Lecture 2: Teichmüller geodesics and the curve complex

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1. Singular euclidean metrics

S denotes a *closed* oriented surface of genus $g \geq 2$.
Recall: Tow simple closed curves $\alpha, \beta \in \mathcal{C}(S)$ with $d(\alpha, \beta) \geq 3$ *fill up* S.
Define a *singular euclidean metric* using α, β.

The *width* of an *annulus* in a surface with singular euclidean metric is the distance between its boundary curves.

Proposition 1: There is $w > 0$: Every area one singular euclidean metric on S has an annulus of width $\geq w$.

Proof (Bowditch): Use *isoperimetric inequality*:
In the euclidean plane,

$$\mathrm{length}(\gamma) \geq 2\sqrt{\pi \mathrm{area}(\text{disc enclosed by } \gamma)}.$$

In piecewise euclidean metric with one p-pronged singularity:
γ a Jordan curve enclosing the singularity
$\Rightarrow$ divide the enclosed disc along boundaries of sectors
$\Rightarrow$ use euclidean isoperimetric inequality on sectors to deduce:
Exists $f : \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{area}(D) \leq f(\mathrm{length}(\partial D))$$

for every disc D in the singular euclidean surface.
A *spine* of S is a *embedded graph* G s.th. $S - G =$ union of discs.
Consequence: The length of a spine is uniformly bounded from below.

Let c be a shortest simple closed geodesic on S (of length η_0).

Fact: If η_0 is large then c is the core curve of a cylinder of width $\geq \chi$ ($\chi > 0$ a universal constant).

Namely: Let

$$\epsilon_0 = \sup\{\epsilon \mid N_\epsilon(c) = \mathrm{annulus}\}$$

$\Rightarrow$ there is a geodesic arc of length $2\epsilon_0$ with both endpoints on c
$\Rightarrow$ if $\epsilon_0 < \eta_0/4$ then c is not shortest.

Second case:

Assume η_0 is small.
Choose $\eta_1 > 0$ universal (depending on S).
Choose a maximal collection of simple arcs, length $\leq 2\eta_1$, with endpoints on c
$\Rightarrow$ the *number* of the arcs is bounded by topology
$\Rightarrow$ the *total length* uniformly bounded
$\Rightarrow$ the collection can not be a spine of $S - N_{\eta_1}(c)$
$\Rightarrow$ there exists a nontrivial component of $S - N_{\eta_1}(c)$
$\Rightarrow$ there exists a nontrivial simple closed curve d in $S - N_{\eta_1}(c)$.

Choose $f : S \to [0, \infty)$, distance decreasing, $f(c) = 0, f(d) = \eta_1$
$\Rightarrow$ $f^{-1}[a, b]$ is not a disc
$\Rightarrow$ find annulus with large width in preimages of disjoint intervals. □

Conclusion: $\forall\alpha, \beta$, $\forall a, b > 0$ with $abi(\alpha, \beta) = 1$ there is δ:

$$i(\rho, \delta) \leq \mathrm{const}\big(ai(\rho, \alpha) + bi(\rho, \beta)\big)\forall\rho.$$

Also: The *length* of δ with respect to the singular euclidean metric q defined by c, d is uniformly bounded.

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Corollary 2:

1. The diameter in $\mathcal{CG}(S)$ of the set of short q-curves is uniformly bounded.
2. The distance in $\mathcal{CG}(S)$ of a short q-curve and a short curve for the hyperbolic metric defined by q is uniformly bounded.

Find a path $c_0, \ldots, c_k \subset \mathcal{C}(S)$ with

1. $c_0 = \alpha, c_k = \beta$.
2. $d(c_i, c_{i+1}) \leq k$.
3. $i(\alpha, c_i) i(c_i, \beta) \leq k i(\alpha, \beta)$.

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Construction: For $t \in \mathbb{R}$ consider the euclidean metric q_t defined by $(e^t c, e^{-t} d/i(c,d))$. Then:

1. t very large $\Rightarrow$ c is q_t-short.
2. t very small $\Rightarrow$ d is q_t-short.

Define $\eta_{c,d}(t) =$ a short curve for q_t.
Then: Unique up to a uniformly bounded error.
These curve vary "coarsely continuously" with t.

Corollary 3: There is $k > 0$ and for $\alpha, \beta, \gamma \in \mathcal{C}(S)$ there is $\delta \in \mathcal{C}(S)$:

$$\delta \in U_k(\eta_{\alpha,\beta}(\mathbb{R})) \cap U_k(\eta_{\beta,\gamma}(\mathbb{R})) \cap U_k(\eta_{\gamma,\alpha}(\mathbb{R})).$$

Proof: Let a, b, c s.th.

$$abi(\alpha, \beta) = bci(\beta, \gamma) = cai(\gamma, \alpha) = 1.$$

Let $\delta \in \mathcal{C}(S)$,

$$i(\delta, \rho) \leq \mathrm{const} \max\{ai(\alpha, \rho), bi(\beta, \rho)\} \forall \rho.$$

Then $i(\delta, \gamma) \leq \mathrm{const}/c$ shows:

$$\max\{ci(\gamma, \delta), ai(\alpha, \delta)\} \leq \mathrm{const}$$

$\Rightarrow$ δ is short for the metric defined by $a\alpha, c\gamma$ and similarly for the metric defined by $b\beta, c\gamma$. □

Very important fact: If we replace c by a multi-curve containing c as a component then we find *nearby short curves*.

Homework: Prove this.

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Proposition 4: For any two points $c, d \in \mathcal{C}(S)$ there is a path $\eta_{c,d} : [0,1] \to \mathcal{CG}(S)$ s.th.

1. If $d(c,d) = 1$ then $\mathrm{diam}(\eta[0,1]) \leq C$.
2. $c, d \in \mathcal{C}(S), s < t \in [0,1] \Rightarrow$ Hausdorff-distance $d_H(\eta_{c,d}[s,t], \eta_{\eta_{c,d}(s),\eta_{c,d}(t)}[0,1]) \leq C$.
3. For a, b, c, $\eta_{[a,b]}[0,1] \subset N_C(\eta_{[b,c]}[0,1]) \cup \eta_{[c,a]}[0,1])$.

3. Hyperbolicity of the curve graph

Definition 1: A geodesic metric space X is δ-hyperbolic if $\forall$ triangles with geodesic sides a, b, c: $c \subset N_\delta(a \cup b)$.

Basic property: In a hyperbolic geodesic metric space, an L-quasi-geodesic is contained in a uniformly bounded neighborhood of a geodesic.

Theorem 5 (Masur-Minsky): The curve graph is hyperbolic.

3. Teichmüller geodesics

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Example: A *weighted simple multi-curve* is a geodesic multicurve whose components carry a positive weight.

Theorem 6 (Thurston):

1. The space $\mathcal{PML}$ of projective measured geodesic laminations equipped with the weak*-topology is homeomorphic to S^{6g-7}.
2. The intersection form i extends continuously to a symmetric function on the space $\mathcal{ML}$ of measured geodesic laminations.
3. If $\eta, \zeta \in \mathcal{ML}$ *fill up* S, i.e. if $i(c,\eta) + i(c,\zeta) > 0$ for all simple closed curves then (η, ζ) defines a *complex structure* $R(\eta, \zeta)$ on S and a *singular euclidean metric*.
4. If moreover $i(\eta, \zeta) = 1$ then $t \to R(e^t\eta, e^{-t}\zeta)$ is a *Teichmüller geodesic*.
5. (Teichmüller) Any two points in the *Teichmüller space* $\mathcal{T}(S)$ can be connected by a unique Teichmüller geodesic segment depending smoothly on the point.
6. Teichmüller space can naturally and $\mathcal{M}(S)$-equivariantly be compactified by adding $\mathcal{PML}$.

Homework: Recall that a mapping class is *periodic* if the subgroup of $\mathcal{M}(S)$ it generates is finite, and it is *pseudo-Anosov* if the subgroup it generates acts on $\mathcal{C}(S)$ with unbounded orbits. Show: A reducible mapping class preserves a multi-curve.