Lecture 1: Teichmüller geodesics and the curve complex

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May 28, 2007

1. Introduction

S denotes a *closed* oriented surface of genus $g \geq 2$.

Differential-geometric facts:
Fact 1: S admits a hyperbolic metric (constant curvature -1).
Fact 2: There is a constant $\chi_0 = \chi_0(S)$ such that for every hyperbolic metric g on S there is a *simple closed* g-geodesic of length at most χ_0.

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Definition 1: The mapping class group $\mathcal{M}(S)$ is the group of isotopy classes of orientation preserving diffeomorphisms of S.

Basic example of a closed 3-manifold:
The *mapping torus* of $\phi \in \mathcal{M}(S)$:

$$M = S \times [0,1]/\sim \text{ where } (x,1) \sim (\phi(x),0).$$

Basic facts: 1) M is a $K(\pi,1)$-space.
2) There is an exact sequence

$$0 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 0$$

i.e. $\pi_1(M)$ is a $\mathbb{Z}$-extension of $\pi_1(S)$.
In particular: $\pi_1(M)$ admits $\pi_1(S)$ as a normal subgroup and the natural $\mathbb{Z}$-cover of M is diffeomorphic to $S \times \mathbb{R}$.

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Basic question: How are
- the properties of ϕ
- the "geometry" of M (for a suitable choice of a metric)
- the topology of M

related?

Idea: Try to use an "easy to understand" combinatorial model for the geometry of M and relate this model to the topology of M.

Definition 2: An L-quasi-isometric embedding of a metric space (X, d) into a metric space (Y, d) is a map $F : X \to Y$ such that

$$d(x, y)/L - L \leq d(Fx, Fy) \leq Ld(x, y) + L$$

for all $x, y \in X$. F is an L-quasi-isometry if moreover for every $y \in Y$ there is $x \in X : d(F(x), y) \leq L$.

Basic observations:

1. Let Γ be a finitely generated group with finite *symmetric* generating set $\mathcal{G}$. The *word norm* $|g|$ of $g \in \Gamma$ is the minimum of a word in the generators representing g.

$$d(g,h) = |g^{-1}h|$$

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2. Let M be a closed 3-manifold with universal covering $\mathbb{R}^3$ $\Rightarrow$ the fundamental group $\pi_1(M)$ of M acts on $\tilde{M} = \mathbb{R}^3$ freely, properly discontinuously and cocompactly.
If g is any Riemannian metric on M then there is a $\pi_1(M)$-equivariant quasi-isometry $F : \tilde{M} \to \Gamma$.
The lifts to $\tilde{M}$ of any two metrics on M are quasi-isometric.

2. Let M be the mapping torus of $\phi \in \mathcal{M}(S)$. Choose a hyperbolic metric on S and a metric on M so that $S \to S \times \{0\}/\sim$ is an isometric embedding. Let $\hat{M}$ be the $\mathbb{Z}$-cover of M.

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Let c be a simple closed geodesic on S of length at most χ_0. Then for each $i \in \mathbb{Z}$, the minimal length of a closed curve in $\hat{M}$ representing $\phi^i(c)$ is *uniformly bounded, independent of* i.

Question: Can we recover from the "collection of shorts curves" in $\hat{M}$ the mapping class ϕ and hence the topology of M?

2. The complex of curves

Definition 3: The complex of curves is the simplicial complex whose *vertex set* $\mathcal{C}(S)$ is the set of all nontrivial free homotopy classes of simple closed curves on S. A collection $c_1, \ldots, c_k \subset \mathcal{C}(S)$ spans a simplex if and only if $c_1, \ldots, c_k$ can be realized disjointly. The curve graph $\mathcal{CG}(S)$ is the one-skeleton of the curve complex.

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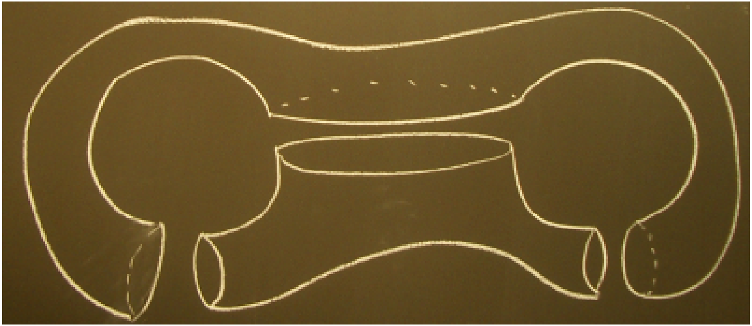
Facts: The complex of curves is connected and of dimension $3g - 4$.
A simplex of maximal dimension is a *pants decomposition* of S: After cutting S open along the curves of the simplex we obtain $2g - 2$ *pairs of pants*.
The curve graph is a naturally a *locally infinite metric graph*.

A pants decomposition for a surface of genus 2



Fact: If $c, d \in \mathcal{C}(S)$ and if $d(c, d) \geq 3$ then c, d fill up S, i.e. c, d decompose S into topological discs.

Definition 4: The intersection number $i(c, d)$ between two simple closed curves c, d is the minimal number of intersections between two curves freely homotopic to c, d.

Proposition 1: There is a number $\kappa = \kappa(S)$ such that $d(c,d) \leq \kappa \log i(c,d) + \kappa$ for all $c, d \in \mathcal{C}(S)$.

Proof: Let c, d be simple closed curves on S with minimal intersection number in their free homotopy classes.
d intersects $S - c$ in simple arcs with both endpoints on c.
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Let δ be an arc whose class contains the maximal number of components and let

$$b = \text{ a component of } \partial(N(c \cup \delta)).$$

Then $i(c,b) = 0 \Rightarrow d(c,b) = 1$ and
$i(b,d) \le (m-1)i(c,d)/m$. $\square$

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Recall: The *Hausdorff-distance* $d_H(A, B)$ of compact subsets A, B of S is

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Facts: 1) The space $\mathcal{L}(S)$ of geodesic laminations with the Hausdorff topology is compact.
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Definition 6: A geodesic lamination λ is maximal if $S - \lambda$ is a union of ideal triangles. It is complete if in addition λ can be approximated in the Hausdorff topology by simple closed geodesics. It is minimal if every half-leaf is dense.

Proposition 2: The diameter of $\mathcal{CG}(S)$ is infinite.

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Proof (Luo): By contradiction: *Assume* that $\mathrm{diam}(\mathcal{CG}(S)) = D < \infty$.
Let μ be a complete minimal geodesic lamination and let $c_i \to \mu$ $(i \to \infty)$ in the Hausdorff topology.
$d(c_0, c_i) \leq D \Rightarrow$ assume e.g. that $d(c_0, c_i) = N \forall i$.
Choose $b_i, d(c_0, b_i) = N - 1, d(b_i, c_i) = 1 \forall i$.
$b_i \to \mu$ *because* $i(b_i, c_i) = 0$ means: $b_i \cup c_i$ is as geodesic lamination $\Rightarrow$ converges up to passing to a subsequence to a geodesic lamination.
Repeat with $(b_i) \Rightarrow$ after N steps conclude that $c_0 \to \mu$, a contradiction. □

3. Pseudo-Anosov mapping classes

The mapping class group acts on $\mathcal{CG}(S)$ as a group of simplicial isometries.

Definition 6: A mapping class $\phi \in \mathcal{M}(S)$ is pseudo-Anosov if the orbit of $<\phi> < \mathcal{M}(S)$ on $\mathcal{CG}(S)$ is *unbounded*.
A mapping class ϕ is periodic if $|<\phi>| < \infty$. A mapping class which is neither periodic nor pseudo-Anosov is reducible.

Our basic example:

1) $\phi \in \mathcal{M}(S)$ periodic $\Leftrightarrow$ the mapping torus $M = \mathrm{Maptorus}(\phi)$ of ϕ has a *finite* covering diffeomorphic to $S \times S^1$

$\Rightarrow$ for *every* smooth metric on M and every $L > 0$ the number of elements in $\pi_1(\hat{M}) = \pi_1(S)$ which can be realized by a curve of length at most L in M is *finite*.

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Also: By the *Nielsen realization problem*, ϕ can be realized as an biholomorphic map of a hyperbolic surface

$\Rightarrow S \to S/ < \phi >$ is a branched covering $\Rightarrow M$ is foliated into smooth circles $\Leftrightarrow M$ is a *Seifert fibered space*.

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2) ϕ pseudo-Anosov $\Rightarrow$ for $M = \mathrm{Maptorus}(\phi)$ and every smooth metric on M there is some $L > 0$ such that the number of elements of $\pi_1(\hat{M})$ which can be realized by a curve of length at most L is *infinite*.

Afternoon discussion:
Construction of a minimal and complete geodesic lamination on a closed surface S.