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LOCAL MARKED LENGTH SPECTRUM RIGIDITY
[after Guillarmou and Lefeuvre]

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INTRODUCTION

The search for characterizing a smooth Riemannian metric on a smooth closed manifold M by easy to define geometric quantities has a long and fruitful history, usually described as rigidity problems.

A particularly appealing rigidity problem can be formulated as follows. Consider a closed manifold M of dimension $n \geq 2$, equipped with a Riemannian metric g_0 of non-positive sectional curvature. By the Hadamard Cartan theorem, the universal covering $\tilde{M}$ of M is diffeomorphic to $\mathbb{R}^n$ and hence M is a classifying space for its fundamental group $\pi_1(M)$. Each nontrivial conjugacy class c in $\pi_1(M)$ can be represented by a closed geodesic γ_c, of minimal length $L(\gamma_c)$ in the corresponding free homotopy class. If we denote by $\mathcal{C}$ the set of all conjugacy classes in $\pi_1(M)$, then the metric g_0 determines a function $L_{g_0} : \mathcal{C} \to (0, \infty)$ by defining $L_{g_0}(c) = L_{g_0}(\gamma_c)$ $(c \in \mathcal{C})$. This function is called the *marked length spectrum* of g_0. It also makes sense for metrics on M which are not nonpositively curved.

The following conjecture was formulated by Burns and Katok **BK85** but may have been known earlier.

Conjecture 0.1. — *Let g_0 be a negatively curved Riemannian metric on a closed manifold M. If g is another metric on M so that $L_g = L_{g_0} : \mathcal{C} \to (0, \infty)$, then g, g_0 are strongly isometric.*

Here two metrics g, g_0 are called *strongly isometric* if there exists a diffeomorphism ϕ isotopic to the identity such that $\phi^* g = g_0$. The following major progress towards this conjecture is the main result of **GL19**.

Theorem 0.2 (Guillarmou and Lefeuvre). — *Let g_0 be a smooth nonpositively curved metric on a closed manifold M of dimension n whose geodesic flow is Anosov. Then there exists a neighborhood U of g_0 in the C^N-topology for some $N > 3n/2 + 8$ such that any metric in U with the same marked length spectrum as g_0 is strongly isometric to g_0.*

One may also consider a similar question where we replace the function L_{g_0} by the *unmarked length spectrum*, that is, we just look at the *set* of lengths of closed geodesics on M, viewed as a subset of $(0,\infty)$ with no additional structure. However, this question has a negative answer, already for closed hyperbolic surfaces. The first examples of non-isometric hyperbolic surfaces with the same unmarked length spectrum are due to Vigneras **V80**.

The goal of this survey is to give a short historical account on partial results towards the marked length spectrum conjecture and to outline the main steps of the proof of Theorem 0.2, giving a more detailed explanation of its assumptions along the way.

1. EARLIER RESULTS TOWARDS THE MARKED LENGTH SPECTRUM CONJECTURE

Nonpositively curved Riemannian metrics on closed oriented surfaces of genus $h \geq 2$ have always been considered as a test case for the understanding of negatively curved metrics on manifolds of all dimensions, although the analogy is problematic due to the fact that by uniformization, any smooth metric g on such a surface S is *conformally equivalent* to a hyperbolic metric. That is, there is a smooth function ρ on S so that the metric $e^\rho g$ is of constant curvature -1. This gives strong additional constraints which do not exist in higher dimension.

In contrast to hyperbolic metrics on closed manifolds of dimension at least 3, a hyperbolic metric on a surface S of genus $h \geq 2$ is not unique up to isometry: There is an entire moduli space of isometry classes of hyperbolic metrics on S of dimension $6h-6$. Such hyperbolic metrics can be constructed explicitly, and there is a collection of $6h-5$ conjugacy classes of *simple* closed curves on S, that is, curves without self-intersection, whose lengths completely determine the hyperbolic metric **Sch93**.

Understanding the marked length spectrum of a negatively curved metric g on S in a fixed conformal class is already interesting. The corresponding rigidity question was answered affirmatively by Katok **K88**. His argument immediately extends to the following

Theorem 1.1 (Katok). — *Let g, g_0 be two smooth conformally equivalent Riemannian metrics of negative curvature on a closed manifold M of dimension $n \geq 2$. If g, g_0 have the same marked length spectrum then they are isometric.*

The proof of this result is quite short. We present a sketch as it rests on two basic principles which are important cornerstones for later progress. For this and for later use, define the *geodesic flow* Φ^t on the unit tangent bundle $P : T^1M \to M$ of a Riemannian manifold (M,g) by $\Phi^t v = \gamma_v'(t)$ where γ_v is the geodesic with initial velocity v. The flow Φ^t preserves the *Lebesgue Liouville measure* μ, which is locally defined by a smooth volume form on T^1M whose integration over the fibers of the bundle T^1M equals the volume element of the metric g on M. Periodic orbits of Φ^t are precisely the unit tangent lines of closed geodesics.

If M is closed and the metric on M is negatively curved, then the flow Φ^t is an *Anosov flow*: Let X be its generator. There exists a $d\Phi^t$-invariant decomposition

$$TT^1M = E^+ \oplus E^- \oplus \mathbb{R}X, \tag{1}$$

and there exists a number $\alpha > 0$ with

$$\|d\Phi^{\mp t}w\| \leq e^{-\alpha t}\|w\|$$

for every $w \in E^{\pm}$, with a suitable choice $\| \, \|$ of a norm on TT^1M defined by some Riemannian metric. The decomposition (1) is called the *Anosov splitting*. It is known to be Hölder continuous, but in general, it is not smooth.

The Anosov property for Φ^t has the following two consequences. First, the normalized Lebesgue Liouville measure $\hat{\mu} = \mu/\mu(T^1M)$ is *ergodic* for Φ^t. This means that whenever $A \subset T^1M$ is a Φ^t-invariant Borel set, then either $\hat{\mu}(A) = 0$ or $\hat{\mu}(T^1M - A) = 0$. In particular, by the *Birkhoff ergodic theorem*, for any L^2-integrable function f on T^1M and for $\hat{\mu}$-almost every $v \in T^1M$, we have

$$\int f d\hat{\mu} = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\Phi^s v)ds.$$

Here the existence of the limit on the right hand side of this equation is part of the statement of the theorem.

Furthermore, the following *Anosov closing lemma* holds true. Let d be any distance function on T^1M defined by a Riemannian metric. Then for any $\delta > 0$, there are numbers $\epsilon = \epsilon(\delta) > 0$, and $T_0 = T_0(\delta) > 0$ with the following property. If for some $v \in T^1M$ and some $T > T_0$, we have $d(v, \Phi^T v) < \epsilon$, then there exists a periodic orbit η for Φ^t, of period $L(\eta) \in [T - \delta, T + \delta]$, such that $d(\Phi^t v, \eta(t)) < \delta$ for all $t \in [0, T]$.

Since continuous functions on compact spaces are uniformly continuous, one obtains as a consequence of the Birkhoff ergodic theorem and the Anosov closing lemma the following.

Corollary 1.2. — *Let $f : T^1M \to \mathbb{R}$ be a continuous function. Then for every $\epsilon > 0$ and $T_0 > 0$, there exists a periodic point v for Φ^t of period $T > T_0$ such that*

$$\left|\frac{1}{T}\int_0^T f(\Phi^t v)dt - \int f d\hat{\mu}\right| < \epsilon.$$

Sketch of a proof of Theorem 1.1. — Let g, g_0 be negatively curved metrics on the same closed manifold M such that $g = \rho g_0$ for a smooth function ρ on M. Assume that g, g_0 have the same marked length spectrum. By perhaps exchanging g and g_0 we may assume that $\mathrm{vol}(M, g) \leq \mathrm{vol}(M, g_0)$ (here vol denotes the volume).

Denote by $P : T^1M_0 \to M$ the unit tangent bundle of M for the metric g_0, equipped with the Lebesgue Liouville measure μ, and let ω be the volume element of g_0 on M.

Then $\rho^{n/2}\omega$ is the volume element for the metric g on M and hence naturality under pull-back shows that

(2) $$\int_{T^1M_0}(P^*\rho)^{n/2}d\mu = \mathrm{vol}(S^{n-1})\mathrm{vol}(M,g) = \mathrm{vol}(T^1M,g) \le \mu(T^1M_0).$$

On the other hand, the integral of the function $\rho^{1/2}$ over each closed geodesic γ for the metric g_0, parameterized by arc length, is the g-length of γ. As the marked length spectra of g and g_0 coincide, this length is not smaller than the g_0-length $L_{g_0}(\gamma)$ of γ. Thus if we denote by Φ^t the geodesic flow on T^1M_0, then for every periodic orbit η of Φ^t, we have

(3) $$\int_\eta (P^*\rho)^{1/2}dt \ge L(\eta)$$

where $L(\eta)$ is the period of the orbit (which is just the length of the corresponding closed geodesic for g_0).

Write $\hat\mu = \mu/\mu(T^1M_0)$. By Corollary 1.2, since the function $(P^*\rho)^{1/2}$ on T^1M_0 is continuous and fulfills the inequality (3) for all periodic orbits η for Φ^t, we have

(4) $$\int_{T^1M_0}(P^*\rho)^{1/2}d\hat\mu \ge 1.$$

Together with inequality (2), this shows that $\int_{T^1M_0}(P^*\rho)^{1/2}d\hat\mu \ge 1 \ge \int_{T^1M_0}(P^*\rho)^{n/2}d\hat\mu$. It now follows from the Hölder inequality that this is possible only if the function ρ is constant and hence if g, g_0 are isometric. □

The proof of Theorem 1.1 motivates the following extension of Conjecture 0.1.

Conjecture 1.3. — *Let g, g_0 be two negatively curved metrics on a closed manifold M. If $L_g(c) \ge L_{g_0}(c)$ for each conjugacy class $c \in \mathcal{C}$, then* $\mathrm{vol}(M,g) \ge \mathrm{vol}(M,g_0)$, *with equality only if g, g_0 are strongly isometric.*

Shortly after the appearance of the article of Katok, Conjecture 0.1 for surfaces was settled by Otal **O90** and, independently, Croke **C90**. They showed

Theorem 1.4 (Croke, Otal). — *Let g, g_0 be two smooth nonpositively curved metrics on a closed surface of genus $g \ge 2$. If g, g_0 have the same marked length spectrum, then they are isometric.*

The approach of both authors is similar and rests on the following two facts. The first fact is valid in all dimensions.

Fact 1: If two metrics g, g_0 on a closed manifold M of dimension $n \ge 2$ are non-positively curved, have the same marked length spectrum and Anosov geodesic flows Φ^t on their unit tangent bundles T^1M, T^1M_0, then these geodesic flows are *time preserving conjugate*: There exists a Hölder continuous map $F : T^1M \to T^1M_0$ such that $\Phi^t \circ F = \Phi^t$. The map F gives information on the coupling of lengths of periodic orbits which are in a suitable sense close to each other.

Fact 2: For a closed surface S with non-positively curved metric g, one can reconstruct the Lebesgue Liouville measure on the unit tangent bundle T^1S of S from the marked length spectrum using the fact that for surfaces, geodesics which intersect transversely a given open geodesic segment in the universal covering $\tilde{S}$ of S form an open subset of the space of all geodesics on $\tilde{S}$ whose measure (for the projection to the space of geodesics on $\tilde{S}$ of the Liouville measure on the unit tangent bundle of $\tilde{S}$) equals π times the length of the segment. As a consequence, if two such metrics g, g_0 have the same marked length spectrum, then the time preserving conjugacy between their geodesic flows conjugates the Liouville measures for g, g_0, and the volumes of S with respect to g, g_0 coincide.

Embarking from these two facts, the proof of the marked length spectrum rigidity theorem for surfaces uses an ingenious and fairly elementary but purely 2-dimensional construction.

The only global result which is known in all dimensions is the following special case of Conjecture 0.1 **H99**.

Theorem 1.5 (Hamenstädt). — *Let (M, g_0) be a closed rank 1 locally symmetric manifold. If g is another negatively curved metric on M with the same marked length spectrum as g_0, then the metrics g, g_0 are strongly isometric.*

The proof of this result consists of two independent steps. The first step resembles the approach for surfaces. Namely, it is shown that whenever g, g_0 are metrics on M with Anosov geodesic flow and such that the Anosov splitting for g_0 is of class C^1, and if the metrics have the same marked length spectrum, then the volumes of g, g_0 coincide. In fact, a time preserving conjugacy between the geodesic flows for g, g_0 maps the Lebesgue Liouville measure for g to the Lebesgue Liouville measure for g_0. Examples of metrics with C^1-Anosov splitting are locally symmetric metrics or metrics whose sectional curvature is strictly 1/4-pinched.

With this information, the rigidity statement follows from the following deep theorem of Besson, Courtois and Gallot **BCG95**. For its formulation, define the *volume entropy* of a negatively curved metric on a closed manifold M to be the quantity

$$h_{\mathrm{vol}} = \lim_{R\to\infty} \frac{1}{R} \log \mathrm{vol}(B(x,R))$$

where $B(x, R)$ is the ball of radius R about x in the universal covering $\tilde{M}$ of M. The limit is known to exist and to be independent of the basepoint x.

Theorem 1.6 (Besson, Courtois and Gallot). — *Let g_0 be a negatively curved locally symmetric metric on a closed Riemannian manifold M of dimension at least 3 and let g be another metric of negative curvature. If the volume entropy of g is not bigger than the volume entropy of g_0, then* $\mathrm{vol}(M, g) \geq \mathrm{vol}(M, g_0)$*, with equality if and only if g and g_0 are isometric.*

Now for a metric g with the same marked length spectrum as the locally symmetric metric g_0, the volume entropies (which are the topological entropies of the geodesic flows) coincide, and the volumes also coincide by Theorem 1.5. Hence the metrics are isometric by Theorem 1.6.

It took twenty more years for a complete solution of a local version of Conjecture 0.1. The approach of Guillarmou and Lefeuvre **GL19** introduces new tools towards this question, mainly from microlocal analysis. The remainder of this note is devoted to a discussion of the main parts of the work of Guillarmou and Lefeuvre. This is organized into three sections, each of which focusses on a different aspect of the proof.

2. CONTROLLING THE ACTION OF THE DIFFEOMORPHISM GROUP

Let M be a smooth closed manifold of dimension $n \geq 2$. Denote by $\mathrm{Diff}_0^{N+1,\alpha}(M)$ the group of diffeomorphisms of M which are isotopic to the identity, equipped with the $C^{N+1,\alpha}$-topology, where $N \geq 1$ is some fixed integer. This topology is defined using an auxiliary Riemannian metric to measure norms of differentials, and a suitable covering by charts to obtain a Hölder structure (see **P16** for more information on the latter). If g is any non-positively curved metric on M and if $\phi \in \mathrm{Diff}_0^{N+1,\alpha}(M)$, then the g-length of any conjugacy class in the fundamental group of M coincides with its $\phi^* g$-length. Thus to understand the marked length spectrum rigidity question, it is necessary to understand the action of $\mathrm{Diff}_0^{N+1,\alpha}(M)$ on the space of Riemannian metrics.

It is well known that through any given metric g_0, it is possible to construct a slice transverse to the orbit of the group $\mathrm{Diff}_0^{N+1,\alpha}(M)$ in an open neighborhood of g_0 in the space of metrics, equipped with the $C^{N,\alpha}$-topology. Such a slice is by no means unique, and the first step will be to find a slice which is adapted to the marked length spectrum problem.

The space of smooth metrics is an open convex subset of the vector space of smooth section of the 2-fold symmetric tensor product S^2T^*M of T^*M. Let ∇ be the Levi Civita connection of g_0 and let $\sigma : \otimes^{m+1}T^*M \to S^{m+1}T^*M$ be the symmetrization operator. The *symmetrized derivative* is defined by

$$\delta^* = \sigma \circ \nabla : C^\infty(M; S^m T^*M) \to C^\infty(M; S^{m+1}T^*M).$$

Note that if $\rho \in C^\infty(M; T^*M)$ is a smooth one-form, then

$$\delta^* \rho = -\frac{1}{2} L_{\rho^\sharp} g_0 \tag{5}$$

where $\rho^\sharp$ is the vector field dual to ρ and, as usual, $L_{\rho^\sharp}$ denotes the Lie derivative.

The *divergence operator* is the formal adjoint

$$\delta f = -\mathrm{Tr}(\nabla f)$$

of δ^* where the trace is taken in the first two variables. A symmetric tensor field f is *divergence free* if $\delta f = 0$.

The formula (5) shows that the image of the operator δ^*, acting on 1-forms, consists precisely of the infinitesimal deformations of the metric with a 1-parameter group of diffeomorphisms. It is now natural to try to find a slice in the space of all metrics transverse to the orbit of the diffeomorphism group by first constructing a good closed subspace in the vector space of sections of S^2T^*M, equipped with a suitable structure of a Banach space, which is complementary to the image of δ^*. Note that the vector space of smooth sections of S^2T^*M is just the tangent space of the space of metrics on M.

Elliptic theory provides a natural approach to this problem. Namely, the second order self adjoint (for the standard L^2-metric) differential operator

$$\delta\delta^* : C^\infty(M; S^2T^*M) \to C^\infty(M; S^2T^*M)$$

is elliptic (**Sh94** p.84), and this is used by Sharafutinov **Sh94** and Croke and Sharafutinov to show

Theorem 2.1 (Theorem 2.2 of **CS98**). — *For $N > 1, \alpha \in (0,1)$, a tensor field $f \in C^{N,\alpha}(M; S^2T^*M)$ admits a unique decomposition*

$$f = f^s + \delta^* p, \quad \delta f^s = 0.$$

Furthermore, there exists a number $C_1 = C_1(N, \alpha) > 0$ such that

$$\|f^s\|_{C^{N,\alpha}(M;S^2T^*M)} \le C_1 \|f\|_{C^{N,\alpha}(M;S^2T^*M)}.$$

The mechanism behind this result is that as

$$\langle \delta\delta^* v, v\rangle = \langle \delta^* v, \delta^* v\rangle$$

for all smooth sections v of T^*M, the kernel of δ^* coincides with the kernel of $\delta\delta^*$, and this is a finite dimensional vector space consisting of smooth one-forms by ellipticity of the operator $\delta\delta^*$. Here $\langle,\rangle$ denotes the L^2-inner product.

As a consequence, the restriction of the operator δ to $\mathrm{Im}(\delta^*)$ is injective. As $\mathrm{Im}(\delta) = \mathrm{Im}(\delta\delta^*)$, and as the kernel of $\delta\delta^*$ is a finite dimensional vector space of smooth sections, there exists a unique p orthogonal to this kernel so that $\delta\delta^* p = \delta f$. Then putting $f^s = f - \delta^* p$ defines a decomposition as claimed.

Theorem 2.1 was used by Croke and Sharafutinov to establish an infinitesimal version of Theorem 0.2 which rules out nontrivial deformations of a given Riemannian metric preserving the marked length spectrum. For the formulation of their result and later use let T^1M be the unit tangent bundle of the metric g_0 and define an evaluation operator

$$\pi : C^{N,\alpha}(M; S^2T^*M) \to C^{N,\alpha}(T^1M), \quad \pi f(v) = f(v,v).$$

Theorem 2.2 (Corollary 1.6 of **CS98**). — *Let (M, g_0) be a closed non-positively curved manifold, with Anosov geodesic flow, and let $f \in C^{N,\alpha}(M; S^2T^*M)$ be such that*

$\int_\gamma \pi f(\gamma') = 0$ *for every closed geodesic* γ *on* M. *Then there exists a smooth* 1*-parameter group of diffeomorphisms* ψ_t *such that*

$$f = \frac{d}{dt}|_{t=0}\psi_t^* g_0.$$

Remark 2.3. — Corollary 1.6 of **CS98** is stated for negatively curved metrics on M. However, the proof only uses non-positive curvature and an Anosov geodesic flow.

Let $\mathrm{Met}^{N,\alpha}$ be the convex cone of all Riemannian metrics on M of class $C^{N,\alpha}$. Its tangent space at the smooth metric g_0 is the vector space $C^{N,\alpha}(M; S^2T^*M)$. Theorem 2.1 shows that there exists a direct decomposition of this tangent space into the infinitesimal deformations by diffeomorphisms, and the kernel of the divergence operator δ. Note that δ depends on the metric g_0.

To pass from infinitesimal to local, it is natural to use an implicit function theorem for Banach spaces. The first step towards this goal is to establish a local version of Theorem 2.1 and show that near g_0, metrics g for which the tensor field $g - g_0$ is divergence free define a slice transverse to the orbit of the diffeomorphism group.

Namely, the natural map

$$\mathrm{Diff}_0^{N+1,\alpha}(M) \times \mathrm{Met}^{N,\alpha} \to \mathrm{Met}^{N,\alpha} \tag{6}$$

is smooth. The vector space $V \subset C^{N,\alpha}(M; S^2T^*M)$ of divergence free symmetric 2-tensors is closed and transverse to the image of the tangent space of $\mathrm{Diff}_0^{N+1,\alpha}(M)$ at the identity. Thus by the inverse function theorem for Banach manifolds, there exists a neighborhood $\mathcal{U}$ of (Id, g_0) in $\mathrm{Diff}_0^{N+1,\alpha}(M) \times V$ such that the restriction to $\mathcal{U}$ of the map defined in equation (6) is a diffeomorphism onto its image, which is a neighborhood of g_0 in $\mathrm{Met}^{N,\alpha}$. This then leads to the following slice result.

Proposition 2.4 (Lemma 4.1 of **GL19**). — *For any* $N > 1$ *and* $\alpha \in (0,1)$ *there exist numbers* $\epsilon > 0$ *and* $C_2 > 0$ *such that for any* g *satisfying* $\|g - g_0\|_{C^{N,\alpha}} < \epsilon$, *there exists a unique* $\phi \in \mathrm{Diff}_0^{N+1,\alpha}(M)$ *close to* Id *such that* $\phi^* g - g_0$ *is divergence free with respect to* g_0 *and* $\|\phi^* g - g_0\|_{C^{N,\alpha}} < C_2 \|g - g_0\|_{C^{N,\alpha}}$. *Moreover,* $g \hookrightarrow \phi$ *is smooth.*

As a consequence, it is enough to study metrics in the slice obtained from g_0 by adding a divergence free symmetric tensor field. We call this slice the slice of divergence free metrics (note that it depends on g_0). Theorem 2.2 provides an infinitesimal rigidity result, but this does not suffice to control the marked length spectrum on metrics near g_0 in the slice. Establishing a method obtain such a control is the main novelty of the work of Guillarmou and Lefeuvre. The remainder of this section introduces the strategy used.

First, to be able to work in the tangent space of $\mathrm{Met}^{N,\alpha}$ at a nonpositively curved metric g_0 whose geodesic flow is Anosov, it is convenient to normalize the marked length spectrum at g_0 as follows.

Let as before $\mathcal{C}$ be the set of all conjugacy classes of $\pi_1(M)$. Define

$$\mathcal{L} : \mathrm{Met}^{N,\alpha} \subset C^{N,\alpha}(M; S^2T^*M) \to \ell^\infty(\mathcal{C}), \quad \mathcal{L}(g)(c) = \frac{L_g(c)}{L_{g_0}(c)}.$$

The map $\mathcal{L}$ is a map between Banach spaces, and standard estimates show that it is differentiable. A calculation establishes the formula

$$d\mathcal{L}_{g_0}(h)(c) = \frac{1}{2L_{g_0}(c)} \int_0^{L_{g_0}(c)} h_{\gamma_c(t)}(\gamma_c'(z), \gamma_c'(t))dt \tag{7}$$

where γ_c is the geodesic for g_0 in the free homotopy class c. This expression is called the X*-ray transform*. It contains the infinitesimal information on the behavior of the marked length spectrum under a smooth deformation of the metric g_0.

Denote by $\mathbf{1}$ the function which gives the value 1 to each $c \in \mathcal{C}$. From formula (7) and an estimate established with fairly standard methods one obtains

Proposition 2.5 (Proposition 2.1 of **GL19**). — *There exists a constant $C_3 = C_3(g_0) > 0$ such that*

$$\|\mathcal{L}(g) - \mathbf{1} - d\mathcal{L}_{g_0}(g - g_0)\|_{\ell^\infty} \le C_3 \|g - g_0\|^2_{C^3(M;S^2T^*M)}$$

for all g in a sufficiently small neighborhood of g_0 in the C^3-topology.

Let us inspect the terms in the estimate of Proposition 2.5. A tangent vector at g_0 of the space of smooth metrics on M is a smooth section h of S^2T^*M. At each point $x \in M$, the matrix $h(x)$ is symmetric and hence it can be diagonalized with respect to g_0. This means that there is an orthonormal basis $e_1, \dots, e_n$ of eigenvectors for h, with corresponding eigenvalues $a_1, \dots, a_n$. Then for any unit vector $v \in T^1_x M$, we have

$$|h(x)(v, v)| \le \max\{|a_i| \mid i\} = \|h(x)\|$$

where $\|h(x)\|$ is the spectral norm of the symmetric matrix $h(x)$. As

$$\|d\mathcal{L}_{g_0}(h)\|_{\ell^\infty} \le \frac{1}{2} \sup\{\|h(x)\| \mid x\} \le \frac{1}{2} \|h\|_{C^0(M;S^2T^*M)}$$

by equation (7), Proposition 2.5 makes the approximation of the X-ray transform near g_0 by its derivative quantitative.

This is however not enough for an application of the inverse function theorem to a neighborhood of g_0 in the slice of divergence free metrics towards the local marked length spectrum rigidity. Namely, although it follows from Theorem 2.2 that on a tangent vector to this slice at g_0, the differential of the X-ray transform does not vanish, the result is not quantitative. Indeed, the standard tool for obtaining a norm control which is sufficient for an application of the implicit function theorem is to *invert* the derivative of the map considered and establish a norm control for this inverse.

The idea for obtaining such a control is as follows. Instead of working directly with the X-ray transform, one may work with distributions on T^1M contained in a suitably chosen Sobolev space. The distributions of interest are the functions πf where f is a divergence free section of S^2T^*M. Instead of inverting the X-ray transform, one may try to invert the generator X of the geodesic flow, viewed as a first order differential operator on T^1M, on a suitably chosen space of distributions, and, similarly to the statement of Theorem 2.1, use this to analyze the kernel of X consisting of invariant distributions which give rise to nontrivial contributions for the X-ray transform.

That this program can be made precise and quantitative using tools from microlocal analysis is the main achievement of the article **GL19** and leads to its main technical result. In its formulation, the diffeomorphism ϕ is as in Proposition 2.5, that is, it is such that $\phi^*g - g_0$ is divergence free. Furthermore, $H^{-1-s}(M; S^2T^*M)$ is the L^2-based Sobolev space of sections of S^2T^*M.

THEOREM 2.6 (Theorem 3 of **GL19**). — *Let (M, g_0) be as before and let $N > 3n/2+8$. For all small $s > 0$, there are $\nu = \mathcal{O}(s)$ and $C_4 = C_4(g_0) > 0$ such that the following holds: There exists $\epsilon > 0$ such that for any C^N-metric g satisfying $\|g - g_0\|_{C^N(M)} < \epsilon$, there is a diffeomorphism ϕ close to* Id *such that*

$$\|\phi^*g - g_0\|_{H^{-1-s}(M;S^2T^1M)} \leq C_4 \cdot \|g - g_0\|_{C^N(M)}^{(1+\nu)/2} \cdot \|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty(\mathcal{C})}^{(1-\nu)/2}$$

where $\mathbf{1}(c) = 1$ for each $c \in \mathcal{C}$.

Theorem 2.6 immediately implies Theorem 0.2. Namely, if g_0, ϵ are in the statement of Theorem 2.6 and if $\|g - g_0\|_{C^N(M)} < \epsilon$ is such that g, g_0 have the same marked length spectrum, then $\mathcal{L}(g) = \mathbf{1}$. Thus if ϕ is a diffeomorphism as in Theorem 2.6, then Theorem 2.6 shows that

$$\|\phi^*g - g_0\|_{H^{-1-s}(M;S^2T^*M))} = 0.$$

But g_0, ϕ^*g are metrics of class C^N for $N > 3/2n + 8$ and hence this is possible only if $\phi^*g = g_0$, that is, if g and g_0 are strongly isometric.

The remaining two sections are devoted to discuss the main ingredients of the proof of Theorem 2.6.

3. INVERSION ON A SPACE OF DISTRIBUTIONS

The assumption that the geodesic flow Φ^t for the metric g_0 is an Anosov flow enters in the proof of Theorem 2.2 in the form of a so-called *Livshitz theorem* for a cohomology equation defined by the flow. This can be viewed as a structural result for the kernel of the linear operator $d\mathcal{L}_{g_0}$, acting on sufficiently regular sections of S^2T^*M. It is based on an analysis of the behavior of the restriction of a function on T^1M to *strong stable* and *strong unstable* manifolds for the Anosov flow, which are leaves of a foliation of T^1M tangent to the subbundles of TT^1M arising in the Anosov splitting.

To make this precise let us denote as before by X the generator of the geodesic flow. View the vector field X as a first order linear differential operator on smooth functions on T^1M. If $F : T^1M \to \mathbb{R}$ is smooth, then the function $f = X(F)$ satisfies $\int_\gamma f(\Phi^t v)dt = 0$ for each periodic orbit γ for the geodesic flow.

Vice versa, by **LMM86**, the latter property characterizes the range of X: If $f : T^1M \to \mathbb{R}$ is sufficiently regular, and if the integral of f over each periodic orbit for Φ^t vanishes, then there exists a function F of controlled regularity such that $f = X(F)$.

The case of interest is when $f = \pi\sigma$ for a smooth section σ of S^2T^*M. Namely, the X-ray transform of a section σ of S^2T^*M is up to the factor $1/2$ obtained by integration of the function $\pi\sigma$ with respect to the countably many Φ^t-invariant Borel probability measures on T^1M which are the normalizations of the Lebesgue measures on the periodic orbits. Each such measure in turn can be viewed as a distribution on T^1M, and this distribution lies in the kernel of the adjoint of the operator X.

Now by invariance of the Lebesgue Liouville measure μ under the geodesic flow, we have $\langle X\rho, \psi\rangle = -\langle \rho, X\psi\rangle$ for any smooth functions ρ, ψ where $\langle , \rangle$ is the L^2 inner product on T^1M. Hence the densely defined operator $-iX$ is self-adjoint on L^2.

The spectral theorem for self-adjoint operators tells us that $\mathrm{Spec}_{L^2}(-iX) \in \mathbb{R}$. This implies that for $\mathrm{Re}(\lambda) > 0$, the resolvents

$$R_-(\lambda) = (-X - \lambda)^{-1}, R_+(\lambda) = (-X + \lambda)^{-1}$$

are well defined on $L^2(T^1M)$, and they can be determined explicitly. Namely, we have

$$R_+(\lambda)f(y) = \int_0^\infty e^{-\lambda t} f(\Phi^t(y))dy, \quad R_-(\lambda)f(y) = -\int_{-\infty}^0 e^{\lambda t} f(\Phi^t(y))dt. \tag{8}$$

In particular, these resolvents depend analytically on λ.

We are interested in inverting the operator $-X$, that is, in the case $\lambda = 0$. To this end one can try to meromorphically extend the resolvent operators across the imaginary line and study its behavior at $\lambda = 0$. For this it is necessary to work with a suitably defined space of distributions on which such an extension acts and which is invariant under the extension. The setup for this construction is the following theorem of Faure and Sjöstrand (Theorem 1.4 of **FS11**).

Theorem 3.1 (Faure and Sjöstrand). — *There exists a number $c > 0$, and for all $s > 0$ and $r < 0$, there is a Hilbert space $\mathcal{H}^{r,s}$ such that $-X$ defines a maximal closed unbounded operator on $\mathcal{H}^{r,s}$. Furthermore, $H^s(T^1M) \subset \mathcal{H}^{r,s} \subset H^r(T^1M)$, and if $\mathrm{Re}(\lambda) > -c\,\min(|r|, s)$, then*

$$-X - \lambda : \mathrm{Dom}(X) \cap \mathcal{H}^{r,s} \to \mathcal{H}^{r,s}$$

(here $\mathrm{Dom}(X) = \{u \in \mathcal{H}^{r,s}; Xu \in \mathcal{H}^{r,s}\}$) is an unbounded Fredholm operator of index 0 depending analytically on λ. The operator $-X - \lambda$ is invertible for $\mathrm{Re}(\lambda)$ large enough on these spaces, the inverse coincides with $R_-(\lambda)$ when acting on $H^s(T^1M)$, and it extends meromorphically to the half-plane $\mathrm{Re}(\lambda) > -c\min(|r|, s)$, with poles of finite multiplicity as a bounded operator on $\mathcal{H}^{r,s}$. For $-X + \lambda$, the same holds with a Sobolev space $\mathcal{H}^{s,r}$ satisfying the same properties.

As a consequence of this theorem, near $\lambda = 0$, the operators $-X - \lambda$ are all Fredholm operators of index 0, defined on the same subdomain of a fixed Hilbert space, with range in thi sHilbert space. Furthermore, this Hilbert space is determined by two numbers

$s > 0, r < 0$, it is a subspace of the Sobolev space $H^r(T^1M)$, and for small enough $s > 0$, it contains the densely embedded space of L^2-functions on which the resolvent of $-X - \lambda$ for $\mathrm{Re}(\lambda) \neq 0$ can be written down explicitly.

The explicit formula (8) allows to estimate for $\mathrm{Re}(\lambda) > 0$ the expression

(9)
$$\lambda\langle R_+(\lambda)u, v\rangle = \int_0^T \int_{T^1M} \lambda e^{-\lambda t} u(\Phi^t y)v(y)d\hat{\mu}(y)dt + \int_T^\infty \int_{T^1M} \lambda e^{-\lambda t} u(\Phi^t y)v(y)d\hat{\mu}(y)dt$$

where as before, $\hat{\mu}$ denotes the normalized Lebesgue Liouville measure. Now the geodesic flow on T^1M is contact Anosov and hence *mixing* for $\hat{\mu}$, that is, if $u, v \in L^2(T^1M)$ then we have

$$\int_{T^1M} u(\Phi^t y)v(y)d\hat{\mu}(y) \to \langle u, 1\rangle\langle v, 1\rangle \quad (t \to \infty).$$

Thus for large T, the second summand on the right hand side of equation (9) is arbitrarily close to $e^{-\lambda T}\langle u, 1\rangle\langle v, 1\rangle$, where in this estimate, the number T can be chosen independent of λ, while the first term is bounded from above by $(1 - e^{-\lambda T})\|u\|_{L^2}\|v\|_{L^2}$. Letting λ tend to zero and using the same reasoning for $R_-(\lambda)$ then shows the following

Lemma 3.2 (Lemma 2.5 of **G17**). — *The only pole of $R_\pm(\lambda)$ on the imaginary line $i\mathbb{R}$ is $\lambda = 0$, and it is a simple pole of residue $\pm 1 \otimes 1$.*

Here the notation $1 \otimes 1$ stands for the L^2-orthogonal projection onto the space of constant functions, which by ergodicity equals the eigenspace of X acting on L^2-functions for the eigenvalue 0.

One can now look at the second term R_0 in the Laurent expansion for R_+ at 0 acting on the Hilbert space $\mathcal{H}^{r,s}$ and the second term $-R_0^*$ in the Laurent expansion of R_- at 0 acting on the Hilbert space $\mathcal{H}^{s,r}$ $(s > 0, r < 0)$ which appear in Theorem 3.1. These operators are characterized by

$$R_+(\lambda) = \frac{1 \otimes 1}{\lambda} + R_0 + \mathcal{O}(\lambda), \quad R_-(\lambda) = -\frac{1 \otimes 1}{\lambda} - R_0^* + \mathcal{O}(\lambda).$$

Multiplying the first equation with $-X + \lambda$ from the left yields

$$\mathrm{Id} = 1 \otimes 1 + (-X + \lambda)R_0 + \mathcal{O}(\lambda)$$

and hence, putting $\lambda = 0$, we have

$$\mathrm{Id} - 1 \otimes 1 = -XR_0 = -R_0X = XR_0^* = R_0^*X$$

as operators $C^\infty(T^1M) \to C^{-\infty}(T^1M)$. This identity extends to those Sobolev spaces on which the operators are bounded. The following result of Guillarmou establishes the important properties of these operators.

Theorem 3.3 (Theorem 2.6 of **G17**). — *For all $s > 0$ and $r < 0$, the operator $\Pi = R_0 + R_0^* : H^s(T^1M) \to H^{-r}(T^1M)$ is bounded and satisfies*

$$X\Pi f = 0 \textit{ for all } f \in H^s(T^1M),$$
$$\textit{and } \Pi X f = 0 \textit{ for all } f \in H^{s+1}(T^1M).$$

Furthermore, if $f \in H^s(T^1M)$ *is orthogonal to the constant functions, then for* $u_+ = -R_0 f \in \mathcal{H}^{s,r}, u_- = R_0^* f \in \mathcal{H}^{r,s}$ *we have* $Xu_+ = Xu_- = f$*. Moreover,* $f \in \ker(\Pi) \cap H^s(T^1M)$ *if and only if there exists* $s' > 0$ *and* $u \in H^{s'}(T^1M)$ *such that* $Xu = f$*. In this case the solution* u *is in fact contained in* $H^s(T^1M)$*.*

This theorem identifies the intersection of the range of the differential operator X with the Sobolev space $H^s(T^1M)$ with the kernel of the pseudo-differential operator Π. Furthermore, its image is contained in the kernel of X and hence in the space of invariant distributions which should control an extension of the X-ray transform. As Π is bounded, one may hope to quantify the idea that functions of the form $\pi\sigma$ where σ is a sufficiently regular divergence free section of S^2T^*M give rise to non-trivial X-ray transforms.

Since the operator $\Pi : H^s(T^1M) \to H^{-r}(T^1M)$ is a bounded pseudo-differential operator of degree -1, and is constructed from the resolvent of an elliptic operator, it is natural to expect that it is elliptic as well. The evaluation operator $\pi : C^k(M; S^2T^*M) \to C^k(T^1M)$, $\pi\sigma(v) = \sigma(v, v)$, extends to a map $H^s(M; S^2T^*M) \to H^s(T^1M)$, again denoted by π, and by Theorem 2.2 and Theorem 3.3, the image under π of the space of divergence free sections of S^2T^*M is complementary to the kernel of Π, at least whenever these sections are sufficiently regular.

The strategy is now to show that the restriction of Π to the closure in $H^s(T^1M)$ of the set of functions of the form πf is injective, where f is a divergence free section of S^2T^*M. The following statement says that this holds indeed true, with controlled norm.

Theorem 3.4 (Lemma 3.3 of **GL19**). — *There exists a number* $C_5 > 0$ *such that*

$$\|f\|_{H^{-s-1}(M;S^2T^*M)} \leq C_5(\|\Pi\pi f\|_{H^{-s}(T^1M)} + |\langle \pi f, 1\rangle|_{L^2(T^1M)})$$

for all $f \in H^{-s}(M; S^2T^*M) \cap \ker \delta$ *and* $s > 0$*.*

The proof of Theorem 3.4 uses in an essential way additional insight into the restriction of the operator Π to the space πf where f is a divergence free section of S^2T^*M. Namely, denote by $\pi_* : C^{-\infty}(T^1M) \to C^{-\infty}(S^2T^*M)$ the push-forward on distributions defined by

$$\langle \pi_* u, f\rangle = \langle u, \pi f\rangle.$$

Building on the earlier works **FS11** and **DZ16**, Theorem 3.5 of **G17** shows that the operator $\Pi_0 = \pi_*\Pi\pi$ is a self-adjoint pseudo-differential operator of order -1 on S^2T^*M, and its restriction to the kernel of δ is elliptic in the following sense. There exist pseudo-differential operators P, S, R of order $1, -2, -\infty$ so that

$$P\Pi_0 = \mathrm{Id} + \delta^* S\delta + R.$$

Lemma 3.6 of **G17** then shows that in the situation at hand, the restriction of $\Pi\pi$ to the space of divergence free distributions which vanish on constants is injective, and this is used together with boundedness of pseudodifferential operators on Sobolev spaces to establish Theorem 3.4.

4. STABILITY ESTIMATES

Theorem 3.4 is not sufficient to complete the proof of Theorem 2.6. Namely, although Π is a pseudo-differential operator of degree -1 and bounded, and it is constructed from the resolvents of $-X \pm \lambda$, it is a priori unclear how the norm of $\Pi\pi f$ in $H^{-s}(T^1M)$ relates to the ℓ^∞-norm of $\mathcal{L} - \mathbf{1}$. Thus it remains to convert the term on the right hand side of Theorem 3.4 to a term involving the X-ray transform, provided that the divergence free section f of S^2T^*M is sufficiently regular.

This is done by interpolation between Sobolev spaces. The link to the X-ray transform is through Corollary 1.2 and a conversion of an L^1-norm using a positivity statement. The latter builds on the following result of **LT05**.

Theorem 4.1 (Lopes and Thieullen). — *Let $\alpha \in (0,1]$ and let X be the generator of the geodesic flow of g_0. There exists $C_6 = C_6(g_0) > 0$ and $\beta \in (0,1)$ such that for any $u \in C^\alpha(T^1M)$ satisfying*

$$\int_\gamma u \geq 0 \qquad \textit{for every periodic orbit } \gamma \textit{ for } \Phi^t,$$

there exist $h \in C^{\alpha\beta}(T^1M)$ and $F \in C^{\alpha\beta}(T^1M)$ such that $F \geq 0$ and $u + Xh = F$. Moreover,

$$\|F\|_{C^{\alpha\beta}} \leq C\|u\|_{C^\alpha}.$$

We now apply this result to πf for a section $f \in C^{N,\alpha}(M; S^2T^*M)$ with $\delta f = 0$. Theorem 3.4 states that

$$\|f\|_{H^{-s-1}(M;S^2T^*M)} \leq C_5(\|\Pi\pi f\|_{H^{-s}(T^1M)} + |\langle \pi f, 1\rangle|_{L^2(T^1M)}). \tag{10}$$

If $\int_\gamma \pi f \geq 0$ for every periodic orbit γ for Φ^t (which is the case if $g_0 + f$ and g_0 have the same marked length spectrum), then Theorem 4.1 shows the existence of a function h satisfying $Xh = F - \pi f$ where $F \geq 0$. By the norm bounds in Theorem 4.1, we have $Xh \in C^{\alpha\beta}(T^1M)$ and hence the function Xh is contained in the domain of the operator Π and in fact lies in the kernel of Π. Thus in the inequality (10), one may replace πf by $\pi f + Xh$.

On the other hand, by Theorem 3.3 for $s > 0$ and $r = -s$, the operator Π is bounded as an operator $H^s(T^1M) \to H^{-s}(T^1M)$ and therefore we obtain

$$\|f\|_{H^{-s-1}(M;S^2T^*M)} \leq C_7\|\pi f + Xh\|_{H^s(T^1M)}$$

for a universal constant $C_7 > 0$.

For ease of notations, in the remainder of this section, $C > 0$ denotes a constant which is universal but whose precise value may change from line to line.

Standard interpolation between Sobolev spaces yields

$$\|f\|_{H^{-s-1}(M;S^2T^*M)} \leq C \cdot \|\pi f + Xh\|^{\nu}_{H^{\beta}(T^1M)} \cdot \|\pi f + Xh\|^{1-\nu}_{L^2} \tag{11}$$

where $\nu = \frac{s}{\beta}$. Furthermore, we have

$$\|\pi f + Xh\|_{L^2} \leq \|\pi f + Xh\|^{1/2}_{L^\infty} \cdot \|\pi f + Xh\|^{1/2}_{L^1}. \tag{12}$$

Since the function $\pi f + Xh$ is *non-negative*, we also have

$$\|\pi f + Xh\|_{L^1} = \int_{T^1M} (\pi f + Xh) d\mu = \int_{T^1M} \pi f d\mu. \tag{13}$$

Now Corollary 1.2 and the formula for the X-ray transform shows that

$$|\int \pi f d\mu| \leq (2\mathrm{vol}(T^1M)) \cdot \|d\mathcal{L}_{g_0}(f)\|_{\ell^\infty(\mathcal{C})}. \tag{14}$$

Inequalities (12, 13, 14) together yield

$$\|\pi f + Xh\|_{L^2} \leq C \cdot \|\pi f + Xh\|^{1/2}_{L^\infty} \cdot \|d\mathcal{L}_{g_0}(f)\|^{1/2}_{\ell^\infty(\mathcal{C})}$$

and

$$\|f\|_{H^{-1-s}(M;S^2T^*M)} \leq C \cdot \|\pi f + Xh\|^{\nu}_{H^{\beta}(T^1M)} \cdot \|\pi f + Xh\|^{(1-\nu)/2}_{L^\infty} \cdot \|d\mathcal{L}_{g_0}(f)\|^{(1-\nu)/2}_{\ell^\infty(\mathcal{C})}. \tag{15}$$

Replacing the first two terms on the right hand side of this inequality by $\|f\|^{(1+\nu)/2}_{C^\alpha}$ where $\alpha > \beta$ is as in Theorem 4.1 is the statement of Theorem 5 of **GL19**

For the completion of Theorem 2.6 in the special case that $\int_\gamma \pi f \geq 0$ for every periodic orbit γ for Φ^T, which suffices for the completion of the proof of Theorem 0.2, one may now use Proposition 2.5 to replace the term $\|d\mathcal{L}_{g_0}(f)\|_{\ell^\infty(\mathcal{C})}$ by $\|\mathcal{L}(g_0 + f) - \mathbf{1}\|_{\ell^\infty(\mathcal{C})}$. This is not possible directly as it induces an error term, quadratic in the C^3-norm of f. Instead, taking advantage of the freedom to choose s and ν, the completion of the proof of Theorem 2.6 requires some additional estimates but no fundamental new idea.

As another beautiful application of this line of ideas, one obtains a local version of Conjecture 1.3.

THEOREM 4.2 (Theorem 2 of **GL19**). — *Let (M, g_0) be a closed manifold of non-positive curvature whose geodesic flow is Anosov and let $N > \frac{n}{2} + 2$. There exists $\epsilon > 0$ such that for any smooth metric g satisfying $\|g - g_0\| < \epsilon$, the following holds true. If $L_g(\gamma) \geq L_{g_0}(\gamma)$ for all conjugacy classes $c \in \mathcal{C}$ of $\pi_1(M)$, then* $\mathrm{vol}(M, g) \geq \mathrm{vol}(M, g_0)$, *with equality only if there exists a diffeomorphism ϕ of M with $\phi^* g = g_0$.*

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