

Lecture 5: The ending lamination conjecture in the case of a positive injectivity radius

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Hyperbolic 3-manifolds diffeomorphic to $S \times \mathbb{R}$

S denotes a *closed* oriented surface of genus $g \geq 2$ with set $\mathcal{C}(S)$ of free homotopy classes of simple closed curves defining the vertex set of the curve graph $\mathcal{CG}(S)$.

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$\rho : \Gamma \rightarrow PSL(2, \mathbb{C})$ a discrete faithful representation s.th.

i) $M = \mathbf{H}^3/\Gamma$ is diffeomorphic to $S \times \mathbb{R}$

ii) There is $\epsilon > 0$: $\forall x \in M, \text{inj}(x) \geq \epsilon$.

Recall: The *convex core* $\mathbf{Core}(M)$ of M is the quotient under Γ of the *convex hull of the limit set* Λ of Γ .

3 Cases:

1. $\mathbf{Core}(M)$ is compact
 - $\Leftrightarrow \Gamma$ is *convex cocompact*
 - $\Leftrightarrow \Gamma$ is *quasifuchsian*
 - \Leftrightarrow the limit set Λ is an embedded circle in S^2
 - \Leftrightarrow if $\Omega = S^2 - \Lambda$ then Ω/Γ consists of a pair of Riemann surfaces diffeomorphic to S which determine M up to isometry.
2. $\partial\mathbf{Core}(M)$ is a connected surface diffeomorphic to S
 - $\Leftrightarrow M$ has precisely one geometrically finite end
 - $\Leftrightarrow \Omega = S^2 - \Lambda$ is a disc, Ω/Γ is a Riemann surface diffeomorphic to S .
3. $\Lambda = S^2$, i.e M is *completely degenerate*.

Definition 1: An end E is **simply degenerate** if it is contained in $\text{Core}(M)$.

Fact: Every free homotopy class in S can be represented by a unique geodesic in M .

Theorem (Bonahon-Thurston): For every simply degenerate end E there is a sequence (c_i) of simple closed curves whose geodesic representatives in M exit the end.

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Idea: Try to control the curves c_i .

Definition 2: A **pleated surface** in M is a map $f : (S, x) \rightarrow M$ for some hyperbolic surface structure $x \in \mathcal{T}(S)$ which is a homotopy equivalence and s.th.

1. f is a path-isometry.
2. There is a geodesic lamination λ on S s.th. $f(\lambda)$ is geodesic in M and $f|_{S - \lambda}$ is totally geodesic.

A lamination λ which arises as above from a pleated surface S is called *realized*.

Proposition 1:

1. Every simple multicurve on c is realized.
2. The diameter in M of a pleated surface is uniformly bounded.

Important question: For which $x \in \mathcal{T}(S)$ there is a pleated surface $f : (S, x) \rightarrow M$?

Proposition 2 (Minsky): $\forall a > 0$ exists $b = b(a, S, \epsilon) > 0$: If $M \sim S \times \mathbb{R}$, $\text{inj}(M) \geq \epsilon$, $g : (S, \sigma) \rightarrow M$, $h : (S, \rho) \rightarrow M$ pleated surfaces with $d_M(g(S), h(S)) \leq a$

$\Rightarrow d_{\mathcal{T}(S)}(\sigma, \rho) \leq b$.

Proof: Choose X_g filling short curve systems for g .

Then: $\text{inj}(M) \geq \epsilon$

$\Rightarrow d(\alpha, \text{geodesic representative of } \alpha \text{ in } M) \leq \text{const}$

\Rightarrow enough to show: If $g : (S, \sigma) \rightarrow M$ is a pleated surface

$d(g(S), \alpha) \leq a$ for some closed geodesic in M of length $\leq \kappa$ then $\text{length}(\alpha \text{ on } g(S)) \leq b$.

Otherwise: Ex. sequence (M_i) of 3-manifolds homeomorphic to $S \times \mathbb{R}$, $\text{inj}(M_i) \geq \epsilon$, sequence of geodesics α_i on M_i of length $\leq \kappa$, sequence of pleated surfaces $g : (S_i, \sigma_i) \rightarrow M_i$ with $d_{M_i}(g_i(S), \alpha_i) \leq a$, $\ell_{\sigma_i}(\alpha_i) \rightarrow \infty$.

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Thurston: A sequence of representations $\rho_i : \Gamma \rightarrow PSL(2, \mathbb{C})$ converging *algebraically* with $\text{inf} \geq \epsilon$ converges *strongly*.

A sequence of pleated surfaces $g_i(S_i, \sigma_i) \rightarrow M_i$ as above has convergent subsequence

\Rightarrow get lengthbound for α_i on (S, σ_i) from the length of α on the limiting pleated surface.

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For each j , α_j, α_{j+1} are contained in a pleated surface (S, σ_j) with image set $S_j \subset M$.

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$$S_j \cap S_{j+1} \neq \emptyset \Rightarrow d_{\mathcal{T}(S)}(\sigma_j, \sigma_{j+1}) \leq b$$

\Rightarrow the assignment $j \rightarrow \sigma_j$ is a *quasi-convex curve* in $\mathcal{T}(S) \Rightarrow$ ex.

Teichmüller geodesic arc γ_i in $\mathcal{T}(S)_\epsilon$ containing (σ_j) in a uniformly bounded neighborhood.

$$\Rightarrow d_M(c_0, c_i) \leq \text{const} d_{\mathcal{C}(S)}(c_0, c_i)$$

$$\Rightarrow d_{\mathcal{C}(S)}(c_0, c_i) \rightarrow \infty \quad (i \rightarrow \infty).$$

By passing to a limit: Can assume the geodesics γ_i converge to Teichmüller geodesic ray $\gamma : [0, \infty)$ "following pleated surfaces".

Masur: Horizontal measured geodesic lamination of γ is uniquely ergodic; support is the *ending lamination* of E .

By construction: Every point in E is contained in uniformly bounded neighborhood of the pleated surfaces defining γ !!

Conclusion: The ending lamination is unique.

Definition 3: The **end-invariants of M** consists of:

- 1) If M is convex cocompact: The Riemann surface Ω/Γ .
- 2) If M has one degenerated end E : The Riemann surface Ω/Γ and the ending lamination of E .
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Theorem 3 (Minsky 1994): Let M_1, M_2 be two hyperbolic 3-manifolds diffeomorphic to $S \times \mathbb{R}$, $\text{inj}(M_i) \geq \epsilon$. If the end-invariants of M_1, M_2 coincide then M_1 and M_2 are isometric.