Lecture 1: Teichmüller geodesics and the curve complex

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1. Introduction

$S$ denotes a *closed* oriented surface of genus $g \geq 2$.

**Differential-geometric facts:**

**Fact 1:** $S$ admits a hyperbolic metric (constant curvature $-1$).

**Fact 2:** There is a constant $\chi_0 = \chi_0(S)$ such that for every hyperbolic metric $g$ on $S$ there is a *simple closed* $g$-geodesic of length at most $\chi_0$. 

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Basic example of a closed 3-manifold:
The *mapping torus* of $\phi \in \mathcal{M}(S)$:

\[ M = S \times [0, 1]/ \sim \text{ where } (x, 1) \sim (\phi(x), 0). \]

**Basic facts:**
1) $M$ is a $K(\pi, 1)$-space.
2) There is an exact sequence

\[ 0 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 0 \]

i.e. $\pi_1(M)$ is a $\mathbb{Z}$-extension of $\pi_1(S)$.

In particular: $\pi_1(M)$ admits $\pi_1(S)$ as a normal subgroup and the natural $\mathbb{Z}$-cover of $M$ is diffeomorphic to $S \times \mathbb{R}$. 
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**Basic question:** How are
- the properties of $\phi$
- the "geometry" of $M$ (for a suitable choice of a metric)
- the topology of $M$
related?
Idea: Try to use an “easy to understand” combinatorial model for the geometry of $M$ and relate this model to the topology of $M$.

Definition 2: An $L$-quasi-isometric embedding of a metric space $(X, d)$ into a metric space $(Y, d)$ is a map $F : X \to Y$ such that

$$d(x, y)/L - L \leq d(Fx, Fy) \leq Ld(x, y) + L$$

for all $x, y \in X$. $F$ is an $L$-quasi-isometry if moreover for every $y \in Y$ there is $x \in X : d(F(x), y) \leq L$. 
Basic observations:

1. Let $\Gamma$ be a finitely generated group with finite symmetric generating set $G$. The word norm $|g|$ of $g \in \Gamma$ is the minimum of a word in the generators representing $g$.

   $$d(g, h) = |g^{-1}h|$$

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2. Let $M$ be a closed 3-manifold with universal covering $\mathbb{R}^3 \Rightarrow$ the fundamental group $\pi_1(M)$ of $M$ acts on $\tilde{M} = \mathbb{R}^3$ freely, properly discontinuously and cocompactly. If $g$ is any Riemannian metric on $M$ then there is a $\pi_1(M)$-equivariant quasi-isometry $F : \tilde{M} \to \Gamma$. The lifts to $\tilde{M}$ of any two metrics on $M$ are quasi-isometric.
2. Let $M$ be the mapping torus of $\phi \in \mathcal{M}(S)$. Choose a hyperbolic metric on $S$ and a metric on $M$ so that $S \to S \times \{0\}/\sim$ is an isometric embedding. Let $\hat{M}$ be the $\mathbb{Z}$-cover of $M$. 

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2. Let $M$ be the mapping torus of $\phi \in \mathcal{M}(S)$. Choose a hyperbolic metric on $S$ and a metric on $M$ so that $S \to S \times \{0\}/\sim$ is an isometric embedding. Let $\hat{M}$ be the $\mathbb{Z}$-cover of $M$.

Let $c$ be a simple closed geodesic on $S$ of length at most $\chi_0$. Then for each $i \in \mathbb{Z}$, the minimal length of a closed curve in $\hat{M}$ representing $\phi^i(c)$ is uniformly bounded, independent of $i$.

**Question**: Can we recover from the ”collection of shorts curves” in $\hat{M}$ the mapping class $\phi$ and hence the topology of $M$?
2. The complex of curves

Definition 3: The complex of curves is the simplicial complex whose vertex set $\mathcal{C}(S)$ is the set of all nontrivial free homotopy classes of simple closed curves on $S$. A collection $c_1, \ldots, c_k \subset \mathcal{C}(S)$ spans a simplex if and only if $c_1, \ldots, c_k$ can be realized disjointly. The curve graph $\mathcal{CG}(S)$ is the one-skeleton of the curve complex.
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Facts: The complex of curves is connected and of dimension $3g - 4$. A simplex of maximal dimension is a pants decomposition of $S$: After cutting $S$ open along the curves of the simplex we obtain $2g - 2$ pairs of pants. The curve graph is a naturally a locally infinite metric graph.
A pants decomposition for a surface of genus 2
Fact: If $c, d \in \mathcal{C}(S)$ and if $d(c, d) \geq 3$ then $c, d$ fill up $S$, i.e. $c, d$ decompose $S$ into topological discs.

Definition 4: The intersection number $i(c, d)$ between two simple closed curves $c, d$ is the minimal number of intersections between two curves freely homotopic to $c, d$. 
**Proposition 1:** There is a number $\kappa = \kappa(S)$ such that $d(c, d) \leq \kappa \log i(c, d) + \kappa$ for all $c, d \in C(S)$.

**Proof:** Let $c, d$ be simple closed curves on $S$ with minimal intersection number in their free homotopy classes. $d$ intersects $S - c$ in simple arcs with both endpoints on $c$. There are at most $m$ homotopy classes rel $c$ of such arcs.
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Proof: Let $c, d$ be simple closed curves on $S$ with minimal intersection number in their free homotopy classes. $d$ intersects $S - c$ in simple arcs with both endpoints on $c$. There are at most $m$ homotopy classes rel $c$ of such arcs. Let $\delta$ be an arc whose class contains the maximal number of components and let

$$b = \text{ a component of } \partial(N(c \cup \delta)).$$

Then $i(c, b) = 0 \Rightarrow d(c, b) = 1$ and $i(b, d) \leq (m - 1)i(c, d)/m$. \qed
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**Recall:** The Hausdorff-distance $d_H(A, B)$ of compact subsets $A, B$ of $S$ is

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset U_\epsilon(B), B \subset U_\epsilon(A)\}.$$ 

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**Facts:** 1) The space $\mathcal{L}(S)$ of geodesic laminations with the Hausdorff topology is compact.  
2) For every geodesic lamination $\lambda$, $S - \lambda$ is a hyperbolic surface with geodesic boundary and area $4\pi(g - 1)$.
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**Definition 6:** A geodesic lamination $\lambda$ is *maximal* if $S - \lambda$ is a union of ideal triangles. It is *complete* if in addition $\lambda$ can be approximated in the Hausdorff topology by simple closed geodesics. It is *minimal* if every half-leaf is dense.
Proposition 2: The diameter of $CG(S)$ is infinite.
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Proof (Luo): By contradiction: Assume that $\text{diam}(CG(S)) = D < \infty$.
Let $\mu$ be a complete minimal geodesic lamination and let $c_i \to \mu$ ($i \to \infty$) in the Hausdorff topology.
$d(c_0, c_i) \leq D \Rightarrow$ assume e.g. that $d(c_0, c_i) = N \forall i$.
Choose $b_i$, $d(c_0, b_i) = N - 1$, $d(b_i, c_i) = 1 \forall i$.
$b_i \to \mu$ because $i(b_i, c_i) = 0$ means: $b_i \cup c_i$ is as geodesic lamination $\Rightarrow$ converges up to passing to a subsequence to a geodesic lamination.
Repeat with $(b_i) \Rightarrow$ after $N$ steps conclude that $c_0 \to \mu$, a contradiction. $\square$
3. Pseudo-Anosov mapping classes

The mapping class group acts on $C\mathcal{G}(S)$ as a group of simplicial isometries.

**Definition 6:** A mapping class $\phi \in \mathcal{M}(S)$ is **pseudo-Anosov** if the orbit of $\langle \phi \rangle \mathcal{M}(S)$ on $C\mathcal{G}(S)$ is *unbounded*. A mapping class $\phi$ is **periodic** if $|\langle \phi \rangle| < \infty$. A mapping class which is neither periodic nor pseudo-Anosov is **reducible**.
Our basic example:

1) $\phi \in \mathcal{M}(S)$ periodic $\iff$ the mapping torus $M = \text{Maptorus}(\phi)$ of $\phi$ has a *finite* covering diffeomorphic to $S \times S^1$

$\Rightarrow$ for every smooth metric on $M$ and every $L > 0$ the number of elements in $\pi_1(\hat{M}) = \pi_1(S)$ which can be realized by a curve of length at most $L$ in $M$ is *finite*.
Our basic example:
1) \( \phi \in \mathcal{M}(S) \) periodic \( \iff \) the mapping torus \( M = \text{Maptorus}(\phi) \) of \( \phi \) has a \textit{finite} covering diffeomorphic to \( S \times S^1 \)
\( \Rightarrow \) for \textit{every} smooth metric on \( M \) and every \( L > 0 \) the number of elements in \( \pi_1(\hat{M}) = \pi_1(S) \) which can be realized by a curve of length at most \( L \) in \( M \) is \textit{finite}.

Also: By the \textit{Nielsen realization problem}, \( \phi \) can be realized as an biholomorphic map of a hyperbolic surface
\( \Rightarrow S \rightarrow S/\langle \phi \rangle \) is a branched covering \( \Rightarrow M \) is foliated into smooth circles \( \iff M \) is a \textit{Seifert fibered space}. 
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$\Rightarrow S \to S/\langle \phi \rangle$ is a branched covering $\Rightarrow M$ is foliated into smooth circles $\iff M$ is a *Seifert fibered space*.

2) $\phi$ pseudo-Anosov $\Rightarrow$ for $M = \text{Maptorus}(\phi)$ and every smooth metric on $M$ there is some $L > 0$ such that the number of elements of $\pi_1(\hat{M})$ which can be realized by a curve of length at most $L$ is *infinite*.
Afternoon discussion:
Construction of a minimal and complete geodesic lamination on a closed surface $S$. 