Lecture 4: Cobounded Teichmüller geodesics and the curve graph

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Cobounded Teichmüller geodesics and the curve graph

S denotes a *closed* oriented surface of genus $g \geq 2$ with set $\mathcal{C}(S)$ of free homotopy classes of simple closed curves defining the vertex set of the curve graph $\mathcal{CG}(S)$.

A pair (λ, μ) of measured geodesic laminations with

$$i(\lambda, \mu) = 1, i(\lambda, c) + i(\mu, c) > 0 \forall c \in \mathcal{C}(S)$$

defines an *area one singular euclidean metric* and a *Teichmüller geodesic*

$$t \to \text{ the complex structure defined by } (e^t \lambda, e^{-t} \mu)$$

in Teichmüller space $\mathcal{T}(S)$.

Definition 1: A sequence of simple closed geodesics $c_i \subset \mathcal{C}(S)$ converges in the coarse Hausdorff topology to a *minimal geodesic lamination* λ *which fills up* S if every accumulation point of (c_i) in the Hausdorff topology contains λ as a subset.

Theorem (Klarreich): If $\gamma : [0, \infty) \to \mathcal{C}(S)$ is a quasi-geodesic then $(\gamma(i))$ converges in the coarse Hausdorff topology to a minimal geodesic lamination which fills up S.

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Recall: There is a map $\Phi : \mathcal{T}(S) \to \mathcal{CG}(S)$:
$\forall x, \Phi(x)$ is simple closed x-geodesic of length at most χ_0 ($\chi_0 > 0$ a Bers constant).

The map Φ is
- *coarsely Lipschitz continuous*: Exists L s.th.

$$d(\Phi x, \Phi y) \leq Ld(x, y) + L \forall x, y$$

(here d is the Teichmüller distance on $\mathcal{T}(S)$.)
- *coarsely $\mathcal{M}(S)$-equivariant*: Exists C s.th.

$$d(g\Phi x, \Phi gx) \leq C \, \forall x \in \mathcal{T}(S), \, \forall g \in \mathcal{M}(S).$$

Definition 2: For $\epsilon > 0$, a quasi-convex curve in $\mathcal{T}(S)_\epsilon$ is a closed subset of $\mathcal{T}(S)$ whose *Hausdorff distance* to the image of a geodesic arc $\zeta : J \to \mathcal{T}(S)_\epsilon$ is at most $1/\epsilon$.

Theorem 1: For every $\nu > 1$ there is a constant $\epsilon = \epsilon(\nu) > 0$:
Let $J \subset \mathbb{R}$ be a closed connected set of diameter at least $1/\epsilon$.
Let $\gamma : J \to \mathcal{T}(S)$ be a ν-quasi-geodesic.
If $\Phi \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{GC}(S)$ then $\gamma(J)$ is a quasi-convex curve in $\mathcal{T}(S)_\epsilon$.

Lemma 2: For every $\nu > 1$ there is a number $\epsilon_0 = \epsilon_0(\nu) > 0$ s.th: Let $\gamma : [0, n] \to \mathcal{T}(S)$ be a ν-quasi-geodesic whose projection $\Phi\gamma$ to $\mathcal{GC}(S)$ is a ν-quasi-geodesic. If $n \geq 1/\epsilon_0$ then $\gamma[0, n] \subset \mathcal{T}(S)_{\epsilon_0}$.

Proof: Wolpert:

$$d(h, h') \geq |\log \ell_h(\alpha) - \log \ell_{h'}(\alpha)| \forall \alpha \in \mathcal{C}(S), \forall h, h' \in \mathcal{T}(S).$$

□

Proof of the theorem:

Step 1: For $\nu > 1$ define a *ν-Lipschitz curve* in $\mathcal{T}(S)$ to be a ν-Lipschitz map $\gamma : J \to \mathcal{T}(S)$.

It is enough to show the statement of the theorem for ν-Lipschitz ν-quasi-geodesics s.th. $\Phi \circ \gamma$ is a ν-quasi-geodesic in $\mathcal{GC}(S)$.

Also: By Lemma 2, assume $\gamma(J) \subset \mathcal{T}(S)_{\epsilon_0}$.

Step 2:
$\mathcal{C}(S)$ is hyperbolic, $\Phi \circ \gamma$ a ν-quasi-geodesic
$\Rightarrow$ there is a geodesic arc ζ in $\mathcal{C}(S)$,

$$d_{\mathrm{Hausdorff}}(\zeta, \Phi \circ \gamma) \leq C.$$

J one-sided infinite, say $[0, \infty) \subset J$
$\Rightarrow$ $\Phi(\gamma(t))$ converges in the coarse Hausdorff topology to a minimal geodesic lamination χ which fills up S.

$\mathcal{ML}$ is the space of measured geodesic laminations. The projective measured lamination $[\alpha]$ defined by $\alpha \in \mathcal{C}(S)$ is *realized* at some $t \in J$ if $\ell_{\gamma(t)}(\alpha) \leq \chi$.

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The projectivization $[\lambda]$ of a measured geodesic lamination λ is realized at an infinite "endpoint" of J if the support of λ equals the coarse Hausdorff limit of the quasi-geodesic $\Phi\gamma(J)$.

Call a projective measured lamination which is realized at a (finite or infinite) endpoint of J an *endpoint lamination*.

Transport the distance on $\mathcal{GC}(S)$ to a *continuous* distance function on $\mathcal{T}(S)$:

Choose a number $R > 2\chi$ and a smooth function $\sigma : [0, \infty) \to [0, 1]$ with $\sigma[0, \chi] \equiv 1$ and $\sigma[R, \infty) \equiv 0$.

For every $h \in \mathcal{T}(S)$ define a finite Borel measure μ_h on $\mathcal{C}(S)$ by

$$\mu_h = \sum_\beta \sigma(\ell_h(\beta))\delta_\beta$$

where δ_β denotes the Dirac mass at β.

Properties: There is $C > 0$ s.th.
1) $1/C \le \mu_h(\mathcal{C}(S)) \le C$.
2) $\mathrm{diam}(\mathrm{supp}(\mu_\mathrm{h})) \le \mathrm{C}$.
3) μ_h depends continuously on h.

Define a new "distance" function ρ on $\mathcal{T}(S)$ by

$$\rho(h, h') = \int_{\mathcal{C}(S)\times\mathcal{C}(S)} d(\cdot,\cdot) d\mu_h \times d\mu'_h / \mu_h(\mathcal{C}(S))\mu_{h'}(\mathcal{C}(S)).$$

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1) ρ is positive and continuous on $\mathcal{T}(S)\times\mathcal{T}(S)$ and invariant under the action of $\mathcal{M}(S)$.
2) There is $a > 0$ s.th.

$$\rho(h,h')/a - a \le d(\Phi(h),\Phi(h')) \le a\rho(h,h') + a.$$

$\Rightarrow$ for $\nu > 1$ there is $p = p(\nu) > 1$:
If $\gamma : J \to \mathcal{T}(S)$ is s.th. $\Phi\gamma$ is a ν-quasi-geodesic
$\Rightarrow$ γ is a p-quasi-geodesic with respect to ρ:

$$\rho(\gamma(s),\gamma(t))/p - p \le |s-t| \le p\rho(\gamma(s),\gamma(t)) + p \forall s,t \in J.$$

For $h \in \mathcal{T}(S), \mu \in \mathcal{ML}$ the product of the transverse measure for μ together with the length element of h defines a measure on $\mathrm{supp}(\mu)$.
Its total mass is called the *h-length* $\ell_h(\mu)$ of μ.

For $p > 1$ define Γ_p to be the set of all triples $(\gamma : J \to \mathcal{T}(S), \lambda_+, \lambda_-)$ s.th.

1. $0 \in J$ and $\mathrm{diam}(J) \geq 1/\epsilon_0$.
2. $\gamma : J \to \mathcal{T}(S)$ is a p-Lipschitz curve which is p-quasi-geodesic w.r.to ρ.
3. $\lambda_+, \lambda_- \in \mathcal{ML}$ are of $\gamma(0)$-length 1, and the projectivizations $[\lambda_+], [\lambda_-]$ are realized at the ends.

Equip Γ_p with the product topology of the weak*-topology on $\mathcal{ML}$ and the compact-open topology for the arc $\gamma \subset \mathcal{T}(S)$.
This topology is metrizable.

Claim: The action of $\mathcal{M}(S)$ on Γ_p is cocompact.
Proof of the claim: By equivariance and cocompactness of the action of $\mathcal{M}(S)$ on $\mathcal{T}(S)_\epsilon$:
Enough to show the claim for the subset of Γ_p consisting of triples with the additional property that $\gamma(0) \in A$, $A \subset \mathcal{T}(S)$ compact.
For this: Use the Arzela-Ascoli theorem.

Each point $(\gamma, \lambda_+, \lambda_-) \in \Gamma_p$ determines the geodesic $\eta([\lambda_+], [\lambda_-])$ in $\mathcal{T}(S)$.
For $(\gamma, \lambda_+, \lambda_-)$ define $\sigma(\gamma, \lambda_+, \lambda_-)$ to be the point on the geodesic $\eta([\lambda_+], [\lambda_-])$ which corresponds to the quadratic differential defined by the measured geodesic laminations λ_+, λ_-.

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The map taking $(\gamma, \lambda_+, \lambda_-)$ to $(\gamma(0), \sigma(\gamma, \lambda_+, \lambda_-)) \in \mathcal{T}(S) \times \mathcal{T}(S)$ is continuous and equivariant with respect to the natural action of $\mathcal{M}(S)$ on Γ_p and on $\mathcal{T}(S) \times \mathcal{T}(S)$
$\Rightarrow$ the action of $\mathcal{M}(S)$ on the image of our map is cocompact
$\Rightarrow d(\gamma(0), \sigma(\gamma, \lambda_+, \lambda_-)) \leq p > 0$.

Let $(\gamma, \lambda_+, \lambda_-) \in \Gamma_p$. For each $s \in J$ define

$$a_-(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_-)}, \quad a_+(s) = \frac{1}{\ell_{\gamma(s)}(\lambda_+)}$$

where: $\ell_{\gamma(s)}(\lambda_\pm)$ is the $\gamma(s)$-length of $\lambda_\pm$.
For $s \in \mathbb{R}$ define $\gamma'(t) = \gamma(t+s)$
$\Rightarrow$ the triple $(\gamma'(0), a_+(s)\lambda_+, a_-(s)\lambda_-)$ lies in the $\mathcal{M}(S)$-cocompact set Γ_p
$\Rightarrow d(\gamma(s), \eta([\lambda_+], [\lambda_-])) \leq p$
$\Rightarrow \gamma$ is contained in the p-neighborhood of $\eta([\lambda_+], [\lambda_-])$. $\square$