2 Segal spaces as invertible infinity-operads

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Operads and dendroidal sets

Slogan: Operads are categories with "many-to-1-morphisms".

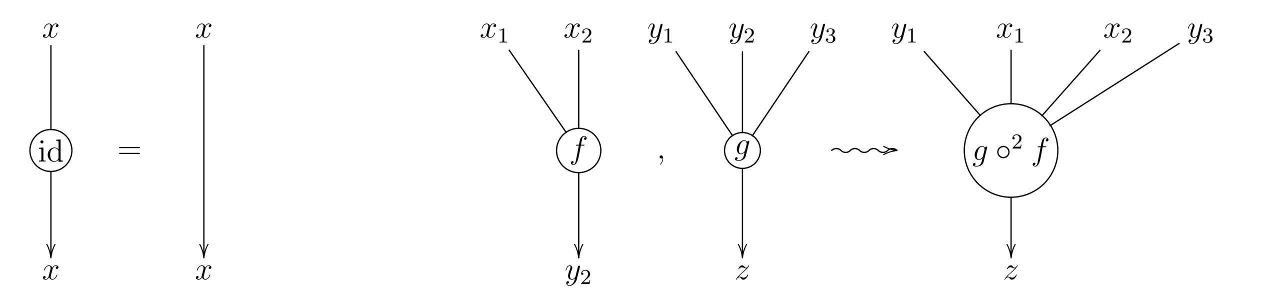
Definition (operads)

An **operad** $\mathcal{O}$ consists of

1. objects (a.k.a. colors) $x, y, z, \dots$
2. a set of n-ary operations of the form $f\colon (x_1, x_2, \dots, x_n) \longrightarrow y$
3. a unital, associative composition law.

Example: vector spaces with multi-linear maps.

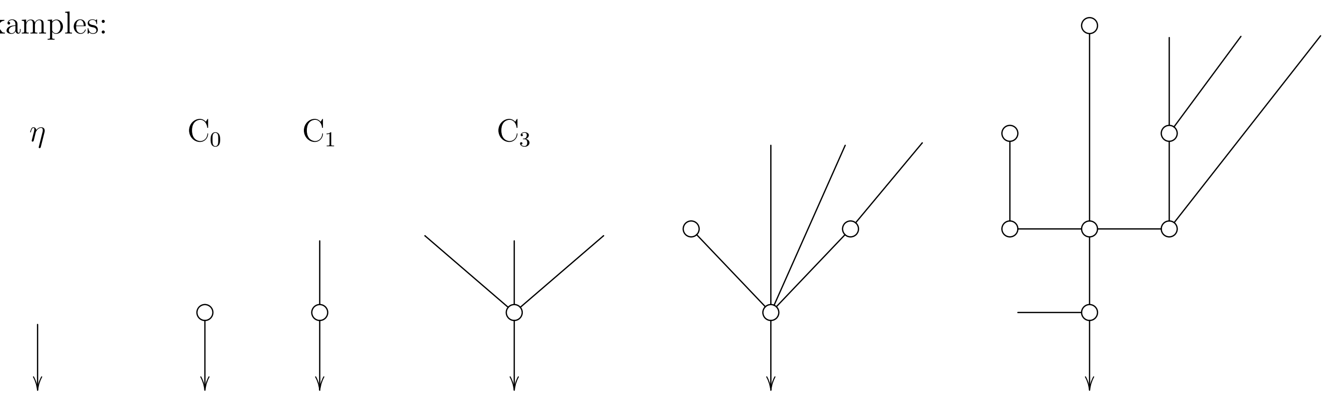
Visual depiction of operations:



- **Op** := category of operads
- **Op** ⊃ **Cat** := category of categories (i.e. operads with only 1-ary operations)

Plane rooted trees give rise to operads with {objects} = {edges}; operations freely generated by vertices.

Examples:



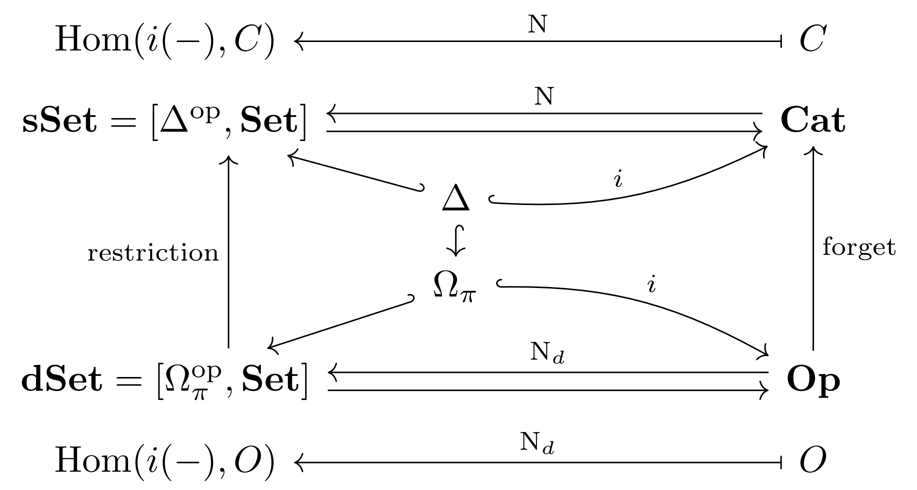
- **Cat** ⊃ Δ := full subcategory spanned by linear orders (i.e. linear trees)
- **Op** ⊃ Ω_π := full subcategory spanned by plane rooted trees.

Definition [MW07] (dendroidal objects)

A **dendroidal** (resp. **simplicial**) **object** in a (∞-)category $\mathcal{C}$ is a functor $\Omega_\pi^{op} \longrightarrow \mathcal{C}$ (resp. $\Delta^{op} \to \mathcal{C}$).

Segal objects

Dendroidal/simplicial sets are dendroidal/simplicial objects in the category of sets.



Easy fact: The functor $N_d\colon \mathbf{Op} \longrightarrow \mathbf{dSet}$ is fully faithful with essential image = {Segal dendroidal sets}

Definition [CM] (Segal dendroidal objects)

A dendroidal object $\mathcal{X}\colon \Omega_\pi^{op} \to \mathcal{C}$ in some (∞-)category $\mathcal{C}$ is called **Segal** if the canonical map

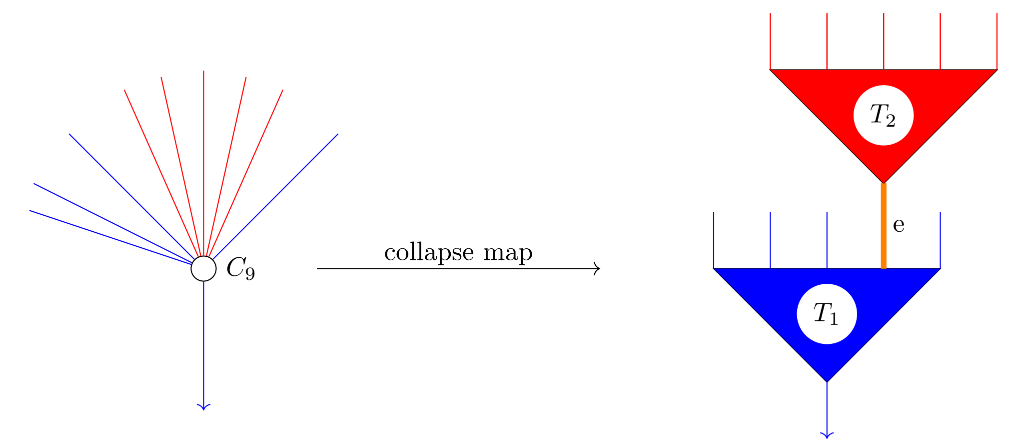
$$\mathcal{X}(T_1 \cup_e T_2) \xrightarrow{\simeq} \mathcal{X}(T_1) \times_{\mathcal{X}(e)} \mathcal{X}(T_2)$$

is an equivalence in $\mathcal{C}$ for every grafting $T_1 \cup_e T_2$ of two trees T_1 and T_2 along an edge e.

Given a Segal dendroidal set $\mathcal{X}\colon \Omega_\pi \to \mathbf{Set}$ we can recover the operad by:

- set of colors = $\mathcal{X}(\eta)$
- set of n-ary operations = $\mathcal{X}(C_n)$
- composition: $\mathcal{X}(C_n) \times_{\mathcal{X}(e)} \mathcal{X}(C_m) \xleftarrow{\simeq} \mathcal{X}(C_n \cup_e C_m) \xrightarrow{\mathcal{X}(\text{collapse map})} \mathcal{X}(C_{n+m-1})$

Example: a collapse map and a grafting of trees.



The localization functor $\mathcal{L}$

Construction

The functor $\mathcal{L}_\pi\colon \Omega_\pi^{op} \longrightarrow \Delta$ maps each tree to the linearly ordered set of "areas between the branches".

- $\mathcal{L}_\pi$ sends the corolla $C_n \in \Omega_\pi$ to $[n] \in \Delta$
- $\mathcal{L}_\pi$ sends all collapse maps to isomorphisms

Theorem [Wal]

The functor $\mathcal{L}_\pi\colon \Omega_\pi \longrightarrow \Delta$ exhibits the simplex category Δ as the ∞-categorical localization of Ω_π at the set of collapse maps.

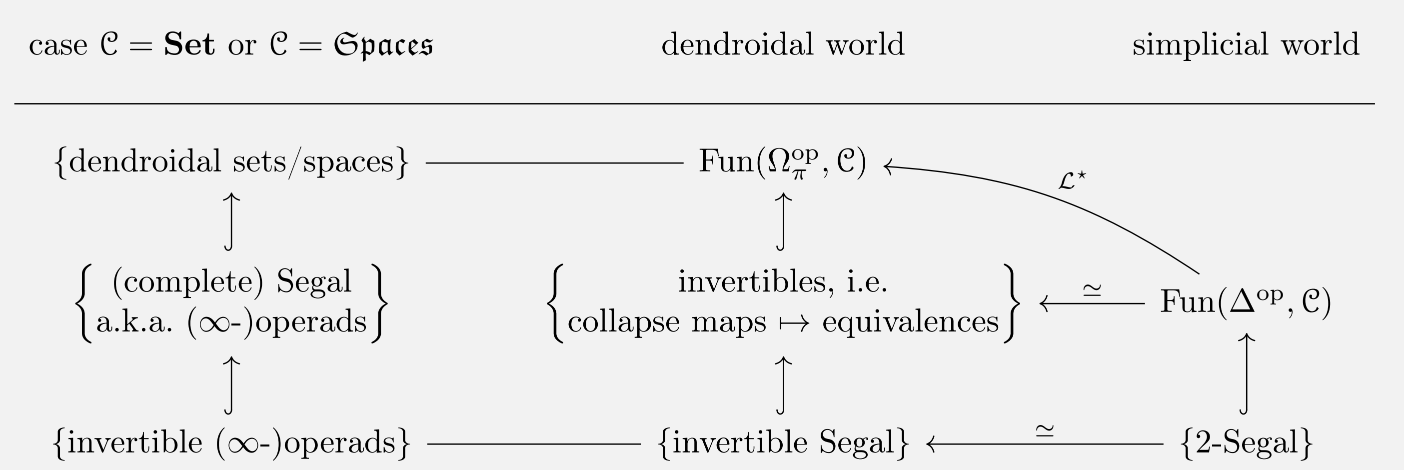
Fact [CM]: complete Segal dendroidal spaces are a model for ∞-operads:

$$\{\infty\text{-operads}\} \simeq \{\text{complete Segal dendroidal spaces}\} \subset \mathrm{Fun}(\Omega_\pi^{op}, \mathfrak{Spaces})$$

The 2-Segal condition on simplicial objects $\mathcal{X}\colon \Delta^{op} \to \mathcal{C}$ (due to Dyckerhoff-Kapranov [DK]) captures associativity and unitality of the "multivalued composition" $\mathcal{X}_{\{0,1\}} \times_{\mathcal{X}_{\{1\}}} \mathcal{X}_{\{1,2\}} \longleftarrow \mathcal{X}_{\{0,1,2\}} \longrightarrow \mathcal{X}_{\{0,2\}}$.
Examples:

- Waldhausen's S-construction from algebraic K-theory yields categorifications of Hall algebras
- categorifications of convolution algebras (e.g. Hecke algebras)

Overview

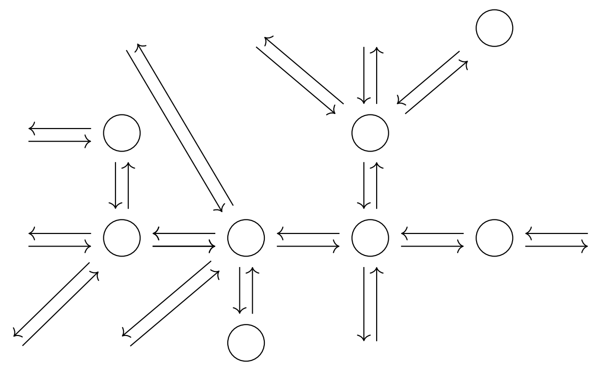


Variants: structured operads

additonal structure	category	type of tree	localization functor maps a tree too...
none	Δ	plane, rooted	linearly ordered set of areas between the branches
cyclic	Λ	plane, rootable	cyclic set of areas between the branches
symmetric	$\mathbf{Fin}_\star^{op}$	rooted	set of leaves plus basepoint (=root)
cyclic symmetric	$\mathbf{Fin}_{\neq\varnothing}^{op}$	rootable	non-empty set of incoming arrows

- Λ = Connes' cyclic category
- $\mathbf{Fin}_\star$ = category of pointed finite sets
- $\mathbf{Fin}_{\neq\varnothing}$ = category of non-empty finite sets

Example: plane rootable tree ⇝ cyclic operad; arrows = objects; duality reverses arrows



References

[CM] Denis-Charles Cisinski and Ieke Moerdijk. Dendroidal Segal spaces and ∞-operads. arXiv:1010.4956v2.

[DK] Tobias Dyckerhoff and Mikhail Kapranov. Higher Segal spaces I. arXiv:1212.3563v1.

[MW07] Ieke Moerdijk and Ittay Weiss. Dendroidal sets. *Algebraic & Geometric Topology*, 7:1441–1470, 2007.

[Wal] Tashi Walde. 2-Segal spaces as invertible ∞-operads. arXiv:1709.09935.

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