from p_1 to p_2 with respect to the orientation defined by ω, and let $\mathfrak{a}$ be the vector $(-1,1)$ at p_1, p_2. (Strictly speaking, we view $(-1,1)$ as a vector in $\mathbb{R}^k$ whose remaining coordinates vanish. The above construction then yields a tangent vector of $\hat{\mathcal{Q}}$. However, the choice of α singles out two zeros of ω and hence the corresponding Schiffer variation makes sense in $\mathcal{Q}$). Then the arc $t \to \Lambda^t_{\mathfrak{a}}\omega$ limits on a differential ζ for which the points p_1, p_2 coalesce, and there are no other identifications of zeros. The differential ζ is obtained from ω by collapsing the saddle connection α. Recall that by assumption, α is the only saddle connection which connects p_1 and p_2. Furthermore, ζ is contained in a component of a stratum in the boundary of $\mathcal{Q}$, and we call such a component a *finite core face* of the principal boundary of $\mathcal{Q}$. If k is the number of zeros of a differential in $\mathcal{Q}$, then the number of zeros of a differential in a finite core face of $\mathcal{Q}$ equals $k-1$. In particular, the (real) dimension of a finite core face of $\mathcal{Q}$ equals $\dim(\mathcal{Q})-2$. Iteration of this construction gives rise to faces of the principal boundary of higher codimension; these faces are not called core faces.

A second degeneration which gives rise to a point in the principal boundary is the contraction of two or more homologous saddle connections (connecting the same zeros and of the same length). In this case the resulting surface is a surface consisting of two or more smooth components which are connected at nodes. The sum of the genera of these surfaces equals g. The resulting abelian differential does not vanish identically on any of the smooth components of this surface with nodes, and it has a regular point or a zero at a node. We call a component of abelian differentials on a surface with nodes arising in this way an *infinite face* of $\mathcal{Q}$.

We call the infinite face a *core face* if it consists of abelian differentials on surfaces comprised of two smooth components which are attached at a single separating node. Note that there are up to $\lfloor g/2 \rfloor$ core faces which correspond to the *type* of the decomposition, i.e. to a decomposition $g = g_1 + g_2$ with $g_1, g_2 \geq 1$. The node is viewed as a marked point on each of the smooth components of the surface with nodes. We call an abelian differential in an infinite core face *regular* if the node is not a zero for the restriction of the abelian differential to a smooth component of the surface. By convention, this also means that the differential does not vanish identically on any smooth component of the underlying surface with nodes. A point which is not regular is called *singular*. The set of singular points is of real codimension two. We refer to [EMZ03] for a detailed discussion of these concepts.

To summarize, there is a decomposition of the principal boundary of $\mathcal{Q}$ into *faces* (see p.76 of [EMZ03]). Each face either is a component of a stratum in the adherence of $\mathcal{Q}$ with fewer zeros, or it corresponds to a *configuration* which consists of a decomposition of the surface into surfaces S_i of genus g_i with $\sum_i g_i = g$, a combinatorial configuration of attachment data which organizes the glueing at the nodes and numbers $b_j \geq 0$ which describe the order of the zero of the differentials at the node (see [EMZ03] for more details). If we let $\overline{\mathcal{Q}}$ be the union of $\mathcal{Q}$ with its principal boundary, then the core faces are the faces in $\overline{\mathcal{Q}}$ of real codimension two.

3. Ergodicity

In this section we restrict the discussion to the principal stratum $\mathcal{Q}$ (which is well known to be connected). Denote by $\overline{\mathcal{Q}}$ the union of $\mathcal{Q}$ with its principal boundary.

Note that $\overline{\mathcal{Q}}$ properly contains the entire moduli space of area one abelian differentials. Furthermore, if Σ is a surface with a single separating node in the boundary of the Deligne Mumford compactification of moduli space whose smooth part consists of two surfaces S_1, S_2 of genus $g_1, g_2 \geq 1$, respectively, with $g_1 + g_2 = g$, then the homology of Σ can naturally be identified with the homology of S. In particular, if ω is a regular abelian differential in an infinite core face of the principal boundary of $\mathcal{Q}$ then the absolute periods of ω are defined.

For $\epsilon > 0$ let $B(\epsilon)$ be the disk of radius ϵ in the complex plane. The disk is invariant under the holomorphic involution $z \to -z$. We denote by $\hat{B}(\epsilon) = B(\epsilon)/\pm \mathrm{Id}$ its quotient (which is homeomorphic to a disk as well).

In Section 9 of [EMZ03] the reader can find versions of the following proposition for more general components of strata of abelian differentials. However, these versions are more complicated.

Proposition 3.1. *Let $\mathcal{F}$ be an infinite core face of the principal boundary of $\mathcal{Q}$ and let $\omega \in \mathcal{F}$ be a regular point. Then there is a number $\epsilon > 0$, and there is a neighborhood V of ω in $\mathcal{F}$, a neighborhood U of ω in $\overline{\mathcal{Q}}$, and a homeomorphism $\varphi : V \times \hat{B}(\epsilon) \to U$ with the following properties.*

(1) $\varphi(x, 0) = x$ *for all* $x \in V$.
(2) $\varphi(\{x\} \times \hat{B}(\epsilon)) \subset \mathcal{AP}(x)$.

Proof. A regular point $\omega \in \mathcal{F}$ is defined by two abelian differentials ω_1, ω_2 on surfaces S_1, S_2 attached at a marked point p_1, p_2. The marked point p_i is a regular point for ω_i. In particular, there is a canonical holomorphic coordinate near p_i which maps p_i to zero and so that in this coordinate, the differential ω_i equals the differential dz.

Using this identification of a neighborhood of p_i with a neighborhood of the complex plane, take a vector γ in the complex plane of sufficiently small length $r < \epsilon$, slit S_1, S_2 open at the marked points in direction of γ and glue the abelian differentials ω_1, ω_2 along the slits as described in Example 2.1. The resulting differential $\hat{\varphi}(\omega, \gamma)$ has the same absolute periods as ω, and $\hat{\varphi}(\omega, 0) = \omega$. Furthermore, we have $\hat{\varphi}(\omega, \gamma) = \hat{\varphi}(\omega, -\gamma)$ and therefore $\hat{\varphi}$ descends to a map $\varphi : V \times B(\epsilon)/\pm \mathrm{Id} \to \overline{\mathcal{Q}}$ where V is a neighborhood of ω in $\mathcal{F}$.

By Lemma 9.8 of [EMZ03], for a sufficiently small neighborhood V of ω and for sufficiently small $\epsilon > 0$, the map φ is a homeomorphism of $V \times B(\epsilon)/\pm \mathrm{Id}$ onto a neighborhood U of ω in $\overline{\mathcal{Q}}$ with the properties stated in the proposition. □

The measure in the statement of the following lemma is the Masur-Veech measure.

Lemma 3.2. *For almost every $\omega \in \mathcal{Q}$, the leaf $\mathcal{AP}(\omega)$ intersects every infinite core face of the principal boundary of $\mathcal{Q}$ in regular points.*

Proof. We say that a translation surface ω (ie a Riemann surface X equipped with a singular euclidean metric defined by a holomorphic one-form ω) has an *isolated bigon* of type (g_1, g_2) where $g_1 + g_2 = g$ if it admits a pair of homologous saddle connections α_1, α_2 connecting two zeros p_1, p_2 with the following property. There is no other saddle connection parallel to α_i, moreover $\alpha_1 \cup \alpha_2$ decomposes S into a surface of genus g_1 and a surface of genus g_2. Since the Teichmüller flow preserves saddle connections and only changes their direction and length, the set of points $\omega \in \mathcal{Q}$ which admit an isolated bigon of type (g_1, g_2) is invariant under the Teichmüller flow.

The set of directions of a translation surface containing a saddle connection is countable and hence of measure zero. Thus Proposition 3.1 shows that for all $g_1, g_2 \geq 1$ with $g_1 + g_2 = g$, the set of all points $\omega \in \mathcal{Q}$ which admit an isolated bigon of type (g_1, g_2) has positive Masur-Veech measure (see also [EMZ03] for details). By invariance of the set of translation surfaces with isolated bigon of type (g_1, g_2) under the Teichmüller flow on $\mathcal{Q}$ and ergodicity of the Masur-Veech measure, we conclude that this set has full measure.

Let $\omega \in \mathcal{Q}$ and assume that there is some $e^{is} \in S^1$ with the property that $e^{is}\omega$ has an isolated horizontal bigon. Assume that this bigon is defined by a pair (α_1, α_2) of homologous saddle connections with endpoints at the zeros p_1, p_2 of ω. We assume that the points p_1, p_2 are ordered in such a way that with respect to the orientation defined by $e^{is}\omega$, the saddle connections α_1, α_2 connect p_1 to p_2. Then the flow line $t \to \Lambda^t_{\mathfrak{a}}(e^{is}\omega)$ defined by the vector $\mathfrak{a}$ with coordinates (weights) $(-1, 1)$ at the points p_1, p_2 and vanishing weight at every other zero of ω collapses the pair (α_1, α_2) of horizontal saddle connections of $e^{is}\omega$ to a point. (As before, strictly speaking this flow is only defined on the cover $\hat{\mathcal{Q}}$ of $\mathcal{Q}$, however since we fix two zeros determined by the isolated bigon, it makes sense in $\mathcal{Q}$). This means that there is some $\tau > 0$ such that $\Lambda_{\mathfrak{a}}^t(e^{is}\omega)$ is defined for $0 \leq t < \tau$, and the surfaces $\Lambda_{\mathfrak{a}}^t(e^{is}\omega)$ converge as $t \nearrow \tau$ to a surface in the infinite core face of the principal boundary of type (g_1, g_2). The lemma follows from the observation that the path $t \to e^{-is}\Lambda_{\mathfrak{a}}^t(e^{is}\omega)$ is contained in $\mathcal{AP}(\omega)$. $\square$

A subset of $\mathcal{Q}$ is *saturated for the absolute period foliation* if it is a union of leaves.

Corollary 3.3. *The set $\mathcal{S}$ of points $\omega \in \mathcal{Q}$ such that $\mathcal{AP}(\omega)$ intersects every infinite core face of the principal boundary of $\mathcal{Q}$ is saturated for the absolute period foliation and of full Masur-Veech measure.*

Remark 3.4. As the results of [EMZ03] used in the proof of Corollary 3.3 are also valid for strata of differentials with at least two zeros, the corollary holds true for those strata as well.

A smooth foliation of $\mathcal{Q}$ is *ergodic* for the Masur-Veech measure if every Borel set $A \subset \mathcal{Q}$ which is saturated for the foliation either has full measure or measure zero. Note that this notion of ergodicity only depends on the measure class of the Masur-Veech measure λ, which is the Lebesgue measure class. Thus for the purpose of showing ergodicity, we may replace the Masur-Veech measure by another measure in its class, and we will do so at several instances to facilitate the argument. To reflect the irrelevance of the choice of an explicit representative, we shall from now on talk about a Lebesgue measure if we mean the Masur-Veech measure or any measure in its measure class.

A finite core face of the principal boundary of a stratum is a component $\mathcal{P}$ of another stratum. Hence if $g \geq 3$ and if $\mathcal{P}$ consists of differentials with more than one zero, then the absolute period foliation of $\mathcal{P}$ is defined.

For an infinite core face $\mathcal{F}$ of the principal boundary, the absolute period foliation is defined as well. Namely, such an infinite core face consists of abelian differentials on closed surfaces S_1, S_2 of genus $g_1 \geq 1, g_2 \geq 1$ and $g_1 + g_2 = g$. Write $S_1 \sqcup S_2$ for the surface obtained by attaching S_1 and S_2 at a single point, viewed as a surface with a node. The moduli spaces of abelian differentials on S_1, S_2 determine a moduli space of abelian differentials on $S_1 \sqcup S_2$ and an absolute period foliation. We require that the area of an abelian differential on $S_1 \sqcup S_2$ equals one and hence the areas of S_1 and S_2 for the differential add up to one.

Define
$$\mathcal{Q}_{S_1 \sqcup S_2} = \{((\omega_1, p_1), (\omega_2, p_2))\}$$
where ω_i is a nontrivial abelian differential on S_i with simple zeros and one marked (regular) point p_i and such that the areas of the differentials ω_1, ω_2 sum up to one. Then $\mathcal{Q}_{S_1 \sqcup S_2}$ naturally has the structure of a real analytic orbifold. Furthermore, we have
$$\mathcal{Q}_{S_1 \sqcup S_2} = \cup_{b \in (0,1)} \mathcal{Q}_{S_1 \sqcup S_2}(b, 1-b)$$
where a point in $\mathcal{Q}_{S_1 \sqcup S_2}(b, 1-b)$ gives area b to S_1. If $g_1 \neq g_2$ then the infinite core face $\mathcal{F}$ can naturally be identified with $\mathcal{Q}_{S_1 \sqcup S_2}$, and if $g_1 = g_2$ then $\mathcal{F}$ is the

quotient of $\mathcal{Q}_{S_1 \sqcup S_2}$ under the involution ι which exchanges the two components of the pair.

Note that for each $b \in (0,1)$, $\mathcal{Q}_{S_1 \sqcup S_2}(b, 1-b)$ is a real hypersurface in $\mathcal{Q}_{S_1 \sqcup S_2}$ which is equipped with an absolute period foliation in its own right. Furthermore, $\mathcal{Q}_{S_1 \sqcup S_2}(b, 1-b)$ is equipped with a natural Lebesgue measure class. Thus it makes sense to talk about ergodicity of the absolute period foliation for this measure class, and this is the notion of ergodicity used in the following

Lemma 3.5. *If the absolute period foliation of the principal stratum of S_1, S_2 is ergodic then so is the absolute period foliation of $\mathcal{Q}_{S_1 \sqcup S_2}(b, 1-b)$.*

Proof. Write $b_1 = b$ and $b_2 = 1-b$. Then $\mathcal{Q}_{S_1 \sqcup S_2}(b_1, b_2)$ is the space of pairs $((\omega_1, p_1), (\omega_2, p_2))$ where ω_i is an abelian differential on S_i of area b_i with simple zeros and a marked point p_i (here the node is at the marked point).

Let $\mathcal{Q}_{S_i}(b_i)$ be the principal stratum in the moduli space of abelian differentials on S_i of area b_i. There is a natural node forgetting map

$$P : \mathcal{Q}_{S_1 \sqcup S_2}(b_1, b_2) \to \mathcal{Q}_{S_1}(b_1) \times \mathcal{Q}_{S_2}(b_2).$$

This map is a fibration whose fibre over a point (ω_1, ω_2) can naturally be identified with the product $(S_1, \omega_1) \times (S_2, \omega_2)$ (it consists of the pair of marked points). The fibration respects absolute periods and therefore if $\omega \in \mathcal{Q}_{S_1 \sqcup S_2}(b_1, b_2)$ then $P^{-1}(P\omega) \subset \mathcal{AP}(\omega)$.

Equip $\mathcal{Q}_{S_1}(b_1) \times \mathcal{Q}_{S_2}(b_2)$ with the product of the Masur-Veech measures (for any choice of normalization). This is a measure in the Lebesgue measure class. The fibre of P over a point (ω_1, ω_2) is equipped with a natural Lebesgue measure (the product of the Lebesgue measures defined by the differentials ω_i on the surfaces S_i). There is a natural Lebesgue measure μ on $\mathcal{Q}_{S_1 \sqcup S_2}(b_1, b_2)$ so that $P_*\mu$ equals the chosen Lebesgue measure on the base, with the above family of Lebesgue measures as conditional measures on the fibres (see [EMZ03] for details).

The above discussion now shows that a Borel set $A \subset \mathcal{Q}_{S_1 \sqcup S_2}(b_1, b_2)$ which is saturated for the absolute period foliation maps to a Borel subset of $\mathcal{Q}_{S_1}(b_1) \times \mathcal{Q}_{S_2}(b_2)$ which is saturated for the absolute period foliation, and it coincides with $P^{-1}(PA)$ up to a set of measure zero. Thus ergodicity of the absolute period foliation on $\mathcal{Q}_{S_i}(b_i)$ implies ergodicity of the absolute period foliation on $\mathcal{Q}_{S_1 \sqcup S_2}(b_1, b_2)$. $\square$

By Corollary 3.3, the set $\mathcal{S}$ of points $\omega \in \mathcal{Q}$ such that the closure of $\mathcal{AP}(\omega)$ in $\overline{\mathcal{Q}}$ intersects the infinite core face $\mathcal{F}$ defined by the surface with nodes $S_1 \sqcup S_2$ has full measure. In particular, if $A \subset \mathcal{Q}$ is a Borel set of positive Lebesgue measure which is saturated for the absolute period foliation, then $A \cap \mathcal{S}$ is a Borel set of positive Lebesgue measure which is saturated for the absolute period foliation and which intersects any neighborhood of $\mathcal{F}$ in $\overline{\mathcal{Q}}$ in a set of positive Lebesgue measure.

Recalling Proposition 3.1, for a fixed infinite core face $\mathcal{F}$ denote by $\partial A \subset \mathcal{F}$ the Borel set of all regular points with the following property. For each $x \in \partial A$, there exists a neighborhood V of x in ∂A, a neighborhood U of x in $\overline{\mathcal{Q}}$, a number $\epsilon > 0$ and a homeomorphism $\varphi : V \times \hat{B}(\epsilon) \to U$ such that $\varphi(\{x\} \times \hat{B}(\epsilon)) - \{x\} \subset A$.

Since $A\cap\mathcal{S}$ is saturated for the absolute period foliation, it follows from Proposition 3.1 that ∂A is saturated for the absolute period foliation. Namely, in any neighborhood of the form $U=\varphi(V\times\hat{B}(\epsilon))\subset\overline{\mathcal{Q}}$ described in Proposition 3.1, the set A is saturated for the local foliation into the leaves $\varphi(\{x\}\times\hat{B}(\epsilon))\cap\mathcal{Q}$ and hence

$$U\cap A=\varphi\big((V\cap\partial A)\times\hat{B}(\epsilon)\big)\cap\mathcal{Q}.$$

In particular, as $A\cap\mathcal{S}$ intersects any neighborhood of $\mathcal{F}$ in a set of positive Lebesgue measure and as the Lebesgue measure class on $U\cap\mathcal{Q}$ is the class of a product measure, the set ∂A is of positive Lebesgue measure.

We observed above that the core face $\mathcal{F}$ is a (perhaps trivial) quotient of $\mathcal{Q}_{S_1\sqcup S_2}$. Let $\Pi:\mathcal{Q}_{S_1\sqcup S_2}\to\mathcal{F}$ be the natural projection. Using the above discussion we are now able to give a more precise information on the set A.

Corollary 3.6. *Let $A\subset\mathcal{Q}$ be a Borel set which is saturated for the absolute period foliation and of positive Lebesgue measure. Let $\mathcal{F}$ be an infinite core face defined by a surface with nodes $S_1\sqcup S_2$. If the absolute period foliation of the principal stratum of S_1,S_2 is ergodic then there is a Borel set $C\subset(0,1)$ of positive Lebesgue measure such that up to a set of measure zero, we have*

$$\Pi^{-1}(\partial A)=\cup_{s\in C}\mathcal{Q}_{S_1\sqcup S_2}(s,1-s).$$

Furthermore, A has full measure in $\mathcal{Q}$ if and only if ∂A has full measure in $\mathcal{F}$.

Proof. The above discussion shows that the set $\Pi^{-1}(\partial A)\subset\mathcal{Q}_{S_1\sqcup S_2}$ is saturated for the absolute period foliation and of positive Lebesgue measure. If the absolute period foliation of S_1,S_2 is ergodic then Lemma 3.5 shows that there is a Borel set $C\subset(0,1)$ of positive Lebesgue measure such that $\Pi^{-1}(\partial A)=\cup_{s\in C}\mathcal{Q}_{S_1\sqcup S_2}(s,1-s)$.

By Proposition 3.1, if A has full Lebesgue measure in $\mathcal{Q}$ then ∂A has full measure on $\mathcal{F}$.

Vice versa, suppose that ∂A has full measure in $\mathcal{F}$. Proposition 3.1 then shows that there is an open neighborhood Z of $\mathcal{F}$ in $\mathcal{Q}$ such that $A\cap Z$ has full Lebesgue measure. As A is saturated for the absolute period foliation, Corollary 3.3 now implies that A has full Lebesgue measure. ☐

Before we complete the proof of the theorem from the introduction we establish a simple measure theoretic lemma which is needed in its proof. For its formulation, let

$$D=\{(b_1,b_2,b_3)\in\mathbb{R}^3\mid b_i>0,b_1+b_2+b_3=1\}\tag{1}$$

be the standard open two-simplex in $\mathbb{R}^3$ equipped with the natural Lebesgue measure.

Lemma 3.7. *Let $D_0\subset D$ be a Borel set of positive Lebesgue measure. Assume that there are Borel sets $C_1,C_3\subset(0,1)$ such that*

$$D_0=\{(b_1,b_2,b_3)\in D\mid b_1\in C_1\}=\{(b_1,b_2,b_3)\in D\mid b_3\in C_3\}.$$

Then $D-D_0$ has vanishing Lebesgue measure.

Proof. Let λ be the Lebesgue measure on $(0,1)$. Define

$$\begin{aligned}\mathrm{essup}(C_i) &= \sup\{a>0 \mid \lambda(C_i\cap[a,1])>0\}\in(0,1] \text{ and}\\ \mathrm{essinf}(C_i) &= \inf\{a>0 \mid \lambda(C_i\cap[0,a])>0\}\in[0,1) \quad (i=1,3).\end{aligned}$$

It follows from the two descriptions of the set D_0 in the lemma that λ-almost every $b<1-\mathrm{essinf}(C_1)$ is contained in C_3. Namely, on the one hand, for a fixed density point $b_1\in C_1$, in a triple (b_1,b_2,b_3) which is a density point of D_0 we can choose b_2 as small as we wish, on the other hand, we also can choose b_2 as close to $1-b_1$ as we wish.

As a consequence, up to a set of measure zero, the set C_3 contains the interval $(0,c)$ for $c=1-\mathrm{essinf}(C_1)>0$, in particular we have $\mathrm{essinf}(C_3)=0$. By exchanging the roles of C_1 and C_3 we conclude that $\mathrm{essinf}(C_1)=0$ as well. But $C_3\supset(0,c)$ for $c=1-\mathrm{essinf}(C_1)$ and therefore $C_3=(0,1)=C_1$. This yields that up to a set of measure zero, we have indeed $D_0=D$. ☐

We are now ready to show

Theorem 3.8. *The absolute period foliation $\mathcal{AP}(\mathcal{Q})$ of the principal stratum in genus $g\geq 2$ is ergodic.*

Proof. We use induction on the genus g of S. The case $g=2,3$ is due to McMullen [McM14]. Let $g\geq 6$ and assume that the proposition holds true for $g-4$ and for $g-2$.

Let $\mathcal{Q}$ be the principal stratum of abelian differentials on a surface of genus g. Let $A\subset\mathcal{Q}$ be a Borel subset which is saturated for the absolute period foliation and which is of positive Lebesgue measure. Then the same holds true for $A\cap\mathcal{S}$ where $\mathcal{S}\subset\mathcal{Q}$ is as in Corollary 3.3.

Let S_1,S_3 be a surface of genus two and let $S_1\sqcup S_2\sqcup S_3$ be the surface with two nodes obtained by attaching S_1,S_3 to a surface S_2 of genus $g-4$ at a single point each. The surface $S_1\sqcup S_2\sqcup S_3$ has three smooth components which are closed surfaces of genus 2 and $g-4$, respectively, with one or two marked points.

The surface with nodes $S_1\sqcup S_2\sqcup S_3$ defines a face $\mathcal{G}$ of the principal boundary of $\mathcal{Q}$. This face can be identified with the moduli space of area one abelian differentials on $S_1\sqcup S_2\sqcup S_3$ with only simple zeros and which do not vanish identically on a smooth component. This moduli space can be described as follows. Let $\mathcal{Q}_{S_1\sqcup S_2\sqcup S_3}$ be the space of triples $((\omega_1,p_1),(\omega_2,p_2^1,p_2^2),(\omega_3,p_3))$ where ω_i is an abelian differential of area $b_i>0$ on the surface S_i with only simple zeros and such that $\sum_i b_i=1$, and where p_i^j are distinct marked points on the surface S_i. Then $\mathcal{G}$ is the quotient of $\mathcal{Q}_{S_1\sqcup S_2\sqcup S_3}$ under the involution ι which exchanges the first and third entry in the triple and exchanges the points p_2^1 and p_2^2. We refer to [EMZ03] for a more detailed description of the faces of the principal boundary of $\mathcal{Q}$.

Write

$$\mathcal{Q}_{S_1\sqcup S_2\sqcup S_3}=\cup_{b_i>0,b_1+b_2+b_3=1}\mathcal{Q}_{S_1\sqcup S_2\sqcup S_3}(b_1,b_2,b_3)$$

where an abelian differential in the space $\mathcal{Q}_{S_1\sqcup S_2\sqcup S_3}(b_1,b_2,b_3)$ gives area b_i to S_i. Note that the map F which associates to a differential $\zeta\in\mathcal{Q}_{S_1\sqcup S_2\sqcup S_3}$ the triple

$F(\zeta) = (b_1, b_2, b_3) \in \mathbb{R}^3$ determined by the requirement that $\zeta \in \mathcal{Q}_{S_1 \sqcup S_2 \sqcup S_3}(F(\zeta))$ is continuous.

Since any closed Riemann surface of genus $h \geq 2$ admits abelian differentials with simple zeros, there is an open subset U of $\overline{\mathcal{Q}}$ (here $\overline{\mathcal{Q}}$ denotes as before the union of $\mathcal{Q}$ with its principal boundary) of the form

$$U = (V \times \hat{B}(\epsilon) \times \hat{B}(\epsilon))/\mathcal{J}$$

with $V \subset \mathcal{Q}_{S_1 \sqcup S_2 \sqcup S_3}$ open and invariant under the involution ι, and where $\mathcal{J}$ acts as ι on V and as the standard involution on $\hat{B}(\epsilon) \times \hat{B}(\epsilon)$ exchanging the two components of a point in the product. Moreover, for each $x \in V$, the set $(\{x, \iota x\} \times \hat{B}(\epsilon) \times \hat{B}(\epsilon))/\mathcal{J}$ is contained in a leaf of the absolute period foliation. The existence of such an open set U follows from the construction in Proposition 3.1 (compare also the discussion in [EMZ03]).

Namely, a regular point in the face defined by the surface with nodes $S_1 \sqcup S_2 \sqcup S_3$ consists of a triple $(\omega_1, \omega_2, \omega_3)$ of abelian differentials on surfaces S_1, S_2, S_3 attached at marked points (where S_1, S_3 contains a single marked point, and S_2 contains two marked points). These marked points are regular for ω_i. Two applications of the construction in the proof of Proposition 3.1 at the two nodes then yields an open set U in $\overline{\mathcal{Q}}$ with the required properties.

As the set A is saturated for the absolute period foliation, it follows from Proposition 3.1 and its proof and from two applications of Corollary 3.6 that there is a ι-invariant Borel subset Z of $\mathcal{Q}_{S_1 \sqcup S_2 \sqcup S_3}$ of positive Lebesgue measure which is saturated for the absolute period foliation and such that up to a set of measure zero, the intersection of A with the set U equals

$$\big(((V \cap Z) \times \hat{B}(\epsilon) \times \hat{B}(\epsilon))/\mathcal{J}\big) \cap \mathcal{Q}.$$

By Corollary 3.6, it now suffices to show that the set Z has full Lebesgue measure.

By induction hypothesis and the case $g = 2$ established in [McM14], the absolute period foliation of the principal stratum for each of the surfaces S_i is ergodic. Note that as we assume that $g \geq 6$, the genus of S_2 is at least two. By Lemma 3.5, applied to three surfaces instead of two (which follows from exactly the same argument), this implies that there is a Borel subset D_0 of the set $D = \{(b_1, b_2, b_3) \in \mathbb{R}^3 \mid b_i > 0, b_1 + b_2 + b_3 = 1\}$ such that up to a set of measure zero, the subset Z of $\mathcal{Q}_{S_1 \sqcup S_2 \sqcup S_3}$ satisfies

$$Z = \cup_{x \in D_0} \mathcal{Q}_{S_1 \sqcup S_2 \sqcup S_3}(x).$$

Moreover, the Lebesgue measure of D_0 is positive. Our goal is to show that the Lebesgue measure of $D - D_0$ vanishes.

Let Σ be a surface of genus $g-2$ (which should be viewed as a smooth component of a surface with a single node in the Deligne Mumford compactification of the moduli space of S whose second smooth component is the surface S_1). The surface with nodes $S_2 \sqcup S_3$ determines an infinite core face of the moduli space of abelian differentials on Σ. There is then a natural identification of $V \times (\hat{B}(\epsilon) - \{0\})$ with an open subset of the principal stratum in the moduli space of abelian differentials on $S_1 \sqcup \Sigma$ which is obtained by opening the second node as described in the proof of Proposition 3.1.

By Corollary 3.6 and the induction hypothesis, applied to S_1 and Σ, there is a Borel set $C_1 \subset (0,1)$ of positive Lebesgue measure such that

$$Z = \cup_{b\in C_1} \mathcal{Q}_{S_1 \sqcup \Sigma}(b, 1-b).$$

As a consequence, the set D_0 is of the form

$$D_0 = \{(b_1, b_2, b_3) \mid b_1 \in C_1, b_1 + b_2 + b_3 = 1\}. \tag{2}$$

Exchanging the roles of S_1 and S_3 shows that on the other hand, there is a Borel set $C_3 \subset (0,1)$ of positive Lebesgue measure such that

$$D_0 = \{(b_1, b_2, b_3) \mid b_3 \in C_3, b_1 + b_2 + b_3 = 1\}. \tag{3}$$

Since the Lebesgue measure of D_0 is positive, Lemma 3.7 yields that $D - D_0$ is a set of measure zero and therefore Z has full Lebesgue measure. The induction step now follows from Corollary 3.6 as advertised before.

To complete the proof of the theorem we are left with showing ergodicity of the absolute period foliation for $g = 4$ and $g = 5$.

We begin with the case $g = 4$. We can not use the argument from the induction step directly, but we will use a similar argument. To this end let now S be a surface of genus 4 and consider a core face $\mathcal{F}$ of the principal boundary of the moduli space of abelian differentials on S defined by a surface with nodes $S_1 \sqcup S_2$ where S_i is a surface of genus 2. Using the above notations, the face $\mathcal{F}$ can be identified with the quotient of the space $\mathcal{Q}_{S_1 \sqcup S_2}$ under the involution ι which exchanges the two factors. As above, our analysis is local, in a neighborhood of $\mathcal{F}$, and hence we may ignore the presence of the involution ι and argue directly in the space $\mathcal{Q}_{S_1 \sqcup S_2}$, replacing Borel sets in $\mathcal{Q}_{S_1 \sqcup S_2}/\iota$ by their preimages in $\mathcal{Q}_{S_1 \sqcup S_2}$.

Let $A \subset \mathcal{S} \subset \mathcal{Q}$ be a Borel set which is saturated for the absolute period foliation and of positive Lebesgue measure. Choose an open neighborhood $U = V \times \hat{B}(\epsilon)$ in $\overline{\mathcal{Q}}$ of a regular point in $\mathcal{F}$. Using the fact that the absolute period foliation of the principal stratum on a surface of genus 2 is ergodic [McM14], Corollary 3.6 shows that there is a Borel set $E \subset (0,1)$ of positive Lebesgue measure such that

$$A \cap U = \big((\cup_{b\in E} \mathcal{Q}_{S_1 \sqcup S_2}(b, 1-b) \cap V)/\iota \times B(\epsilon)\big) \cap \mathcal{Q}. \tag{4}$$

Note that this identity holds true for all open subsets of $\overline{\mathcal{Q}}$ with the properties described in Proposition 3.1. Following the strategy used in the induction step, our goal is to show that E has full Lebesgue measure.

Replace $A \cap U$ by its preimage under the involution ι. By abuse of notation, we denote this set again by A. Decompose the surface S_2 into two tori T_2, T_3 and examine the corresponding core face of the principal boundary of S_2. If the absolute period foliation on a torus were ergodic, we could use exactly the argument from the induction step. However, this is not the case, so we have to be more careful.

Consider the face $\mathcal{G} = \mathcal{Q}_{S_1 \sqcup T_2 \sqcup T_3}$ of the principal boundary of $\mathcal{Q}$ determined by the decomposition $S_1 \sqcup T_2 \sqcup T_3$. There is an open subset O of $\overline{\mathcal{Q}}$ of the form

$$O = W \times \hat{B}(\epsilon) \times \hat{B}(\epsilon)$$

where $W \subset \mathcal{Q}_{S_1 \sqcup T_2 \sqcup T_3}$. Identity (4) shows that up to a set of measure zero, we have

$$A \cap O = \big((\cup_{b \in E, b_2+b_3=1-b} \mathcal{Q}_{S_1 \sqcup T_2 \sqcup T_3}(b, b_2, b_3) \cap W) \times \hat{B}(\epsilon) \times \hat{B}(\epsilon) \big) \cap O. \tag{5}$$

In other words, using the above notation, the following holds true. Let

$$D_0 = \{(b_1, b_2, b_3) \mid b_1 \in E, b_1 + b_2 + b_3 = 1\} \subset D$$

where the set D is defined in (1); then

$$A \cap O = \big((\cup_{x \in D_0} \mathcal{Q}_{S_1 \sqcup T_2 \sqcup T_3}(x) \cap W) \times \hat{B}(\epsilon) \times \hat{B}(\epsilon) \big) \cap O. \tag{6}$$

In particular, $A \cap O$ is saturated for the foliation of O into sets of the form $\{\omega\} \times \mathcal{T}(s) \times \hat{B}(\epsilon)$ where ω is an abelian differential of area $1-s$ on a surface Σ of genus three obtained by opening up the first node in the surface with nodes $S_1 \sqcup T_2 \sqcup T_3$ and where $\mathcal{T}(s)$ denotes the moduli space of area s abelian differentials on flat tori with one marked point.

This observation allows to use the argument from the induction step. Namely, the surface with nodes $\Sigma \sqcup T_3$ determines a core face of the principal stratum $\mathcal{Q}$ for S. Since the absolute period foliation of the principal stratum in genus 3 is ergodic [McM14] and since A is saturated for the foliation of O into the sets of the form $\{\omega\} \times \mathcal{T}(s) \times \hat{B}(\epsilon)$, there is a Borel set $C \subset (0,1)$ of positive Lebesgue measure such that

$$A \cap O = \big((\cup_{s \in C} \mathcal{Q}_{\Sigma \sqcup T_3}(1-s, s) \cap W) \times \hat{B}(\epsilon) \times \hat{B}(\epsilon) \big) \cap O. \tag{7}$$

Comparing with the equation (5) for $A \cap O$, an application of Lemma 3.7 now shows that C has full measure in $(0,1)$. Then Corollary 3.6 yields that A has full measure as well which is what we wanted to show.

Note that the argument given for $g = 4$ applies to any $g \geq 4$ (we included the somewhat simpler argument for the case $g \geq 6$ to give a more transparent exposition of the main idea). In particular, this argument applies to the case $g = 5$. This completes the discussion of the base of the induction and finishes the proof. □

Remark 3.9. Our proof of ergodicity of the absolute period foliation of the principal stratum is also valid for other strata provided that two conditions are satisfied. First we have to be careful about the distributions of zeros on the smooth components of the core faces of the principal boundary. More importantly, to start the induction step we have to establish ergodicity for two base cases of distinct genus. In case of the principal boundary, these two cases are due to McMullen and are proved with completely different methods which can not be applied if the genus is at least 4. In fact I do not know a single example of a component of a stratum with more than one zero which is not principal and where ergodicity of the absolute period foliation is known.

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