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# INTRODUCTION TO RIGID GEOMETRY

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# CONTENTS

# CHAPTER 1

# AFFINOID ALGEBRAS AND AFFINOID SPACES

## 1.1. Non-archimedean fields

***Definition 1.1.1.*** — An absolute value (or a valuation) on a field $k$ is a map $|\cdot| : K \to \mathbb{R}$ such that

1. $|a| \geq 0$ and $|a| = 0$ iff $a = 0$;
2. $|ab| = |a||b|$;
3. $|a + b| \leq |a| + |b|$.

If the third condition can be replaced by the stronger condition

(3)' $|a + b| \leq \max\{|a|, |b|\}$

then we say the valuation is non-archimedean; otherwise, we say it is archimedean. We say the valuation $|\cdot|$ is trivial, if $|a| = 1$ for $a \neq 0$.

Two valuations $|\cdot|_1$ and $|\cdot|_2$ on $k$ are called equivalent, if there exists $r > 0$ such that $|\cdot|_2 = |\cdot|_1^r$.

An absolute value $|\cdot|$ defines a distance function on $K$: for $a, b \in k$, we put $d(a, b) = |a - b|$. This makes $K$ a metric space, and we can talk about the completion of $K$ with respect to this metric, denoted usually by $\hat{K}$. It is clear that equivalent absolute values define the same topology on $K$, and hence give rise to the same completion.

***Example 1.1.2.*** — (1) $\mathbb{R}$ and $\mathbb{C}$ with the usual absolute values are complete archimedean valued fields. A non-trivial fact is that any complete archimedean field is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ [**Iya75**, Theorem 3.1].

(2) For each prime number $p$, one can define a $p$-adic valuation on $\mathbb{Q}$ as

$$|p^n \frac{a}{b}| = p^{-n}, \quad \text{for } (a, b) = 1, (p, ab) = 1.$$

We have also the usual archimedean absolute value $|\cdot|_\infty$ on $\mathbb{Q}$. A famous theorem of Ostrowski claims that *the absolute values $|\cdot|_p, |\cdot|_q$ are not equivalent distinct $p, q \leq \infty$, and any non-trivial absolute value on $\mathbb{Q}$ is equivalent to some of $|\cdot|_p$ with $p \leq \infty$.*

The completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$ is $\mathbb{R}$ and its completion w.r.t. $|\cdot|_p$ is usually denoted by $\mathbb{Q}_p$. Every elements of $\mathbb{Q}_p$ can be uniquely expressed as

$$x = \sum_{n>>-\infty}^{\infty} a_n p^n$$

with $a_n \in \{0, 1, \cdots, p-1\}$.

(3) For any field $k$, there is a natural non-trivial non-arhimedean valuation on $K = k((z))$. Namely, for $f(z) = \sum_{n=n_0}^{\infty} a_n z^n \in K$ with $a_{n_0} \neq 0$, we put

$$|f(z)| = \rho^{n_0}$$

for some real number $\rho$ with $0 < \rho < 1$.

In this course, we will focus on complete non-trivial non-archimedean fields. Let $K$ be such a field. Then the set

$$\mathcal{O}_K = \{x \in K; |x| \leq 1\}$$

is a subring of $K$, called the integral ring of $K$, and the set

$$\mathfrak{m}_K = \{x \in K; |x| < 1\}$$

is a maximal ideal of $\mathcal{O}_K$. (Indeed, if $x \in \mathcal{O}_K \backslash \mathfrak{m}_K$, then $|x^{-1}| = |x|^{-1} = 1$, hence $x$ is invertible in $\mathcal{O}_K$.) The quotient $k = \mathcal{O}_K/\mathfrak{m}_K$ is called the residue field of $K$. We say that $K$ is a discrete valuation field, if $|K^\times|$ is a discrete subgroup of $\mathbb{R}^\times$.

***Proposition 1.1.3.*** — *The valuation field $K$ is discrete if and only if $\mathfrak{m}_K$ is a principal ideal. In this case, if $\mathfrak{m}_K = \pi \cdot \mathcal{O}_K$, then every element of $K^\times$ is written uniquely as $x = \pi^n u$ for $n \in \mathbb{Z}$ and $u \in \mathcal{O}_K^\times$, and any non-trivial $\mathcal{O}_K$-submodule of $K$ is of the form $\pi^n \mathcal{O}_K$ for some $n \in \mathbb{Z}$.*

*Proof.* — Assume first that $|K^\times|$ is discrete. Then the set $|K^\times| \cap (0,1)$ has a maximal element, say $\rho$. Let $\pi \in \mathfrak{m}_K$ with $|\pi| = \rho$. We will prove that $\mathfrak{m}_K = (\pi)$. Indeed, for any $x \in \mathfrak{m}_K$, we have $|x| \leq \rho = |\pi|$, so $|x/\pi| \leq 1$, i.e. $x/\pi \in \mathcal{O}_K$ and $x \in (\pi)$.

Conversely, if $\mathfrak{m}_K = (\pi)$ is principal, then we claim that $|K^\times| = |\pi|^{\mathbb{Z}}$. First, we show that $|K^\times| \cap (|\pi|, |\pi|^{-1}) = \{1\}$. Indeed, if there exists $|x| \in |K^\times|$ with $|\pi| < |x| < 1$, then $x/\pi \in \mathfrak{m}_K$ but $x \notin (\pi)$. Similarly, if there exists $|y| \in |K^\times|$ with $1 < |y| < |\pi|^{-1}$, then $y^{-1} \in \mathfrak{m}_K$ but $y^{-1} \notin (\pi)$. This shows already that $|K^\times|$ is discrete. Since every discrete subgroup of $\mathbb{R}^\times$ is of the form $\gamma^{\mathbb{Z}}$, we conclude that $|K^\times| = |\pi|^{\mathbb{Z}}$.

Assume now that $K$ is discrete, so that $|K^\times| = |\pi|^{\mathbb{Z}}$. Therefore, for any $x \in K^\times$, there exists $n \in \mathbb{Z}$ such that $|x| = |\pi|^n$, so

$$x = \pi^n \cdot u, \text{ with } u = \frac{x}{\pi^n} \in \mathcal{O}_K^\times.$$

Now let $I \subseteq K$ be an $\mathcal{O}_K$-submodule. Note that if $|\pi|^n \in |I|$, then $\pi^n \cdot \mathcal{O}_K \subseteq I$. Indeed, let $a \in I$ with $|a| = |\pi|^n$. Then for any $x \in \pi^n \cdot \mathcal{O}_K$, one has $|x/a| \leq 1$, i.e. $x = a \cdot x/a \in a \cdot \mathcal{O}_K$. Therefore, if $|I| \subseteq \mathbb{R}_{>0}$ is unbounded, one must have $I = K$. Otherwise, $|I|$ has a maximal value, say $|\pi|^n$, and $\pi^n \mathcal{O}_K \subseteq I$. On the other hand, for any $x \in I$, one has $|x| \leq |\pi|^n$, hence $x/\pi^n \in \mathcal{O}_K$, i.e. $x \in \pi^n \cdot \mathcal{O}_K$. □

***Proposition 1.1.4.*** — *If $K'/K$ is a finite extension, then $|\cdot|$ on $K$ extends uniquely to a non-archimedean valuation on $K'$: For $a \in K'$, we put*

$$|x| = |N_{K'/K}(x)|^{1/[K':K]}.$$

*Moreover, $K'$ is complete with respect to this absolute value, and if $K$ is a discrete valuation field, then so is $K'$.*

We will not prove this Proposition in this course. Here, the most complicated point of the proof is to show that $|N_{K'/K}(x)|^{1/[K':K]}$ is indeed a non-archimedean absolute value. One has to use Hensel's Lemma. See [**Iya75**, Theorem 2.3, 4.4] for more details.

Therefore, the absolute value on $K$ extends uniquely to an algebraic closure $\overline{K}/K$. However, note that $\overline{K}$ is in general not complete with respect to this absolute value. For instance, let $\mathbb{C}_p$ denote the completion of $\overline{\mathbb{Q}}_p$. We consider the element

$$x = \sum_{(n,p)=1} \zeta_n p^n \in \mathbb{C}_p,$$

where $\zeta_n$ is a primitive $n$-th root of unity. Then by Krasner's Lemma, if $x \in \overline{\mathbb{Q}}_p$, then one would have $\zeta_n \in \mathbb{Q}_p(x)$ for all $n$ with $p \nmid n$. This contradicts with $[\mathbb{Q}_p(x) : \mathbb{Q}_p] < \infty$.

By definition, the Galois action of $\mathrm{Gal}(\overline{K}/K)$ on $\overline{K}$ keeps the absolute value, i.e. $|\sigma(x)| = |x|$ for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$ and $x \in \overline{K}$. This allows us to extend the action of $\mathrm{Gal}(\overline{K}/K)$ to $\widehat{\overline{K}}$ by continuity.

Over $\mathbb{R}$ and $\mathbb{C}$, there is a good theory of real or complex analytic geometric. In view of Ostrowski's theorem, it is natural to ask whether there exists an analoguous theory of analytic geometry over $\mathbb{Q}_p$, or more generally over a non-archimedean field $K$. Let us start with the following simple observation:

***Lemma 1.1.5.*** — *If $a, b \in K$ with $|a| \neq |b|$, then $|a+b| = \max\{|a|, |b|\}$.*

*Proof.* — We may assume that $|a| > |b|$. Then $|a+b| \leq |a|$. On the other hand, one has

$$|a| = |(a+b) - b| \leq \max\{|a+b|, |b|\}.$$

But $|a| > |b|$, hence $|a| \leq |a+b|$. □

***Corollary 1.1.6.*** — *A series $\sum_{n=0}^{\infty} a_n$ with $a_n \in K$ is convergent if and only if $a_n \to 0$ when $n \to \infty$.*

*Proof.* — The convergence of $\sum_{n=0}^{\infty} a_n$ means that $S_n = \sum_{i=0}^{n} a_i$ is a Cauchy sequence. Hence $a_n = S_n - S_{n-1}$ tends to 0 as $n \to \infty$. Conversely, if $a_n \to 0$, then $S_n$ is a Cauchy sequence. □

For $x \in K$ and $r \in \mathbb{R}_{>0}$, we put

$$D^-(x, r) = \{y \in K; |y - x| < r\}, \quad D^+(x, r) = \{y \in K; |y - x| \leq r\}.$$

They can be viewed as non-archimedean analogues of open and closed disks centered at $x$ with radius $r$. By definition, $\{D^-(x,r); r > 0\}$ is a fundamental system of open neighborhoods of $x$. However, one has, for any $0 < r' < r$ and $? = +, -$,

$$D^?(x,r) = \bigcup_{y \in D^?(x,r)} D^-(y,r') \quad \text{and} \quad K \backslash D^-(x,r) = \bigcup_{y \in K, |y-x| \geq r} D^-(y,r').$$

This shows that $D^-(x,r)$ and $D^+(x,r)$ are both open and closed in $K$! This is a fundamental difference between the topology of non-archimedean fields with the archimedean case. Despite of this property, we still call $D^+(x,r)$ (resp. $D^-(x,r)$) the closed (resp. open) disk with center $x$ and radius $r$.

***Corollary 1.1.7.*** — *The topology on $K$ is totally disconnected, i.e. the connected component of every point in $K$ is the set consisting of only that point.*

*Proof.* — Let $A$ be a subset of $K$ containing at least two distinct elements $x, y$ equipped with the induced topology. For any $r \in \mathbb{R}_{>0}$ with $r < |x - y|$. Then one has a partition

$$A = D^-(x,r) \cap A \coprod (K \backslash D^-(x,r)) \cap A$$

with both parts both open and closed. Hence, $A$ can not be connected. □

Because of this "weird" topology on $K$, it is not clear how to define a good theory of analytic geometric over non-archimedean fields. The classical notion of differentiable functions does not work well in this context. However, according to the philosophy of algebraic geometry, the study of an affine algebraic variety is equivalent to the studying its ring of algebraic functions. One may thus first try to figure out what should be the good notion of analytic functions on some simple geometric objects (like $D^+(x,r)$ and $D^-(x,r)$) over non-archimedean fields like.

Tate's idea was to mimic Weierstrass' definition of holomorphic functions. Consider first the $d$-dimensional closed unit disk:

$$\mathbb{B}^d(\overline{K}) = \{x = (x_1, \cdots, x_d) \in \overline{K}^d; |x| := \max_{1 \leq i \leq d}(|x_i|) \leq 1\}.$$

We have the following easy

***Lemma 1.1.8.*** — *A formal power series*

$$f(T) = \sum_{\mu \in \mathbb{N}^d} a_\mu \underline{T}^\mu \in K[[T_1, \cdots, T_d]]$$

*is convergent at every point of $\mathbb{B}^d(\overline{K})$ if and only if $a_\mu \to 0$ when $|\mu| := \sum_{i=1}^d \mu_i \to \infty$.*

*Proof.* — Exercise. □

We introduce thus the algebra

$$A_d := K\langle T_1, \cdots, T_d \rangle = \{f(T) = \sum_{\mu \in \mathbb{N}^d} a_\mu \underline{T}^{\underline{n}}; a_\mu \in K, a_\mu \to 0 \text{ as } |\mu| \to \infty\},$$

called the *d variables Tate algebra* with coefficients in $K$. We also call the elements of $K\langle T_1, \cdots, T_d\rangle$ *convergent power series.* This can be viewed as analytic functions on $\mathbb{B}^d$. There is a natural Gauss norm on $A\langle T_1, \cdots, T_d\rangle$ defined by

$$\| \sum_{\mu\in\mathbb{N}^d} a_\mu \underline{T}^\mu \| := \max_{\mu\in\mathbb{N}^d} |a_\mu|.$$

Note that the maximum is well defined, since $a_\mu \to 0$ when $|\mu| \to \infty$. We equip $A_d$ with the topology defined by this norm. This is the major complication of the Tate-algebras compared with the usual polynomial algebras. However, we will see later that such algebras have similar properties as usual polynomial algebras, and one can even forget the topology when we deal with the

## 1.2. Interlude: Banach $K$-algebras and modules

***Definition 1.2.1.*** — (1) A normed space $E$ over a non-archimedean field $K$ is a vector space $E$ over $K$ equipped with a norm $\|\cdot\| : E \to \mathbb{R}$ such that

- $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$;
- $\|ax\| = |a|\|x\|$ for $a \in K$ and $x \in E$;
- $\|x + y\| \leq \max\{\|x\|, \|y\|\}$.

(2) A normed space $E$ over $K$ is a Banach space if it is complete with respect to its norm.

Recall that if $E$ is a $K$-Banach space, and $F \subseteq E$ be a subspace, then there exists a unique induced structure of Banach space on the quotient $E/F$ if and only if $F$ is closed in $E$.

We fix an element $\pi \in \mathfrak{m}_K$. Let $E$ be a $K$-Banach space, an $\mathcal{O}_K$-lattice of $E$ is an $\mathcal{O}_K$-submodule $E_0 \subseteq E$ such that $(\pi^n E_0)_{n\geq 0}$ form a fundamental system of open neighborhoods of 0 in $E$. A linear operator $\phi : E \to F$ of $K$-Banach spaces is continuous if and only if it maps lattices of $E$ to lattices of $F$.

Classical theorems on Banach spaces in functional analysis over archimedean fields still hold for non-archimedean fields:

***Theorem 1.2.2.*** — *Let $E, F$ be $K$-Banach spaces, and $f : E \to F$ be a linear map.*

*(1) If $f$ is continuous and surjective, then $f$ is open. In particular, if $f$ is bijective, then it is an isomorphism.*

*(2) If $f$ is continuous and injective, then $f$ sends $E$ isomorphically onto a closed subspace of $F$.*

*(3) The map $f$ is continuous if and only if the graph of $f$ is closed in $E \times F$, i.e. if $x_i$ is a sequence convergent to $x$ in $E$ and $f(x_i) \to y$ as $i \to \infty$, then $f(x) = y$.*

*(4) Let $\mathcal{C}$ be a collection of continuous linear operators $T : E \to F$. If for all $x \in E$ one has*

$$\sup_{T\in\mathcal{C}} \|T(x)\| < \infty,$$

*then* $\sup_{T\in\mathcal{C}, \|x\|=1} \|T(x)\| < \infty$.

***Definition 1.2.3.*** — (1) A Banach algebra $A$ over $K$ is a $K$-algebra equipped with a structure of Banach space over $K$ such that $\|xy\| = \|x\| \cdot \|y\|$ for any $x, y \in A$.

(2) A normed $A$-module is an $A$-module $M$ equipped with a norm $\|\cdot\|$ such that the action of $A$ on $M$ is continuous, i.e.

$$A \times M \to M, \quad (a, m) \mapsto am$$

is continuous. A complete normed $A$-module is also called a Banach $A$-module.

***Proposition 1.2.4.*** — *The Tate algebra $A_d = K\langle T_1, \cdots, T_d\rangle$ is a Banach $K$-algebra with respect to its Gauss norm.*

*Proof.* — We first check that $A_d$ is a normed algebra such that $\|\cdot\|$ is multiplicative. The non-trivial part is to show $\|fg\| = \|f\| \cdot \|g\|$ for $f, g \in A_d$. Let $\prec$ denote the lexicographic order on $\mathbb{N}^d$, i.e. $\mu \prec \nu$ if and only if $\mu_1 \leq \nu_1$ or $\mu_1 = \nu_1$ and $\mu_2 \leq \nu_2$, ... Write $f = \sum_\mu a_\mu T^\mu$ and $g = \sum_{\nu \in \mathbb{N}^d} b_\nu T^\nu$. Let $\mu_0 \in \mathbb{N}^d$ (resp. $\nu_0 \in \mathbb{N}^d$) be the minimal index such that $|a_{\mu_0}| = \|f\|$ and $|b_{\nu_0}| = \|g\|$. Then we see that $|a_{\mu_0} b_{\nu_0}| = \|fg\|$.

We prove that $A_d$ is complete. Assume that $(f_i)_{i \geq 0}$ is a Cauchy sequence in $A_d$, i.e. for any $\varepsilon > 0$, there exists $N$ such that $\|f_i - f_j\| \leq \varepsilon$ for $i, j \geq N$. Write $f_i = \sum_\mu a_{i,\mu} T^\mu$. Then for each fixed $\mu \in \mathbb{N}^d$, the sequence $(a_{i,\mu})_{i \geq 0}$ is a Cauchy sequence; let $a_\mu$ be its limit and put $f = \sum_\mu a_\mu T^\mu$. Then we see easily that $\lim_{i \to \infty} f_i = f$ and $f \in A_d$. □

**1.2.5. $\pi$-adic $\mathcal{O}_K$-algebras.** — A topological $\mathcal{O}_K$-algebra $B$ is called $\pi$-*adic* if $\{\pi^n B : n \in \mathbb{Z}_{\geq 0}\}$ form a fundamental system of open neighborhoods of 0 in $B$. Let $N$ be a $B$-module, we equip it with the $\pi$-adic topology. We say that $N$ is separated, if $\bigcap_{n \geq 0} \pi^n N = 0$, and is complete if

$$N = \varprojlim_n N/\pi^n N.$$

Hence, a complete $\mathcal{O}_K$-algebra is always separated. Note that a separated quotient of a complete $B$-module is complete. In particular, we have the notion of complete separated $\pi$-adic $\mathcal{O}_K$-algebra.

***Lemma 1.2.6* (Nakayama Lemma).** — *Let $B$ be a complete $\pi$-adic $\mathcal{O}_K$-algebra, and $N$ be a separated $B$-module. Assume that there exist elements $e_1, \cdots, e_n \in N$ such that*

$$N = \sum_{i=1}^n Be_i + \pi N.$$

*Then we have $N = \sum_{i=1}^n Be_i$. In particular, $N$ is quotient of $B^n$ and hence complete.*

*Proof.* — For any $x \in N$, one can write $x = \sum_i b_{1,i} e_i + \pi x_1$. Similarly, one has $x_1 = \sum_i b_{2,i} e_i + \pi x_2$, hence $x = \sum_i (b_{1,i} + \pi b_{2,i}) e_i + \pi^2 x_2$. Repeating the process, one gets

$$x = \sum_i (b_{1,i} + \pi b_{2,i} \pi + \cdots + b_{n,i} \pi^{n-1}) e_i + \pi^n x_n.$$

Put $b_i = \sum_{j=1}^\infty b_{j,i} \pi^{j-1}$ and $x' = \sum_i b_i e_i$, which makes sense since $B$ is complete. Then $x - x' \in \bigcap_{n \geq 0} \pi^n N = 0$ by the separateness of $N$. □

Let $A$ be a Banach $K$-algebra $A$. Denote by $A^\circ$ its subset of elements with $\|x\| \leq 1$. Then $A^\circ$ is a complete separated $\pi$-adic open $\mathcal{O}_K$-subalgebra of $A$, and $A = A \otimes_{\mathcal{O}_K} K$. For instance, we have $K^\circ = \mathcal{O}_K$ and

$$A_d^\circ = \mathcal{O}_K\langle T_1, \cdots, T_d\rangle = \{f = \sum_\mu a_\mu T^\mu \in \mathcal{O}_K[[T_1, \cdots, T_d]]; a_\mu \to 0 \text{ as } |\mu| \to \infty\}.$$

Let $M$ be a normed $A$-module. An $A^\circ$-lattice of $M$ is an $A^\circ$-submodule $M^\circ \subseteq M$ such that $M = K \cdot M^\circ$ and $(\pi^n M^\circ)_{n\in\mathbb{Z}}$ form a fundamental system of open neighborhoods of 0 in $M$. Note that an $A^\circ$-lattice always exists, and $M^\circ$ is always separated.

***Corollary 1.2.7.*** — *Let $A$ be a Banach $K$-algebra, and $M$ be a normed $A$-module such that $\widehat{M}$ is finite over $A$. Then $M = \widehat{M}$, i.e. $M$ is complete.*

*Proof.* — We have $\widehat{M} = \sum_{i=1}^n Ae_i$ with $e_i \in \widehat{M}$. By open mapping theorem, the Banach space $\widehat{M}$ is identified with a quotient of $A^n$. Hence, $\widehat{M}^\circ = \sum_i A^\circ e_i$ is an $A^\circ$-lattice of $M$. Since $M \subset \widehat{M}$ is dense, one has $e_i \in f_i + \pi\widehat{M}^\circ$ for some $f_i \in M^\circ$. Put $M^\circ := \sum_{i=1}^n f_i A^\circ$. Then one has $\widehat{M}^\circ \subseteq \sum_i f_i A^\circ + \pi\widehat{M}^\circ$. By Lemma 1.2.6, we get $\widehat{M}^\circ = M^\circ$. Hence, $M^\circ$ is complete, and so is $M$.

□

***Theorem 1.2.8.*** — *Let $A$ be a noetherian Banach $K$-algebra, and $M$ be a finite $A$-module (without topology). Then there exists a unique topology on $M$ such that*

1. *$M$ is a Banach $A$-module,*
2. *every $A$-submodule of $M$ is closed for this topology,*
3. *and if $f : M \to N$ is a morphism of finitely generated $A$-modules, then $f$ is automatically continuous, and open if $f$ is surjective.*

*Proof.* — The uniqueness follows from the open mapping theorem. Since $M$ is finite over $A$, we choose a surjection $\phi : A^n \to M$. Since $A$ is noetherian and complete, the closure of $\mathrm{Ker}(\phi)$ in $A^n$ is complete and finite over $A$. Hence by Corollary 1.2.7, so is $\mathrm{Ker}(\phi)$ itself. Hence, $\mathrm{Ker}(\phi)$ is a closed subspace of $A^n$, and the quotient is a natural Banach $A$-module. The same arguments show that every submodule of $M$ is closed. It remains to prove (3). Consider the graph $\Gamma_f \subseteq M \times N$. It is an $A$-submodule, hence closed by (2). The continuity of $f$ now follows easily from the closed graph theorem.

□

## 1.3. Basic properties of Tate algebras

Let $K$ be a complete non-archimedean field, $\mathcal{O}_K$ be its valuation ring, and $k = \mathcal{O}_K/\mathfrak{m}_K$ be the residue field.

We fix a $\pi \in \mathfrak{m}_K$. Let $A$ be a complete separated $\pi$-adic $\mathcal{O}_K$-algebra or a $K$-Banach algebra. We can define similarly

$$A\langle T_1, \cdots, T_n\rangle := \{f(T) \in \sum_{\mu\in\mathbb{N}^d} a_\mu T^\mu; a_\mu \to 0 \text{ as } |\mu| \to \infty\}.$$

the ring of convergent power series with coefficients in $A$. We say a topological $A$-algebra $B$ is *topologically of finite type* (or *t.f.t.* $A$-algebra for short) over $A$ if there is a surjection

$$A\langle T_1, \cdots, T_n\rangle \to B$$

for some integer $n \geq 1$. Note that if $B$ is complete a t.f.t. $K$-algebra, $B$ has a natural structure of $K$-Banach space induced from $K\langle T_1, \cdots, T_n\rangle$.

For an $\mathcal{O}_K$-algebra $A$, we denote by $\tilde{A} = A/\mathfrak{m}_K A$ the reduction modulo $\mathfrak{m}_K$.

***Lemma 1.3.1.*** — *Let $R$ be a complete $\pi$-adic $\mathcal{O}_K$-algebra and $S$ be a t.f.t. $R$-algebra. Suppose that $\tilde{S}$ is finitely generated as a $\tilde{R}$-module. Then $S$ is finitely generated as a $R$-module.*

*Proof.* — By assumption, there exists a surjection

$$R\langle T_1, \cdots, T_n\rangle \to S.$$

Let $t_i \in S$ denote the image of $T_i$ for $1 \leq i \leq n$. We have to show that each $t_i$ is integral over $R$. By Lemma 1.2.6, it suffices to show that $S/\pi S$ is a finitely generated $R/\pi R$-module, or equivalently if $\bar{t}_i \in S/\pi S$ denotes the image of $t_i$, each $\bar{t}_i$ is integral over $R/\pi R$. Let $\tilde{t}_i \in \tilde{S}$ be the reduction modulo $\mathfrak{m}_K$. By assumption, each $\tilde{t}_i$ is integral over $\tilde{R}$, that is there is a monic polynomial

$$P_i(X) = X^{n_i} + a_1 X^{n_i-1} + \cdots + a_{n_i-1}X + a_{n_i} \in R[X]$$

such that $P_i(t_i) \in \mathfrak{m}_K S$. We may write $P_i(t_i) = \lambda_i g_i$ with $\lambda_i \in \mathfrak{m}_K$ and $g_i \in S$. Take an integer $n$ sufficiently large so that $\lambda_i^n \in \pi\mathcal{O}_K$. Then one has $P_i^n(t_i) \in \pi S$, and it follows that $\bar{t}_i$ is integral over $R/\pi R$. □

***Lemma 1.3.2.*** — *Let $B^\circ$ be a complete t.f.t. $\mathcal{O}_K$-algebra, and $B = B^\circ \otimes_{\mathcal{O}_K} K$. Let $t_1, \cdots, t_d \in B^\circ$ be such that their images in $B^\circ/\mathfrak{m}_K B^\circ$ are algebraically independent over $k$. The homomorphism*

$$\phi : K\langle T_1, \cdots, T_d\rangle \to B \quad \textit{with} \qquad T_i \mapsto t_i$$

*is an isomorphism of $K\langle T_1, \cdots, T_d\rangle$ onto a closed subalgebra of $B$.*

*Proof.* — Let $f = \sum_\mu a_\mu T^\mu \in \mathrm{Ker}(\phi)$. Up to multiplying $f$ by a scalar in $K$, we may assume that $\|f\| = 1$. Hence, we have $\tilde{f} \neq 0$ and $\tilde{f}(\tilde{t}_1, \cdots, \tilde{t}_d) = 0$, which contradicts with the algebraic independence of $t_i$'s. Note that $\phi$ is certainly continuous and $B$ is a $K$-Banach space, hence $\phi$ is an isomorphism onto a closed subalgbra of $A$. □

***Theorem 1.3.3* (Noether normalization).** — *Let $\mathfrak{a}$ be a closed ideal of the Tate algebra $A_n$, and $B = A_n/\mathfrak{a}$. Then there exists an integer $d$ and an injective algebra homomorphism $A_d \to B$ such that $B$ is a finitely generated $A_d$-module.*

*Proof.* — Recall that $A_n^\circ = \mathcal{O}_K\langle T_1, \cdots, T_n\rangle$. Let $B^\circ \subseteq B$ be the image of $A_n^\circ$, which is an $\mathcal{O}_K$-lattice of $B$. Then the reduction $\tilde{B}^\circ$ is a quotient $\tilde{A}_n^\circ \cong k[T_1, \cdots, T_n]$. By Noether's normalization theorem for polynomial algebras, there exist $\tilde{y}_1, \cdots, \tilde{y}_d \in \tilde{B}^\circ$ such that $\tilde{y}_1, \cdots, \tilde{y}_d$ are algebraically independent and $\tilde{B}^\circ$ is a finitely generated module over the subalgebra $k[\tilde{y}_1, \cdots, \tilde{y}_d]$. Let $y_1, \cdots, y_d \in B^\circ$ be lifts of $\tilde{y}_1, \cdots, \tilde{y}_d$. We get an $\mathcal{O}_K$-algebra homomorphism $A_d^\circ \to B^\circ$ sending $T_i$ to $y_i$. By Lemma 3.3.12, $B^\circ$ is a finitely generated $A_d^\circ$-module, and hence $B$ is finite over $A_d$. By Lemma 1.3.2, the map $A_d \to B$ identifies $A_d$ as a closed subalgebra of $B$.

□

***Remark 1.3.4.*** — We will see in the Theorem below that every ideal of $A_n$ is closed, hence the condition on $\mathfrak{a}$ in the Proposition is redundant.

***Theorem 1.3.5.*** — *The Tate algebra $A_n$ is noetherian. Every ideal of $A_n$ is closed, and every maximal ideal of $A_n$ is a closed subspace over $K$ of finite codimension.*

*Proof.* — We prove the noetherianness of $A_n$ by induction on $n$. The case $n = 0$ being trivial, we suppose that $A_d$ is noetherian for $d \leq n-1$. We claim that $A_n/fA_n$ is noetherian for all non-zero $f \in A_n$. We assume the claim for the moment and finish the proof as follows. Let $\mathfrak{a} \subseteq A_n$ be a non-zero ideal. Take a non-zero element $f \in \mathfrak{a}$. Let $\bar{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in $A_n/fA_n$. By the noetherianness of $A_n/fA_n$, $\bar{\mathfrak{a}}$ is finitely generated, and hence so is $\mathfrak{a}$.

We prove now the claim. The map $g \mapsto fg$ defines an isometry of $A_n$ onto $fA_n$, so that $fA_n$ is complete hence closed in $A_n$. Applying Theorem 1.3.3 to the ideal $fA_n$, we get an injective homomorphism $A_d \to B := A_n/fA_n$ such that $B$ is a finitely generated $A_d$-module. We may assume that $\|f\| = 1$, and hence $\tilde{f} \neq 0$ in $\tilde{A}_n^\circ$. Since $\tilde{B}^\circ$ is finite over $\tilde{A}^\circ$, we see that $d \leq n-1$. By induction hypothesis, $A_d$ is noetherian, so is $B$.

By Theorem 1.2.8, every ideal of $A_n$ is closed. Let $\mathfrak{p} \subseteq A_n$ be a maximal ideal. By Theorem 1.3.3, there is an injective homomorphism $A_d \to k(\mathfrak{p}) = A_n/\mathfrak{p}$ so that $k(\mathfrak{p})$ is finite over $A_d$. The fact that $k(\mathfrak{p})$ is a field implies that so is $A_d$, which forces $d = 0$.

□

***Corollary 1.3.6.*** — *There is a natural surjective map*

$$\mathbb{B}^n(\overline{K}) \to \mathrm{Max}(A_n), \quad x \mapsto \mathfrak{m}_x = \{f \in A_n; f(x) = 0\}.$$

*where* $\mathrm{Max}(A_n)$ *denotes the set of the maximal ideals of* $A_n$.

*Proof.* — Clearly, for each $x \in \mathbb{B}^n(\overline{K})$, $\mathfrak{m}_x$ is a maximal ideal of $A_n$. Conversely, for any maximal ideal $\mathfrak{m} \subseteq A_n$, the residue field $k(\mathfrak{m}) := A/\mathfrak{m}$ is a finite extension of $K$. We choose an embedding $k(\mathfrak{m}) \hookrightarrow \overline{K}$, and denote by $\phi : A_n \to \overline{K}$ the composite homomorphism. Since $\phi$ is continuous, there is a constant $C > 0$, such that $|\phi(f)| \leq C$ for all $f \in A_n$ with $\|f\| \leq 1$. This implies that $|\phi(T_i)| \leq 1$. Let $x_i = \phi(T_i) \in \overline{K}$, then $x = (x_1, \cdots, x_n) \in \mathbb{B}^n(\overline{K})$ and $\mathfrak{m}_x = \mathfrak{m}$.

□

***Remark 1.3.7.*** — We have actually a bijection between the orbits of $\mathbb{B}^n(\overline{K})$ under $\mathrm{Gal}(\overline{K}/K)$ with $\mathrm{Max}(A_n)$.

***Proposition 1.3.8* (Maximum Modulus Principle).** — *For any $f \in A_n$, there exists a point $x \in \mathbb{B}^n(\overline{K})$ such that*

$$|f(x)| = \|f\|.$$

*Proof.* — It is trivial that $|f(x)| \leq \|f\|$. Up to multiplying $f$ by a scalar, we may assume that $\|f\| = 1$. The reduction $\tilde{f} \in k[T_1, \cdots, T_n]$ is non-zero. There exists $\tilde{x} \in \overline{k}^n$ such that $\tilde{f}(\tilde{x}) \neq 0$. Let $x \in \mathcal{O}_{\overline{K}}^n$ be a lift of $\tilde{x}$. Then $|f(x)| = 1$. □

***Proposition 1.3.9.*** — *The ring $A_n$ is Jacobson, i.e. for any ideal $\mathfrak{a}$ of $A_n$, its radical* $\mathrm{rad}(\mathfrak{a})$ *equals the intersection of all maximal ideals $\mathfrak{m} \in \mathrm{Max}(A_n)$ containing $\mathfrak{a}$.*

*Proof.* — It is well known that $\mathrm{rad}(\mathfrak{a})$ is the intersection of all prime ideals of $A_n$ containing $\mathfrak{a}$. Hence, one gets

$$\mathrm{rad}(\mathfrak{a}) \subseteq \bigcap_{\mathfrak{m} \in \mathrm{Max}(A_n) \mathfrak{a} \subseteq \mathfrak{m}} \mathfrak{m}.$$

To prove the Proposition, it suffices to show that every prime ideal $\mathfrak{p}$ of $A_n$ is an intersection of maximal ideals.

Consider first the case $\mathfrak{p} = 0$. If $f \in \bigcap_{\mathfrak{m} \in \mathrm{Max}(A_n)} \mathfrak{m}$, then we have $f(x) = 0$ for all $x \in \mathbb{B}^n(\overline{K})$. It follows from the maximum modulus principle that $\|f\| = 0$, and hence $f = 0$.

Consider now general case. By Theorem 1.3.3, there exist an injective homomorphism $A_d \to B = A_n/\mathfrak{p}$ such that $B$ is finite over $A_d$. We have to show that $\bigcap_{\mathfrak{m} \in \mathrm{Max}(B)} \mathfrak{m} = 0$. Let $f \in \bigcap_{\mathfrak{m} \in \mathrm{Max}(B)} \mathfrak{m}$. Then $f$ is integral over $A_d$, i.e. it satisfies a monic polynomial equation:

$$f^r + a_1 f^{r-1} + \cdots a_{r-1} f + a_r = 0,$$

with $a_i \in A_d$. If $f \neq 0$, we may assume that $a_r \neq 0$. By commutative algebra, for every maximal ideal $\mathfrak{m}_x$ of $A_d$, there exists a maximal ideal $\mathfrak{m}_y \in \mathrm{Max}(B)$ lying above $\mathfrak{m}_x$. Taking the residues of the equation above modulo $\mathfrak{m}_y$, one gets $a_r \in \mathfrak{m}_x$, and hence $a_r = 0$ by the case $\mathfrak{p} = 0$. This contradicts with $a_r \neq 0$.

□

We now discuss the technique of Weierstrass preparation theorem and its consequences.

***Lemma 1.3.10.*** — *Let $f = \sum_{\mu \in \mathbb{N}^n} a_\mu T^\mu \in A_n$. The following statements are equivalent:*

1. *$f$ is a unit in $A_n$.*
2. *$f(x) \neq 0$ for all $x \in \mathbb{B}^n(\overline{K})$.*
3. *One has $|a_0| > |a_\mu|$ for any $\mu \neq 0$.*

*Proof.* — (1)⇔(2): Note that $f$ is a unit in $A_n$ if and only if it is not contained in any maximal ideal, that is $f \neq 0 \mod \mathfrak{m}_x$ for any $\mathfrak{m} \in \mathrm{Max}(A_n)$. The equivalence of (1) and (2) follows immediately from Corollary 1.3.6.

(1)⇒ (3): We may assume that $\|f\| = 1$, that is $f \in \mathcal{O}_K\langle T_1, \cdots, T_n\rangle$. Let $g$ be the inverse of $f$. Then $\|g\| = 1$ and $fg = 1$. Let $\tilde{f}, \tilde{g} \in k[T_1, \cdots, T_n]$ be the reductions modulo $\mathfrak{m}_K$ of $f$ and $g$. One has $\tilde{f}\tilde{g} = 1$. Hence, $\tilde{f}$ is a constant, and statement (3) follows immediately.

(3)⇒(2): In fact, for any $x \in \mathbb{B}^n(\overline{K})$, one has $|f(x)| = |a_0| \neq 0$. □

Note that when $\|f\| = 1$, $f$ is a unit in $A_n$ if and only if its reduction $\tilde{f} \in k[T_1, \cdots, T_n]$ is a constant function.

***Definition 1.3.11.*** — An element

$$f = \sum_{i=0}^{\infty} g_i T_n^i \in A_n = A_{n-1}\langle T_n \rangle, \text{ with } g_i \in A_{n-1}$$

is called $T_n$-distinguished of order $s \in \mathbb{N}$, if $\|f\| = \|g_s\|$ and $\|g_i\| < \|g_s\|$ for all $i > s$.

***Lemma 1.3.12.*** — *Given finitely many elements $f_1, \cdots, f_r \in A_n$, there exists a continuous automorphism $\sigma : A_n \to A_n$ of the form*

$$\sigma(T_i) = \begin{cases} T_i + T_n^{m_i}, & \text{if } i \neq n, \\ T_n & \text{if } i = n, \end{cases}$$

*for some integers $m_i \in \mathbb{N}$ such that $\sigma(f_i)$ are all $T_n$-distinguished. Moreover, we have $\|\sigma(f)\| = \|f\|$ for all $f \in A_n$.*

*Proof.* — It is clear that $\sigma$ sends $A_n^\circ = \mathcal{O}_K\langle T_1, \cdots, T_n\rangle$ to itself. Hence, $\|\sigma(f)\| \leq \|f\|$. Note also that $\sigma^{-1}$ is given by

$$T_i \mapsto T_i - T_n^{m_i}, \quad T_n \mapsto T_n,$$

thus similar arguments show that $\|\sigma^{-1}(f)\| \leq \|f\|$. Hence, $\|\sigma(f)\| = \|f\|$.

Consider first the case $r = 1$. We may assume that $\|f\| = 1$. Let $\tilde{f} = \sum_{\nu \in S} \tilde{a}_\nu T^\nu$ be the reduction of $f$, where $S$ is a subset of $\mathbb{N}^n$. We may assume that $\tilde{a}_\nu \neq 0$ for all $\nu \in S$. Let $t \in \mathbb{N}$ be some integer determined later, and consider the automorphism $\sigma$ with exponents given by $m_1 = t^{n-1}, \cdots, m_{n-1} = t$. The reduction of its action on $f$ is given by

$$\tilde{\sigma}(\tilde{f}) = \sum_{\nu \in S} \tilde{a}_\nu (T_1 + T_n^{t^{n-1}})^{\nu_1} (T_2 + T_n^{t^{n-2}})^{\nu_2} \cdots (T_{n-1} + T_n^t)^{\nu_{n-1}} T_n^{\nu} = \sum_{\nu \in S} T_n^{t^{n-1}\nu_1 + \cdots t\nu_{n-1} + \nu_n} + \tilde{g}.$$

We choose $t$ sufficiently large so that the degree of $T_n$ in $\tilde{g}$ is strictly less than

$$\deg_{T_n}(\tilde{g}) < s := \max_{\nu \in S} \{t^{n-1}\nu_1 + \cdots t\nu_{n-1} + \nu_n\}.$$

Note that, by Vandermonde it is well known that $t^{n-1}\nu_1 + \cdots t\nu_{n-1} + \nu_n$ are distinct for different $\nu \in S$. Let $\nu_0 \in S$ be the index such that the maximum above is obtained, then we get

$$\tilde{\sigma}(\tilde{f}) = \tilde{a}_{\nu_0} T_n^s + \text{something with degree} < s \text{ in } T_n.$$

The coefficient of $T_n^s$ of $\sigma(f)$ has reduction $\tilde{a}_{\nu_0}$, and hence is a unit in $A_{n-1}$ by Lemma 1.3.10, and the coefficient of $T_n^\mu$ for any $\mu > s$ has reduction zero. Hence, $\sigma(f)$ is $T_n$-distinguished.

In the general case, if there are several $f_1, \cdots, f_r$, we can still take $t$ sufficiently large such that the above construction works for all $f_i$. □

***Theorem 1.3.13* (Weierstrass Division).** — *Let $g \in A_n$ be a $T_n$-distinguished element of degree $s$. Then for any $f \in A_n$, there exists a unique series $q \in A_n$ and unique polynomial $r \in A_{n-1}[T_n]$ of degree $r < s$ satisfying*

$$f = qg + r.$$

*Moreover, we have $\|f\| = \max\{\|q\|\|g\|, \|r\|\}$.*

*Proof.* — We may assume that $\|g\| = 1$. Suppose one has an equality $f = qg + r$ as in the statement. We know that $\|f\| \leq \max\{\|g\|\|q\| = \|q\|, \|r\|\}$. If $\|f\|$ is strictly less than the right hand side, we have $\|q\| = \|r\| > \|f\|$. Up to multiplying $q, r, f$ by the same scalar, we may assume that $\|q\| = \|r\| = 1$. One can consider thus the reductions of $q, g$ and $r$. One would have $\tilde{q}\tilde{g} + \tilde{r} = 0$ in $k[T_1, \cdots, T_n]$ with $\deg_{T_n}(\tilde{r}) < \deg_{T_n}(\tilde{q})$. But this implies $\tilde{q} = 0$, which contradicts with $\|q\| = 1$.

It remains to prove the existence of such a division formula. We write

$$g = \sum_{i=0}^{\infty} g_i T_n^i$$

with $g_\mu \in A_{n-1}$, and $\|g_s\| = \|g\| = 1$ and $\|g_i\| < 1$ for all $i > s$. We set $\epsilon = \max\{\|g_i\|; i > s\}$ so that $\epsilon < 1$. We first prove the following weaker statement:

**Claim:** *For any $f \in A_n$, there exists $q, f_1 \in A_n$ and a polynomial $r \in A_{n-1}[T_n]$ of degree $< s$ such that*

$$f = qg + r + f_1$$

*where $\|q\|, \|r\| \leq \|f\|$ and $\|f_1\| \leq \epsilon\|f\|$.*

Assuming the claim, we conclude the proof of the Theorem as follows. Proceeding inductively on $i$, we get a sequence: $f_0 = f$, $f_i = q_i g + r_i + f_{i+1}$, where $\deg_{T_n}(r_i) < s$, $\|q_i\|, \|r_i\| \leq \|f_i\| \leq \epsilon^i\|f\|$ and $\|f_{i+1}\| \leq \epsilon\|f_i\| \leq \epsilon^{i+1}\|f\|$. Then one gets

$$f = (\sum_i q_i)g + (\sum_i r_i).$$

It remains to prove the claim. By approximation, we may assume that $f \in A_{n-1}[T_n]$. Write

$$g = g' + g'' \quad \text{with} \quad g' = \sum_{i=0}^{s} g_i T_n^i,\ g'' = \sum_{i \geq s+1} g_i T_n^i.$$

Then $g' \in A_{n-1}[T_n]$ is $T_n$-distinguished. By the Euclidean division in $A_{n-1}[T_n]$, we get

$$f = qg' + r$$

for some $q \in A_{n-1}[T_n]$ and $r \in A_{n-1}[T_n]$ with $\deg_{T_n}(r) < s$. As seen in the first part of the proof, we have $\|f\| = \max\{\|q\|, \|r\|\}$. From this, we see that

$$f = qg + r + f_1, \quad f_1 = qg''$$

and $\|f_1\| = \|q\|\|g''\| \leq \epsilon\|f\|$. □

***Corollary 1.3.14*** **(Weierstrass Preparation Theorem)**
*Let $g \in A_n$ be a $T_n$-distinguished element of order $s$. Then there exists a unique monic polynomial $P \in A_{n-1}[T_n]$ of degree $s$ and a unit $e \in A_n$ such that $g = Pe$. Moreover, $\|P\| = 1$ so that it is $T_n$-distinguished of order $s$.*

*Proof.* — Write $g = \sum_i g_i T_n^i$ with $g_i \in A_{n-1}$. As usual, we may assume $\|g\| = 1$. Since $g$ is $T_n$-distinguished, we have $\|g_s\| = 1$ and $\|g_i\| < 1$ for $i > s$. By Weierstrass division theorem, we have $T_n^s = qg + r$ with $\deg_{T_n}(r) < s$ and $1 = \max\{\|r\|, \|q\|\}$. Put $P = T_n^s - r$ so that $P = qg$ and $\|P\| = 1$. To prove the corollary, it is enough to see that $q$ is a unit.

Look at the reduction $\tilde{P} = \tilde{q}\tilde{g}$ in $\tilde{A}^{\circ}_{n-1}[T]$ with $A^{\circ}_{n-1} = k[T_1, \cdots, T_{n-1}]$. Note that both $\tilde{P}$ and $\tilde{g}$ are polynomials in $T_n$ of degree $s$ with top coefficients lying in $k^{\times}$. Therefore, $\tilde{q}$ has to lie in $k^{\times}$, which is equivalent to saying that $\tilde{q}$ is a unit in $A_n$.

The uniqueness of $P$ follows from the uniqueness of Weierstrass division by reversing the process above. □

***Proposition 1.3.15.*** — *The ring $A_n$ is a unique factorization domain and hence normal, i.e. it is integrally closed in its field of fractions.*

*Proof.* — We proceed by induction on $n$, and assume that $A_{n-1}$ is a UFD. By the Lemma of Gauss, we know that $A_{n-1}[T_n]$ is a UFD. We need to show that every element $f \in A_n$ can be factored as a product of prime elements, and such a factorization is unique up to units. We may assume that $f$ is not a unit. By Lemma 1.3.12, we may assume that $f$ is $T_n$-distinguished. By Weierstrass preparation theorem, one may assume that $f$ is monic polynomial and $T_n$-distinguished. Since $A_{n-1}[T_n]$ is a UFD, we have a factorization in $A_{n-1}[T_n]$ $f = P_1 \cdots P_r$ with each $P_r$ prime in $A_{n-1}[T_n]$. Since $f$ is monic, we may assume that $P_i$ are all monic and thus $\|P_i\| \geq 1$. Since $\|f\| = 1$, we get $\|P_i\| = 1$, i.e. each $P_i$ is $T_n$-distinguished. To finish the proof, it suffices to show that each $P_i$ is also a prime element in $A_n$. By Weierstrass division, the canonical map

$$A_{n-1}[T_n]/(P_i) \xrightarrow{\sim} A_n/(P_i)$$

is an isomorphism. Hence the conclusion follows.

□

## 1.4. Affinoid $K$-algebras

Let $K$ be a complete non-archimedean field as usual.

***Definition 1.4.1.*** — A topological $K$-algebra $A$ is called an *affinoïd algebra* over $K$, if there exists a surjective algebra homomorphism

$$K\langle T_1, \cdots, T_n\rangle \to A.$$

In other words, an affinoid algebra is a t.f.t. topological $K$-algebra.

By Theorem 1.3.5 and Proposition 1.3.9, the following statements hold:

1. Every affinoid algebra is noetherian, and every ideal in an affinoid algebra is closed.
2. The residue field of any maximal ideal of an affinoid algebra is finite over $K$.
3. Every affinoid algebra is Jacobson.

Note that every affinoid algebras can be equipped with a structure of Banach space over $K$. If $A$ is an affinoid algebra with a surjection with kernel $\mathfrak{a}$. We can equip $A$ with the quotient Banach structure: for any $\bar{f} \in A$, we put

$$\|\bar{f}\|_{\alpha} = \inf_{f \in \alpha^{-1}(\bar{f})} \|f\|.$$

Since $\mathfrak{a}$ is closed in $A_n$, $A$ becomes a Banach $K$-vector space with norm $\|\cdot\|_{\alpha}$.

***Definition 1.4.2.*** — Let $E$ be a Banach $K$-vector space. We say a collection of elements $(e_i)_{i\in I}$, with $I$ finite or countable, is an orthonormal basis of $E$, if every element $x \in E$ writes uniquely as

$$x = \sum_{i\in K} a_i e_i, \quad \text{with } a_i \in K$$

such that $a_i \to 0$ as $i \to \infty$, and $\|x\| = \max_{i\in I}\{|a_i|\}$.

For instance, the set of elements $\{T^\mu, \mu \in \mathbb{N}^n\}$ is an orthonormal basis of the Tate $A_n$-algebra over $K$.

***Proposition 1.4.3.*** — *Let $\mathfrak{a} \subseteq A_n$ be an ideal. Then there exists an orthonormal basis $(e_i)_{i\in I}$ of $A_n$ and a subset $J \subseteq I$ such that $(e_i)_{i\in J}$ is an orthonormal basis of $\mathfrak{a}$.*

*Proof.* — The proof of this proposition is a little complicated. Here, we give only a sketch. Since $A_n$ is noetherian, we may choose generators $f_1, \cdots, f_r \in \mathfrak{a}$ with $\|f_i\| = 1$ such that every element $x \in \mathfrak{a}$ can be written (not necessarily uniquely) as

$$x = \sum_{i=1}^{r} a_i f_i, \quad \text{with } a_i \in A_n.$$

Consider the set $\{T^\mu f_i; \mu \in \mathbb{N}^n, 1 \leq i \leq r\}$. Let

$$\mathfrak{a}^\circ := \sum A_n^\circ f_i \subseteq A_n^\circ = \mathcal{O}_K\langle T_1, \cdots, T_n\rangle$$

and $\tilde{\mathfrak{a}}^\circ \subseteq k[T_1, \cdots, T_n]$ be its reduction. It is clear that $\{T^\mu \tilde{f}_i; \mu \in \mathbb{N}^n, 1 \leq i \leq r\}$ generate $\tilde{\mathfrak{a}}^\circ$ as a $k$-subspace of $k[T_1, \cdots, T_n]$. We can choose a subset $\{e_j; j \in J\}\tilde{J} \subseteq \{T^\mu f_i; \mu \in \mathbb{N}^n, 1 \leq i \leq r\}$ such that their residue classes form a basis of $\tilde{\mathfrak{a}}^\circ$ over $k$. We enlarge the collection $(e_j)_{j\in J}$ by adding some elements of the form $T^\mu$ to get a collection $(e_i)_{i\in I}$ such that the residue classes of $e_i$ form a basis $k[T_1, \cdots, T_n]$ over $k$. Then we claim that such a basis $(e_i)_{i\in I}$ with subset $(e_j)_{j\in J}$ satisfies the requirement of the Proposition. When $K$ is discrete valued with uniformizer $\pi$, then one can prove by induction on $m$ that $A_n^\circ/(\pi^m)$ is generated by the images of $(e_i)_{i\in I}$ over $\mathcal{O}_K/\pi^m$, and that $\mathfrak{a}^\circ/\pi^m$ is generated by the images of $(e_j)_{j\in J}$. When $K$ is not discrete valued, see [**Bo08**, 1.3 Theorem 6]. □

***Corollary 1.4.4.*** — *Let $A$ be an affinoid algebra with surjection $\alpha : A_n \to A$. For any $\bar{f} \in A$, there exists an $f \in \alpha^{-1}(\bar{f})$ such that*

$$\|\bar{f}\|_\alpha = \|f\|.$$

*In particular, the image of $\|\cdot\|_\alpha$ is equal to $|K^\times|$.*

We now prove a fundamental result on the category of affinoid algebras.

***Proposition 1.4.5.*** — *Every $K$-algebra homomorphism of affinoïd algebras over $K$ is continuous.*

*Proof.* — Let $\phi : A \to B$ be a $K$-algebra homomorphism of affinoid algebras. We distinguish several cases:

*Case 1*: Both $A$ and $B$ are finite dimensional over $K$. The statement follows from Theorem 1.2.8.

*Case 2*: $B$ is finite dimensional over $K$ and $A$ is arbitrary. One may reduce to the previous case by factoring $\phi$ as the composite $A \to A/\operatorname{Ker}(\phi) \to B$.

*Case 3*: Both $A$ and $B$ are arbitrary affinoid algebras. By closed graph theorem, it suffices to show that if $f_i$ is sequence in $A$ with limit 0 such that their images $g_i = \phi(f_i)$ has limit $g$ in $B$, then we have $g = 0$. For any $\mathfrak{m} \in \operatorname{Max}(B)$ and any positive integer $n$, applying Case 2 to the composition $A \to B \to B/\mathfrak{m}^n$, we get $g \in \mathfrak{m}^n$. Hence, we have $g \in \bigcap_{n\geq 0, \mathfrak{m}\in\operatorname{Max}(B)} \mathfrak{m}^n$. We conclude by the following Lemma 1.4.6 in commutative algebra.

□

***Lemma 1.4.6.*** — *Let $B$ be a commutative noetherian ring. Then we have*

$$\bigcap_{\mathfrak{m}\in\operatorname{Max}(B), n\geq 0} \mathfrak{m}^n = 0.$$

*Proof.* — Note that for any $\mathfrak{m} \in \operatorname{Max}(B)$ one has $\widehat{B}_{\mathfrak{m}} = \varprojlim_n B/\mathfrak{m}^n$. Here, $\widehat{B}_{\mathfrak{m}}$ means the completion of the localization of $B$ at $\mathfrak{m}$. To prove the Lemma, it suffices to show that the canonical map

$$B \xrightarrow{f} \prod_{\mathfrak{m}\in\operatorname{Max}(B)} B_{\mathfrak{m}} \xrightarrow{g} \prod_{\mathfrak{m}\in\operatorname{Max}(B)} \widehat{B}_{\mathfrak{m}}$$

is injective. Let $b \in \operatorname{Ker}(f)$, and $\operatorname{Ann}(b)$ be its annihilator. The vanishing of the image of $b$ in $B_{\mathfrak{m}}$ implies that $\mathfrak{b}$ contains an element not in $\mathfrak{m}$. Hence, $\operatorname{Ann}(b)$ can not be contained in any maximal ideal of $B$, that is $\operatorname{Ann}(b) = B$ or equivalently $b = 0$. The injectivity of the map $g$ follows from the noetherian property of $B$. □

***Corollary 1.4.7.*** — *The Banach norm $\|\cdot\|_\alpha$ on $A$ for different projections are equivalent to each other, i.e. if $\beta : A_m \to A$ is another surjection, then there exist constants $c_1, c_2 > 0$ such that $c_1\|f\|_\beta \leq \|f\|_\alpha \leq c_2\|f\|_\beta$. In particular, the topology on $A$ is independent of the surjection $A_n \to A$.*

We denote by $\mathbf{Aff}(K)$ the category of affinoid algebras over $K$. In fancy words, Proposition 1.4.5 says that $\mathbf{Aff}(K)$ is a full subcategory of the category of $K$-algebras (without topology).

**1.4.8.** — Let $A \in \mathbf{Aff}(K)$. For a maximal ideal $x \in \operatorname{Max}(A)$ and $f \in A$, we denote by $K_x$ the residue field of $A$ at $x$, and by $f(x)$ the residue class of $f$ in $K_x$. As $K_x$ is finite over $K$, the valuation on $K$ extends uniquely to $K_x$. For any $f \in A$, we put

$$\|f\|_{\sup} = \sup_{x\in\operatorname{Max}(A)} |f(x)|,$$

and we call $\|\cdot\|$ the supremum norm of $A$. Note that, in general, $\|\cdot\|_{\sup}$ is not a norm for $A$, but just a semi-norm, because we have $\|f\| = 0$ if $f$ is nilpotent.

***Proposition 1.4.9.*** — *Let $A$ be an affinoid algebra.*

1. *We have $\|f^n\|_{\sup} = \|f\|^n$ for all $f \in A$ and $n \in \mathbb{N}$.*
2. *We have $\|f\|_{\sup} = 0$ if and only if $f$ is nilpotent in $A$.*
3. *For any homomorphism $\varphi : B \to A$ of affinoid $K$-algebras, we have $\|\varphi(b)\|_{\sup} \leq \|b\|_{\sup}$.*
4. *If $A = A_n$ is the Tate algebra, then $\|\cdot\|_{\sup}$ coincides with the Gauss norm.*
5. *For any surjection $\alpha : A_n \to A$ and $f \in A$, we have $\|f\|_{\sup} \leq \|f\|_\alpha$.*

*Proof.* — (1) follows from $|f^n(x)| = |f(x)|^n$. For (2), we have

$$\|f\|_{\sup} = 0 \Leftrightarrow f(x) = 0, \forall x \in \mathrm{Max}(A) \Leftrightarrow f \in \bigcap_{\mathfrak{m} \in \mathrm{Max}(A)} \mathfrak{m} = \mathrm{rad}(A).$$

For (3), note that if $\mathfrak{m} \in \mathrm{Max}(A)$, then $\varphi^{-1}(\mathfrak{m}) \subseteq B$ is also a maximal ideal. So

$$\|\varphi(b)\|_{\sup} = \sup_{\mathfrak{m} \in \mathrm{Max}(A)} |\varphi(b)(\mathfrak{m})| = \sup_{\mathfrak{m} \in \mathrm{Max}(A)} |b(\varphi^{-1}(\mathfrak{m}))| \leq \sup_{\mathfrak{n} \in \mathrm{Max}(B)} |b(\mathfrak{n})| = \|b\|_{\sup}.$$

(4) When $A = A_n$, recall that $\mathrm{Max}(A_n)$ are all of the form $\mathfrak{m}_x = \{f \in A_n | f(x) = 0\}$ for some $x \in \mathbb{B}^n(\overline{K})$. So for any $\mathfrak{m} \in \mathrm{Max}(A_n)$, we have $|f(\mathfrak{m})| \leq \|f\|$. On the other hand, the maximum modulus principle implies that there exists $x_0 \in \mathbb{B}^n(\overline{K})$ such that $\|f\| = |f(x_0)|$, i.e. the residue class of $f$ at $\mathfrak{m}_{x_0}$ has norm $\|f\|$. So we have $\|f\| \leq \|f\|_{\sup}$. On the other hand, since for any $x \in \mathbb{B}^n(\overline{K})$, we have $|f(x)| \leq \|f\|$.

(5) For any $f \in A_n$ with image $\bar{f} = \alpha(f) \in A$, we have

$$\|\bar{f}\|_{\sup} \leq \|f\|_{\sup} = \|f\|$$

by (3) and (4). Hence, $\|\bar{f}\|_{\sup} \leq \|f\|_\alpha$. □

***Proposition 1.4.10.*** — *Let $A$ be an affinoid algebra over $K$, and $f \in A$. Then the following statements are equivalent:*

1. *$\|f\|_{\sup} < 1$ for all $x \in \mathrm{Max}(A)$.*
2. *$1 - fT$ in invertible in $A\langle T\rangle$.*
3. *$f$ is topologically nilpotent in $A$, i.e. $\|f^n\|_\alpha \to 0$ as $n \to \infty$ for any surjection $\alpha : A_n \to A$.*

*Proof.* — (1) $\Leftrightarrow$ (2): It suffices to show that $1 - fT$ is not contained in any maximal ideal of $A\langle T\rangle$. Let $y \in \mathrm{Max}(A\langle T\rangle)$ and $x \in \mathrm{Max}(A)$ be induced by $y$ via the inclusion $A \hookrightarrow A\langle T\rangle$. Then $(1 - fT)(y) = 1 - f(x)T(y)$. Since $T$ is power bounded in $A\langle T\rangle$, the powers of $T(y)$ are also bounded in $K_y$, hence $|T(y)| \leq 1$. Since $|f(x)| < 1$, it follows that $|1 - f(x)T(y)| = 1$, in particular $1 - fT$ is not contained in $y$.

(2) $\Leftrightarrow$ (3): If $1 - fT$ is invertible, let $g = \sum_{i=0}^\infty g_i T^i$ be its inverse, with $g_i \in A, g_i \to 0$. By comparing the coefficients of $T^i$ in $(1 - fT)g = 1$, we get $g_0 = 1$, $g_1 = fg_0, \cdots$ etc. Thus one gets $g_i = f^i$ for any $i$. Hence $f^i \to 0$ as $i \to \infty$. □

***Lemma 1.4.11.*** — *Let*

$$p(T) = T^r + a_1 T^{r-1} + \cdots + a_r = \prod_{i=1}^r (T - \alpha_i) \in K[T]$$

*be a polynomial with roots $\alpha_1, \cdots, \alpha_r \in \overline{K}$. Then*

$$\max_{1\leq i\leq r} |\alpha_i| = \max_{1\leq j\leq r} |c_j|^{1/j}.$$

*Proof.* — Since $c_j$ is the $j$-th elementary function of $\alpha_i$'s, we have

$$|c_j| \leq (\max_i |\alpha_i|)^j \Leftrightarrow |c_j|^{1/j} \leq \max_i |\alpha_i|.$$

On the other hand, assume that $|\alpha_1| = \cdots = |\alpha_{r_1}| > |\alpha_i|$ for any $i > r_1$. Then

$$|c_{r_1}| = |\alpha_1 \cdots \alpha_{r_1}| = (\max_i |\alpha_i|)^{r_1},$$

hence the statement. □

***Lemma 1.4.12.*** — *Let $A$ be an integral affinoid algebra over $K$, $A_d \hookrightarrow A$ be an injective homomorphism such that $A$ is finite over $A_d$, and $f \in A$.*

1. *Let $p_f(T) = T^r + a_1T^{r-1} + \cdots + a_r \in A_d[T]$ be the minimal polynomial of $f$ over the fraction field of $A_d$. Then one has $a_i \in A_d$ for all $1 \leq i \leq r$, and we have $A_d[f] \cong A_d[T]/(p_f)$.*
2. *Let $x \in \mathrm{Max}(A_d)$, and $y_1, \cdots, y_s$ be the maximal ideals lying above $x$. Then*
$$\max_{1\leq i\leq s} |f(y_i)| = \max_{1\leq j\leq r} |a_j(x)|^{1/j}.$$
3. *We have $\|f\|_{\sup} = \max_j \|a_j\|^{1/j}$.*

*Proof.* — (1) Denote by $F$ the fraction field of $A_d$. Since $f$ is integral over $A_d$, there exists a monic polynomial $h \in A_d[T]$ such that $h(f) = 0$. Then $p_f|h$ in $F[T]$. Since $A_d$ is a UFD, Gauss Lemma for $A_d$ implies that $p_f \in A_d[T]$. Note that $A_d[T]$ is a UFD and $p_f$ is irreducible in $F[T]$, then $p_f$ is a primitive element in $A_d[T]$. Consider the surjective $\phi : A_d[T] \to A$ sending $T \mapsto f$. If $g(T) \in \mathrm{Ker}(\phi)$, then one has $p_f|g$ in $F[T]$, and hence in $A_d[T]$ by Gauss Lemma.

(2) Recall that if $A \subseteq B$ is an integral extension of rings, then the natural map $\mathrm{Max}(B) \to \mathrm{Max}(A)$ is surjective. We have thus two surjections

$$\mathrm{Max}(A) \to \mathrm{Max}(A_d[f]) \to \mathrm{Max}(A_d).$$

We may assume that $A = A_d[f] \cong A_d[T]/(p_f)$. Denote by $K_x = A_d/x$ the residue field at $x$, and let $\bar{p}_f$ be the residue class of $p_f$ in $K_x[T]$, and $\alpha_1, \cdots, \alpha_r$ be the roots of $\bar{p}_f$ in $\overline{K}$. Then the maximal ideals of $A$ lying above $x$ are given by $\mathrm{Spec}(A \otimes_{A_d} K_x) = \mathrm{Spec}(K_x[T]/(\bar{p}_f))$. Hence, there is a bijection between $f(y_1), \cdots, f(y_s)$ and the conjugacy classes over $K_x$ of $\alpha_1, \cdots, \alpha_r$. It follows from the previous Lemma that

$$\max_{1\leq i\leq s} |f(y_i)| = \max_{i=1,\cdots,r} |\alpha_i| = \max_{1\leq j\leq r} |a_j(x)|^{1/j}.$$

Statement (3) follows from (2), since $\mathrm{Max}(A) \to \mathrm{Max}(A_d)$ is a surjection. □

***Proposition 1.4.13.*** — *For any $f \in A \in \mathbf{Aff}(K)$, the following statements are equivalent:*

1. *$\|f\|_{\sup} \leq 1$.*

2. *For any algebra homomorphism $\phi : A_d = K\langle T_1, \cdots, T_d\rangle \to A$ such that $A$ is finite over $\phi(A_d)$, then $f$ is integral over $\phi(A_d^\circ)$.*
3. *$f$ is power bounded, i.e. the set $\{f^n : n \in \mathbb{N}\}$ is bounded in $A$.*

*Proof.* — (2) $\Rightarrow$ (3): We fix a residue norm $\|\cdot\|_\alpha$ on $A$. Take any homomorphism $\phi : A_d \hookrightarrow A$ such that $A$ is finite over $\phi(A_d)$ (whose existence is guaranteed by Theorem 1.3.3). Since $\phi$ is continuous, $\phi(A_d^\circ)$ is bounded, i.e. there exists a constant $C > 0$ such that $\|a\|_\alpha \leq C$ for all $a \in \phi(A_d^\circ)$. Since $f$ is integral over $\phi(A_d^\circ)$ by (2), it satisfies a monic equation

$$f^n + a_1 f^{n-1} + \cdots + a_n = 0$$

with $a_i \in \phi(A_d^\circ)$. Then any power of $f$ can be written a linear combination of $1, f, \cdots, f^{n-1}$ with coefficients in $\phi(A_d)$, it is easy to see that $\{f^m : m \in \mathbb{N}\}$ is contained in $\sum_i \phi(A_d^\circ) f^i$, which is bounded.

(3) $\Rightarrow$ (1): Since $f$ is power bounded in $A$, so is $f(x)$ for any $x \in \mathrm{Max}(A)$. This implies that $|f(x)| \leq 1$ for any $x \in \mathrm{Max}(A)$.

(1) $\Rightarrow$ (2): First, we reduce to the case that $A$ is integral. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ be the minimal prime ideals of $A$. If the satement is proved for integral $A$, there exists monic polynomials $h_i(T) \in A_d[T]$ such that $h_i(f) \mod 0 \mod \mathfrak{p}_i$ for each $1 \leq i \leq s$. Putting $h = \prod_i h_i$, then $h(f) \in \bigcap_i \mathfrak{p}_i = \mathrm{rad}(A)$. Hence, there exists an integer $N$ such that $h^N(f) = 0$.

Secondly, we reduce to the case that $\phi$ is injective. Consider the subalgebra $\phi(A_d) \subseteq A$. By Noether's normalization Theorem 1.3.3, there exists a map $\psi : A_{d'} \to A_d$ such that the composite

$$\phi' : A_{d'} \xrightarrow{\psi} A_d \to A_d/\operatorname{Ker}(\phi) = \phi(A_d) \hookrightarrow A$$

is injective, and $\phi(A_d)$ (hence $A$) is finite over $\phi'(A_{d'})$. Since $\psi$ is continuous, $\psi(T_i)$ is power bounded, i.e. we have $\|\psi(T_i)\| \leq 1$. So we have $\phi'(A_{d'}^\circ) \subseteq \phi(A_d^\circ)$. If we can show that $f$ is integral over $\phi'(A_{d'}^\circ)$, we are done. Up to replacing $\phi$ by $\phi'$, we may assume that $\phi$ is injective.

Let $p_f(T) = T^r + a_1 T^{r-1} + \cdots + a_r$ be the minimal polynomial of $f$ over the fraction field of $A_d$. By Lemma 1.4.12(1) and (3), one has $a_i \in A_d$ and

$$\max_{1\leq j\leq r} \|a_j\|^{1/j} = \|f\|_{\sup} \leq 1.$$

□

***Corollary 1.4.14.*** — *Let $\phi : A \to B$ be a homomorphism of affinoid $K$-algebras. Given $f_1, \cdots, f_r \in B$, there exists a homomorphism $\psi : A\langle T_1, \cdots, T_n\rangle \to B$ such that $\psi|_A = \phi$ and $\psi(T_i) = f_i$ if and only if $\|f_i\|_{\sup} \leq 1$ for all $i$.*

*Proof.* — If such a $\psi$ exists, then $\|\psi(T_i)\|_{\sup} \leq \|T_i\|_{\sup} = 1$ by Proposition 1.4.9(3). Conversely, if $\|f_i\|_{\sup} \leq 1$, then $f_i$ is power bounded by the Proposition. Then, for every $g = \sum_\mu a_\mu T^\mu$ with $a_\mu \in A$ the image $\psi(g) = \sum_\mu \phi(a_\mu) f^\mu$ is well-defined. □

## 1.5. Affinoid spaces and affinoid subdomains

Let $A$ be an affinoid algebra over $K$. We put $\mathrm{Sp}(A) = \mathrm{Max}(A)$, and call it the affinoid space associated to $A$. As in the case of algebraic geometry, one should view $A$ as the ring of functions over $\mathrm{Sp}(A)$. For any $f \in A$ and $x \in \mathrm{Sp}(A)$, we denote by $f(x)$ the residue class of $f$ modulo the maximal ideal of $A$ corresponding to $x$.

**1.5.1. Zariski topology on** $\mathrm{Sp}(A)$**.** — We define the Zariski topology on $\mathrm{Sp}(A)$ such that the closed subsets of $\mathrm{Sp}(A)$ are given by

$$V(\mathfrak{a}) = \{x \in \mathrm{Sp}(A); f(x) = 0, \forall f \in \mathfrak{a}\}.$$

The Zariski topology on $\mathrm{Sp}(A)$ satisfy similar properties as the Zariski topology for affine schemes.

***Lemma 1.5.2.*** — 1. *One has $V(\mathfrak{a}) \supset V(\mathfrak{b})$ for ideals $\mathfrak{a} \subseteq \mathfrak{b}$.*

2. *One has $V(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} V(\mathfrak{a}_i)$ for any family of ideals $(\mathfrak{a}_i)_{i\in I}$ of $A$.*
3. *$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.*
4. *For any $f \in A$, we put*

$$D_f = \{x \in \mathrm{Sp}(A); f(x) \neq 0\}.$$

*Then the collection of sets $\{D_f; f \in A\}$ form a basis of Zariski-open subsets of $A$, that is every open subset is a union (actually a finite union) of $D_f$'s.*

*Proof.* — Exercise. □

For any subset $Y \subset \mathrm{Sp}(A)$, we put

$$I(Y) = \{f \in A; f(y) = 0 \text{for all } y \in Y\} = \bigcap_{y\in Y} \mathfrak{m}_y.$$

***Proposition 1.5.3.*** — 1. *For any subset $Y \subseteq \mathrm{Sp}(A)$, we have $V(I(Y)) = \overline{Y}$.*

2. *(Hilbert Nullstellensatz) If $\mathfrak{a} \subseteq A$ is an ideal, then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

*Proof.* — The proof is very similar as in algebraic geometry.

(1) Write $\mathfrak{a} = I(Y)$. Then $f \in \mathfrak{a}$ iff $Y \subseteq V(f)$. Since $\overline{Y}$ is the intersection of closed subsets containing $Y$ and each such a closed subset is the intersection of some $V(f)$'s with $f \in \mathfrak{a}$, one has

$$\overline{Y} = \bigcap_{f\in A, Y\subseteq V(f)} V(f) = \bigcap_{f\in\mathfrak{a}} V(f) = V(\mathfrak{a}).$$

(2) We have

$$\begin{aligned} I(V(\mathfrak{a})) &= \{f \in A; f(x) = 0 \text{ for all } x \in V(\mathfrak{a})\} \\ &= \bigcap_{x\in V(\mathfrak{a})} \mathfrak{m}_x = \bigcap_{\mathfrak{a}\subseteq\mathfrak{m}_x} \mathfrak{m}_x = \sqrt{\mathfrak{a}}, \end{aligned}$$

where the last step uses the fact that the affinoid algebra $A$ is Jacobson. □

The following two corollaries are immediate consequences of the Proposition.

***Corollary 1.5.4.*** — *The maps $V(\cdot)$ and $I(\cdot)$ induce a bijection between the set of reduced ideals in $A$ and the set of Zariski-closed subsets of* $\mathrm{Sp}(A)$.

***Corollary 1.5.5.*** — *Let $\{f_i; i \in I\} \subseteq A$. The following are equivalent:*

1. *The $f_i$ have no common zeros in* $\mathrm{Sp}(A)$.
2. *The $f_i$ generate the unit ideal of $A$.*

As in algebraic geometry, if $\phi : A \to B$ is a homomorphism of affinoid $K$-algebras, $\phi$ induces a morphism $\phi^a : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ given by $\mathfrak{m}_x \mapsto \phi^{-1}(\mathfrak{m}_x)$, which is continuous for the Zariski topology. In the sequel, we will call $\phi^a$ (together with $\phi$) a morphism of affinoid spaces.

The Zariski topology is too coarse for the study of affinoid spaces. First, it is exactly the same theory as algebraic geometry, and it does not use essentially the topology of the non-archimedean topology; secondly, the open subset $D_f \subseteq \mathrm{Sp}(A)$ corresponds to the $A$-algebra $A[\frac{1}{f}]$, which is not complete.

**1.5.6. Canonical topology.** — Let $A$ be an affinoid algebra over $K$, and $X = \mathrm{Sp}(A)$. For $f \in A$ and $\epsilon \in \mathbb{R}_{>0}$, we put

$$X(f, \epsilon) = \{x \in X; |f(x)| \leq \epsilon\}.$$

***Definition 1.5.7.*** — The canonical topology on $\mathrm{Sp}(A)$ is the topology generated by all sets of type $X(f, \epsilon)$, i.e. a subset $U \subseteq \mathrm{Sp}(A)$ is open for the canonical topology if and only if it is a union of sets of type $X(f_1, \epsilon_1) \cap \cdots \cap X(f_r, \epsilon_r)$.

We write $X(f) = X(f, 1)$ and $X(f_1, \cdots, f_r) = X(f_1) \cap \cdots \cap X(f_r)$.

***Lemma 1.5.8.*** — *The canonical topology is generated by the subsets of type $X(f)$, i.e. every open subset of* $\mathrm{Sp}(A)$ *is a finite union of subsets of type $X(f_1, \cdots, f_r)$.*

*Proof.* — Note that $|f(x)| \in |\overline{K}^*|$ for all $f \in A$ and $x \in \mathrm{Sp}(A)$. For all $f \in A$ and $\epsilon \in \mathbb{R}_{>0}$, one has

$$X(f, \epsilon) = \bigcup_{\epsilon' \leq \epsilon, \epsilon' \in |\overline{K}^*|} X(f, \epsilon')$$

As $\epsilon' \in |\overline{K}^*|$, there exists $c \in K^*$ and an integer $s > 0$ such that $\epsilon'^s = |c|$. Hence, one has

$$X(f, \epsilon') = X(f^s, \epsilon'^s) = X(c^{-1} f^s).$$

□

***Lemma 1.5.9.*** — *Let $f \in A$ and $x \in \mathrm{Sp}(A)$ with $|f(x)| = \epsilon > 0$. Then there exists an element $g \in A$ such that $g(x) = 0$ and $|f(y)| = \epsilon$ for all $y \in X(g)$. In particular, $D(g)$ is an open neighborhood of $x$ contained in $\{x \in X; |f(x)| = \epsilon\}$.*

*Proof.* — Let $\mathfrak{m}_x \subseteq A$ be the maximal ideal corresponding to $x$, and write $\bar{f} = f(x)$ for the residue class of $f$ in $K_x = A/\mathfrak{m}_x$. Let

$$P(T) = T^n + a_1 T^{n-1} + \cdots + a_n \in K[T]$$

be the minimal polynomial of $\bar{f}$ over $K$. Let $\alpha_1, \cdots, \alpha_n \in \overline{K}$ be the roots of $P(T)$ so that

$$P(T) = \prod_{i=1}^{n}(T - \alpha_i).$$

Note that $|f(x)| = |\alpha_i| = \epsilon$ for all $i$.

Let $g = P(f) \in A$. Then $g(x) = P(\bar{f}) = 0$ and we claim that for all $y \in \mathrm{Sp}(A)$ with $|g(y)| < \epsilon^n$, one has $|f(y)| = \epsilon$. Indeed, assume that there exists $y \in \mathrm{Sp}(A)$ with $|g(y)| < \epsilon^n$ such that $|f(y)| \neq \epsilon$. Then

$$|f(y) - \alpha_i| = \max\{|f(y)|, |\alpha_i|\} \geq |\alpha_i| = \epsilon$$

and hence

$$|g(y)| = \prod_i |f(y) - \alpha_i| \geq \epsilon^n$$

which contradicts with the choice of $y$. Now we take some $c \in K^*$ such that $|c| < \epsilon^n$. Then one has $|f(y)| = \epsilon$ for all $y \in X(c^{-1}g)$. □

***Corollary 1.5.10.*** — *For any $\epsilon > 0$ and $f \in A$, the following subsets are open in the canonical topology:*

$$\{x \in \mathrm{Sp}(A); f(x) \neq 0\}, \quad \{x \in \mathrm{Sp}(A); |f(x)| < \epsilon\},$$
$$\{x \in \mathrm{Sp}(A); |f(x)| = \epsilon\}, \quad \{x \in \mathrm{Sp}(A); |f(x)| > \epsilon\}.$$

*Proof.* — It follows immediately from the Lemma that $\{x \in \mathrm{Sp}(A); |f(x)| = \epsilon\}$ is open in the canonical topology. For $\{x \in \mathrm{Sp}(A); |f(x)| < \epsilon\}$, the statement follows from

$$\{x \in \mathrm{Sp}(A); |f(x)| < \epsilon\} = \bigcup_{\epsilon' < \epsilon} X(f, \epsilon')$$

□

***Definition 1.5.11.*** — Let $X = \mathrm{Sp}(A)$ be an affinoid space. A subset $U \subseteq X$ is called an affinoid subdomain if there exists a morphism of affinoid $K$-spaces $\iota : X' = \mathrm{Sp}(A') \to X = \mathrm{Sp}(A)$ such that $\iota(X') \subset U$ and the following universal property is satisfied: For any morphism of affinoid $K$-spaces $f : Y \to X$ with $f(Y) \subseteq U$, there exists a unique morphism $f' : Y \to X'$ such that $f = \iota \circ f'$.

***Example 1.5.12.*** — 1. For $f_1, \cdots, f_r \in A$, the subset

$$X(f_1, \cdots, f_r) = \{x \in X; |f_i(x)| \leq 1\}$$

is an affinoid subdomain with $\iota$ given by the natural morphism $\iota : X' = \mathrm{Sp}(A\langle f_1, \cdots, f_r\rangle) \to X$ where $A\langle f_1, \cdots, f_r\rangle = A\langle T_1, \cdots, T_r\rangle/(T_i - f_i)$. We will call such subsets Weierstrass domains of $X$. Indeed, if $\phi : Y = \mathrm{Sp}(B) \to X$ is a morphism of affinoid $K$-spaces with $\phi(Y) \subseteq X(f_1, \cdots, f_r)$, then one has $|\phi^*(f_i)(y)| = |f(\phi(y))| \leq 1$ for any $y \in \mathrm{Sp}(B)$. This implies that $|\phi^* f_i|_{\sup} \leq 1$, i.e. $\phi^*(f_i)$ is power bounded. Hence, there exists a morphism $\psi^* : A\langle T_1, \cdots, T_r\rangle \to B$ such that $\psi^*|_A = \phi^*$ and $\psi^*(T_i) = \phi^* f_i$, i.e. the morphism $\psi^*$ factors through the quotient $A\langle T_1, \cdots, T_r\rangle/(T_i - f_i)$.

2. For $f_1, \cdots, f_r, g_1, \cdots, g_r \in A$, the subset
$$X(f_1, \cdots, f_r, g_1^{-1}, \cdots, g_s^{-1}) = \{x \in X; |f_i(x)| \leq 1, |g_j(x)| \geq 1\}$$
is an affinoid subdomain of $X$ with $\iota$ induced by the natural morphism
$$A \to A' = A\langle T_1, \cdots, T_r, S_1, \cdots, S_r\rangle/(T_i - f_i, 1 - g_i S_i).$$
We such subset of $X$ a Laurent subdomain of $X$. Indeed, if $\phi : Y = \mathrm{Sp}(B) \to X$ is a morphism of affinoid $K$-spaces such that $\phi(Y) \subseteq X(f_1, \cdots, f_r, g_1^{-1}, \cdots, g_s^{-1})$, then the same argument as in (1) shows that $\|\phi^*(f_i)\|_{\sup} \leq 1$ and $\phi^*(g_i)$ is invertible with $\|\phi^*(g_i)^{-1}\|_{\sup} \leq 1$. Hence, there exists a unique morphism $\psi^* : A\langle T_1, \cdots, T_r, S_1, \cdots, S_r\rangle/(T_i - f_i, 1 - g_i S_i) \to B$ which extends $\phi^* : A \to B$ and $\psi^*(T_i) = f_i$ and $\psi^*(S_i) = g_i$ for each $1 \leq i \leq r$.
3. Let $f_0, \cdots, f_r \in A$ which generate $A$ as an ideal. The subset
$$X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0}) = \{x \in X; |f_i(x)| \leq |f_0(x)|\}$$
is an affinoid subdomain of $X$, and we call such a subset a rational domain of $X$. The morphism $\iota$ corresponding to $X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0})$ is given by the natural morphism
$$\iota : X' = \mathrm{Sp}\big(A\langle T_1, \cdots, T_r\rangle/(f_i - f_0 T_i)\big) \to X.$$
Since each $T_i$ is power bounded in $A\langle T_1, \cdots, T_r\rangle/(f_i - f_0 T_i)$, one has $|T_i(x')| \leq 1$ and hence $|f_i(\iota(x'))| \leq |f_0(\iota(x'))|$ for all $x' \in X'$. Hence, we have $\iota(X') \subseteq X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0})$.
   To see that $X' \to X$ satisfies the universal property, we first note that $f_0$ does not vanish at any point of $X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0})$. Indeed, if $f_0$ vanishes at some $x \in X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0})$, then $|f_i(x)| \leq |f_0(x)| = 0$ for all $i$, which contradicts that $f_0, f_1, \cdots f_r$ has no common zeros in $X$. If $\phi : Y = \mathrm{Sp}(B) \to X = \mathrm{Sp}(A)$ is a morphism of affinoid $K$-spaces with $\phi(Y) \subseteq X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0})$, then $\phi^*(f_0), \cdots, \phi^*(f_r)$ have no common zeros in $Y$, $\phi^*(f_0)$ is invertible, and $\|\frac{\phi^*(f_i)}{\phi^*(f_0)}\|_{\sup} \leq 1$. Therefore, there exists a unique morphism
$$\psi^* : A\langle T_1, \cdots, T_r\rangle/(f_i - f_0 T_i) \to B$$
extending $\phi^*$ and $\psi^*(T_i) = \frac{\phi^*(f_i)}{\phi^*(f_0)}$.

***Proposition 1.5.13.*** — *Let $\iota : X' = \mathrm{Sp}(A') \to X = \mathrm{Sp}(A)$ be a morphism representing an affinoid subdomain $U \subseteq X$. Then the following hold:*
1. *$\iota$ is injective with image $U = \iota(X')$.*
2. *For any $x \in X'$, we have $\mathfrak{m}_x = \mathfrak{m}_{\iota(x)} A'$, and for any $n \in \mathbb{N}$, $\iota^*$ induces an isomorphism $A/\mathfrak{m}_{\iota(x)}^n \cong A'/\mathfrak{m}_x^n$*

*Proof.* — (1) The injectivity of $\iota$ follows easily from the universal property of $\iota$.

(2) For any $y \in U$ and $n \in \mathbb{N}$, there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota^*} & A' \\ \big\downarrow{\scriptstyle\pi} & \overset{\alpha}{\swarrow} & \big\downarrow{\scriptstyle\pi'} \\ A/\mathfrak{m}_y^n & \xrightarrow{\sigma} & A'/\mathfrak{m}_y^n A'. \end{array}$$

Note that $\mathrm{Sp}(A/\mathfrak{m}_y^n)$ is a one-point affinoid $K$-space with the natural map $\mathrm{Sp}(A/\mathfrak{m}_y^n) \to X$ mapping to $y \in U$. Hence, by the universal property of $\iota$, there exists a unique homomorphism $\alpha : A' \to A/\mathfrak{m}_y^n$ such that the upper-left triangle is commutative. We claim that the right-lower triangle is commutative as well. By the universal property of $\iota$, $\pi'$ is the unique homomorphism such that $\pi' \circ \iota^* = \sigma \circ \pi = \sigma \circ \alpha \circ \iota^*$. Hence, one deduces that $\pi' = \sigma \circ \alpha$.

We show now that $\sigma$ is an isomorphism. By taking $n = 1$, this implies $\mathfrak{m}_y A' = \mathfrak{m}_x$ where $x \in X'$ with $\iota(x) = y$. Note that the surjectivity of $\pi$ implies that of $\alpha$. By $\pi' = \sigma \circ \alpha$, we get $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\pi') = \mathfrak{m}_y^n A'$. On the other hand, it is clear that $\mathfrak{m}_y^n \subseteq \mathrm{Ker}(\alpha)$ as $\alpha \circ \iota^* = \pi$. Hence it follows that $\mathrm{Ker}(\pi') = \mathrm{Ker}(\alpha)$, and thus $\sigma$ is an isomorphism. □

In the sequel, if $U$ is an affinoid subdomain represented by $\iota : X' = \mathrm{Sp}(A') \to X = \mathrm{Sp}(A)$, we will identify $X'$ with $U$.

***Corollary 1.5.14.*** — *Let $U = \mathrm{Sp}(A') \subseteq X = \mathrm{Sp}(A)$ be an affinoid subdomain. Then $B$ is flat over $A$.*

*Proof.* — It suffices to show that for $x \in U$, the local ring $A'_x$ is flat over $A_x$. However, Proposition 1.5.13(2), we have an isomorphism of completed local rings $\hat{A}'_x \cong \hat{A}_x$. Since $A'_x$ and $A_x$ are both noetherian, $\hat{A}'_x$ is faithfully flat over $A_x$ and $\hat{A}_x$ is flat over $A_x$. It follows immediately that $A'_x$ is flat over $A_x$. □

**1.5.15. Completed tensor products.** — Let us recall first the completed tensor product of Banach $K$-spaces. Let $A$ be a Banach $K$-algebra, and $E, F$ be two Banach $A$-module. Consider $E \otimes_A F$. For any $x \in E \otimes_A F$, we put

$$\|x\| = \inf_{x=\sum_{i=1}^r a_i \otimes b_i} \max_i \|e_i\| \|f_i\|$$

where $x = \sum_i e_i \otimes f_i$ runs through all possible representations of $x$. This defines a semi-norm on $E \otimes_A F$, and we define $E \hat{\otimes}_A F$ as the completion of $E \otimes_A F$. We have two natural maps $i_1 : E \to E \hat{\otimes}_A F$ and $i_2 : F \to E \hat{\otimes}_A F$ The completed tensor product has following universal property: if $M$ is an Banach $A$-module and $\phi : E \to M$ and $\psi : F \to M$ are two continuous $A$-linear maps, then there exists a unique $A$-linear map $\phi \hat{\otimes} \psi : E \hat{\otimes}_A F \to M$ such that $\phi = (\phi \hat{\otimes} \psi) \circ i_1$ and $\psi = (\phi \hat{\otimes} \psi) \circ i_2$.

***Lemma 1.5.16.*** — *Let $p : Y = \mathrm{Sp}(B) \to X = \mathrm{Sp}(A)$ and $q : Z = \mathrm{Sp}(C) \to X$ be two morphisms of affinoid spaces. The completed tensor product $B \otimes_A C$ is also an affinoid* *algebra, and the commutative diagram*

$$\begin{array}{ccc} \mathrm{Sp}(B\hat{\otimes}_A C) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

*is Cartesian in the category of affinoid spaces.*

*Proof.* — Via the homomorphisms $A \to B$ and $A \to C$, one can write $B = A\langle T_1, \cdots, T_r\rangle/\mathfrak{a}$ and $C \cong A\langle T_1, \cdots, T_s\rangle/\mathfrak{b}$. If $\mathfrak{a} = (f_1, \cdots, f_m)$ and $\mathfrak{b} = (g_1, \cdots, g_n)$, then

$$B \otimes_A C \cong A\langle T_1 \otimes 1, \cdots, T_r \otimes 1, 1 \otimes T_1, \cdots, 1 \otimes T_s\rangle/(f_i \otimes 1, 1 \otimes g_j : 1 \leq i \leq m, 1 \leq j \leq n).$$

So if $A$ is a quotient of the Tate algebra $A_\ell$, then $B \otimes_A C$ is the quotient of the Tate algebra $A_{\ell+r+s}$, hence an affinoid algebra.

Now if $f : X' = \mathrm{Sp}(D) \to Y$ and $g : X' \to Z$ are two morphisms of affinoid spaces such that $p \circ f = q \circ g$, then the unique $A$-linear homomorphism $B\hat{\otimes}_A C \to D$ is a homomorphism of $A$-algebras, which induced a unique morphism of affinoid spaces $X' \to Y \times_X Z$. □

***Corollary 1.5.17.*** — *Let $\iota : X' \hookrightarrow X$ be an affinoid subdomain, and $\phi : Y \to X$ be an arbitrary morphism then $Y' = \phi^{-1}(X') \hookrightarrow Y$ is an affinoid subdomain of $Y$.*

*Proof.* — The inclusion $Y' \hookrightarrow Y$ can be identified with the natural projection $X' \times_X Y \to Y$. The universal property for affinoid subdomain follows easily from the Cartesian property of $Y' = X' \times_X Y$.

□

***Example 1.5.18.*** — Let $\phi : Y = \mathrm{Sp}(B) \to X = \mathrm{Sp}(A)$ be a morphism of affinoid spaces with associated morphism of affinoid algebras $\phi^* : A \to B$. Then one has

$$\phi^{-1}X(f_1, \cdots, f_r) = Y(\phi^*(f_1), \cdots, \phi^*(f_r)),$$
$$\phi^{-1}X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0}) = Y(\frac{\phi^* f_1}{\phi^* f_0}, \cdots, \frac{\phi^* f_r}{\phi^* f_0})$$

***Lemma 1.5.19* (Transitivity of affinoid subdomains).** — *Let $X = \mathrm{Sp}(A)$ be an affinoid space. If $U \hookrightarrow X$ is an affinoid subdomain and $V$ is an affinoid subdomain of $U$, then $V$ is an affinoid subdomain of $X$.*

*Proof.* — This is tautology, and the proof is left as an exercise. □

***Proposition 1.5.20.*** — *Let $U$ and $V$ be affinoid subdomains of $X = \mathrm{Sp}(A)$. Then $U \cap V$ is also an affinoid subdomain of $X$. If $U$ and $V$ are Weierstrass, resp. Laurent, resp. rational subdomains, then the same is for $U \cap V$.*

*Proof.* — If $i : U \hookrightarrow X$ is the natural inclusion, then $i^{-1}(V) = U \cap V$ is an affinoid subdomain of $V$ by Corollary 1.5.17.

The statements for Weierstrass and Laurent domains are clear from the definition. Consider here the case of intersection of rational domains. Let

$$U = X(\frac{f_1}{f_0}, \cdots, \frac{f_r}{f_0}), \quad V = X(\frac{g_1}{g_0}, \cdots, \frac{g_s}{g_0})$$

be two rational subdomains, where $f_i, g_j \in A$ satisfying $(f_0, \cdots, f_r) = 1$ and $(g_0, \cdots, g_s) = 1$. We claim that

$$U \cap V = X(\frac{f_i g_j}{f_0 g_0}; i = 0, \cdots, r; j = 0, \cdots, s).$$

For simplicity, we denote the right hand side by $W$. It is clear that $U \cap V \subseteq W$. Conversely, for $x \in W$, one has

$$|f_i(x)g_j(x)| \le |f_0(x)||g_0(x)|, \quad \forall 0 \le i \le r, 0 \le j \le s.$$

Note that $f_0(x)g_0(x) \neq 0$; otherwise, one would have $f_i(x) = g_j(x) = 0$ for all $i, j$. Taking $i = 0$, one gets $|g_j(x)| \le |g_0(x)|$, and taking $j = 0$, one gets $|f_i(x)| \le |f_0(x)|$. This shows that all $x \in W$ belong to $U \cap V$. □

***Proposition 1.5.21.*** — *Let $\phi : Y = \mathrm{Sp}(B) \to X = \mathrm{Sp}(A)$ be a morphism of affinoid $K$-spaces, and let $x \in X$ be a point corresponding to the maximal ideal $\mathfrak{m} \subseteq A$.*

1. *Assume that $\phi^*$ induces a surjection $A/\mathfrak{m} \twoheadrightarrow B/\mathfrak{m}B$. Then there exists a special affinoid subdomain $X' \hookrightarrow X$ containing $x$ such that the induced morphism $\varphi : Y' = \phi^{-1}(X') \to X'$ is a closed immersion in the sense that the corresponding homomorphism of affinoid $K$-algebras is a surjection.*
2. *Assume that $\phi^*$ induces isomorphisms $A/\mathfrak{m}^n \cong B/\mathfrak{m}^n B$ for all $n \in \mathbb{N}$. Then there exists a special affinoid subdomain $X' \hookrightarrow X$ containing $x$ such that the induced morphism $Y' = \phi^{-1}(X') \to X'$ is an isomorphism.*

*Proof.* — (1) Note that one has either $B/\mathfrak{m}B = 0$ or $A/\mathfrak{m} \cong B/\mathfrak{m}B$. Consider first the case $B/\mathfrak{m}B = 0$, i.e. $\phi^{-1}(x)$ is empty. Then there exists $m_1, \cdots, m_r \in \mathfrak{m}$ and $b_1, \cdots, b_r \in B$ such that

$$\phi^*(m_1)b_1 + \cdots + \phi^*(m_r)b_r = 1.$$

One has to find a special affinoid subdomain $X' \subseteq X$ such that $\phi^{-1}(X')$ is empty. Let $c \in K^*$ be such that

$$|c|^{-1} > \max_i \{\|b_i\|_{\sup}\}.$$

We claim that one can take $X' = X(c^{-1}m_1, \cdots, c^{-1}m_r)$. Indeed, given $x \in X(c^{-1}m_1, \cdots, c^{-1}m_r)$, if $y \in \phi^{-1}(x)$, then

$$|\phi^*(m_i)(y)b_i(y)| = |m_i(x)||b_i(y)| \le |c| \max_i \{\|b_i\|_{\sup}\} < 1,$$

which contradicts with $\sum_i b_i \phi^*(m_i) = 1$.

We suppose now $A/\mathfrak{m} \cong B/\mathfrak{m}B$. Choose power bounded elements $b_i \in B$ for $i = 1, \cdots, r$ such that one has a surjection

$$\Phi^* : A\langle T_1, \cdots, T_r\rangle \to B, \quad T_i \mapsto b_i$$

extending $\phi^* : A \to B$. Let $m_1, \cdots, m_s$ be generators of $\mathfrak{m}$. Since $\phi^*$ induces an isomorphism $A/\mathfrak{m} \cong B/\mathfrak{m}B$, there exist $a_i \in A$ and $c_{i,j} \in B$ such that

$$b_i - \phi^*(a_i) = \sum_{j=1}^{s} c_{ij}\phi^*(m_j). \tag{1.5.21.1}$$

We choose residue norm $\|\cdot\|$ on $A$, and the surjection $\Phi^*$ induces a residue norm on $B$, still denoted by $\|\cdot\|$. Then one has $\|b_i\| \leq 1$. Note that one can always multiply some constants on both sides of (1.5.21.1). We may assume thus that $\|c_{ij}\| \leq 1$. Put $A' = A\langle c^{-1}m_1, \cdots, c^{-1}m_s\rangle$ for some $c \in K^*$ with $|c| < 1$. We claim that $X' = X(c^{-1}m_1, \cdots, c^{-1}m_s) = \mathrm{Sp}(A')$ satisfies the requirement of the statement. Indeed, by construction of $\Phi^*$, one has a commutative diagram

$$\begin{array}{ccccc} & & A\langle T_1, \cdots, T_r\rangle & \xrightarrow{\Phi^*} & B \\ & & \downarrow & & \downarrow \\ A' & \longrightarrow & A'\langle T_1, \cdots, T_r\rangle & \xrightarrow{\Phi'^*} & B\langle c^{-1}\phi^* m_1, \cdots, c^{-1}\phi^*(m_s)\rangle = B' \end{array}$$

where $\Phi^*$ and $\Phi'^*$ are surjective. We need to show that the composed map of the lower row is surjective. By the surjectivity of $\Phi'^*$, every element $b' \in B'$ can be written as

$$b' = \sum_{\mu} \phi^*(a'_\mu) b_\mu = \sum_{\mu} \phi^*(a'_\mu)\big(\phi(a_1) + \sum_j c_{1j}\phi^*(m_j)\big)^{\mu_1} \cdots \big(\phi(a_r) + \sum_j c_{rj}\phi^*(m_j)\big)^{\mu_r}$$

with $\|c_{i,j}\| \leq \|b'\|$. One can applying the argument to $c_{i,j}$, and show that the resulting series is convergent.

(2) By (1), we may assume that $\phi^* : A \to B$ is surjective. As $A/\mathfrak{m}^n \cong B/\mathfrak{m}^n B$ for all $n \geq 0$, we get

$$\mathrm{Ker}(\phi^*) \subseteq \mathfrak{a} := \bigcap_{n \geq 1} \mathfrak{m}^n.$$

Since $A$ is noetherian, $\mathfrak{a}$ is a finitely generated over $A$, and we have $\mathfrak{m} \cdot \mathfrak{a} = \mathfrak{a}$. Hence, by Krull's intersection theorem, the ideal $\mathfrak{a}$ is annihilated by $f = 1 - a$ with $a \in \mathfrak{m}$. Since $A \to A\langle \frac{1}{f}\rangle$ factors through $A[\frac{1}{f}]$, it follows that $\mathrm{Ker}(\phi^*)$ is contained in the kernel of $A \to A\langle \frac{1}{f}\rangle$, and we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi^*} & B \\ \downarrow & \overset{\alpha^*}{\swarrow} & \downarrow \\ A\langle \frac{1}{f}\rangle & \xrightarrow{\phi'^*} & B\langle \frac{1}{\phi^*(f)}\rangle. \end{array}$$

such that the square and the upper left triangle is commutative. It then follows from the surjectivity of $\phi^*$ that the lower right triangle is also commutative. We have to show that $\phi'^*$ is an isomorphism. Indeed, the surjectivity of $\phi^*$ implies the surjectivity of $\phi'^*$. To show the injectivity of $\phi'^*$, we note that $\alpha^*(\phi^*(f)) = f$ is invertible and power bounded in $A\langle \frac{1}{f}\rangle$.

By the universal property of $B\langle\frac{1}{\phi^*(f)}\rangle$, there exists a homomorphism $\psi^* : B\langle\frac{1}{\phi^*(f)}\rangle \to A\langle\frac{1}{f}\rangle$ such that $\alpha^*$ coincides with the composition

$$B \to B\langle\frac{1}{\phi^*(f)}\rangle \xrightarrow{\psi^*} A\langle\frac{1}{f}\rangle$$

Now it is easy to see that $\psi^* \circ \phi'^*$ is the identity on $A\langle\frac{1}{f}\rangle$, in particular, $\phi'^*$ is injective. □

***Corollary 1.5.22.*** — *Let $U \subseteq X = \mathrm{Sp}(A)$ be an affinoid subdomain of $X$. Then $U$ is open for the canonical topology on $X$, and the induced topology on $U$ coincides with the canonical topology of $U$.*

*Proof.* — By the Proposition, for any $x \in U$, there exists a special affinoid subdomain $V$ with $x \in V \subseteq U$. Check directly that any special affinoid subdomain is open for the canonical topology. □

**1.5.23. Local rings.** — Let $X = \mathrm{Sp}(A)$ be an affinoid space. Similarly to the case of algebraic geometry, for any affinoid subdomain $U \subseteq X$, we denote by $\mathcal{O}_X(U)$ the affionid algebra corresponding to $U$. For instance, if $U = X(f)$, then $\mathcal{O}_X(U) = A\langle f\rangle$. Then the functor $\mathcal{O}_X : U \mapsto \mathcal{O}_X(U)$ gives a presheaf on the category of affinoid subdomains of $X$ (with morphisms given by natural inclusions). We put

$$\mathcal{O}_{X,x} = \varinjlim_{x\in U} \mathcal{O}_X(U)$$

where $U$ runs through the affinoid subdomains of $X$ containing $x$.

***Proposition 1.5.24.*** — *Let $x \in X$ be a point corresponding to the maximal ideal $\mathfrak{m} \subseteq A = \mathcal{O}_X(X)$. Then $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}\mathcal{O}_{X,x}$.*

*Proof.* — Let $U \subseteq X$ be an affinoid subdomain. By Proposition 1.5.13, there exists an isomorphism $\mathcal{O}_X/\mathfrak{m} \xrightarrow{\sim} \mathcal{O}_X(U)/\mathfrak{m}\mathcal{O}_X(U)$. Passing to direct limit, one gets an isomorphism

$$\mathcal{O}_X(X)/\mathfrak{m} \cong \mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x}$$

which shows that $\mathfrak{m}\mathcal{O}_{X,x}$ is a maximal ideal of $\mathcal{O}_{X,x}$.

It remains to see that $\mathfrak{n} := \mathfrak{m}\mathcal{O}_{X,x}$ is the only maximal ideal of $\mathcal{O}_{X,x}$. Let $f_x \in \mathcal{O}_{X,x}\backslash\mathfrak{n}$. Then $f_x$ is represented by an some element $f \in \mathcal{O}_X(U)$ for some affinoid subdomain $U \subseteq X$. Then $f(x) \neq 0$, and up to multiplying $f$ by a scalar in $K$, we may assume that $|f(x)| \geq 1$. Then the Laurent domain $U(f^{-1})$ is an affinoid subdomain of $X$ containing $x$, and $f$ is invertible in $\mathcal{O}_X(U(f^{-1}))$, hence $f_x$ is invertible in $\mathcal{O}_{X,x}$. □

***Proposition 1.5.25.*** — *The canonical map $A \to \mathcal{O}_{X,x}$ factors as*

$$A \to A_{\mathfrak{m}} \to \mathcal{O}_{X,x}$$

*where $A_{\mathfrak{m}}$ is the localization of $A$ at $\mathfrak{m}$, and it induces an isomorphisms*

$$A/\mathfrak{m}^n \xrightarrow{\sim} A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}$$

*for all integers $n \geq 1$. In particular, one has isomorphisms of completions*

$$\varprojlim_n A/\mathfrak{m}^n \cong \widehat{A}_{\mathfrak{m}} \cong \widehat{\mathcal{O}}_{X,x}.$$

*Proof.* — The first statement is clear. The isomorphism $A/\mathfrak{m}^n A \cong A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}}$ comes from the exactness of the localization functor, and the fact that $A/\mathfrak{m}^n \cong (A/\mathfrak{m}^n)_{\mathfrak{m}}$. For any affinoid subdomain $U \subseteq X$, we have

$$A/\mathfrak{m}^n \cong \mathcal{O}_X(U)/\mathfrak{m}^n\mathcal{O}_X(U)$$

by Proposition 1.5.13. The isomorphism $A/\mathfrak{m}^n \cong \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}$ follows by taking the direct limit on $U$.

□

***Corollary 1.5.26.*** — *An affinoid function $f \in A = \mathcal{O}_X(X)$ is trivial if and only if the image of $f$ at $\mathcal{O}_{X,x}$ is trivial for all $x \in X$.*

*Proof.* — The statement follows from the injections

$$A \hookrightarrow \prod_{\mathfrak{m}\in \mathrm{Sp}(A)} A_{\mathfrak{m}} \hookrightarrow \prod_{x\in X} \mathcal{O}_{X,x}.$$

The first injectivity holds for arbitrary noetherian rings, and follows from the consideration of annihilator of an element in the kernel of $A \to \prod_{\mathfrak{m}} A_{\mathfrak{m}}$. The second injection follows from the isomorphism $\widehat{A}_{\mathfrak{m}} \cong \widehat{\mathcal{O}}_{X,x}$ and the injectivity of $A_{\mathfrak{m}} \hookrightarrow \widehat{A}_{\mathfrak{m}}$ (Krull's intersection theorem). □

***Corollary 1.5.27.*** — *Let $X = \bigcup_{i\in I} X_i$ be a covering by affinoid subdomains. Then the the restriction map*

$$\mathcal{O}_X(X) \to \prod_{i\in I} \mathcal{O}_X(X_i)$$

*is injective.*

*Proof.* — Indeed, this follows from the previous corollary and the fact that $\mathcal{O}_X(X) \hookrightarrow \prod_{x\in X} \mathcal{O}_{X,x}$ factors through $\prod_{i\in I} \mathcal{O}_X(X_i)$. □

***Proposition 1.5.28.*** — *For any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is noetherian.*

*Proof.* — Let $\mathfrak{m} \subseteq A$ be the maximal ideal corresponding to $x \in X$. We show first that $\mathcal{O}_{X,x}$ is separated for the $\mathfrak{m}\mathcal{O}_{X,x}$-adic topology, i.e. $\bigcap_{n\geq 0} \mathfrak{m}^n\mathcal{O}_{X,x} = 0$. In fact, if $f \in \bigcap_{n\geq 0} \mathfrak{m}^n\mathcal{O}_{X,x}$, there exists an affinoid subdomain $U \subseteq X$ such that $f$ is defined over $U$. Since $\mathcal{O}_X(U)/\mathfrak{m}^n\mathcal{O}_X(U) \cong \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}$ by Proposition 1.5.25, it follows that $f \in \mathfrak{m}^n\mathcal{O}_X(U)$ for all $n \geq 1$. Write $U = \mathrm{Sp}(A')$ and $\mathfrak{m}' = \mathfrak{m}A'$. Then the image of $f$ in $A'_{\mathfrak{m}'}$ lies in $\bigcap_{n\geq 1} \mathfrak{m}'^n$, which is zero by Krull's intersection theorem. Hence, the image of $f$ in $\mathcal{O}_{X,x}$ is also zero.

Let $\mathfrak{a}_x$ be a finitely generated ideal of $\mathcal{O}_{X,x}$. Up to replacing $X$ by an affinoid subdomain, we may assume that there exists a (finitely generated) ideal $\mathfrak{a} \subseteq A$ such that $\mathfrak{a}_x = \mathfrak{a}\cdot\mathcal{O}_{X,x}$. Then $\mathcal{O}_{X,x}/\mathfrak{a}_x$ is the local ring at $x$ of the affinoid subspace $\mathrm{Sp}(A/\mathfrak{a})$. Therefore, $\mathcal{O}_{X,x}/\mathfrak{a}_x$ is separated for the $\mathfrak{m}$-adic topology, i.e. the canonical map

$$\mathcal{O}_{X,x}/\mathfrak{a}_x \hookrightarrow (\widehat{\mathcal{O}_{X,x}/\mathfrak{a}_x}) \cong \widehat{\mathcal{O}}_{X,x}/\hat{\mathfrak{a}}_x$$

is an inclusion, where $\hat{\mathfrak{a}}_x$ is the closure of $\mathfrak{a}_x$. We deduce thus $\hat{\mathfrak{a}}_x \cap \mathcal{O}_{X,x} = \mathfrak{a}_x$ for any finitely generated ideal $\mathfrak{a}_x \subseteq \mathcal{O}_{X,x}$.

To show that $\mathcal{O}_{X,x}$ is noetherian, it suffices to show that every ascending chain of finitely generated ideals in $\mathcal{O}_{X,x}$

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subseteq \cdots$$

is stable. Since $\hat{\mathcal{O}}_{X,x} = \hat{A}_{\mathfrak{m}}$ is noetherian (because so is $A_{\mathfrak{m}}$), the following chain of ideals in $\hat{\mathcal{O}}_{X,x}$:

$$\hat{\mathfrak{a}}_1 \subset \hat{\mathfrak{a}}_2 \subseteq \cdots$$

is stable. Then the statement follows from $\mathfrak{a}_i = \hat{\mathfrak{a}}_i \cap \mathcal{O}_{X,x}$. □

## 1.6. Tate's acyclicity theorem

Let $X = \mathrm{Sp}(A)$ be an affinoid space over $K$, and $\mathbf{Aff}(X)$ be the category of affinoid subdomains on $X$ with morphisms given by natural inclusions. We say a family of morphism $U_i \to U$ with $i \in I$ in $\mathbf{Aff}(X)$ is a covering if $\bigcup_{i\in I} U_i = U$.

***Definition 1.6.1.*** — Let $\mathcal{F}$ be a presheaf on $\mathbf{Aff}(X)$, and $\mathfrak{U} = (U_i)_{i\in I}$ be a covering of $U$ in $\mathbf{Aff}(X)$. We say that $\mathcal{F}$ is sheafy for $\mathfrak{U}$, if the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}_X(U_i) \to \prod_{i,j} \mathcal{F}_X(U_i \cap U_j), \tag{1.6.1.1}$$

where the first morphism is given by $f \mapsto (f|_{U_i})_{i\in I}$, and the second is $(f_i)_{i\in I} \mapsto f_i|_{U_i\cap U_j} - f_j|_{U_i\cap U_j}$.

Consider the presheaf $\mathcal{O}_X$ of affinoid functions on $\mathbf{Aff}(X)$. By Corollary 1.5.27, the first morphism in (1.6.1.1) is indeed injective for $\mathcal{O}_X$. However, the exactness at $\prod_{i\in I} \mathcal{O}_X(U_i)$ is not true for general infinite coverings.

***Example 1.6.2.*** — Let $X = \mathrm{Sp}(A)$ with $K\langle T_1, \cdots, T_n\rangle$ be the 1-dimensional unit ball. Choose $\pi \in K$ with $0 < |\pi| < 1$. For any $i \geq 1$, we put

$$U_i = \mathrm{Sp}(A\langle \frac{T_1^i}{\pi}, \cdots, \frac{T_n^i}{\pi} : i \geq 1\rangle)$$

Then one has the covering

$$X = \bigcup_{i\geq 1} U_i \bigcup U_0$$

with $U_0 = \mathrm{Sp}(A\rangle \frac{1}{T_1}, \cdots, \frac{1}{T_n}\rangle$. Then the sequence

$$0 \to \mathcal{O}_X(X) \to \prod_{i\geq 0} \mathcal{O}_X(U_i) \to \prod_{i,j\geq 0} \mathcal{O}_X(U_i \cap U_j)$$

is not exact in the middle.

It is Tate's insight that one has to restrict to finite coverings of affinoids subdomains.

***Theorem 1.6.3* (Tate).** — *Let $X$ be an affinoid $K$-space. The presheaf $\mathcal{O}_X$ is sheafy for any* finite *covering $\mathfrak{U} = (U_i)_{i\in I}$ in $\mathbf{Aff}(X)$.*

The proof of this Theorem reduces first to some simple cases of coverings by special affinoid subdomains, and then we perform an explicit computation. It is clear that we just need to prove the Theorem for coverings of $X$. Let $\mathfrak{U} = (U_i)_{i\in I}$ and $\mathfrak{V} = (V_j)_{j\in J}$ be two coverings of $X$. We say that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ if there exists a map $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for all $j \in J$.

***Lemma 1.6.4.*** — *Let $\mathfrak{U} = (U_i)_{i\in J}$ and $\mathfrak{V} = (V_j)_{j\in J}$ be two coverings of $X$ by affinoid subdomains, where $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. Let $\mathcal{F}$ be a presheaf such that for any $U \subseteq X$ the canonical map*

$$\mathcal{F}(U) \to \prod_{j\in J} \mathcal{F}(U \cap V_j)$$

*is also injective. Then, if $\mathcal{F}$ is sheafy for $\mathfrak{V}$, then so is it for $\mathfrak{U}$.*

*Proof.* — We just prove the exactness at $\prod_{i\in I} \mathcal{F}(U_i)$. Let $(f_i)_{i\in I} \in \prod_i \mathcal{F}(U_i)$ such that $f_i|_{U_i\cap U_{i'}} = f_{i'}|_{U_i\cap U_{i'}}$. Choose $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for $j \in J$. We define $g_j = f_{\tau(j)}|_{V_j}$ for $j \in J$. Then, for $j, j' \in J$, one has

$$g_j|_{V_j\cap V_{j'}} = f_{\tau(j)}|_{V_j\cap V_{j'}} = f_{\tau(j')}|_{V_j\cap V_{j'}} = g_{j'}|_{V_j\cap V_{j'}}.$$

By assumption, $\mathcal{F}$ is sheafy for $\mathfrak{V}$. Hence, there exists $g \in \mathcal{F}(X)$ such that $g|_{V_j} = g_j$ for $j \in J$. We claim that $g|_{U_i} = f_i$ for each $i \in I$. Indeed, it suffices to show that $(g|_{U_i} - f_i)|_{V_j\cap U_i} = 0$ for all $j \in J$. By definition, one has

$$(g|_{U_i})|_{U_i\cap V_j} = (g|_{V_j})|_{U_i\cap V_j} = f_{\tau(j)}|_{U_i\cap V_j},$$

which coincides with $(f_i|_{U_i\cap U_{\tau(j)}})|_{U_i\cap V_j}$.

□

***Lemma 1.6.5.*** — *Let $\mathfrak{U} = (U_i)_{i\in I}$ and $\mathfrak{V} = (V_j)_{j\in J}$ be two coverings of $X$ by affinoid subdomains. Assume that $\mathcal{F}$ is sheafy for $\mathfrak{V}$ and $\mathfrak{U}|_{V_j}$ for each $j \in J$. Then $\mathcal{F}$ is sheafy for $\mathfrak{U}$.*

*Proof.* — Let $(f_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i)$ such that $f_i|_{U_i\cap U_{i'}} = f_{i'}|_{U_i\cap U_{i'}}$ for $i, i' \in I$. One has to construct an $f \in \mathcal{F}(X)$ such that $f|_{U_i} = f_i$ for all $i \in I$. The construction consists of two steps. First, one uses the fact that $\mathcal{F}$ is sheafy for $\mathcal{U}|_{V_j}$ to construct a section $g_j \in \mathcal{F}(V_j)$ for all $j \in J$ by gluing $f_i|_{U_i\cap V_j}$ for $i \in I$. Then, the condition that $\mathcal{F}$ is sheafy for $\mathfrak{V}$ allows us to construct $f$.

□

We can start the proof of Theorem 1.6.3. The first step is to reduce to the case of rational coverings. We will need the following Theorem by Gerritzen and Grauert.

***Theorem 1.6.6* (Gerritzen-Grauert).** — *Any affinoid subdomain of $X$ is a finite union of rational subdomains of $X$.*

The proof of this Theorem is quite technical, and we refer the reader to [**Bo08**, §1.6, Theorem 10]. Hence, to prove Tate theorem, it suffices to show $\mathcal{O}_X$ is sheafy for any finite coverings of $X$ by rational affinoid subdomains. We consider some special affinoid coverings of $X$. Let $f_0, f_1, \cdots, f_r \in A$ without common zeros. For $i = 0, \cdots, r$, we put

$$U_i = X(\frac{f_0}{f_i}, \cdots, \frac{f_r}{f_i}).$$

Then $(U_i)_{i=0,\cdots,r}$ is a covering of $X$ in $\mathbf{Aff}(X)$.

***Proposition 1.6.7.*** — *Any finite affinoid covering $\mathfrak{U} = (U_i)_{1\leq i\leq n}$ of $X$ admits a rational covering as refinement.*

*Proof.* — By Gerritzen-Grauert's theorem, we may assume that each $U_i$ is a rational subdomain, say

$$U_i = X(\frac{f_1^{(i)}}{f_0^{(i)}}, \cdots, \frac{f_{r_i}^{(i)}}{f_0^{(i)}}), \quad 1 \leq i \leq n$$

Let $I$ be the set of tuples $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n$ with $0 \leq \nu_i \leq r_i$, and put

$$f_\nu = f_{\nu_1,\cdots,\nu_n} = \prod_{i=1}^n f_{\nu_i}^{(i)}, \quad \text{for } \nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n.$$

Let $I' \subseteq I$ be the subset of tuples such that $\nu_i = 0$ for at least one $i$. We claim that $(f_\nu)_{\nu\in I'}$ have no common zeros and thus generate a rational covering of $X$. Let $x \in X$ be a point such that all such functions vanish. Since $\mathfrak{U}$ is a covering of $X$, there exists $U_j$, say $U_n$, such that $x \in U_n$. Then we have $f_0^{(n)}(x) \neq 0$. Hence, this implies that

$$\prod_{i=1}^{n-1} f_{\nu_i}^{(i)}(x) = 0, \quad 0 \leq \nu_i \leq r_i.$$

This is impossible, because $f_0^{(i)}, \cdots, f_{r_i}^{(i)}$ have no common zeros for each $i$.

Now, we show that the covering $(f_\nu)_{\nu\in I'}$ is a refinement of $\mathfrak{U}$. Fix $\nu = (\nu_1, \cdots, \nu_n) \in I'$ with $\nu_n = 0$. We want to show that

$$X_\nu(\frac{f_\mu}{f_\nu}; \mu = (\mu_0, \cdots, \mu_n) \in I')$$

is contained in $U_n$. For $x \in X_\nu$ and an index $\mu_n$ with $0 \leq \mu_n \leq r_n$, one has to show that

$$|f_{\mu_n}^{(n)}(x)| \leq |f_0^{(n)}(x)|.$$

Note that $x$ belongs to one of $U_i$. If $x \in U_n$, there is nothing to prove. If $x \in U_i$ with $i < n$, we may assume that $i = 1$, so that one has

$$|f_\mu^{(1)}(x)| \leq |f_0^{(1)}(x)|$$

for all $\mu \leq r_1$. Then, one has $(0, \nu_2, \cdots, \nu_{n-1}, \mu_n) \in I'$, and hence

$$|f_0^{(1)} f_{\nu_2}^{(2)} \cdots f_{\nu_{n-1}}^{(n-1)} f_{\mu_n}^{(n)}(x)| \leq |f_{\nu_1}^{(1)}(x) f_{\nu_2}^{(2)}(x) \cdots f_0^{(n)}(x)| \leq |f_0^{(1)}(x)| \cdot |f_{\nu_2}^{(2)} \cdots f_0^{(n)}(x)|.$$

It follows that $|f_{\mu_n}^{(n)}(x)| \leq |f_0^{(n)}(x)|$.

□

Let $f_1, \cdots, f_n \in A$. Then

$$X(f_1^{\epsilon_1}, \cdots, f_n^{\epsilon_n}), \quad \epsilon_i \in \{1, -1\}$$

form a covering of $X$; we call it the Laurent covering generated by $f_1, \cdots, f_n \in A$.

***Lemma 1.6.8.*** — *Let $\mathfrak{U}$ be a rational covering of $X$. There exists a Laurent covering $\mathfrak{V}$ of $X$ such that for each $V \in \mathfrak{V}$, $\mathfrak{U}|_V$ is a rational covering generated by units on $V$.*

*Proof.* — Let $f_1, \cdots, f_n \in A$ be the elements that generate the rational covering $\mathfrak{U}$. By the maximum modulus principle, there exists $c \in K^*$ such that

$$|c| < \inf_{x\in X}(\max_{0\leq i\leq n}|f_i(x)|).$$

We claim that the Laurent covering $\mathfrak{V}$ generated by $c^{-1}f_0, \cdots, c^{-1}f_n$ satisfy the requirement. A general element of $\mathfrak{V}$ has the form

$$V = X((c^{-1}f_0)^{\epsilon_0}, \cdots, (c^{-1}f_n)^{\epsilon_n})$$

with $\epsilon_i \in \{1, -1\}$. We may assume that $\epsilon_0 = \cdots = \epsilon_s = 1$ and $\epsilon_{s+1} = \cdots = \epsilon_n = -1$. Consider the covering $\mathfrak{U}|_V$. Note first that

$$V \cap X(\frac{f_0}{f_i}, \cdots, \frac{f_n}{f_i}) = \emptyset$$

for $0 \leq i \leq s$. Indeed, if $x \in V \cap X(\frac{f_0}{f_i}, \cdots, \frac{f_n}{f_i})$, one has $|f_i(x)| = \max_{0\leq j\leq n}|f_j(x)|$, and thus

$$|f_i(x)| \leq |c| < \inf_{y\in X}|f_i(y)|$$

which is a contradiction. Thus, the covering $\mathfrak{U}|_V$ is given by $V \cap X(\frac{f_0}{f_j}, \cdots, \frac{f_n}{f_j})$ for $s+1 \leq j \leq n$. Let $x \in V \cap X(\frac{f_0}{f_j}, \cdots, \frac{f_n}{f_j})$ with $s+1 \leq j \leq n$. Then $|f_j(x)| = \max_\ell(|f_\ell(x)|)$, and the condition $|f_i(x)| \leq |c|$ for $i = 0, \cdots, s$ implies automatically that $|f_i(x)| \leq |f_j(x)|$. Therefore, one has

$$V \cap X(\frac{f_0}{f_j}, \cdots, \frac{f_n}{f_j}) = V \cap X(\frac{f_{s+1}}{f_j}, \cdots, \frac{f_n}{f_j}).$$

Note that $f_{s+1}, \cdots, f_n$ are invertible over $V$. The claim follows. □

***Lemma 1.6.9.*** — *Let $\mathfrak{U}$ be a rational covering of $X$ generated by units of $A$. Then there exists a Laurent covering which refines $\mathfrak{U}$.*

*Proof.* — Let $f_1, \cdots, f_n \in A$ be the units that generate $\mathfrak{U}$. Then the Laurent covering generated by $f_i^{-1}f_j$ is refinement of $\mathfrak{U}$. □

In summary, we have proven

***Proposition 1.6.10.*** — *Let $\mathcal{F}$ be a separated presheaf on $\mathbf{Aff}(X)$. If $\mathcal{F}$ is sheafy for all Laurent coverings then $\mathcal{F}$ is sheafy for all finite affinoid coverings of $X$.*

*Proof.* — This is evident from Proposition 1.6.7 and Lemmas 1.6.4, 1.6.8 and 1.6.9. □

*Proof of Theorem 1.6.3.* — By Proposition 1.6.10, it suffices to show that $\mathcal{O}_X$ is sheafy for any finite Laurent covering of $X$. By induction, it suffices to show that $\mathcal{O}_X$ is sheafy generated by one element $f \in A$. Then Theorem 1.6.3 follows from the following □

***Lemma 1.6.11.*** — *For $f \in A$, one has an exact sequence:*

$$0 \to A \to A\langle f\rangle \oplus A\langle \frac{1}{f}\rangle \xrightarrow{\alpha} A\langle f, \frac{1}{f}\rangle \to 0,$$

*where $\alpha(g,h) = g - h$.*

*Proof.* — The injectivity of $A \hookrightarrow A\langle f\rangle \oplus A\langle \frac{1}{f}\rangle$ and the surjectivity of $A\langle f\rangle \oplus A\langle \frac{1}{f}\rangle \to A\langle f, \frac{1}{f}\rangle$ are both evident. It remains to prove the exactness at $A\langle f\rangle \oplus A\langle \frac{1}{f}\rangle$. Recall that

$$A\langle f\rangle = A\langle T\rangle/(T-f), \quad A\langle \frac{1}{f}\rangle \cong A\langle S\rangle/(1-fS)$$

and $A\langle f, \frac{1}{f}\rangle = A\langle S, S^{-1}\rangle/(1-fS)$, and the morphism $A\langle f\rangle \oplus A\langle \frac{1}{f}\rangle \to A\langle f, \frac{1}{f}\rangle$ is induced by $T \mapsto S^{-1}$. Let $g = \sum_{i=0}^{\infty} a_i T^i$ and $h = \sum_{j=0}^{\infty} b_j S^j$ with $a_i, b_j \in A$. Assume that there exists $\sum_{i=-\infty}^{\infty} c_i S^i$ with $c_i \in A$ such that

$$\sum_{i=0}^{\infty} a_i S^{-i} - \sum_{j=0}^{\infty} b_j S^j = (1-fS)\sum_{i=-\infty}^{\infty} c_i S^i.$$

One has to show that there exists $d \in A$ such that $d \equiv g \mod (T-f)$ and $d \equiv h \mod (1-fS)$. Up to subtracting $a_0$ from both $g$ and $h$, we may assume that $a_0 = 0$.One has

$$\begin{cases} a_i = c_{-i} - fc_{-i-1}, & i \geq 1; \\ -b_j = c_j - fc_{j-1}, & j \geq 0. \end{cases}$$

Therefore, one gets

$$g = \sum_{i\geq 1}^{\infty}(c_{-i} - fc_{-i-1})T^i = (T-f)\sum_{i\geq 1}^{\infty} c_{-i}T^{i-1} + fc_{-1},$$

and

$$h = \sum_{j\geq 0}^{\infty}(fc_{j-1} - c_j)S^j = fc_{-1} - (1-fS)\sum_{j=0}^{\infty} c_j S^j.$$

Hence, one may take $d = fc_{-1}$.

□

***Corollary 1.6.12.*** — *Let $M$ be a finite $A$-module, and $f \in A$. Put $M\langle f\rangle := M \otimes_A A\langle f\rangle$, and similar meaning for $M\langle \frac{1}{f}\rangle$ and $M\langle f, \frac{1}{f}\rangle$. Then one has an exact sequence*

$$0 \to M \to M\langle f\rangle \oplus M\langle \frac{1}{f}\rangle \to M\langle f, \frac{1}{f}\rangle \to 0.$$

*Proof.* — This corollary follows from Lemma 1.6.11 and the fact that $A\langle f, \frac{1}{f}\rangle$ is flat over $A$. □

# CHAPTER 2

# RIGID ANALYTIC SPACES

## 2.1. Grothendieck topology on affinoid spaces

First, we recall some generalities on Grothendieck topology.

***Definition 2.1.1.*** — A Grothendieck topology consists of a category $\mathfrak{C}$ and a set $\mathrm{Cov}(\mathfrak{C})$ of families $(U_i \to U; i \in I)$ of morphisms in $\mathfrak{C}$ such that the following conditions are satisfied:

1. Any isomorphism $V \xrightarrow{\sim} U$ in $\mathfrak{C}$ belongs to $\mathrm{Cov}(\mathfrak{C})$.
2. If $(U_i)_{i\in I} \to U$ belongs to $\mathrm{Cov}(\mathfrak{C})$ and $(V_{i,j})_{j\in J_i} \to U_i$ belongs to $\mathrm{Cov}(\mathfrak{C})$ for each $i \in I$, then so does the family $(V_{i,j} \to U : i \in I, j \in J_i)$.
3. If $(U_i)_{i\in I} \to U$ belongs to $\mathrm{Cov}(\mathfrak{C})$, then for any object $V$ in $\mathfrak{C}$ the fibre product $U_i \times_U V$ exists and $(U_i \times_U V)_{i\in I} \to V$ belongs to $\mathrm{Cov}(\mathfrak{C})$.

The families of morphisms in $\mathrm{Cov}(\mathfrak{C})$ will be called coverings of the Grothendieck topology $(\mathfrak{C}, \mathrm{Cov}(\mathfrak{C}))$.

***Example 2.1.2.*** — (1) Let $X$ be a topological space, $\mathfrak{C}$ be the category of open subsets of $X$ with morphisms given by natural inclusions. We define $\mathrm{Cov}(\mathfrak{C})$ be the set of families $(U_i)_{i\in I} \to I$ of open immersions such that $\bigcup_{i\in I} U_i = U$. Then $(\mathfrak{C}, \mathrm{Cov}(\mathfrak{C}))$ form a Grothendieck topology.

(2) Let $X$ be a scheme. Denote by $\mathrm{Et}(X)$ be the category of étale schemes over $X$, and $\mathrm{Cov}(\mathrm{Et}(X))$ consist of the families of morphisms $f_i : U_i \to U$ for $i \in I$ such that $\bigcup_{i\in I} f_i(U_i) = U$. Then the pair $(\mathrm{Et}(X), \mathrm{Cov}(\mathrm{Et}(X)))$ is a Grothendieck topology.

The notion of presheaves and sheaves can be naturally generalized to the setup of Grothendieck topology.

***Definition 2.1.3.*** — Let $\mathfrak{T} = (\mathfrak{C}, \mathrm{Cov}(\mathfrak{C}))$ be a Grothendieck topology. A presheaf on $\mathfrak{T}$ is a contravariant functor $\mathcal{F} : \mathfrak{C} \to \mathbf{Sets}$. A presheaf $\mathcal{F}$ on $\mathfrak{T}$ is called a sheaf if for any covering $(U_i)_{i\in I} \to U$ in $\mathrm{Cov}(\mathfrak{C})$, the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact.

From now on, let $X = \mathrm{Sp}(A)$ be an affinoid $K$-space. We will define a Grothendieck topology associated to $X$.

***Definition 2.1.4.*** — (1) An open subset $U \subseteq X$ is called *admissible* if there exists a (not necessarily finite) covering $\bigcup_{i\in I} U_i = U$ by affinoid subdomains such that for all morphism $\varphi : Y \to X$ of affinoid $K$-spaces such that the covering $(\varphi^{-1}(U_i))_{i\in I}$ of $Y$ admits a refinement by finite affinoid subdomains.

(2) We denote by $\mathfrak{C}_X$ the category of admissible open subsets of $X$ with morphisms given by inclusions. A family of morphisms $(U_i)_{i\in I} \to U$ in $\mathfrak{C}$ is called an *admissible covering* if $\bigcup_{i\in I} U_i = U$ and for any morphism of affinoid spaces $\varphi : Y \to X$ with $\varphi(Y) \subseteq U$ the covering $\varphi^{-1}(U_i)$ of $Y$ admits a refinement by finitely many affinoid subdomains. We denote by $\mathrm{Cov}(\mathfrak{C}_X)$ the set of admissible covering in $\mathfrak{C}_X$.

***Lemma 2.1.5.*** — *The pair $(\mathfrak{C}_X, \mathrm{Cov}(\mathfrak{C}_X))$ is a Grothendieck topology such that the following completeness condition is satisfied:*

(G0) *One has $\emptyset, X \in \mathfrak{C}_X$.*

(G1) *Let $(U_i \to U; i \in I)$ be an admissible covering in* $\mathrm{Cov}(\mathfrak{C}_X)$. *If $V \subseteq U$ is a subset such that $U_i \cap V$ is admissible open for all $i \in I$, then $V$ is also admissible open.*

(G2) *Let $(U_i \to U; i \in I)$ be a covering of $U$ by admissible open subsets of an admissible open subset $U$. Assume that $(U_i)_{i\in I}$ admits an admissible covering of $U$ as refinement. Then $(U_i \to U; i \in I)$ is also an admissible covering.*

*Proof.* — We verify first that $(\mathfrak{C}_X, \mathrm{Cov}(\mathfrak{C}_X))$ is a Grothendieck topology. Condition (1) in Definition 2.1.1 is trivial. Let $U = \bigcup_{i\in I} U_i$ be an admissible covering of $U$, and $U_i = \bigcup_{j\in J_i} V_{i,j}$ be an admissible covering of $U_i$ for each $i \in I$. One want to show that the covering $U = \bigcup_{i\in I}(\bigcup_{j\in J_i} V_{i,j})$ is admissible. Consider a morphism of affinoid spaces $\varphi : Y \to X$ such that $\varphi(Y) \subseteq U$. One needs to check that $(\varphi^{-1}(V_{i,j}) \to Y : i \in I, j \in J_i)$ admits a refinement by finite affinoid subdomains. By the admissibility of $U = \bigcup_{i\in I} U_i$, there exists a refinement of $Y = \bigcup_i \varphi^{-1}(U_i)$ by affinoid subdomains $Y = \bigcup_{i'\in I'} W_{i'}$ for some finite set $I'$. Choose a morphism $\tau : I' \to I$ such that $W_{i'} \subseteq \varphi^{-1}(U_{\tau(i')})$. For each fixed $i' \in I$, the admissibility of the covering $U_{\tau(i')} = \bigcup_{j\in J_{\tau(i')}} V_{\tau(i'),j}$ implies that the covering $W_{i'} = \bigcup_{j\in J_{\tau(i')}} \varphi^{-1}(V_{\tau(i'),j}) \cap W_{i'}$ admits a refinement by affinoid subdomains

$$W_{i'} = \bigcup_{j'\in J'_{i'}} Z_{i',j'}$$

for some finite set $J'_{i'}$. Then the finite covering $(Z_{i',j'} \subseteq Y : i' \in I', j' \in J'_{i'})$ is a refinement of $Y = \bigcup_{i,j} \varphi^{-1}(V_{i,j})$. This verifies condition 2.1.1(2). The third condition of 2.1.1 can be checked in a similar way, and we leave it as an exercise.

We now show that $(\mathfrak{C}_X, \mathrm{Cov}(\mathfrak{C}_X))$ verifies conditions (G0)-(G2). The condition (G0) is trivial. We verify here only (G1), and the verification of (G2) is similar. By assumption that $V \cap U_i$ is admissible, there exists a covering $V \cap U_i = \bigcup_{j\in J_i} V_{i,j}$ by affinoid subdomains such that if $\psi : Z \to X$ is a morphism of affinoid spaces such that $\psi(Z) \subseteq U \cap V_i$, the covering $\psi^{-1}(V_{i,j})$ of $Z$ admits a finite covering by affinoid subdomains as refinement. We claim that $V = \bigcup_{i\in I}\bigcup_{j\in J_i} V_{i,j}$ is a covering that verifies the condition in the definition of admissible open subsets. The arguments here are exactly the same as checking condition (2) of Definition 2.1.1.

□

We will call $(\mathfrak{C}_X, \mathrm{Cov}(\mathfrak{C}_X))$ the Grothendieck topology on $X$.

***Example 2.1.6.*** — (1) Let $X = \mathrm{Sp}(A)$ with $A = K\langle T\rangle$ be the 1-dimensional closed unit ball, and $\pi \in K$ with $0 < |\pi| < 1$. Let $U \subset X$ be the open unit ball, i.e. $U = X\backslash V$ with $V = \mathrm{Sp}(K\langle T, T^{-1}\rangle)$. We have in fact

$$U = \bigcup_{n\geq 1} U_n, \quad \text{with } U_n = \mathrm{Sp}(A\langle \frac{T^n}{\pi}\rangle).$$

To see that $U$ is admissible, we consider a morphism of affinoid $K$-spaces $\varphi : Y \to X$ such that $\varphi(Y) \subseteq U$. Let $f = \varphi^*(T) \in B := \mathcal{O}_Y(Y)$. We have $|f(y)| = |T(\varphi(y))| < 1$ for all $y \in Y$. By the maximum modulus principle, there exists an integer $m \geq 1$ such that $|f(y)| \leq |\pi|^{\frac{1}{m}}$. Then one has $\varphi(Y) \subseteq U_m$; in particular the covering $Y = \bigcup_{n\geq 1}\varphi^{-1}(U_n)$ has a refinement by finite affinoid subdomains.

For any integer $n \geq 1$, we put $V_n = \mathrm{Sp}(A\langle\frac{\pi}{T^n}\rangle)$. Then one has $V_{n+1} \subseteq V_n$ and $\bigcap_{n\geq 1} V_n = V$. For any $n \geq 1$, the covering $X = U \bigcup V_n$ is an admissible covering, but $X = U\bigcup V$ is not an admissible covering.

(2) Let $X = \mathrm{Sp}(A)$ be a general affinoid $K$-space. For any $f \in A$ and $\epsilon \in \mathbb{R}_{>0}$, then the subsets

$$X(f,\epsilon)^- := \{x \in X; |f(x)| < \epsilon\}, \quad \{x \in X; |f(x)| > \epsilon\}$$

are all admissible. Indeed, one has

$$X(f,\epsilon)^- = \bigcup_{\epsilon'\in|\overline{K}^\times|,\epsilon'<\epsilon} X(f,\epsilon'),$$

where $X(f,\epsilon') = \{x \in X; |f(x)| \leq \epsilon'\}$ is an affinoid subdomain. The same arguments as (1) show that any morphism of affinoid spaces $\varphi : Y \to X$ with $\varphi(Y) \subseteq X(f,\epsilon)^-$ must factor through one of $X(f,\epsilon')$. Similar arguments show that $\{x \in X||f(x)| > \epsilon\}$ is admissible as well.

***Proposition 2.1.7.*** — *Let $\mathcal{F}$ be a presheaf on the category of affinoid subdomains of $X$. Assume that $\mathcal{F}$ is sheafy for all finite affinoid coverings. Then $\mathcal{F}$ extends uniquely to a sheaf for the Grothendieck topology on $X$.*

*Proof.* — Indeed, for an admissible open subset $U \subseteq X$, we choose a covering $U = \bigcup_{i\in I} U_i$ with each $U_i$ affinoid subdomain. We put

$$\mathcal{F}(U) = \mathrm{Ker}(\prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{i,j}\mathcal{F}(U_i\cap U_j)).$$

Since $\mathcal{F}$ is sheafy for finite affinoid coverings, $\mathcal{F}(U)$ is independent of the choice of $(U_i)_{i\in I}$. This extends the presheaf $\mathcal{F}$ to the category $\mathfrak{C}_X$.

Let $(U_i \to U, i \in I)$ be an admissible covering in $\mathrm{Cov}(\mathfrak{C}_X)$. We have to show that

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,i'\in I} \mathcal{F}(U_i \cap U_{i'})$$

is exact. Choose an admissible covering $(V_{i,j} \to U_i; j \in J_i)$ for each $U_i$ such that $V_{i,j}$ is affinoid subdomain. By definition, one has

$$\mathcal{F}(U_i) = \mathrm{Ker}(\prod_{j\in J_i} \mathcal{F}(V_{i,j}) \rightrightarrows \prod_{j,j'\in J_i} \mathcal{F}(V_{i,j} \cap V_{i,j'})).$$

Note that $U = \bigcup_{i\in I} \bigcup_{j\in J_i} V_{i,j}$ is also an admissible covering of $U$, and for fixed $i, i' \in I$, $(V_{i,j} \cap V_{i',j'} \to U_i \cap U_{i'}; j \in J_i, j' \in J_{i'})$ is an admissible covering. Hence, one has an injection

$$\mathcal{F}(U_i \cap U_{i'}) \hookrightarrow \prod_{j\in J_i, j'\in J_{i'}} \mathcal{F}(V_{i,j} \cap V_{i',j'}).$$

It follows that $\mathrm{Ker}(\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,i'\in I} \mathcal{F}(U_i \cap U_{i'}))$ coincides with

$$\mathrm{Ker}(\prod_{i\in I, j\in J_i} \mathcal{F}(V_{i,j}) \rightrightarrows \prod_{i,i'\in I} \prod_{j\in J_i, j'\in J_{i'}} \mathcal{F}(V_{i,j} \cap V_{i',j'})).$$

Finally, since $(V_{i,j} \to U; i \in I, j \in J_i)$ is an admissible covering of $U$, the kernel above is equal to $\mathcal{F}(U)$.

□

***Remark 2.1.8.*** — Actually, let $\mathfrak{C}'_X$ be the category of affinoid subdomains of $X$, and $\mathrm{Cov}(\mathfrak{C}_X)$ be the set of finite covering of affinoid subdomains. Then $(\mathbf{Aff}(X), \mathrm{Cov}(\mathbf{Aff}(X)))$ form a Grothendieck topology. This new topology is obviously weaker than $(\mathfrak{C}_X, \mathrm{Cov}(\mathfrak{C}_X))$, i.e. $\mathbf{Aff}(X)$ is a fully faithful subcategory of $\mathcal{C}_X$ and $\mathrm{Cov}(\mathbf{Aff}(X))$ is a subset of $\mathrm{Cov}(\mathfrak{C}_X)$. Then this Proposition says that even though these two topologies are not the same, however the associated categories of sheaves are the same.

By Tate's acyclicity theorem, $\mathcal{O}_X$ is a presheaf on the category of affinoid subdomains of $X$ which is sheafy for finite affinoid coverings. Lemma 2.1.7 implies that $\mathcal{O}_X$ extends uniquely to a sheaf on the Grothendieck topology of $X$, still denoted by $\mathcal{O}_X$. More generally, let $M$ be a finite $\mathcal{O}_X(X)$-module. We define a sheaf $\tilde{M}$ on the Grothendieck topology of $X$ as follows. For any affinoid subdomain $U \subseteq X$, we put

$$\tilde{M}(U) := \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(X)} M.$$

***Proposition 2.1.9.*** — *The presheaf $\tilde{M}$ is sheafy for all finite coverings of affinoid subdomains, so that $\tilde{M}$ extends to a sheaf on the Grothendieck topology of $X$.*

*Proof.* — The arguments are is the same as Theorem 1.6.3. By Proposition 1.6.10, one reduces to showing that $\tilde{M}$ is sheafy for all Laurent coverings. By induction, it suffices to do this for Laurent coverings generated by one element. Then one concludes by Corollary 1.6.12. □

Obviously, the sheaf $\tilde{M}$ should be viewed as an analogue of coherent sheaf on affinoid space.

## 2.2. Rigid analytic spaces

Let us first establish the setup for rigid analytic spaces.

***Definition 2.2.1.*** — (1) A $G$-topological space is a topological space $X$ equipped with a Grothendieck topology $(\mathfrak{C}_X, \mathrm{Cov}(\mathfrak{C}_X))$ such that $\mathfrak{C}_X$ is given by a subcategory of open subsets of $X$. The open subsets of $X$ in $\mathfrak{C}_X$ are called admissible open, and the coverings in $\mathrm{Cov}(\mathfrak{C}_X)$ are called admissible coverings.

(2) A morphism of $G$-topological space $\varphi : Y \to X$ is called continuous, if the inverse image keeps admissible open subsets and admissible coverings, i.e.

(a) if $U \in \mathfrak{C}_X$, then $f^{-1}(U) \in \mathfrak{C}_Y$;
(b) if $(U_i \to U; i \in I) \in \mathrm{Cov}(\mathfrak{C}_X)$, then $(f^{-1}(U_i) \to f^{-1}(U); i \in I) \in \mathrm{Cov}(\mathfrak{C}_Y)$.

Let $\varphi : Y \to X$ be a continuous morphism of $G$-topological spaces. For a presheaf $\mathcal{F}$ on $Y$, we define its push-forward $\varphi_*\mathcal{F}$ by

$$(\varphi_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

If $\mathcal{F}$ is a sheaf, then the continuity of $\varphi$ implies that $\varphi_*\mathcal{F}$ is also a sheaf. Similarly, for a sheaf $\mathcal{G}$ on $X$, we can define its pullback $\varphi^{-1}\mathcal{G}$ by

$$(\varphi^{-1}\mathcal{G})(V) = \varprojlim_{U \supset \varphi(V)} \mathcal{G}(U),$$

where $U$ runs through all admissible open subset of $X$ containing $\varphi(V)$. As in the case of schemes, $f^{-1}$ is the left adjoint to $f_*$, i.e. one has

$$\mathrm{Hom}_Y(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{G}, f_*\mathcal{F}).$$

***Definition 2.2.2.*** — (1) A $G$-ringed $K$-space is a pair $(X, \mathcal{O}_X)$, where $X$ is a $G$-topological space, and $\mathcal{O}_X$ is a sheaf of $K$-algebras on it. We say $(X, \mathcal{O}_X)$ is a locally $G$-ringed $K$-space if, in addition, the stalks

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U} \mathcal{O}_X(U)$$

are local rings, where $U$ runs through the admissible open subsets containing $x$.

(2) A morphism of $G$-ringed $K$-spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a pair $(\varphi, \varphi^*)$, where $\varphi : Y \to X$ is a continuous morphism of $G$-topological spaces, and $\varphi^* : \mathcal{O}_X \to \varphi_*(\mathcal{O}_Y)$ is a morphism of sheaves of rings on $X$. Assume that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally $G$-ringed spaces. We say $(\varphi, \varphi^*)$ is a morphism of locally $G$-ringed spaces if, for any $y \in Y$ with image $x = \varphi(y) \in X$, the induced ring homomorphism

$$\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$$

is local.

Let $X = \mathrm{Sp}(A)$ be an affinoid $K$-space. According to the discussion of the previous section, we have a locally $G$-ringed space $(X, \mathcal{O}_X)$. The construction $X \mapsto (X, \mathcal{O}_X)$ from the category of affinoid spaces to that of locally $G$-ringed space is functorial. Indeed, let $\varphi : Y = \mathrm{Sp}(B) \to X = \mathrm{Sp}(A)$ be a morphism of affinoid $K$-spaces (i.e. a morphism of affinoid $K$-algebras $A \to B$). Then it induces a morphism of sheaves of rings $\varphi^* : \mathcal{O}_X \to \varphi_*(\mathcal{O}_Y)$ as follows. For any affinoid subdomain $U = \mathrm{Sp}(A')$ of $X$, the inverse image $f^{-1}(U)$ is an affinoid subdomain of $Y$ with affinoid algebra $A'\hat{\otimes}_A B$. The morphism $\varphi^* : A \to B$ induces thus a ring homomorphism

$$\mathcal{O}_X(U) = A' \to A'\hat{\otimes}_A B = \mathcal{O}_Y(f^{-1}(U)).$$

For a general admissible open subset, we choose an admissible covering $U = \bigcup_{i\in I} U_i$ with each $U_i$ affinoid subdomain. Then $\mathcal{O}_X(U_i) \to \mathcal{O}_Y(f^{-1}(U_i))$ for all $i \in I$ induce a ring homomorphism

$$\varphi_U^* : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U)) = (\varphi_*\mathcal{O}_Y)(U).$$

It is easy to see that $\varphi_U^*$ are compatible with the restrictions on $U$, so that $\varphi^*$ defines a morphism of sheaves of rings $\mathcal{O}_X \to \varphi_*\mathcal{O}_Y$. For any $y \in Y$ with $x = \varphi(x)$, if $\mathfrak{m}_y \subseteq B$ and $\mathfrak{m}_x \subseteq A$ denote respectively the corresponding maximal ideals, then

***Proposition 2.2.3.*** — *The functor from the category of affinoid $K$-spaces to the category of locally $G$-ring spaces is fully faithful.*

*Proof.* — A morphism of affinoid algebra $A \to B$ can be certainly recovered by a morphism of locally $G$-ringed spaces $(\varphi, \varphi^*)$ by taking global sections.

Conversely, given a morphism $(\varphi, \varphi^*) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally $G$-ringed space, we have a homomorphism of $K$-affinoid algebras $\varphi_X^* : A = \mathcal{O}_X(X) \to \mathcal{O}_Y(Y) = B$. We have to show that $(\varphi, \varphi^*)$ is the morphism induced by $\varphi_X^*$.

Let $y \in Y$ with $x = \varphi(y)$. We have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi_X^*} & B \\ \downarrow & & \downarrow \\ \mathcal{O}_{X,x} & \xrightarrow{\varphi_y^*} & \mathcal{O}_{Y,y}. \end{array}$$

Let $\mathfrak{m}_x \subseteq A$ and $\mathfrak{m}_y \subseteq B$ be respectively the maximal ideals corresponding to $x$ and $y$. Then the maximal ideals of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are respectively $\mathfrak{m}_x\mathcal{O}_{X,x}$ and $\mathfrak{m}_y\mathcal{O}_{Y,y}$. Since $\varphi_y^*$ is local, we have $\mathfrak{m}_x\mathcal{O}_{X,x} = \varphi_y^{*,-1}(\mathfrak{m}_y\mathcal{O}_{Y,y})$ so that $\mathfrak{m}_x = \varphi_X^{*,-1}(\mathfrak{m}_y)$. This shows that the morphism of topological spaces $\varphi : X \to Y$ is induced by $\varphi_X^*$. It remains to see that $\varphi^*$ is determined by $\varphi_X^*$. One can restrict to affionid subdomains. For any affinoid subdomain $U = \mathrm{Sp}(A') \subseteq X$, we have a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\qquad\varphi_X^*\qquad} & B \\ \downarrow & & \downarrow \\ A' = \mathcal{O}_X(U) & \xrightarrow{\varphi_U^*} & \mathcal{O}_Y(f^{-1}(U)) = A'\hat{\otimes}_A B, \end{array}$$

where the vertical maps being restrictions. By the universal property of affinoid subdomain, there exists a unique morphism $\varphi_U^*$ that makes the diagram commute, namely, the natural map $a' \mapsto a' \otimes 1$ induced by $\varphi_X^* : A \to B$. □

This Proposition allows us to view the category of affinoid spaces as a subcategory of locally $G$-ringed spaces. Roughly speaking, a rigid $K$-space is a locally $G$-ringed space which locally looks like an affinoid space.

***Definition 2.2.4.*** — A rigid (analytic) $K$-space is a locally $G$-ringed space $(X, \mathcal{O}_X)$ such that

(i) the $G$-topology of $X$ satisfies conditions (G0)-(G2) in Lemma 2.1.5, and
(ii) $X$ admits an admissible covering $(X_i)_{i\in I}$ such that $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid $K$-space for all $i \in I$.

In the sequel, if there is no risk of confusions, we will use $X$ (instead of $(X, \mathcal{O}_X)$) to denote a rigid analytic $K$-space.

***Example 2.2.5.*** — Any admissible open subsets of an affinoid space is a rigid $K$-space. For instance, the open unit ball $X = \mathrm{Sp}(K\langle T\rangle)\backslash\mathrm{Sp}(K\langle T, T^{-1}\rangle)$ is a rigid $K$-space.

***Proposition 2.2.6.*** — *Let $Y$ be a rigid $K$-space, and $X$ be an affinoid $K$-space. Then the canonical map*

$$\mathrm{Hom}(Y, X) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{O}_X(X), \mathcal{O}_Y(Y)), \quad (\varphi, \varphi^*) \mapsto \varphi_X^*$$

*is an isomorphism.*

*Proof.* — The statement is known when $Y$ is affinoid by Proposition 2.2.3. For a general $Y$, we choose an admissible covering $Y = \bigcup_{i\in I} Y_i$ by affinoid spaces. Then one has a canonical isomorphism

$$\mathrm{Hom}(Y_i, X) \cong \mathrm{Hom}(\mathcal{O}_X(X), \mathcal{O}_Y(Y_i))$$

for each $i \in I$. Hence, if $\varphi_X^*$ is an element of $\mathrm{Hom}(\mathcal{O}_X(X), \mathcal{O}_Y(Y))$, it induces a morphism of locally $G$-ringed spaces $\varphi_i : Y_i \to X$. Moreover, $\varphi_i|_{Y_i\cap Y_j}$ must coincide with $\varphi_j|_{Y_i\cap Y_j}$ for all $i, j \in I$. Indeed, if $U \subseteq Y_i \cap Y_j$ is an affonoid $K$-space, then $\varphi_i|_U$ and $\varphi_j|_U$ both coincide with the morphism $U \to X$ induced by the homomorphism of affinoid $K$-algebras $\mathcal{O}_X(X) \to \mathcal{O}_Y(Y) \to \mathcal{O}_Y(U)$. Therefore, this allows us to glue $\varphi_i$ together to get a morphism of $G$-topological spaces $\varphi : X \to Y$, and the sheafy property allows us to glue $\varphi_i^*$ together to get a morphism of ring sheaves $\varphi^* : \mathcal{O}_X \to \varphi_*\mathcal{O}_Y$. □

***Proposition 2.2.7.*** — *The fibre product exists in the category of rigid analytic $K$-spaces.*

*Proof.* — Let $\varphi : Y \to X$ and $\psi : Z \to X$ be two morphisms of rigid $K$-spaces. When $X, Y, Z$ are affinoid spaces, the fibre product $Y \times_X Z$ is constructed as $\mathrm{Sp}(\mathcal{O}_Y(Y)\hat{\otimes}_{\mathcal{O}_X(X)}\mathcal{O}_Z(Z))$. In general, the fibre product is obtained by usual glueing process. □

## 2.3. Rigid analytification

In this section, we will construct a functor from the category of $K$-schemes locally of finite type to the category of rigid analytic $K$-spaces, called the rigid analytification. This is a rigid analogue of taking the complex analytic manifold associated to an algebraic varieties over complex numbers.

We start by constructing the rigid analytic analogue of the $n$-dimensional affine spaces $\mathbb{A}^n_K$. Choose $\pi \in K$ with $0 < |\pi| < 1$. For any integer $r \geq 0$, put

$$\mathbb{B}(0, |\pi|^{-r}) = \mathrm{Sp}(A_n^{(r)}), \quad \text{with } A_n^{(r)} = K\langle \pi^r T_1, \cdots, \pi^r T_n \rangle.$$

Geometrically, $\mathbb{B}(0, |\pi|^{-r})$ is the space of $n$-dimensional ball centered at 0 with radius $|\pi|^{-r}$. We have a natural inclusion of affinoid spaces $\mathbb{B}(0, |\pi|^{-r}) \hookrightarrow \mathbb{B}(0, |\pi|^{-r-1})$, which corresponds to the natural homomorphism of

$$A_n^{(r+1)} \subseteq A_n^{(r)}.$$

We define the locally $G$-ringed space

$$\mathbb{A}_K^{n,\mathrm{an}} := \bigcup_r \mathbb{B}(0, |\pi|^{-r}).$$

Then $\mathbb{A}_K^{n,\mathrm{an}}$ is a rigid $K$-space which is not affinoid. The following Lemma says that in some sense, $\mathbb{A}^n_K$ and $\mathbb{A}_K^{n,\mathrm{an}}$ have the same set of closed points.

***Lemma 2.3.1.*** — *The inclusions*

$$A_n^{(0)} \supseteq A_n^{(1)} \supseteq \cdots \supseteq A_n^{(r)} \supseteq \cdots \supseteq K[T_1, \cdots, T_n]$$

*induces inclusions of maximal ideals*

$$\mathrm{Max}(A_n^{(0)}) \subseteq \mathrm{Max}(A_n^{(1)}) \subseteq \cdots \subseteq \mathrm{Max}(A_n^{(r)}) \subseteq \cdots \subseteq \mathrm{Max}(K[T_1, \cdots, T_n])$$

*such that* $\mathrm{Max}(K[T_1, \cdots, T_n]) = \bigcup_{r \geq 0} \mathrm{Max}(A_n^{(r)})$.

*Proof.* — This is easy. Indeed, one has to check two assertions:

1. Let $\mathfrak{m}' \subseteq A_n^{(0)}$ be a maximal ideal. Then $\mathfrak{m} = \mathfrak{m}' \cap K[T_1, \cdots, T_n]$ is a maximal ideal of $K[T_1, \cdots, T_n]$ such that $\mathfrak{m}' = \mathfrak{m} A_n^{(0)}$.
2. Given a maximal ideal $\mathfrak{m}' \subseteq K[T_1, \cdots, T_n]$, there exists an integer $r \geq 0$ such that $\mathfrak{m}' A_n^{(r)}$ is a maximal ideal of $A_n^{(r)}$.

To prove (1), the first part of the assertion is easy. Here, we only show that $\mathfrak{m} = \mathfrak{m}' A_n^{(0)}$. It is clear that $\mathfrak{m} \supseteq \mathfrak{m}' A_n^{(0)}$. We have a composed map:

$$K[T_1, \cdots, T_n]/\mathfrak{m} \to A_n^{(0)}/\mathfrak{m} A_n^{(0)} \to A_n^{(0)}/\mathfrak{m}'.$$

Since $K[T_1, \cdots, T_n]$ is dense in $A_n^{(0)}$, it follows that both morphisms above are surjective. As $\mathfrak{m}' = \mathfrak{m} \cap K[T_1, \cdots, T_n]$, the composed map $K[T_1, \cdots, T_n]/\mathfrak{m}' \to A_n^{(0)}/\mathfrak{m}$ is also injective. Hence, all the morphisms above are all isomorphisms. □

***Corollary 2.3.2.*** — *The natural inlucsion*

$$K[T_1, \cdots, T_n] \hookrightarrow A_n^{(0)} = K\langle T_1, \cdots, T_n\rangle$$

*is flat.*

*Proof.* — Write $A = K[T_1, \cdots, T_n]$ for simplicity. It suffices to show that the natural inclusion $A \hookrightarrow A_n^{(0)}$ induces, for all maximal ideal $\mathfrak{m}' \subseteq A_n^{(0)}$, an isomorphism of complete local rings $\hat{A}_{\mathfrak{m}} \to \hat{A}^{(0)}_{n,\mathfrak{m}'}$, where $\mathfrak{m} = \mathfrak{m}' \cap A$. For any integer $r \geq 1$, we claim that the canonical map

$$\psi : A/\mathfrak{m}^r \to A_n^{(0)}/\mathfrak{m}'^r$$

is an isomorphism.

Indeed, we have a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A_n^{(0)} \\ \downarrow & \overset{\alpha}{\swarrow} & \downarrow \beta \\ A/\mathfrak{m}^r & \xrightarrow{\ \psi\ } & A'/\mathfrak{m}'^r, \end{array}$$

where the vertical arrows are surjective. Note that $A/\mathfrak{m}^r$ is finite dimensional over $K$, and can be viewed as an affinoid $K$-algebra. The universal property of Tate algebra $A_n^{(0)}$ implies the existence of a morphism $\alpha : A_n^{(0)} \to A/\mathfrak{m}^r$ such that the two triangles in the diagram are commutative. To finish the proof, it is enough to show that $\psi$ is an isomorphism, or equivalently the natural inclusion $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$ is actually an equality. Indeed, one has $\mathrm{Ker}(\beta) = \mathfrak{m}'r = \mathfrak{m}^r A_n^{(0)} \subseteq \mathrm{Ker}(\alpha)$, because $\mathfrak{m}^r \subseteq \mathrm{Ker}(\alpha)$. □

Now one can consider the analytification of an affine $K$-scheme of finite type. Let $X = \mathrm{Spec}(K[T_1, \cdots, T_n]/\mathfrak{a})$. We consider the morphism of algebras

$$K[T_1, \cdots, T_n]/\mathfrak{a} \to \cdots \to A_n^{(r)}/(\mathfrak{a}) \to A_n^{(r-1)}/(\mathfrak{a}) \to \cdots \to A^{(0)}/(\mathfrak{a})$$

and the associated maps of maximal spectra:

$$\mathrm{Max}(A_n^{(0)}/(\mathfrak{a})) \subseteq \mathrm{Max}(A_n^{(1)}/(\mathfrak{a})) \subseteq \cdots \subseteq \mathrm{Max}(K[T_1, \cdots, T_n]/\mathfrak{a}).$$

By Lemma 2.3.1, one see easily that $\mathrm{Max}(K[T_1, \cdots, T_n]/\mathfrak{a})$ is the union of $\mathrm{Max}(A_n^{(r)}/(\mathfrak{a}))$ for $r \geq 0$. We put

$$X^{\mathrm{an}} = \bigcup_{r \geq 0} \mathrm{Sp}(A_n^{(r)}/(\mathfrak{a})), \tag{2.3.2.1}$$

and we call it the analytification of $X$. Of course, one has to verify that such a definition does not depend on the presentation of the affine algebra of $X$ as the quotient of $K[T_1, \cdots, T_n]$. This can be done by checking that $X^{\mathrm{an}}$ verifies certain universal property.

***Definition 2.3.3.*** — Let $(X, \mathcal{O}_X)$ be a $K$-scheme locally of finite type. A rigid analytification of $(X, \mathcal{O}_X)$ is a rigid $K$-space $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ together with a morphism of locally $G$-ringed spaces $(\iota, \iota^*) : (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$ such that the following universal property is satisfied: Given a rigid $K$-space $(Y, \mathcal{O}_Y)$ and morphism of locally G-ringed $K$-spaces

$(f, f^*) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ there exists a unique $(g, g^*) : (Y, \mathcal{O}_Y) \to (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ such that $(f, f^*) = (\iota, \iota^*) \circ (g, g^*)$.

Note that the universal property of the rigid analytification of a $K$-scheme of finite type is unique.

***Lemma 2.3.4.*** — *Let $Z$ be an affine $K$-scheme, and $Y$ be a rigid analytic $K$-space. Then the set of morphisms of locally G-ringed spaces $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ corresponds bijectively to the set of $K$-algebra homomorphisms $\mathcal{O}_Z(Z) \to \mathcal{O}_Y(Y)$.*

*Proof.* — Suppose we are given a morphism of $K$-algebras $\sigma : \mathcal{O}_Z(Z) \to \mathcal{O}_Y(Y)$ one can reconstruct a morphism of locally $G$-ringed spaces $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ such that $\sigma = \varphi_X^*$. By standard glueing arguments, one may assume that $Y = \mathrm{Sp}(B)$ is affine. To simplify the notation, write $A = \mathcal{O}_Z(Z)$. For a maximal ideal $\mathfrak{m} \subseteq B$, its inverse image $\sigma^{-1}(\mathfrak{m})$ must be also maximal, since $B/\mathfrak{m}$ is finite dimensional over $K$. This induces a morphism of sets

$$\varphi : Y = \mathrm{Max}(B) \to \mathrm{Max}(A) \subseteq \mathrm{Spec}(A) = Z.$$

One verifies now that $\varphi$ is continuous for the $G$-topologies. It suffices to show that for any $f \in A$, the inverse image of $D_f = \mathrm{Spec}(A[1/f])$ in $Y$ is admissible. In fact, we have

$$\varphi^{-1}(D_f) = \{y \in Y | \sigma(f)(y) \neq 0\} = \bigcup_{c \in K^\times} Y(\frac{c}{f})$$

is admissible open. Finally, it remains to show that given a $\sigma$, there is at most one morphism of locally $G$-ringed spaces $(\varphi, \varphi^*) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ such that $\varphi_Y^* = \sigma$. The arguments are exactly the same as Proposition 2.2.3.

□

***Corollary 2.3.5.*** — *Let $X = \mathrm{Spec}(K[T_1, \cdots, T_n]/\mathfrak{a})$ be an affine $K$-scheme of finite type. Then the rigid analytic $K$-space $X^{\mathrm{an}}$ constructed in* (2.3.2.1) *is the analytification of $X$.*

*Proof.* — By Lemma 2.3.4, the natural morphisms of $K$-algebras

$$K[T_1, \cdots, T_n]/\mathfrak{a} \to \cdots \to A_n^{(r)}/(\mathfrak{a}) \to A_n^{(r-1)}/(\mathfrak{a}) \to \cdots \to A^{(0)}/(\mathfrak{a})$$

induces a morphism of locally $G$-ringed $K$-spaces $(\iota, \iota^*) : (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$.

It remains to show that every morphism $(f, f^*) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally G-ringed $K$-spaces factor through uniquely via $(\iota, \iota^*)$. We may assume that $Y = \mathrm{Sp}(B)$ is affinoid. The morphism $f$ corresponds to a morphism of $K$-algebras:

$$\sigma = f_Y^* : K[T_1, \cdots, T_n]/\mathfrak{a} \to B.$$

One has to show that there exists an integer $r$ and morphism $\sigma^{(r)} : A_n^{(r)}/(\mathfrak{a}) \to B$ such that $\sigma$ factors as

$$K[T_1, \cdots, T_n]/\mathfrak{a} \to A_n^{(r)}/(\mathfrak{a}) \xrightarrow{\sigma^{(r)}} B.$$

Consider the composed map $\tau : K[T_1, \cdots, T_n] \to K[T_1, \cdots, T_n]/\mathfrak{a} \xrightarrow{\sigma} B$. Let $g_i = \tau(T_i)$. Then by maximum modulos principle, there exists an $r \geq 0$ such that

$$\|g_i\|_{\sup} \leq |\pi|^{-r}.$$

Hence, $\tau$ extends to a morphism $\tau^{(r)} : A_n^{(r)} \to B$ which factors through $A_n^{(r)}/(\mathfrak{a})$. The resulting morphism $A_n^{(r)}/(\mathfrak{a}) \to B$ is what we wanted. □

***Proposition 2.3.6.*** — *Each scheme $X$ locally of finite type over $K$ admits a (unique) rigid analytification $X^{\mathrm{an}}$. Moreover, $X \mapsto X^{\mathrm{an}}$ is a functor from the category of $K$-schemes locally of finite type to the category of rigid $K$-spaces.*

*Proof.* — The case for affine $K$-scheme of finite type has been check in Corollary 2.3.5. The general case follows easily by glueing. The second part follows from the universal property of rigid analytification. □

***Proposition 2.3.7.*** — *Let $X$ be a $K$-scheme locally of finite type, and $\iota : X^{\mathrm{an}} \to X$ be its rigid analytification. Then the canonical morphism of sheaves of rings $\iota^* : \mathcal{O}_X \to \mathcal{O}_{X^{\mathrm{an}}}$ is flat.*

*Proof.* — It is clear that we can assume that $X = \mathrm{Spec}(A)$ is affine with $A = K[T_1, \cdots, T_n]/\mathfrak{a}$. To suffices to show that the canonical morphism $K[T_1, \cdots, T_n]/\mathfrak{a} \to K\langle T_1, \cdots, T_n\rangle/(\mathfrak{a})$ is flat, which follows from the flatness of $K[T_1, \cdots, T_n] \to K\langle T_1, \cdots, T_n\rangle$ by tensoring. □

## 2.4. Generalities on cohomology

**2.4.1. Abelian categories and derived functors.** — We recall some general properties on abelian categories and derived functors. A good reference for this part is [**Gro57**] and [**Ar62**]. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has enough injective objects, if every object of $\mathcal{A}$ embeds into an injective object.

***Theorem 2.4.2* (Grothendieck).** — *Assume that $\mathcal{A}$ verifies the following axioms:*

(AB3) *For any family of objects $(A_i)_{i\in I}$ of $\mathcal{A}$, the direct sum $\oplus_{i\in I} A_i$ exists.*

(AB5) *If $(A_i)_{i\in I}$ is a filtred family of subobjects of an object $A$ of $\mathcal{A}$ and if $B$ is another subobject of $A$, then*
$$\sum_{i\in I}(A_i \cap B) = (\sum_{i\in I} A_i) \cap B.$$

(Generator) *There exists an object $U$ of $\mathcal{A}$ such that for every object $B$ of $\mathcal{A}$ there exists an index set $I$ and a epimorphism $\oplus_{i\in I} U \to B$.*

*Then $\mathcal{A}$ has enough injective objects.*

Assume from now on that $\mathcal{A}$ has enough injective objects. Then every object of $\mathcal{A}$ admits a resolution by injective objects, i.e., for every object $M$ of $\mathcal{A}$, there exists a complex $I^\bullet = (I^0 \to I^1 \to \cdots)$ together with a morphism $M \to A^0$ such that

1. each $I^j$ is an injective object, and
2. the complex $0 \to M \to I^0 \to I^1 \to \cdots$ is acyclic.

If $M \to N$ is a morphism in $\mathcal{A}$, then one can choose resolutions $M \to I^\bullet$ and $N \to J^\bullet$ so that the morphism $M \to N$ extends to a morphism of complexes $I^\bullet \to J^\bullet$, and different extensions differ by a homotopy of complexes.

Let $\mathcal{B}$ be another abelian category, and let $f : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor. For any object $M$ of $\mathcal{A}$, we choose an injective resolution $M \to I^\bullet$. Let $f(I^\bullet)$ be the complex obtained by applying the functor $f$. For any integer $q \geq 0$, we define

$$R^q f(M) := H^q(f(I^\bullet)),$$

which can be easily seen to be independent of the resolution $I^\bullet$. Then $M \mapsto R^q f(M)$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$, called the $q$-th *derived functor* of $f$. We have $R^0 f = f$, since $f$ is left exact. It is clear that $R^q f = 0$ for all $q > 0$ if $f$ is exact.

Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be two additive left exact functors. Assume that both $\mathcal{A}$ and $\mathcal{B}$ contain enough injective objects and $f$ send injective objects of $\mathcal{A}$ to injective objective objects of $\mathcal{B}$. Then there is a Leray-spectral sequence

$$E_2^{p,q} = R^p g R^q f(M) \Rightarrow R^{p+q}(gf)(M).$$

A special case that $f$ send injective objects to injective objects is the following

***Lemma 2.4.3.*** — *If $f$ admits a left adjoint functor $f^! : \mathcal{B} \to \mathcal{A}$, which is exact, then $f$ sends injective objects of $\mathcal{A}$ to injective objects of $\mathcal{B}$.*

*Proof.* — Let $I$ be an injective object of $\mathcal{A}$, then the functor $\mathrm{Hom}(\_, f(I)) = \mathrm{Hom}(f^!\_, I)$ is the composition of two exact functors, hence exact. □

**2.4.4. Čech cohomology of presheaves.** — Let $X = (\mathfrak{C}_X, \mathrm{Cov}(\mathfrak{C}_X))$ be a Grothendieck topology. Let $\mathcal{F}$ be an abelian presheaf on $X$. Let $\mathfrak{U} = (U_i \to U)_{i \in I} \in \mathrm{Cov}(\mathfrak{C}_X)$ be an admissible covering in $X$. For any tuple $(i_0, \cdots, i_n) \in I^n$, we put $U_{i_0, \cdots, i_n} = U_{i_0} \times_U \cdots \times_U U_{i_n}$, and define the Čech complex $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ whose $n$-th term is

$$\mathcal{C}^n(\mathfrak{U}, \mathcal{F}) = \begin{cases} 0 & \text{if } n < 0, \\ \prod_{i_0, i_1, \cdots, i_n \in I} \mathcal{F}(U_{i_0, \cdots, i_n}) & \text{if } n \geq 0. \end{cases}$$

The morphism $d^n : \mathcal{C}^n(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^{n+1}(\mathfrak{U}, \mathcal{F})$ is given by

$$d^n(f)_{i_0, \cdots, i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j f_{i_0, \cdots, i_{j-1}, i_{j+1}, \cdots, i_{n+1}}|_{U_{i_0, \cdots, i_n}}.$$

For $n \geq 0$, we put

$$\check{H}^n(\mathfrak{U}, \mathcal{F}) = H^n(\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})).$$

We say that the presheaf $\mathcal{F}$ is $\mathfrak{U}$-*acyclic* if $\check{H}^n(\mathfrak{U}, \mathcal{F}) = 0$ for all $n \geq 1$.

***Proposition 2.4.5*** **([Ar62], Theorem 1.4.1).** — *For every admissible covering $\mathfrak{U} = (U_i \to U)_{i \in I}$, $\check{H}^0(U, \_)$ is left exact, and the functor $\check{H}^q(\mathfrak{U}, \_)$ is the $q$-th derived functor of $\check{H}^0(\mathfrak{U}, \_)$.*

*Proof.* — The left exactness of $\check{H}^0(\mathfrak{U}, \_)$ follows from the left exactness of Ker. First, we prove that every injective presheaf $I$ is $\mathfrak{U}$-acyclic. For any admissible open subset $U$ of $X$, let $\mathfrak{Z}_U$ be the presheaf such that

$$\mathfrak{Z}_U(V) = \mathbb{Z}^{\mathrm{Hom}(V,U)} = \bigoplus_{\phi: V \to U} \mathbb{Z}, \quad \text{for any } V \in \mathfrak{C}_X.$$

Then for any abelian presheaf $\mathcal{F}$ on $X$, we have

$$\mathrm{Hom}(\mathfrak{Z}_U, \mathcal{F}) = \mathcal{F}(U).$$

So the Čech complex $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ is obtained by applying the functor $\mathrm{Hom}(\_, \mathcal{F})$ to the complex

$$\bigoplus_{i \in I} \mathfrak{Z}_{U_i} \leftarrow \bigoplus_{i,j} \mathfrak{Z}_{U_i \times U_j} \leftarrow \cdots,$$

where the transition map is defined as follows. If $S = \coprod_i \mathrm{Hom}(V, U_i)$, then

$$\bigoplus_{i_0, \cdots, i_n \in I} \mathfrak{Z}_{U_{i_0, \cdots, i_n}}(V) = \mathbb{Z}^{S^{n+1}}$$

and the $V$-valued points of the complex above is identified with

$$\mathbb{Z}^S \leftarrow \mathbb{Z}^{S \times S} \leftarrow \mathbb{Z}^{S \times S \times S} \leftarrow \cdots \tag{2.4.5.1}$$

with transition maps given by

$$n(s_0, \cdots, s_n) \mapsto \sum_{j=0}^{n} (-1)^j n(s_0, \cdots, s_{j-1}, s_{j+1}, \cdots, s_n).$$

Since $I$ is injective, the functor $\mathrm{Hom}(\_, I)$ is exact. To prove that $I$ is $\mathfrak{U}$-acyclic, it suffices to show that the complex (2.4.5.1) is exact in degree $\leq -1$. But this complex is homotopically trivial in degree $\leq -1$ with homotopy given by

$$n(s_0, s_1, \cdots, s_n) \mapsto m(t, s_0, \cdots, s_n)$$

for some $t \in S$ fixed.

Now, it is formal to deduce that $\check{H}^q(\mathfrak{U}, \_)$ is the $q$-th derived functor of $\check{H}^0(\mathfrak{U}, \_)$. Indeed, let $\mathcal{F}$ be an abelian presheaf on $X$. We choose an injective resolution

$$\mathcal{F} \to I^0 \to I^1 \to \cdots.$$

Then the cohomology of the Čech complex $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ is the same of the simple complex associated to the double complex

$$\mathcal{C}^\bullet(\mathfrak{U}, I^\bullet).$$

Since $I^j$ is $\mathfrak{U}$-acyclic for each $I^j$, we see that the latter has also the same cohomology as the complex

$$\check{H}^0(\mathfrak{U}, I^0) \to \check{H}^0(\mathfrak{U}, I^1) \to \check{H}^0(\mathfrak{U}, I^2) \to \cdots.$$

This means that $\check{H}^q(\mathfrak{U}, \_)$ is the $q$-th derived functor of $\check{H}^0(\mathfrak{U}, \_)$. □

If $\mathfrak{V} = (V_j \to U)$ is another admissible covering of $U$ and $\mathfrak{V}$ refines $\mathfrak{U}$, then one has a natural morphism

$$\check{H}^n(\mathfrak{U}, \mathcal{F}) \to \check{H}^n(\mathfrak{V}, \mathcal{F}).$$

For any admissible open subset $U \subset X$, we define the Čech cohomology of $U$ as

$$\check{H}^n(U, \mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^n(\mathfrak{U}, \mathcal{F}),$$

where $\mathfrak{U}$ runs through all admissible coverings of $U$. Note that there is a canonical map

$$\mathcal{F}(U) \to \check{H}^0(U, \mathcal{F})$$

which is injective if $\mathcal{F}$ is separated, and bijective if $\mathcal{F}$ is a sheaf.

***Corollary 2.4.6.*** — *The functor $\check{H}^q(U, \_)$ is the $q$-th derived functor of $\check{H}^0(U, \_)$.*

**2.4.7. Presheaves and Sheaves for a Grothendieck topology.** — Denote by $\mathbf{Presh}(X)$ the category of abelian presheaves, and by $\mathbf{Sh}(X)$ be the category be the category of abelian sheaves on $X$. Then both $\mathbf{Presh}(X)$ and $\mathbf{Sh}(X)$ have enough injective objects. Let $i : \mathbf{Sh}(X) \to \mathbf{Presh}(X)$ be the natural inclusion of sheaves into presheaves. Then $i$ is left exact, i.e. if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of abelian sheaves on $X$, then

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$$

for any admissible open $U \subseteq X$.

***Proposition 2.4.8.*** — *The functor $i$ admits a left adjoint $\mathcal{F} \mapsto \mathcal{F}^{\#}$. Moreover, the functor $\#$ is exact and $\# \circ i = \mathrm{id}_{\mathbf{Presh}(X)}$.*

*Proof.* — We only give a sketch of the construction of the functor $\#$, since the construction is the same as the case of algebraic geometry. First, we define a functor $+ : \mathbf{Presh}(X) \to \mathbf{Presh}(X)$. Let $\mathcal{F}$ be an abelian presheaf on $X$. We put

$$\mathcal{F}^+(U) := \check{H}^0(U, \mathcal{F}).$$

We claim that

(a) For any presheaf $\mathcal{F}$, the presheaf $\mathcal{F}^+$ is separated, i.e. for all admissible covering $(U_i \to U)_{i \in I}$, the canonical map

$$\mathcal{F}^+(U) \to \prod_{i \in I} \mathcal{F}^+(U_i)$$

is injective.

(b) If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf.

(c) If $\mathcal{F}$ is already a sheaf, then $\mathcal{F} = \mathcal{F}^+$.

It is an immediate consequence of (a) and (b) that $\mathcal{F}^{\#} = \mathcal{F}^{++}$ is a sheaf. The claims (a) and (c) are direct from the definition. For (b), let $(U_i \to U)_{i \in I} \in \mathrm{Cov}(\mathfrak{C}_X)$ and

$$s \in \mathrm{Ker}(\prod_i \mathcal{F}^+(U_i) \to \prod_{i,j} \mathcal{F}^+(U_{i,j})).$$

One has to show that $s$ is the image of some element in $\mathcal{F}^+(U)$. By definition of $\mathcal{F}^+(U_i) = \check{H}^0(U_i, \mathcal{F})$, there exists some admissible covering $(W_{i,\alpha} \to U_i)_\alpha$ and

$$s_i = (s_{i,\alpha}) \in \operatorname{Ker}(\prod_\alpha \mathcal{F}(W_{i,\alpha}) \to \prod_{\alpha,\alpha'} \mathcal{F}(W_{i,\alpha} \times_{U_i} W_{i,\alpha'})).$$

which represents the $i$-component of $s$. By assumption, $s_i|_{U_{i,j}} = (s_{i,\alpha})_\alpha$ and $s_j|_{U_{i,j}} = (s_{j,\beta})_\beta$ represents the same element in $\mathcal{F}^+(U_{i,j})$. This means that they agree in some covering of $U_{i,j}$ which is a common refinement of $(W_{i,\alpha} \times_U U_j \to U_{i,j})_\alpha$ and $(U_i \times_U W_{j,\beta} \to U_{i,j})_\beta$. Since $\mathcal{F}$ is supposed to be separated, this implies that $s_i|_{U_{i,j}}$ and $s_j|_{U_{i,j}}$ already agree at any common refinement. Hence, we have

$$s_{i,\alpha}|_{W_{i,\alpha} \times W_{j,\beta}} = s_{j,\beta}|_{W_{i,\alpha} \times W_{j,\beta}}.$$

If $\mathfrak{W} = (W_{i,\alpha} \to U)_{i,\alpha}$ is the composed covering, then

$$(s_{i,\alpha})|_{i,\alpha} \in \operatorname{Ker}(\prod_{i,\alpha} \mathcal{F}(W_{i,\alpha}) \to \prod_{i,j} \mathcal{F}(W_{i,\alpha} \times_U W_{j,\beta}))$$

defines an element $\check{H}^0(\mathfrak{W}, \mathcal{F})$ which represents $s \in \mathcal{F}^+(U)$.

By construction, there exists a canonical morphism $\iota : \mathcal{F} \to \mathcal{F}^\#$ which satisfies the universal property if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of abelian presheaves on $X$ then there exists a unique morphism of sheaves $\phi^\# : \mathcal{F}^\# \to \mathcal{G}$ such that $\phi = \phi^\# \circ \iota$, i.e. we have a bijection

$$\operatorname{Hom}_{\mathbf{Presh}(X)}(\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}^\#, \mathcal{G}).$$

This shows that # is left adjoint to $i$. It remains to show the exactness of #. It is general property that a functor is right exact if it admits a right adjoint. The left exactness of # follows from the left exactness of +. □

***Corollary 2.4.9.*** — *The functor $i : \mathbf{Presh}(X) \to \mathbf{Sh}(X)$ sends injective objects to injective objects.*

*Proof.* — Since $i$ admits an exact left adjoint, the statement follows immediately from Lemma 2.4.3. □

**2.4.10. Cohomology of sheaves.** — Let $U \in \mathfrak{C}_X$. Taking global sections on $U$

$$\mathcal{F} \mapsto \Gamma(U, \mathcal{F}) = \mathcal{F}(U)$$

is a left exact functor $\mathbf{Sh}(X)$ to the category of abelian groups. Denote by $H^q(U, \_)$ its $q$-th derived functor for all integers $q \geq 0$. For an abelian sheaf $\mathcal{F}$, we call $H^q(U, \mathcal{F})$ the $q$-th cohomology of $U$ with coefficients in $\mathcal{F}$.

Note that for any admissible covering $\mathfrak{U} = (U_i \to U)_{i \in I}$, we have an equality of functors

$$H^0(U, \_) = \check{H}^0(\mathfrak{U}, \_) \circ i : \quad \mathbf{Sh}(X) \longrightarrow \mathbf{Ab}.$$

Since $i$ sends injective objects to injective objects, we have a spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, R^q i(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F})$$

for any $\mathcal{F} \in \mathbf{Sh}(X)$. Here, $R^q i(\mathcal{F})$ is the $q$-th derived functor of $i$ applied to $\mathcal{F}$. Explicitly, $R^q i(\mathcal{F})$ is the presheaf on $X$ that associates to each $V \in \mathfrak{C}_X$ the abelian $H^q(V, \mathcal{F})$. In particular, there exists a canonical map

$$\check{H}^q(U, \mathcal{F}) \to H^q(\mathfrak{U}, \mathcal{F}).$$

Similarly, if we take the direct limit for all the admissible coverings of $U$, then we have an equality of functors

$$H^0(U, \_) = \check{H}^0(U, \_) \circ i : \quad \mathbf{Sh}(X) \longrightarrow \mathbf{Ab},$$

and as well as a spectral sequence

$$E_2^{p,q} = \check{H}^p(U, R^q i(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}). \tag{2.4.10.1}$$

In particular, there exists a canonical map

$$\check{H}^q(U, \mathcal{F}) \to H^q(U, \mathcal{F}).$$

***Lemma 2.4.11.*** — *We have $R^q i(\mathcal{F})^\# = 0$ and $\check{H}^0(U, R^q i\mathcal{F}) = 0$ for all object $U \in \mathfrak{C}_X$ and $q \geq 1$.*

*Proof.* — Since $\mathrm{id}_{\mathbf{Sh}(X)} = \# \circ i$ and $\#$ is exact, it follows that $\# \circ R^q i = R^q \,\mathrm{id}_{\mathbf{Sh}(X)} = 0$ for all $q \geq 1$. By the construction of $\#$, we have $R^q i(\mathcal{F})^{++} = 0$, i.e. for any object $U$ of $\mathfrak{C}_X$, one has

$$R^q i(\mathcal{F})^{++}(U) = \check{H}^0(U, R^q i(\mathcal{F})^+) = 0.$$

Since $R^q i(\mathcal{F})^+$ is separated, it follows that

$$R^q i(\mathcal{F})^+(U) = \check{H}^0(U, R^q i(\mathcal{F})) \to \check{H}^0(U, R^q i(\mathcal{F})^+) = 0$$

is injective, hence $\check{H}^0(U, R^q i(\mathcal{F})) = 0$.

□

***Proposition 2.4.12.*** — *Let $\mathcal{F}$ be an abelian sheaf of $X$.*

1. *For any object $U \in \mathfrak{C}_X$, the canonical map $\check{H}^q(U, \mathcal{F}) \to H^q(U, \mathcal{F})$ is an isomorphism for $q = 0, 1$.*
2. *Let $\mathcal{S}$ be a collection of objects in $\mathfrak{C}_X$ satisfying the following conditions:*
   (a) *The intersection of two objects in $\mathcal{S}$.*
   (b) *We have $\check{H}^q(U, \mathcal{F}) = 0$ for all object $U$ of $\mathcal{S}$ and $q > 0$.*
   (c) *For each object $U$ in $\mathcal{S}$, every admissible covering $(U_i \to U)_{i \in I}$ admits a refinement consisting of objects in $S$.*

   *Then we have $H^n(U, \mathcal{F}) = 0$ for all objects in $U \in \mathcal{S}$ and $n > 0$.*
3. *Let $U$ be an object of $\mathfrak{C}_X$. Assume that $\mathfrak{U} = (U_i \to U)_{i \in I}$ is an object of $\mathrm{Cov}(\mathfrak{C}_X)$ such that $H^q(U_{i_0, \cdots, i_n}, \mathcal{F}) = 0$ for all $i_0, \cdots, i_n \in I$ and $q > 0$, then the canonical map*

   $$\check{H}^q(\mathfrak{U}, \mathcal{F}) \to H^q(U, \mathcal{F})$$

   *is an isomorphism.*

*Proof.* — (1) By the spectral sequence (2.4.10.1), there is a exact sequence

$$0 \to \check{H}^1(U, \mathcal{F}) \to H^1(U, \mathcal{F}) \to \check{H}^0(U, R^1 i\mathcal{F}).$$

But the last term vanishes by Lemma 2.4.11, hence the statement.

(2) We prove by induction on $n$ that $H^n(U, \mathcal{F}) = 0$ for all $n > 0$ and all $U \in \mathcal{S}$. When $n = 1$, this follows from assumption (b) and assertion (1) above. Assume that $n > 1$ and the statement is true up to degree $n - 1$. By the spectral sequence (2.4.10.1), it suffices to show that $\check{H}^{n-q}(U, R^q i(\mathcal{F})) = 0$ for all $1 \leq q \leq n$. For $q = n$, the vanishing of $\check{H}^0(U, R^n i\mathcal{F}) = 0$ follows from Lemma 2.4.11. Assume thus that $1 \leq q \leq n - 1$. By definition, we have

$$\check{H}^{n-q}(U, R^q i(\mathcal{F})) = \varinjlim_{\mathfrak{U}} \check{H}^{n-q}(\mathfrak{U}, R^q i(\mathcal{F})).$$

By assumption (c), the admissible coverings of $U$ consisting of objects in $\mathcal{S}$ form a cofinal system. It suffices to show that $\check{H}^{n-q}(\mathfrak{U}, R^q i\mathcal{F}) = 0$ for every admissible covering $\mathfrak{U}$ of $U$ consisting of objects in $\mathcal{S}$. However, a general term of the Čech complex $\mathcal{C}^\bullet(\mathfrak{U}, R^q i(\mathcal{F}))$ is given by $R^q i\mathcal{F}(U_{i_0,\cdots,i_n}) = H^q(U_{i_0,\cdots,i_n}, \mathcal{F})$. Since $U_{i_0,\cdots,i_n} \in \mathcal{S}$ by assumption (a), this cohomology group vanishes for $1 \leq q \leq n - 1$ by induction hypothesis. Hence, the whole Čech complex $\mathcal{C}^\bullet(\mathfrak{U}, R^q i\mathcal{F})$ is trivial for $1 \leq q \leq n - 1$. This concludes the proof of (2).

(3) By the spectral sequence $E_2^{p,q} = \check{H}^p(\mathfrak{U}, R^q i(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F})$, it suffices to show that $\check{H}^p(\mathfrak{U}, R^q i(\mathcal{F})) = 0$ if $q > 0$. The same arguments as (2) show that the Čech complex $\mathcal{C}^\bullet(\mathfrak{U}, R^q i(\mathcal{F}))$ is trivial for $q > 0$. □

## 2.5. Cohomology of rigid spaces

Let $X = \mathrm{Sp}(A)$ be an affinoid $K$-space equipped with the Grothendieck topology, and let $M$ be an $A$-module of finite type.

Recall that one can attach to $M$ a sheaf $\tilde{M}$ on $X$ with $\tilde{M}(U) = M \otimes_A A'$ for every affinoid suddomain $U = \mathrm{Sp}(A')$ of $X$. In algebraic geometry, a fundamental result of Serre says that the cohomology of an affine scheme with coefficients in a quasi-coherent sheaf vanishes in degree $> 0$. In rigid geometry, we have similarly the following

***Theorem 2.5.1* (Tate).** — *The sheaf $\tilde{M}$ is acyclic for every finite covering $\mathfrak{U} = (U_i \to X)_{i \in I}$ by affinoid subdomains of $X$, i.e. we have*

$$\check{H}^n(\mathfrak{U}, \tilde{M}) = 0 \quad \textit{for } n > 0.$$

This is a general form of Tate's acyclicity Theorem 1.6.3, and the proof is very similar.

***Lemma 2.5.2* ([Se55], No. 29, Prop. 5).** — *Let $\mathfrak{U} = (U_i \to X)_{i \in I}$ and $\mathfrak{V} = (V_j \to X)_{j \in J}$ be two finite coverings of $X$ by affinoid subdomains. Let $\mathcal{F}$ be an abelian sheaf on $X$. Assume that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, and $\mathcal{F}$ is acyclic the restriction of $\mathfrak{V}$ to $U_t$ for each tuple $t = (i_0, \cdots, i_n) \in I^{n+1}$. Then the canonical map*

$$\check{H}^q(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} \check{H}^q(\mathfrak{V}, \mathcal{F})$$

*is an isomorphism for all $q \geq 0$.*

*Sketch of the proof.* — One consider the double complex

$$C^{p,q}(\mathfrak{U},\mathfrak{V};\mathcal{F}) = \prod_{s,t}\Gamma(U_s\cap V_t,\mathcal{F}),$$

where $s$ (respectively $t$) runs through the set of $(p+1)$-tuples $(i_0,\cdots,i_p)\in I^{p+1}$ (respectively $(q+1)$-tuples $(j_0,\cdots,j_t)\in J^{t+1}$), with differentials

$$d_{\mathfrak{U}}: C^{p,q}(\mathfrak{U},\mathfrak{V};\mathcal{F})\to C^{p+1,q}(\mathfrak{U},\mathfrak{V};\mathcal{F}),\quad d_{\mathfrak{V}}: C^{p,q}(\mathfrak{U},\mathfrak{V};\mathcal{F})\to C^{p,q+1}(\mathfrak{U},\mathfrak{V};\mathcal{F})$$

given by the usual Čech differentials for the coverings $(U_s\cap V_t\to V_t)_{s\in I^{p+1}}$ of $V_t$ and the $(U_s\cap V_t\to U_s)_{t\in J^{q+1}}$ of $U_s$ respectively. We have $d_{\mathfrak{U}}d_{\mathfrak{V}} = d_{\mathfrak{V}}d_{\mathfrak{U}}$. There are two natural morphisms of complexes:

$$\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F})\to\mathcal{C}^{\bullet,0}(\mathfrak{U},\mathfrak{V};\mathcal{F}),\quad \mathcal{C}^\bullet(\mathfrak{V},\mathcal{F})\to\mathcal{C}^{0,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F})$$

inducing morphisms of cohomology groups

$$\check{H}^n(\mathfrak{U},\mathcal{F})\to H^n(\underline{s}\mathcal{C}^{\bullet,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F})),\quad \check{H}^n(\mathfrak{V},\mathcal{F})\to H^n(\underline{s}\mathcal{C}^{\bullet,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F})),$$

which are isomorphisms if Here, $\underline{s}\mathcal{C}^{\bullet,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F})$ denote the simple complex associated to the double complex $\mathcal{C}^{\bullet,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F})$. Note that, for each fixed $p\geq 0$, the complex $\mathcal{C}^{p,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F})$ is the product of Čech complexes of $\mathfrak{V}|_{U_s}$ for all $s\in I^{p+1}$. Hence, we get

$$H^q(\mathcal{C}^{p,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F}))\cong\prod_{s\in I^{p+1}}\check{H}^q(U_s,\mathcal{F})$$

which vanishes for $q>0$ according to the assumption. Therefore, the canonical morphism

$$\check{H}^n(\mathfrak{U},\mathcal{F})\to H^n(\underline{s}\mathcal{C}^{\bullet,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F}))$$

is an isomorphism.

On the other hand, since $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, for each tuple $t\in J^{q+1}$, the covering $\mathfrak{U}|_{V_s} = (U_i\cap V_s\to V_s)_{i\in I}$ contains the identity map $V_s\to V_s$. In other words, the identity $(V_s\to V_s)$ is a refinement of $\mathfrak{U}|_{V_s}$, hence it is equivalent to $\mathfrak{U}|_{V_s}$ (since $\mathfrak{U}|_{V_s}$ is clearly a refinement of $(V_s\to V_s)$). According to [**Se55**, §22], one has

$$\check{H}^p(\mathfrak{U}|_{V_s},\mathcal{F}) = 0\quad\forall p>0.$$

This implies that the canonical map

$$\check{H}^n(\mathfrak{V},\mathcal{F})\to H^n(\underline{s}\mathcal{C}^{\bullet,\bullet}(\mathfrak{U},\mathfrak{V};\mathcal{F}))$$

is an isomorphism as well.

□

Using this Lemma, one can reduce the proof of Theorem 2.5.1 to the case when $\mathfrak{U}$ is a Laurent covering generated by $f_1,\cdots,f_r$. By induction, one can further reduce to the case that $r=1$. In this case, Theorem 2.5.1 is equivalent to saying that

$$0\to M\to M\langle f\rangle\oplus M\langle 1/f\rangle\to M\langle f,1/f\rangle\to 0$$

is exact, which was verified in Corollary 1.6.12.

***Corollary 2.5.3.*** — *In the setup of Theorem 2.5.1, we have* $H^n(X,\tilde{M}) = 0$ *for* $n>0$.

*Proof.* — Let $\mathcal{S}$ be the set of affinoid subdomains of $X$. Theorem 2.5.1 implies that the assumptions in Proposition 2.4.12(2) are satisfied. It follows that $H^n(X, \tilde{M}) = 0$. □

**2.5.4. Coherent sheaves.** — If we want to generalize the results above for affinoid spaces to general rigid analytic space, we need to define analogues of the sheaf $\tilde{M}$ on general rigid spaces.

***Definition 2.5.5.*** — Let $X$ be a rigid $K$-space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is coherent, if there exists an affinoid admissible covering $(X_i \to X)_{i\in I}$ of $X$ such that $\mathcal{F}|_{X_i}$ is isomorphic to the sheaf associated to a finite $\mathcal{O}_X(X_i)$-module.

It is easy to check the coherent sheaves in the rigid setting satisfy the same properties as in usual algebraic geometry, for instance, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of abelian sheaves on a rigid spaces, if any two of $\mathcal{F}_i$'s are coherent, then so is the third.

***Theorem 2.5.6*** **(Kiehl).** — *Let $X = \mathrm{Sp}(A)$ be an affinoid $K$-space and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is isomorphic to $\tilde{\mathcal{F}}(X)$.*

***Corollary 2.5.7.*** — *Let $X$ be a rigid analytic space, and $\mathcal{F}$ be a sheaf on $X$. The following conditions are equivalent:*

1. *$\mathcal{F}$ is coherent.*
2. *For any affinoid admissible covering $X_i \to X$, $\mathcal{F}|_{X_i}$ is isomorphic to the sheaf associated to a finite $\mathcal{O}_X(X_i)$-module.*

We sketch the proof Theorem 2.5.6. The if part of Theorem 2.5.6 is obvious. Let $\mathfrak{U} = (U_i \to X)_{i\in I}$ be a finite affinoid covering of $X = \mathrm{Sp}(A)$ such that $\mathcal{F}|_{U_i}$ is the sheaf associated to a finite $\mathcal{O}_X(U_i)$-module, which has to be $\mathcal{F}(U_i)$. We will call such a property being $\mathfrak{U}$-coherent.

***Lemma 2.5.8.*** — *If $\mathcal{F}$ is a $\mathfrak{U}$-coherent sheaf on an affinoid space $X$, then we have $\check{H}^1(\mathfrak{U}, \mathcal{F}) = 0$.*

For the proof of this Lemma, we refer to [**Bo08**, §1.14 Lemma 6]. Here, we just say that as in the proof of Theorem 1.6.3, we use Lemma 2.5.2 to reduce to the case when $\mathfrak{U}$ is the Laurent covering generated by one element $f$. So write $\mathfrak{U} = (U_1, U_2)$ with $U_1 = X(f)$ and $U_2 = X(f^{-1})$. By assumption, $M_i = \mathcal{F}(U_i)$ for $i = 1, 2$ are finite $\mathcal{O}_X(U_i)$-modules, and one has

$$M_{12} := \mathcal{F}(U_1 \cap U_2) \cong M_1 \otimes_{A\langle f\rangle} A\langle f, f^{-1}\rangle \cong M_2 \otimes_{A\langle f^{-1}\rangle} A\langle f, f^{-1}\rangle.$$

One is reduced to showing that the map

$$M_1 \times M_2 \to M_{12}, \quad (g_1, g_2) \mapsto g_1 - g_2$$

is surjective. One finishes the proof by an approximation argument.

Assume now Lemma 2.5.8 holds, and we deduce Theorem 2.5.6 as follows. Write $A_i = \mathcal{O}_X(U_i)$ and $M_i = \mathcal{F}(U_i)$ for each $i \in I$. By assumption, $M_i$ is a finite $A_i$-module, and $\mathcal{F}|_{U_i}$ is the sheaf associated to $M_i$. We have to show that,

(a) $M := \mathcal{F}(X)$ is a finite generated $A$-module, and

(b) $M_i = M \otimes_A A_i$.

Let $x \in X$ be a point, and $\mathfrak{m}_x \subseteq A$ be the corresponding maximal ideal. Consider the subsheaf $\mathfrak{m}_x\mathcal{F} \subseteq \mathcal{F}$ and the quotient. Let $n \geq 1$ be an integer. It is clear that both $\mathfrak{m}_x^n\mathcal{F}$ and $\mathcal{F}$ are $\mathfrak{U}$-coherent. By Lemma 2.5.8, one has $\check{H}^1(\mathfrak{U}, \mathfrak{m}_x^n\mathcal{F}) = 0$, and hence

$$0 \to \mathfrak{m}_x^n\mathcal{F}(X) \to \mathcal{F}(X) \to (\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(X) \to 0.$$

We claim that the natural restriction maps

$$M/\mathfrak{m}_x^n M = (\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(X) \xrightarrow{\sim} (\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(U_j) = M_j/\mathfrak{m}_x^n M_j \xrightarrow{\sim} \mathcal{F}_x/\mathfrak{m}_x^n\mathcal{F}_x$$

are bijective for any index $j \in J$ with $x \in U_j$, where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at $x$. Actually, for each $i \in I$, $\mathcal{F}/\mathfrak{m}_x^n\mathcal{F}|_{U_i}$ is the sheaf associated to the $A_i$-module $M_i/\mathfrak{m}_x^n M_i$. Hence, $\mathcal{F}/\mathfrak{m}_x^n\mathcal{F}|_{U_i}$ is either 0 if $x \notin U_i$, or a skyscraper sheaf at $x$ with fiber $M_i/\mathfrak{m}_x^n M_i \cong \mathcal{F}_x/\mathfrak{m}_x^n\mathcal{F}_x$ if $x \in U_i$. In particular, the restriction map

$$(\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(U_i) \xrightarrow{\sim} (\mathcal{F}/\mathfrak{m}_x^n)(U_i \cap U_j)$$

is always an isomorphism. Consider the commutative diagram

$$\begin{array}{ccccccc}
0 \to & (\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(X) & \longrightarrow & \prod_{i\in I}(\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(U_i) & \longrightarrow & \prod_{i,i'}(\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(U_i \cap U_{i'}) & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & (\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(U_j) & \longrightarrow & \prod_{i\in I}(\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(U_i \cap U_j) & \longrightarrow & \prod_{i,i'}(\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(U_i \cap U_{i'} \cap U_j), &
\end{array}$$

where right two vertical arrows are isomorphisms by the argument above. Hence, it follows thus so is $(\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(X) \to (\mathcal{F}/\mathfrak{m}_x^n\mathcal{F})(U_j)$. This proves the claim.

Consider the canonical map

$$\phi_j : M \otimes_A A_j \to M_j.$$

For any $x \in X_j = \mathrm{Max}(A_j)$, by Nakayama's Lemma, the claim implies that $\mathcal{F}_x$ is generated by the image of $M$. But $\mathcal{F}_x = M_j \otimes_{A_j} \mathcal{O}_{X,x}$, this is equivalent to saying that $\phi_j \otimes_{A_j} \mathcal{O}_{X,x}$ is surjective for all $x \in X_j$. Since $\prod_{x\in X_j} \mathcal{O}_{X,x}$ is faithfully flat over $A_j$, this implies that $\phi_j$ itself is surjective. Since $M_j$ is finitely generated $A_j$-module, we can choose sections $f_1, \cdots, f_r \in M$ such that their images in $M_j$ generate $M_j$ for all $j \in I$. Those sections define thus a surjective map of sheaves

$$\mathcal{O}_X^r \xrightarrow{\theta} \mathcal{F}$$

Let $\mathcal{G}$ denote the kernel of $\theta$. Then $\mathcal{G}$ is also $\mathfrak{U}$-coherent, hence $\check{H}^1(\mathfrak{U}, \mathcal{G}) = 0$, and we get a surjection $A^r \to M$ by taking global sections. This shows that $M$ is a finitely generated $A$-module.

It remains to show that the canonical map

$$\phi_j : M \otimes_A A_j \to M_j$$

is an isomorphism. Recall that $A/\mathfrak{m}_x^n A \cong A_j/\mathfrak{m}_x^n A_j$ for any $x \in U_j$ by Proposition 1.5.21. The claim show that $\phi_j \otimes_{A_j} A_j/\mathfrak{m}_x^n$ is an isomorphism for all $x \in \mathrm{Max}(A_j)$. It follows that $\phi_j \otimes_{A_j} \widehat{A}_{j,x}$ for all $x \in \mathrm{Max}(A_j)$. But $A_j \to \prod_{x\in \mathrm{Max}(A_j)} \widehat{A}_{j,x}$ is faithfully flat, it follows that $\phi_j$ itself is an isomorphism. This finishes the proof of Theorem 2.5.6.

**2.5.9. Cohomology of general rigid $K$-spaces.** — Let $X$ be a rigid $K$-space. We have now a good notion of coherent sheaves on general rigid $K$-spaces. Using Proposition 2.4.12, one can compute the cohomology of a coherent sheaf on $X$ in terms of Čech cohomology.

***Definition 2.5.10.*** — (1) A morphism of rigid $K$-spaces $i : Y \to X$ is a closed embedding, if there exists an affinoid admissible covering $X = \bigcup_{j\in J} X_j$ such that $i|_{X_j} : Y \times_X X_j \to X_j$ is a closed embedding of affinoid $K$-spaces for each $j \in J$, i.e. $Y \times_X X_j$ is affinoid and the induced morphism of affinoid algebras $\mathcal{O}_X(X_j) \to \mathcal{O}_Y(Y \times_X X_j)$ is surjective.

(2) A morphism $f : Y \to X$ of rigid $K$-space is separated, if the diagonal morphism

$$\Delta : X \to X \times_Y X$$

is a closed embedding.

(3) We say a rigid space $X$ is separated, if the structure morphism $X \to \mathrm{Sp}(K)$ is separated.

***Remark 2.5.11.*** — The definition above of closed embedding is compatible with the notion of closed embedding for affinoids, i.e. if $i : Y \to X$ is a closed embedding in the sense above and $X$ is affinoid, then $Y$ is affinoid and the induced map of $K$-algebras $\mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is surjective. Actually, consider the $\mathcal{O}_X$-module $i_*\mathcal{O}_Y$. The assumption implies that $i_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-sheaf, and the canonical map

$$\phi : \mathcal{O}_X \to i_*\mathcal{O}_Y$$

surjective. Hence, $\mathcal{I} = \mathrm{Ker}(\phi)$ is also a coherent $\mathcal{O}_X$-module. By Corollary 2.5.3, one gets $H^1(X, \mathcal{I}) = 0$ and thus a surjection $\mathcal{O}_X(X) \to i_*\mathcal{O}_Y(X) = \mathcal{O}_Y(Y)$.

***Lemma 2.5.12.*** — *Let $X$ be a separated rigid $K$-space. Then the intersection of two admissible affinoid open subsets in $X$ is still affinoid.*

*Proof.* — The proof is similar to the situation in algebraic geometry. Let $U_1, U_2$ be two admissible affinoid open in $X$. Then one has a Cartesian diagram

$$\begin{array}{ccc} U_1 \cap U_2 & \longrightarrow & U_1 \times U_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

By assumption, the bottom horizontal arrow is a closed embedding, so is the upper horizontal one.

□

***Theorem 2.5.13.*** — *Let $X$ be a separated rigid analytic $K$-space, and $\mathcal{F}$ be a coherent sheaf on $X$. If $\mathfrak{U}$ is an affinoid admissible covering of $X$, then one has a canonical isomorphism*

$$\check{H}^q(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} H^q(U, \mathcal{F})$$

*for all $q \geq 0$.*

*Proof.* — This follows from Proposition 2.4.12 and Corollary 2.5.3 The separateness assumption is used to guarantee the intersections $U_{i_0}\cap\cdots\cap U_{i_n}$ are affinoids. □

The following Lemma will be useful in next subsection.

***Lemma 2.5.14.*** — *Let $i: Y\hookrightarrow X$ be a closed embedding of rigid $K$-spaces, and $\mathcal{F}$ be a coherent sheaf on $Y$. Then $i_*\mathcal{F}$ is also a coherent on $X$, and one has a canonical isomorphism*

$$H^q(Y,\mathcal{F})\cong H^q(X,i_*\mathcal{F}),\quad \textit{for all } q\geq 0.$$

*Proof.* — This follows from the fact that $i_*$ is exact, and sends injective abelian sheaves to injective abelian sheaves. □

## 2.6. Kiehl's proper mapping theorem

In this section, we will introduce the analogue of proper morphisms of rigid analytic spaces, and then prove that Kiehl's fundamental theorem on the finiteness of the higher direct image of a coherent sheaf under proper map.

***Definition 2.6.1.*** — Let $\varphi: X\to Y$ be a morphism of rigid analytic varieties with $Y$ affinoid, and let $U,U'\subseteq X$ be admissible open affinoid subspaces. We say that $U$ is relatively compact in $U'$ with respect to $Y$, written as $U\Subset_Y U'$, if there exist $f_1,\cdots,f_r\in\mathcal{O}_X(U')$ such that there exists a surjection of affinoid algebras

$$\mathcal{O}_Y\langle T_1,\cdots,T_n\rangle\to\mathcal{O}_X(U')$$

sending $T_i\mapsto f_i$ and $U\subseteq\{x\in U';|f_i(x)|<1\}$, or equivalently, there exists $\epsilon\in(0,1)\cap|\overline{K}^*|$ such that

$$U\subseteq\{x\in U';|f_i(x)|\leq\epsilon\}$$

***Definition 2.6.2.*** — A morphism of rigid spaces $\varphi: X\to Y$ is called proper if

(i) $\varphi$ is separated.
(ii) There exists an admissible affinoid covering $(Y_i)_{i\in I}$ of $Y$ and for each $i\in I$ two admissible affinoid coverings $(X_{i,j})_{j=1,\cdots,n_i}$ and $(X'_{i,j})_{j=1,\cdots,n_i}$ of $\varphi^{-1}(Y_i)$ such that $X_{i,j}\Subset_{Y_i} X'_{i,j}$.

***Example 2.6.3.*** — (1) Let $X$ be a projective rigid analytic space over $K$. Then $X$ is proper over $K$. Indeed, choose a closed embedding $X\subseteq\mathbb{P}^{n,\mathrm{an}}_K$. For $i=0,\cdots,n$ and $c\in|K^\times|$ with $c\geq 1$, put

$$U_i(c)=\{x=(x_0,\cdots,x_n)\in X||x_j|\leq c|x_i|\}$$

Then $(U_i(c))_{i=0,1\cdots,n}$ form an admissible covering of $X$ for $c\geq 1$, and $U_i(c)\Subset_K U_i(c')$ for $c<c'$. Hence, $X$ is proper over $K$ in the above sense.

(2) Finite morphisms of rigid spaces are proper. Indeed, if $\varphi: X\to Y$ is finite morphism with $Y$ affinoid, there exist $f_1,\cdots,f_r\in\mathcal{O}_X(X)$ such that $\mathcal{O}_X(X)=\sum_i\mathcal{O}_Y(Y)f_i$. Up to scaling $f_i$, we may assume that $f_i$ are power bounded. Consider the homomorphism

$$\theta_c:\mathcal{O}_Y(Y)\langle T_1,\cdots,T_n\rangle\to\mathcal{O}_X(X)$$

sending $T_i \mapsto cf_i$ for any $c \in K^\times$ with $|c| < 1$. Then we see that $\theta_c$ is surjective and $X \subseteq X(c^{-1} \cdot cf_1, \cdots, c^{-1} \cdot cf_n)$ and $X \subseteq_Y X$.

***Theorem 2.6.4* (Kiehl).** — *Let $\varphi : X \to Y$ be a proper morphism of rigid $K$-spaces and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then $R^i\varphi_*(\mathcal{F})$ are coherent $\mathcal{O}_Y$-modules.*

In order to prove Theorem 2.6.4, we need the notion of completely continuous morphisms. Let $B$ be an affinoid $K$-algebra, equipped with a residue norm $\|\cdot\|$. Let $M, N$ be two Banach $B$-modules. Denote by $\mathcal{L}_B(M,N)$ the space of continuous $B$-linear maps. We equip $\mathcal{L}_B(M,N)$ with the strong topology, i.e. the topology defined by the norm

$$\|f\| = \sup_{x \in M, \|x\|=1} \|f(x)\|_N$$

for any $f \in \mathcal{L}_B(M,N)$

***Definition 2.6.5.*** — A continous $B$-linear map $f : M \to N$ is called completely continuous, if it is the limit of $(f_i)_{i\in\mathbb{N}}$ where each $f_i$ is a continuous homomorphism such that $\mathrm{im}(f_i)$ is a finite $B$-submodule of $N$.

If $f : M \to N$ is a completely continuous and $N' \subseteq N$ be a closed $B$-submodule, then $f : f^{-1}(N') \to N'$ is completely continuous.

***Theorem 2.6.6* ([Bo08], §1.17, Theorem 4).** — *Let $f, g : M \to N$ be continuous homomorphisms of Banach $B$-modules. Assume that*

(i) *$f$ is surjective, and*
(ii) *there exists an epimorphism $p : E \to M$ such that the composite $gp$ is completely continuous.*

*Then the image $\mathrm{im}(f+g)$ is closed in $N$ and the cokernel $N/\mathrm{im}(f+g)$ is a finite $B$-module.*

***Proposition 2.6.7.*** — *Let $Y = \mathrm{Sp}(B)$, and $\varphi : X \to Y$ be a morphism of rigid spaces. Let $U, V$ be two admissible affinoid subsets of $X$ such that $V \subseteq_Y U$. Let $\mathcal{F}$ be a coherent sheaf on $X$, and $M \subseteq \mathcal{F}(U)$ (resp. $N \subset \mathcal{F}(V)$) be an $\mathcal{O}_X(U)$-submodule (resp. $\mathcal{O}_X(V)$-submodule) such that restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ induces $M \to N$. Then there exists a Banach $B$-module $E$ together with a $B$-linear surjection $E \to M$ such that the composite*

$$E \to M \xrightarrow{restriction} N$$

*is completely continuous.*

*Proof.* — By definition of relative compactness, there exist a surjection

$$\theta : B\langle T_1, \cdots, T_n \rangle \to \mathcal{O}_X(U)$$

sending $T_i \mapsto f_i$ such that $V \subseteq \{x \in U; |f_i(x)| < 1\}$. Choose a surjection $\mathcal{O}_X(U)^r \to M$. We claim that the surjection

$$E := B\langle T_1, \cdots, T_n \rangle^r \to \mathcal{O}_X(U)^r \to M$$

satisfies the requirement. One has to show that the composite

$$f : E \to M \to N$$

is a limit of morphisms with finite image. It suffices to treat the case $\mathcal{F} = \mathcal{O}_X$ and $r = 1$. Consider the composite

$$\theta_V : B\langle T_1, \cdots, T_n\rangle \to \mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

Then $\theta_V(T_i)$ are topologically nilpotent by the relative compactness of $V$ in $U$. Hence, the multiplication by $\theta_V(T_i)$ on $N$ are also topologically nilpotent. Write

$$B\langle T_1, \cdots, T_n\rangle^r = \widehat{\bigoplus}_{m\geq 0} F_m$$

with $F_m = (\oplus_{\nu\in\mathbb{N}^n, |\nu|=m} BT^{\mu})^r$. Let $f_i : B\langle T_1, \cdots, T_n\rangle^r \to N$ be the morphism whose restriction to $F_m$ equals $f$ for $m \leq i$ and 0 for $m > i$. Then $\mathrm{im}(f_i)$ is a finite $B$-module and $f_i$ has limit $f$. □

We come to the proof of The proof of Theorem 2.6.4. We may assume that $Y = \mathrm{Sp}(B)$ is affinoid. The proof consists of two steps. The first step is given by the following

***Proposition 2.6.8.*** — *Let $\varphi : X \to Y$ be a proper morphism of rigid spaces with $Y$ affinoid. Assume that there exist two finite admissible affinoid coverings $\mathfrak{U} = (U_i)_{i=1,\cdots,s}$ and $\mathfrak{V} = (V_i)_{i=1,\cdots,s}$ of $X$ such that $V_i \Subset_Y U_i$ for each $i$. Then $H^q(X, \mathcal{F})$ is a finitely generated module over $B = \mathcal{O}_Y(Y)$.*

*Proof.* — Since the Čech complex $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ and $\mathcal{C}^\bullet(\mathfrak{V}, \mathcal{F})$ both compute the cohomology groups $H^q(X, \mathcal{F})$, the restriction map

$$\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^\bullet(\mathfrak{V}, \mathcal{F})$$

is a quasi-isomorphism. For each $q \geq 0$, let $Z^q(\mathfrak{U}, \mathcal{F}) \subseteq \mathcal{C}^q(\mathfrak{U}, \mathcal{F})$ be the $q$-cocylces. Then we have a surjective map

$$\mathcal{C}^{q-1}(\mathfrak{V}, \mathcal{F}) \oplus Z^q(\mathfrak{U}, \mathcal{F}) \xrightarrow{d_{\mathfrak{V}}^{q-1} + r^q} Z^q(\mathfrak{V}, \mathcal{F}),$$

where $r^q$ is the restriction to $Z^q(\mathfrak{U}, \mathcal{F})$ of the natural restriction map $\mathcal{C}^q(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^q(\mathfrak{V}, \mathcal{F})$. By Proposition 2.6.7, $r^q$ satisfies the assumption of Theorem 2.6.6 for $g = r^q$. Hence, $d_{\mathfrak{V}}^q = (d_{\mathfrak{V}}^{q-1} + r^q) - r^q$ has closed image in $Z^q(\mathfrak{V}, \mathcal{F})$ and the cokernel $H^q(X, \mathcal{F}) \cong Z^q(\mathfrak{V}, \mathcal{F})/\mathrm{im}(d_{\mathfrak{V}}^{q-1})$ is a finite $B$-module. □

The second step of the proof of Theorem 2.6.4 is the following

***Proposition 2.6.9.*** — *Let $\varphi : X \to Y$ be a proper morphism of rigid spaces with $Y = \mathrm{Sp}(B)$ affinoid, and $\mathcal{F}$ be a coherent sheaf on $X$. Then for any integer $q \geq 0$, $R^q\varphi_*(\mathcal{F})$ is the coherent sheaf associated to the $B$-module $H^q(X, \mathcal{F})$, i.e. if $Y' = \mathrm{Sp}(B')$ is an affinoid subdomain of $Y = \mathrm{Sp}(B)$, then the canonical map*

$$H^q(X, \mathcal{F}) \otimes_B B' \xrightarrow{\sim} H^q(X', \mathcal{F})$$

*is an isomorphism, where $X' = X \times_Y Y'$.*

We proceed by induction on the Krull dimension of $B$. When $\dim(B) = 0$, $Y$ is a finite disjoint union of rigid $K$-spaces supported at a single point, and $B'$ is a direct summand of $B$. The statement is trivial.

Assume that $d = \dim(B) > 0$. It suffices to show that for any maximal ideal $\mathfrak{m}'$ of $B'$ the localized map

$$H^q(X, \mathcal{F}) \otimes_B \hat{B}'_{\mathfrak{m}'} \xrightarrow{\sim} H^q(X', \mathcal{F}) \otimes'_B \hat{B'}_{\mathfrak{m}'} \tag{2.6.9.1}$$

is an isomorphism. By Proposition 1.5.21, there exists a unique maximal ideal $\mathfrak{m}$ of $B$ such that $\mathfrak{m}' = \mathfrak{m}B'$. By Noetherian normalization, there exists a finite injective morphism $A_d \hookrightarrow B$. Then $\mathfrak{n} = \mathfrak{m} \cap A_d$ is a maximal ideal of $A_d$. Take a non-zero $b \in \mathfrak{n} \subseteq \mathfrak{m}$. Then the Krull dimension of $B/(b^i)$ is $d-1$. Put $Y_i = \mathrm{Sp}(B/(b^i))$ for all $i \geq 1$. Then the sheaf $\mathcal{F}/b^i\mathcal{F}$ is supported in $X \times_Y Y_i$, and

$$H^q(X, \mathcal{F}/b^i\mathcal{F}) \cong H^q(X \times_Y Y_i, \mathcal{F}/b^i\mathcal{F})$$

is a finite $B/(b^i)$-module. By induction hypothesis the canonical map

$$H^q(X, \mathcal{F}/b^i\mathcal{F}) \otimes_B B' \xrightarrow{\sim} H^q(X \times_Y Y', \mathcal{F}/b^i\mathcal{F}).$$

is an isomorphism. Consider the canonical map

$$H^q(X, \mathcal{F}) \otimes_B B/(b^i) \to H^q(X, \mathcal{F}/b^i\mathcal{F}),$$

and denote by $D_i, E_i$ its kernel and cokernel respectively. When $i \in \mathbb{N}$ varies, we get a long exact sequence of projective systems of $B$-modules

$$0 \to D_i \to H^q(X, \mathcal{F}) \otimes_B B/(b^i) \to H^q(X, \mathcal{F}/b^i\mathcal{F}) \to E_i \to 0.$$

Since $B'$ is flat over $B$, by tenoring with $B'$, one gets a commutative diagram of maps

$$\begin{array}{ccccccc}
D_i \otimes_B B' & \hookrightarrow & H^q(X, \mathcal{F}) \otimes_B B'/(b^i) & \longrightarrow & H^q(X, \mathcal{F}/b^i\mathcal{F}) \otimes_B B' & \twoheadrightarrow & E_i \otimes_B B' \\
 & & \downarrow & & \downarrow \cong & & \\
 & & H^q(X', \mathcal{F}) \otimes_{B'} B'/(b^i) & \longrightarrow & H^q(X', \mathcal{F}/b^i\mathcal{F}), & &
\end{array}$$

where the right vertical arrow is an isomorphism by the discussion above.

Recall that a projective system $(M_i)_{i \geq 1}$ of $B$-modules is called a null system, if for any $i$ there exists an index $j_0 > i$ such that the image of $M_j$ in $M_i$ is zero for all $j \geq j_0$. Then we have the following

***Lemma 2.6.10.*** — *The projective systems $(D_i)_{i \geq 1}$ and $(E_i)_{i \geq 1}$ are null systems.*

Granting this Lemma for a moment, we finish the proof of Proposition 2.6.9 and Theorem 2.6.4 as follows. Taking projective limits in $i$, one gets a commutative diagram

$$\begin{array}{ccc}
H^q(X, \mathcal{F}) \otimes_B \hat{B}' & \xrightarrow{\cong} & \varprojlim_i [H^q(X, \mathcal{F}/b^i\mathcal{F}) \otimes_B B'] \\
\downarrow & & \downarrow \cong \\
H^q(X', \mathcal{F}) \otimes_{B'} \hat{B}' & \xrightarrow{\cong} & \varprojlim_i H^q(X', \mathcal{F}/b^i\mathcal{F}).
\end{array}$$

Here, $\hat{B}' = \varprojlim_{i\in\mathbb{N}} B'/(b^i)$ denotes the $b$-adic completion of $B'$, the upper horizontal arrow is an isomorphism by Lemma 2.6.10, and so is the lower horizontal arrow by replacing $X$ by $X'$. It follows that the left vertical arrow

$$H^q(X,\mathcal{F})\otimes_B \hat{B}' \xrightarrow{\cong} H^q(X',\mathcal{F})\otimes_{B'} \hat{B}'$$

is also an isomorphism as well. By tensoring with $\hat{B}'_{\mathfrak{m}'}$ over $\hat{B}'$, it follows that (2.6.9.1) is an isomorphism, which finishes the proof of Proposition 2.6.9.

It remains to prove Lemma 2.6.10. We need some preliminary results. Consider the graded ring

$$S = \bigoplus_{i=0}^{\infty} b^i B.$$

This is a Noetherian ring, since $B$ is noetherian and S is generated over $B$ by its homogeneous of degree 1 part, $bB$, which is a finite $B$-module. Recall that a graded $S$-module $M = \bigoplus_{i=0}^{\infty} M_i$ is finitely generated over $S$ if and only if each $M_i$ is finitely generated over $B$ and there exists an integer $i_0 \in \mathbb{N}$ such that

$$M_i = b^{i-i_0} M_{i_0}$$

for all $i \geq i_0$.

***Lemma 2.6.11.*** — *The graded module*

$$M^q(\mathcal{F}) := \bigoplus_{i=0}^{\infty} H^q(X, b^i\mathcal{F})$$

*is finitely generated over $S$ for all $q \geq 0$.*

*Proof.* — Consider first the case that $\mathcal{F}$ is $b$-torsion free. In this case, the multiplication by $b^i$ induces an isomorphism of sheaves $\mathcal{F} \xrightarrow{\sim} b^i\mathcal{F}$, and hence an isomorphism of cohomology groups

$$H^q(X,\mathcal{F}) \xrightarrow{\times b^i} H^q(X, b^i\mathcal{F}).$$

It follows that $M^q(\mathcal{F})$ is generated over $S$ by $H^q(X,\mathcal{F})$, viewed as the degree 0 part of $M^q(\mathcal{F})$.

Consider the general case of $\mathcal{F}$. Let $\mathcal{T} \subseteq \mathcal{F}$ be the $b$-power torsion subsheaf of $\mathcal{F}$. Since $\mathcal{T}$ is a coherent $\mathcal{O}_X$-module, there exists an integer $i_0$ such that $\mathcal{T}$ is killed by $b^{i_0}$. For any $i \in \mathbb{N}$, we put $\mathcal{T}_i = \mathcal{T} \cap b^i\mathcal{F}$. Then the Artin-Rees Lemma implies that there exists an integer $i_1$ such that $\mathcal{T}_j = b^{j-i_1}\mathcal{T}_{i_1}$ for all $j \geq i_1$. Hence, it follows that $\mathcal{T}_i = 0$ if $i \geq 2\max(i_0, i_1)$. In particular, the graded $S$-module

$$N^q := \bigoplus_{i=0}^{\infty} H^q(X, \mathcal{T}_i)$$

is finitely generated over $S$. Note that exact sequence $0 \to \mathcal{T}_i \to b^i\mathcal{F} \to b^i(\mathcal{F}/\mathcal{T}) \to 0$ induces an exact sequence of $S$-modules

$$N^q \to M^q(\mathcal{F}) \to M^q(\mathcal{F}/\mathcal{T}).$$

Note that $\mathcal{F}/\mathcal{T}$ is $b$-torsion free. Hence $M^q(\mathcal{F}/\mathcal{T})$ is a finitely generated $S$-module by the special case discussed above. It follows that $M^q(\mathcal{F})$ is also finitely generated over $S$, since $S$ is Noetherian ring. □

Consider the exact sequence of sheaves

$$0 \to b^i\mathcal{F} \to \mathcal{F} \to \mathcal{F}/b^i\mathcal{F} \to 0.$$

Taking $H^q(X, \_)$, one deduces

$$H^q(X, b^i\mathcal{F}) \to H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}/b^i\mathcal{F}) \to H^{q+1}(X, b^i\mathcal{F}).$$

Let $\overline{H^q(X, b^i\mathcal{F})}$ denotes the image of $H^q(X, b^i\mathcal{F})$ in $H^q(X, \mathcal{F})$, then one has

$$D_i = \overline{H^q(X, b^i\mathcal{F})}/b^i H^q(X, \mathcal{F}).$$

and $E_i = \mathrm{Ker}(H^{q+1}(X, b^i\mathcal{F}) \to H^{q+1}(X, \mathcal{F}))$. By Lemma 2.6.11, there exists an integer $j_0 \in \mathbb{N}$ such that $H^q(X, b^j\mathcal{F}) = b^{j-j_0}H^q(X, b^{j_0}\mathcal{F})$ for all $j \geq j_0$. It follows that for any fixed $i > j_0$, one has

$$\overline{H^q(X, b^j\mathcal{F})} = b^{j-i}\overline{H^q(X, b^i\mathcal{F})} \subseteq b^{j-i}H^q(X, \mathcal{F}).$$

Hence, the image of $D_j$ in $D_i$ is zero if $j \geq 2i$. On the other hand, it is clear that $E_i$ is killed by $b^i$. Applying Lemma 2.6.11 to $M^{q+1}(\mathcal{F})$, we see that

$$H^{q+1}(X, b^j\mathcal{F}) = b^{j-i}H^q(X, b^i\mathcal{F})$$

for $j \geq i$ sufficiently large. It follows that the image of $E_j$ in $E_i$ is contained in $b^{j-i}E_i$, and hence 0 if $j \geq 2i$.

## 2.7. Rigid GAGA

Let $X$ be a $K$-scheme locally of finite type. We associated in Section 2.3 a rigid $K$-space $X^{\mathrm{an}}$ together with a morphism of locally $G$-ringed spaces $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$. Let $\mathcal{F}$ be a coherent sheaf on $X$. We put

$$\mathcal{F}^{\mathrm{an}} := \iota^{-1}\mathcal{F} \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}$$

and call it the rigid analytification of $\mathcal{F}$. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of coherent $\mathcal{O}_X$-modules, then it induces a natural morphism

$$f^{\mathrm{an}} : \mathcal{F}^{\mathrm{an}} \to \mathcal{G}^{\mathrm{an}}$$

of coherent sheaves on $X^{\mathrm{an}}$. In analogy to the case of complex analytic geometry, we have the following analogue of Serre's GAGA theorem in rigid geometry.

***Theorem 2.7.1* (Rigid GAGA).** — *Let $X$ be a projective scheme over $K$.*

1. *For any coherent sheaf $\mathcal{F}$ and integer $n \geq 0$, there exists a canonical isomorphism*

$$H^n(X, \mathcal{F}) \cong H^n(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}).$$

2. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on $X$, then $f \mapsto f^{\mathrm{an}}$ induces a bijection*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}).$$

3. *The functor $\mathcal{F}^{\mathrm{an}}$ is an equivalence of categories between the category of coherent sheaves on $X$ and that of coherent sheaves on $X^{\mathrm{an}}$.*

*Proof.* — The proof is almost copied from [**Se56**].

(1) By Lemma 2.5.14, one reduces to the case $X = \mathbb{P}^n_K$. When $\mathcal{F} = \mathcal{O}_X$, one prove by direct computation that

$$H^q(\mathbb{P}^{n,\mathrm{an}}_K, \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}_K}) = \begin{cases} K & q = 0; \\ 0 & q > 0. \end{cases}$$

Hence, $H^q(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}) \cong H^q(\mathbb{P}^{n,\mathrm{an}}_K, \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}_K})$ is an isomorphism.

When $\mathcal{F} = \mathcal{O}_X(r)$ for $r \in \mathbb{Z}$. We proceed by $n = \dim(X) \geq 0$. When $n = 0$, the assertion is trivial. Assume thus $n \geq 1$ and the assertion is true for $\mathbb{P}^m_K$ with $m < n$. Consider the exact sequence

$$0 \to \mathcal{O}_X(r-1) \xrightarrow{\times x_r} \mathcal{O}_X(r) \to \mathcal{O}_H(r) \to 0$$

where $H \cong \mathbb{P}^{n-1}_K$ is the hyperplane $x_r = 0$. By the induction hypothesis and the five Lemma, the assertion holds for $\mathcal{F} = \mathcal{O}_X(r)$ if and only if it holds for $\mathcal{F} = \mathcal{O}_X(r-1)$. One reduces by induction to the case $r = 0$ discussed above.

Now consider the case when $\mathcal{F}$ is a general coherent sheaf on $\mathbb{P}^n_K$. We proceed by descending induction on $q$ to prove that $H^q(\mathbb{P}^n_K, \mathcal{F}) \cong H^q(\mathbb{P}^{n,\mathrm{an}}_K, \mathcal{F}^{\mathrm{an}})$. One take an exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is a direct sum of lines bundles of the form $\mathcal{O}_{\mathbb{P}^n_K}(r)$. Since $\mathcal{O}_X \to \iota_* \mathcal{O}_{X^{\mathrm{an}}}$ is flat, $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is an exact functor. Hence, one deduces a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} H^q(\mathbb{P}^n_K, \mathcal{G}) & \longrightarrow & H^q(\mathbb{P}^n_K, \mathcal{E}) & \longrightarrow & H^q(\mathbb{P}^n_K, \mathcal{F}) & \longrightarrow & H^{q+1}(\mathbb{P}^n_K, \mathcal{G}) & \longrightarrow & H^{q+1}(\mathbb{P}^n_K, \mathcal{E}) \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_4} & & \downarrow{\scriptstyle \varphi_5} \\ H^q(\mathbb{P}^{n,\mathrm{an}}_K, \mathcal{G}^{\mathrm{an}}) & \longrightarrow & H^q(\mathbb{P}^{n,\mathrm{an}}_K, \mathcal{E}^{\mathrm{an}}) & \longrightarrow & H^q(\mathbb{P}^{n,\mathrm{an}}_K, \mathcal{F}^{\mathrm{an}}) & \longrightarrow & H^{q+1}(\mathbb{P}^{n,\mathrm{an}}_K, \mathcal{G}^{\mathrm{an}}) & \longrightarrow & H^{q+1}(\mathbb{P}^{n,\mathrm{an}}_K, \mathcal{E}^{\mathrm{an}}) \end{array}$$

By induction hypothesis, the maps $\varphi_4, \varphi_5$ are bijective, and $\varphi_2$ is bijective by the previous discussion. By the Five Lemma, we see that $\varphi_3$ is surjective. But this applies to any coherent sheaf on $\mathbb{P}^n_K$, in particular to $\mathcal{G}$. Hence, $\varphi_1$ is surjective as well. It follows by Five Lemma again that $\varphi_3$ is bijective.

(2) Consider the coherent sheaf $\mathcal{A} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ on $X$, and the analytic coherent sheaf $\mathcal{B} = \mathcal{H}om_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$ on $X^{\mathrm{an}}$. We have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^0(X, \mathcal{A}), \quad \mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}) = H^0(X^{\mathrm{an}}, \mathcal{B}).$$

Hence, the statement will follow from (1) and the following

***Lemma 2.7.2.*** — *We have a canonical isomorphism*

$$\mathcal{A}^{\mathrm{an}} \cong \mathcal{B}.$$

*Proof.* — We choose a finite presentation

$$\mathcal{O}_X^r \to \mathcal{O}_X^s \to \mathcal{F} \to 0.$$

Applying the analytification functor, one gets an exact sequence

$$\mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}_{X^{\mathrm{an}}}^s \to \mathcal{F}^{\mathrm{an}} \to 0.$$

Since $\mathcal{O}_{X^{\mathrm{an}}}$ is faithfully flat over $\mathcal{O}_X$, one has a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{A}^{\mathrm{an}} & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^s, \mathcal{G})^{\mathrm{an}} & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^r, \mathcal{G})^{\mathrm{an}} \\ & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} \\ 0 \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X^{\mathrm{an}}}(\mathcal{O}_{X^{\mathrm{an}}}^s, \mathcal{G}^{\mathrm{an}}) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}^r, \mathcal{G}^{\mathrm{an}}) \end{array}$$

It is clear that both $\varphi_2$ and $\varphi_3$ are isomorphisms. We conclude by the Five Lemma. □

(3) Let $\mathcal{M}$ be a coherent sheaf on $\mathbb{P}_K^{n,\mathrm{an}}$. We have to show that $\mathcal{M}$ comes from an algebraic coherent sheaf $\mathcal{M}$ on $\mathbb{P}_K^n$. We proceed by induction on $n \geq 0$. The case of $n = 0$ is trivial. Assume thus that the statement is true for $\mathbb{P}_K^{n-1,\mathrm{an}}$.

The idea is to use a rigid analogue of Serre's vanishing theorem to express $\mathcal{F}$ as the quotient of twists of constant sheaves. The key point is to prove the following two lemmas

***Lemma 2.7.3.*** — *Let $H$ be a hyperplane of $\mathbb{P}_K^n$, and let $\mathcal{E}$ be a coherent sheaf on $H^{\mathrm{an}}$. Then we have $H^q(H^{\mathrm{an}}, \mathcal{E}(n)) = 0$ for $n >> 0$ and $q > 0$.*

*Proof.* — By induction hypothesis, there exists an algebraic coherent sheaf $\mathcal{F}$ such that $\mathcal{E} = \mathcal{F}^{\mathrm{an}}$. Then by the first part of the Theorem, one has a canonical isomorphism $H^q(H, \mathcal{F}) \cong H^q(H^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$. The Lemma now follows from the usual Serre vanishing theorem in algebraic geometry. □

***Lemma 2.7.4.*** — *Let $\mathcal{M}$ be a analytic coherent sheaf on $X^{\mathrm{an}} = \mathbb{P}_K^{n,\mathrm{an}}$. Then there exists an integer $n$ such that $\mathcal{M}$ is generated by $H^0(X^{\mathrm{an}}, \mathcal{M}(n))$, where $\mathcal{M}(n) = \mathcal{M} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{O}_{X^{\mathrm{an}}}(n)$.*

*Proof.* — Since $X^{\mathrm{an}}$ can be covered by finitely many affinoid subspaces and any admissible covering of an affinoid space has a finite refinement by affionids, $X^{\mathrm{an}}$ is quasi-compact in the sense that every adimissible covering of $X^{\mathrm{an}}$ has a finite refinement by affinoid subspaces. It suffices to show that,*for any $x \in X^{\mathrm{an}}$, there there exists an integer $n$, depending maybe on $x$ and $\mathcal{M}$, such that $H^0(X^{\mathrm{an}}, \mathcal{M}(n))$ generate $\mathcal{M}(n)_x$.*

Let $H$ be a hyperplane of $\mathbb{P}^n$ with $x \in H$. Consider the exact sequence of coherent sheaves:

$$0 \to \mathcal{O}_{X^{\mathrm{an}}}(-1) \to \mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}_H \to 0.$$

Tensoring with $\mathcal{M}(n)$, we get

$$0 \to \mathcal{C}(n) \to \mathcal{M}(n-1) \to \mathcal{M}(n) \to \mathcal{M}_H \to 0$$

with $\mathcal{C} = \mathrm{Tor}_1^{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{O}_H, \mathcal{M})$. Let $\mathcal{B}_n$ be the image of $\mathcal{M}(n-1) \to \mathcal{M}(n)$. Then the exact sequence above splits into

$$0 \to \mathcal{C}(n) \to \mathcal{M}(n-1) \to \mathcal{B}_n \to 0, \quad 0 \to \mathcal{B}_n \to \mathcal{M}(n) \to \mathcal{M}_H(n) \to 0.$$

Taking cohomology, we get

$$H^1(X^{\mathrm{an}}, \mathcal{M}(n-1)) \to H^1(X^{\mathrm{an}}, \mathcal{B}_n) \to H^2(X^{\mathrm{an}}, \mathcal{C}(n))$$

and

$$H^1(X^{\mathrm{an}}, \mathcal{B}_n) \to H^1(X^{\mathrm{an}}, \mathcal{M}(n)) \to H^1(X^{\mathrm{an}}, \mathcal{M}_H(n)).$$

Note that both $\mathcal{M}_H(n)$ and $\mathcal{C}(n)$ are supported on $H$. By Lemma 2.7.3, we have $H^2(X^{\mathrm{an}}, \mathcal{C}(n)) = 0$ and $H^1(X^{\mathrm{an}}, \mathcal{M}_H(n)) = 0$ for $n >> 0$. One deduces inequalities

$$\dim H^1(X^{\mathrm{an}}, \mathcal{M}(n-1)) \geq \dim H^1(X^{\mathrm{an}}, \mathcal{B}_n) \geq H^1(X^{\mathrm{an}}, \mathcal{M}(n)).$$

By Kiehl's finiteness theorem 2.6.4, we know that $\dim H^1(X^{\mathrm{an}}, \mathcal{M}(n))$ are finite for all $n \geq 0$. Therefore, there exists an integer $n_0$ such that $H^1(X^{\mathrm{an}}, \mathcal{M}(n))$ is constant for all $n \geq n_0$. In particular, the map

$$H^1(X^{\mathrm{an}}, \mathcal{B}_n) \xrightarrow{\sim} H^1(X^{\mathrm{an}}, \mathcal{M}(n))$$

is an isomorphism for $n \geq n_0$, and hence

$$H^0(X^{\mathrm{an}}, \mathcal{M}(n)) \to H^0(X^{\mathrm{an}}, \mathcal{M}_H(n))$$

is surjective for all $n \geq n_0$. By induction hypothesis, $\mathcal{M}_H$ comes from some algebraic coherent sheaf and hence there exists an integer $n_1 > n_0$ such that $H^0(X^{\mathrm{an}}, \mathcal{M}_H(n))$ generate $\mathcal{M}(n)_x$. □

Finish of the proof of Theorem 2.7.1 (3): Let $\mathcal{M}$ be an analytic coherent sheaf on $X^{\mathrm{an}} = \mathbb{P}_K^{n,\mathrm{an}}$. By Lemma 2.7.4, there exists an integer $n > 0$ such that $\mathcal{M}(n)$ is the quotient of $\mathcal{O}_{X^{\mathrm{an}}}^r$ for some $r \geq 0$. Hence $\mathcal{M}$ is the quotient of some $\mathcal{O}_{X^{\mathrm{an}}}^r(-n)$. Let $\mathcal{N}$ be the kernel of $\mathcal{O}_{X^{\mathrm{an}}}^r(-n) \to \mathcal{M}$. Then $\mathcal{N}$ is a coherent sheaf of $X^{\mathrm{an}}$. Applying the same arguments to $\mathcal{N}$, we see there exists a surjection $\mathcal{O}_{X^{\mathrm{an}}}^s(-m) \to \mathcal{N}$. Let $\phi^{\mathrm{an}}$ denote the composite

$$\phi^{\mathrm{an}} : \mathcal{O}_{X^{\mathrm{an}}}^s(-m) \to \mathcal{N} \hookrightarrow \mathcal{O}_{X^{\mathrm{an}}}^r(-n)$$

By Theorem 2.7.1, $\phi^{\mathrm{an}}$ is the rigid analytification of morphism of algebraic coherent sheaves $\phi : \mathcal{O}_X^s(-m) \to \mathcal{O}_X^r(-n)$. Let $\mathcal{F}$ denote the cokernel of $\phi$, then $\mathcal{M} \cong \mathcal{F}^{\mathrm{an}}$.

□

# CHAPTER 3

# FORMAL GEOMETRY

## 3.1. Formal schemes

Let $A$ be a topological ring. We say that $A$ is *adic* if

1. the topology on $A$ is $I$-adic for some ideal $I \subseteq A$, i.e. for any $a \in A$, $(a + I^n)_{n \geq 0}$ form a basis of open neighborhoods of $a \in A$, and
2. $A$ is separated and complete for this topology, i.e. $A \cong \varprojlim_n A/I^n$.

Such an ideal $I \subseteq A$ is usually called an ideal of definition.

Let $A$ be an adic ring with an ideal of definition $I$. One can associate to $A$ an affine formal scheme

$$X = \mathrm{Spf}(A) = \varinjlim_n \mathrm{Spec}(A/I^n)$$

As a topological space, $X$ is the same as $\mathrm{Spec}(A/I^n)$ for any $n \geq 1$ equipped with the Zariski topology, or equivalently $X$ consists of prime ideals of $A$, which are open in the $I$-adic topology. For any $f \in A$, let $D_f \subset X$ denote the subset where $f$ does not vanishes, i.e. $D_f$ consists of the open prime ideals $\mathfrak{p}$ of $A$ such that $f \notin \mathfrak{p}$. We put also

$$A\langle f \rangle := \varprojlim_n (A/I^n)[f^{-1}]$$

Then one has an isomorphism

$$A\langle f^{-1} \rangle \cong A\langle T \rangle/(1 - fT),$$

where $A\langle T \rangle = \varprojlim_n (A/I^n)[T]$. If $I$ is finitely generated, the topology on $A\langle f^{-1} \rangle$ coincides with the $IA\langle f^{-1} \rangle$-adic topology. Indeed, in this case, the canonical map

$$I^m A\langle f^{-1} \rangle \xrightarrow{\sim} \varprojlim_n I^m (A/I^n)[f^{-1}]$$

is an isomorphism for $m \geq 0$, so that

$$A\langle f^{-1} \rangle / I^m A\langle f^{-1} \rangle \cong A/I^m[f^{-1}].$$

Then one has $D_f = \mathrm{Spf}(A\langle f \rangle)$. As the usual Zariski topology of schemes, the $D_f$ form a basis of open subsets of $X$, and a collection of subsets $\{D_{f_i} : i \in I\}$ covers $X$ if and only if $\sum_{i \in I} (f_i) = A$.

***Lemma 3.1.1.*** — *The presheaf of topological rings on $X$ defined by*

$$D_f \mapsto A\langle f^{-1}\rangle$$

*is a sheaf.*

*Proof.* — If $(D_{f_i})_i$ covers $D_f$, then one has a short exact sequence

$$0 \to A/I^n[f^{-1}] \to \prod_i A/I^n[f_i^{-1}] \to \prod_{i,j} A/I^n[\frac{1}{f_i f_j}].$$

The statement follows from the fact that taking projective limit is a left exact functor. □

For any $x \in X$, let $\mathfrak{p}_x \subseteq A$ be the corresponding open prime ideal. We put

$$\mathcal{O}_{X,x} = \varinjlim_{x\in D_f} A\langle f^{-1}\rangle.$$

***Lemma 3.1.2.*** — *The ring $\mathcal{O}_{X,x}$ is local with maximal ideal $\mathfrak{p}_x\mathcal{O}_{X,x}$.*

*Proof.* — For any $f \in A$ with $x \in D_f$, one has a short exact sequence

$$0 \to \mathfrak{p}_x(A/I^n)[f^{-1}] \to (A/I^n)[f^{-1}] \to (A/\mathfrak{p}_x)[f^{-1}] \to 0.$$

Since the projective system $\mathfrak{p}_x(A/I^n)[f^{-1}]$ is Mittag-Leffler, one deduces

$$0 \to \mathfrak{p}_x\langle f^{-1}\rangle \to A\langle f^{-1}\rangle \to (A/\mathfrak{p}_x)[f^{-1}] \to 0,$$

where $\mathfrak{p}_x\langle f^{-1}\rangle = \varprojlim_n \mathfrak{p}_x(A/I^n)[f^{-1}]$. Taking direct limit for all $f$ with $f \notin \mathfrak{p}_x$, we get

$$0 \to \mathfrak{p}_x\mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \to Q_x \to 0$$

where $Q_x$ denotes the fraction field of $A/\mathfrak{p}_x$. This shows that $\mathfrak{p}_x\mathcal{O}_{X,x}$ a maximal ideal of $\mathcal{O}_{X,x}$.

We have to show that $\mathfrak{p}_x\mathcal{O}_{X,x}$ is the unique maximal ideal of $\mathcal{O}_{X,x}$. Let $g_x \in \mathcal{O}_{X,x}\backslash\mathfrak{p}_x\mathcal{O}_{X,x}$ represented by $g \in A\langle f^{-1}\rangle$ for some $f \in A$ with $f \notin \mathfrak{p}_x$. Then the image $\bar{g} \in A/I[\frac{1}{f}]$ of $g$ is not in $\mathfrak{p}_x(A/I)[f^{-1}]$. Up to multiplying $g$ by some powers of $f$, we may assume that $\bar{g} \in A/I$. Let $g' \in A$ be an arbitrary lift of $\bar{g}$. Then $g' \notin \mathfrak{p}_x$, and we can write $g = g'(1-d)$ in $A\langle\frac{1}{fg'}\rangle$ with $d = 1 - g/g'$. Note that $d \in \mathfrak{p}_x A\langle\frac{1}{fg'}\rangle$, and hence $1-d$ is invertible in $A\langle\frac{1}{fg'}\rangle$ with inverse given by $\sum_{i=0}^{\infty} d^i$. Therefore, the image of $g$ is invertible in $A\langle\frac{1}{fg'}\rangle$ and hence in $\mathcal{O}_{X,x}$.

□

Therefore, one can view the formal scheme $\mathrm{Spf}(A)$ as a locally topologically ringed space with structure sheaf $\mathcal{O}_X$.

***Definition 3.1.3.*** — A formal scheme is a locally topologically ringed space $(X, \mathcal{O}_X)$ such that every point $x \in X$ admits an open neighborhood $U$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine formal scheme $\mathrm{Spf}(A)$ for some adic ring $A$.

***Example 3.1.4.*** — A fundamental construction of formal schemes is given by the formal completion along a closed subscheme. Let $X$ be a locally noetherian scheme, and $Y \subseteq X$ be a closed subscheme defined by an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. We define the formal completion of $X$ along $Y$ as the formal scheme

$$\hat{X}_Y = \varinjlim_n Y_n$$

where $Y_n$ is the closed subscheme of $X$ defined by $\mathcal{I}^n$. In other words, the formal scheme $\hat{X}_Y$ has the same topological space as $Y$, and for each affine open subset $U = \mathrm{Spec}(A) \subseteq X$, the sections of $\mathcal{O}_{\hat{X}_Y}$ at $U \cap Y$ is given by

$$\mathcal{O}_{\hat{X}_Y}(U \cap Y) = \varprojlim_n A/\mathcal{I}^n(U).$$

For instance, the completion of $X = \mathrm{Spec}(\mathbb{Z}[T])$ along the characteristic $p$ fiber $Y = \mathrm{Spec}(\mathbb{F}_p[T])$ is given by

$$\hat{X}_Y = \mathrm{Spf}(\mathbb{Z}_p\langle T\rangle).$$

If $\mathcal{F}$ is a coherent sheaf on $X$, then one can also define the completion of $\mathcal{F}$ along $Y$, denoted by $\hat{\mathcal{F}}$ as follows: for any affine open subset $U$ of $X$, we have

$$\hat{\mathcal{F}}(U \cap Y) = \varprojlim_n \mathcal{F}(U)/\mathcal{I}^n(U)\mathcal{F}(U) = \varprojlim_n (\mathcal{F}/\mathcal{I}^n\mathcal{F})(U).$$

## 3.2. Admissible formal schemes

From now on, we fix a complete non-archimedean field $K$ with ring of integers $\mathcal{O}_K$. We fix also an element $\pi \in \mathcal{O}_K$ with $0 < |\pi| < 1$. For an integer $n \geq 0$, we put

$$\mathcal{O}_K\langle T_1, \cdots, T_n\rangle = \varprojlim_n \mathcal{O}_K/(\pi^n)[T_1, \cdots, T_n].$$

This is a separated complete $\pi$-adic $\mathcal{O}_K$-algebra. This can be viewed as an integral model of the Tate's algebra $K\langle T_1, \cdots, T_n\rangle$. Note that $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$ is not noetherian if $\mathcal{O}_K$ is not discrete valued. This possible failure of noetherianness will be the main technical difficulty in formal geometry over $\mathcal{O}_K$.

***Definition 3.2.1.*** — We say a topological $\mathcal{O}_K$-algebra $A$ is

1. *of topologically finite type* if $A$ is isomorphic to $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle/\mathfrak{a}$, equipped with the $\pi$-adic topology;
2. *of topologically finite presentation* if, in addition to (1), the ideal $\mathfrak{a}$ is finitely generated;
3. *admissible* if $A$ is topologically of finite presentation and $A$ is flat over $\mathcal{O}_K$.

We will need the following theorem by Raynaud and Gruson:

***Theorem 3.2.2*** **([RG71], Th. 3.4.6).** — *Let $R$ be a ring such that* $\mathrm{Ass}(R)$ *is finite. Let $A$ be an $R$-algebra of finite type, and $M$ be a finite $A$-module, which is flat over $R$. Then $M$ is of finite presentation over $A$.*

***Theorem 3.2.3.*** — *Let $A$ be an $\mathcal{O}_K$-algebra topologically of finite type, and $M$ be a finitely generated $A$-module which is flat over $\mathcal{O}_K$. Then $M$ is an $A$-module of finite presentation.*

*Proof.* — Since $M$ is finitely generated over $A$, there exists an exact sequence

$$0 \to N \to A^s \to M \to 0$$

for some integer $s \geq 1$. We have to see that $N$ is finitely generated over $A$. By definition of $\mathcal{O}_K$-algebra topologically of finite type, $A$ can be written as a quotient of $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$ for certain integer $n \geq 1$. Viewing $M$ and $N$ as modules over $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$, we may assume that $A = \mathcal{O}_K\langle T_1, \cdots, T_n\rangle$. As $M$ is flat over $\mathcal{O}_K$, one deduces an exact sequence

$$0 \to N/\pi N \to (A/(\pi))^s \to M/\pi M \to 0.$$

for any $n \in \mathbb{N}$. By Raynaud-Gruson's theorem $M/\pi M$ is an $A/\pi A$-module of finite presentation, hence $N/\pi N$ is finitely generated over $A/\pi A$. Note that $N$ is separated for the $\pi$-adic topology, since $N$ is a submodule of $A^s$, which is $\pi$-adically separated. Hence, by Lemma 1.2.6, it follows that $N$ is also finitely generated. □

***Corollary 3.2.4.*** — *Let $A$ be an $\mathcal{O}_K$-algebra of topologically finite type. If $A$ is flat over $\mathcal{O}_K$, then $A$ is of topologically finite presentation over $\mathcal{O}_K$.*

*Proof.* — By definition, $A$ is a quotient of $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$ for some integer $n \geq 1$ with kernel $\mathfrak{a}$, and equipped with the $\pi$-adic topology. Consider $A$ a finitely generated $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$-module, it follows from the previous Corollary that $\mathfrak{a}$ is also a finitely generated $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$-module. This means that $A$ is of topologically finite presentation. □

We have the following replacement of Artin-Rees.

***Lemma 3.2.5.*** — *Let $A$ be an $\mathcal{O}_K$-algebra of topologically finite type, and $M$ be a finite $A$-module, and $N \subset M$ be a submodule. Then*

1. *The submodule*

$$N_{\text{sat}} := \{x \in M; \textit{there exists } n \in \mathbb{N} \textit{ such that } \pi^n x \in M\}$$

   *is finitely generated.*
2. *The $\pi$-adic topology on $M$ induces the $\pi$-adic topology of $N$.*

*Proof.* — (1) Since $M/N_{\text{sat}}$ is flat over $\mathcal{O}_K$ and finitely generated over $A$, it is of finite presentation by Theorem 3.2.3. Since $M$ is finitely generated, it follows that $N_{\text{sat}}$ is finitely generated as well.

(2) Since $N_{\text{sat}}$ is finitely generated, there exists $n \in \mathbb{N}$ such that $\pi^n N_{\text{sat}} \subset N$. Then one has

$$\pi^{m+n} M \cap N \subset \pi^{n+m} N_{\text{sat}} \subset \pi^m N \subset \pi^m M \cap N$$

for all $m \in \mathbb{N}$. □

***Proposition 3.2.6.*** — *Let $A$ be an $\mathcal{O}_K$-algebra of topologically finite type and $M$ be a finite $A$-module. Then $M$ is $\pi$-adic separated and complete.*

*Proof.* — We may assume that $A = \mathcal{O}_K\langle T_1, \cdots, T_n\rangle$, in particular separated and complete for the $\pi$-adic topology. Being a quotient of a $\pi$-adically complete $A$-module, $M$ is trivially complete. To show the separateness of $M$, we consider $x \in \bigcup_{n\in\mathbb{N}} \pi^n M$ and $N = Ax$. Since the $\pi$-adic topology on $M$ induces the $\pi$-adic topology on $N$, there exists $n \in \mathbb{N}$ such that $N = \pi^n M \cap N \subseteq \pi N$. Hence, there exists an equation of the form $(1-d)x = 0$ with $d \in \pi A$. But it is easy to see that $(1-d)$ is invertible with inverse $\sum_{i=0}^{\infty} d^i$. It follows that $x = 0$. □

***Corollary 3.2.7.*** — *Every $\mathcal{O}_K$-algebra of topologically finite type is separated and complete for the $\pi$-adic topology.*

***Lemma 3.2.8.*** — *Let $A$ be a topological $\mathcal{O}_K$-algebra, which is $\pi$-adically separated complete. Then*

1. *$A$ is of topologically finite type if and only if $A_0$ is of finite type over $\mathcal{O}_K/(\pi)$.*
2. *$A$ is of topologically finite presentation if and only if $A_n$ is of finite presentation over $\mathcal{O}_K/(\pi^{n+1})$*

*Proof.* — The only if part is trivial. We verify here the "if" part. Assume that $A_0$ is of finite type over $\mathcal{O}_K/(\pi)$. Then there exists a surjective homomorphism of $\mathcal{O}_K/(\pi)$-algebras: $\varphi_0 : \mathcal{O}_K/(\pi)[T_1, \cdots, T_n] \to A_0$ for some integer $n \geq 1$. Let $\varphi : \mathcal{O}_K\rangle T_1, \cdots, T_n\rangle \to A$ be any lift of $\varphi_0$. Then $\varphi$ is surjective by Nakayama.

Let $\mathfrak{a} = \mathrm{Ker}(\varphi)$, and consider the exact sequence

$$0 \to \mathfrak{a} \to \mathcal{O}_K\langle T_1, \cdots, T_n\rangle \xrightarrow{\varphi} A \to 0.$$

Then one has an exact sequence

$$0 \to \mathfrak{a}/\mathfrak{a} \cap \pi^{n+1}\mathcal{O}_K\langle T_1, \cdots, T_n\rangle \to \mathcal{O}_K/(\pi^{n+1})[T_1, \cdots, T_n] \to A_n \to 0$$

for every integer $n \geq 0$. Assume that each $A_n$ is of finite presentation over $\mathcal{O}_K/(\pi^{n+1})$. Then $\mathfrak{a}/\mathfrak{a} \cap \pi^{n+1}\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$ is finitely generated ideal of $\mathcal{O}_K/(\pi^{n+1})[T_1, \cdots, T_n]$. By Lemma 3.2.5, there exists an $n \in \mathbb{N}$ such that $\mathfrak{a}/\mathfrak{a} \cap \pi^{n+1}\mathcal{O}_K\langle T_1, \cdots, T_n\rangle \subseteq \pi\mathfrak{a}$. It follows that $\mathfrak{a}/\pi\mathfrak{a}$ is a finitely generated $\mathcal{O}_K/(\pi)[T_1, \cdots, T_n]$. By Nakayama, $\mathfrak{a}$ is finitely generated over $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$. □

***Corollary 3.2.9.*** — *Let $A$ be a topological $\mathcal{O}_K$-algebra, which is separated and complete for the $\pi$-adic topology, and let $f_1, \cdots, f_r$ be elements generating the unit ideal. Then the following assertions are equivalent:*

1. *$A$ is of topologically finite type (resp. of topologically finite presentation, resp. admissible).*
2. *$A\langle f_i^{-1}\rangle$ is of topologically finite type (resp. of topologically of finite presentation, resp. admissible).*

*Proof.* — The statement follows from the corresponding assertion for usual schemes and the previous Lemma. □

Finally, we give a useful criterion for the flatness. For an $\mathcal{O}_K$-algebra of topologically finite type, we put $A_n = A/(\pi^{n+1})$ for all $n \geq 0$. Similar notations apply to $\mathcal{O}_K$-modules.

***Proposition 3.2.10.*** — *Let $\varphi : A \to B$ be a morphism of $\mathcal{O}_K$-algebras of topologically of finite type, and $M$ a finite $B$-module. Then $M$ is flat over $A$ if and only if $M_n$ is flat over $A_n$ for all integer $n \geq 0$.*

*Proof.* — The only if part is trivial since flatness is preserved by base change. Conversely, assume that $M_n$ is flat over $A_n$ for all $n \geq 0$. We have to show that for every finitely generated ideal $I \subseteq A$, the canonical map $g : I \otimes_A M \to M$ is injective. By Lemma 3.2.5(2), for any integer $m \geq 0$, there exists an integer $n \in \mathbb{N}$ such that $\pi^{n+1}A \cap I \subseteq \pi^m I$. Put $N = I/(I \cap \pi^{n+1}A)$ We have a commutative diagram

$$\begin{array}{ccccccc} (\pi^{n+1}A \cap I) \otimes_A M & \longrightarrow & I \otimes_A M & \longrightarrow & N \otimes_A M & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow h & & \\ & & M & \longrightarrow & A_n \otimes_A M & & \end{array}$$

such that the upper row is exact. By the flatness of $M$, $h$ injective. A simple diagram chasing shows that $\mathrm{Ker}(g) \subseteq \mathrm{im}((\pi^{n+1}A \cap I) \otimes_A M) \subseteq \pi^m(I \otimes_A M)$. This holds for any $m \geq 0$, and we conclude by the fact that $I \otimes_A M$ is separated for the $\pi$-adic topology.

□

***Corollary 3.2.11.*** — *Let $A$ be an $\mathcal{O}_K$-algebra of topologically finite type, and $f \in A$. Then $A \to A\langle f^{-1}\rangle$ is flat.*

***Definition 3.2.12.*** — Let $X$ be a formal $\mathcal{O}_K$-scheme. We say that $X$ is locally of topologically finite type (resp. locally of topologically finite presentation, resp. admissible) if there exists an affine covering $X = \bigcup_i X_i$ with $X_i = \mathrm{Spf}(A_i)$ such that each $A_i$ is of topologically finite type (resp. of topologically finite presentation, resp. admissible).

Let $X$ be a formal $\mathcal{O}_K$-scheme locally of topologically finite type. Then $X$ admits a maximal admissible closed formal subscheme $X_{\mathrm{ad}}$. Indeed, let $\mathcal{J} \subseteq \mathcal{O}_X$ be the subsheaf of $\pi$-power elements, i.e. for any open subset $U \subseteq X$, $\mathcal{J}(U)$ consists of elements $x \in \mathcal{O}_X(U)$ killed by a certain power of $\pi$. Let $X_{\mathrm{ad}} \subseteq X$ be the closed formal subscheme defined by ideal sheaf $\mathcal{J}$. Then $X_{\mathrm{ad}}$ is locally of topologically finite type and flat over $\mathcal{O}_K$. Hence, $X_{\mathrm{ad}}$ is admissible by Theorem 3.2.3.

**3.2.13. Rigid generic fiber.** — Let $\mathbf{FS}_{\mathcal{O}_K}$ be the category of $\mathcal{O}_K$-formal schemes locally of finite type, and $\mathbf{Rig}_K$ be the category of rigid spaces over $K$. Then one can construct a functor

$$\mathrm{rig} : \mathbf{FS}_{\mathcal{O}_K} \to \mathbf{Rig}_K$$

as follows. For every affine formal scheme $X = \mathrm{Spf}(A)$ in $\mathbf{FS}_{\mathcal{O}_K}$, $A$ is a quotient of $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$. We put $X^{\mathrm{rig}} := \mathrm{Sp}(A \otimes_{\mathcal{O}_K} K)$, which is an affinoid space over $K$. It is easy to check that such a construction is functorial for $A$, and is compatible with localization, namely for any $f \in A$, we have $\mathrm{Spf}(A\langle f^{-1}\rangle)^{\mathrm{rig}} = \mathrm{Sp}(A \otimes_{\mathcal{O}_K} K\langle f^{-1}\rangle)$. For a general object $X$ in $\mathbf{FS}_{\mathcal{O}_K}$, we choose an affine covering $X = \bigcup X_i$ with $X_i = \mathrm{Spf}(A_i)$, and define $X^{\mathrm{rig}}$ as the rigid space obtained by gluing $X_i^{\mathrm{rig}}$ along $(X_i \cap X_j)^{\mathrm{rig}}$. It is evident that $X^{\mathrm{rig}}$ is canonically identified with $aX_{\mathrm{ad}}^{\mathrm{rig}}$, and a morphism $\varphi : X \to Y$ in $\mathbf{FS}_{\mathcal{O}_K}$ induces naturally a morphism $X^{\mathrm{rig}} \to Y^{\mathrm{rig}}$ in $\mathbf{Rig}_K$. We usually call $X^{\mathrm{rig}}$ the rigid generic fiber of the formal scheme $X$.

***Remark 3.2.14.*** — Let $X$ be a $\mathcal{O}_K$-scheme locally of finite type, and $\mathcal{X}$ be its completion along the special fiber. Then in general, the rigid generic fiber of $\mathcal{X}$ is different from the rigid analytification of the generic fiber $X_K$.

## 3.3. Coherent sheaves on formal schemes

In this section, we will define coherent sheaves on a formal $\mathcal{O}_K$-scheme of topologically finite type. Since such a formal scheme is not locally noetherian, the notion of coherent sheaves is slightly more complicated than the noetherian case.

***Definition 3.3.1.*** — Let $A$ be a ring. An $A$-module $M$ is called *coherent*, if it is finitely generated and every finitely generated submodule of $M$ is of finite presentation, or equivalently the kernel of any morphism of $A$-modules $\varphi : A^r \to M$ is finitely generated. A ring $A$ is called coherent if $A$ is coherent as an module over itself, i.e. if every finitely generated ideal of $A$ is of finite presentation.

***Lemma 3.3.2.*** — *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of $A$-modules. If two of the $M_i$'s are coherent, then so is the third.*

*Proof.* — We prove only that if $M_1$ and $M_3$ are coherent, then so is $M_2$. Indeed, let $M_2'$ be a finitely generated submodule of $M_2$. Let $M_3'$ be the image of $M_2'$ in $M_3$ and $M_1' = M_1 \cap M_2'$. Then one has a new exact sequence

$$0 \to M_1' \to M_2' \to M_3' \to 0.$$

Then $M_3'$ is finitely generated, hence finitely presented by the coherence of $M_3$. One deduces that $M_1'$ is finitely generated, hence finitely presented by the coherence of $M_1$. Choose surjections $\varphi_1 : A^r \to M_1'$ and $\varphi_3 : A^s \to M_3'$ which factor through $M_2'$ so that one obtains a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^r & \longrightarrow & A^{r+s} & \longrightarrow & A^s & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\ 0 & \longrightarrow & M_1' & \longrightarrow & M_2' & \longrightarrow & M_3' & \longrightarrow & 0. \end{array}$$

Then $\mathrm{Ker}(\varphi_1)$ and $\mathrm{Ker}(\varphi_3)$ are both finitely generated, hence so is $\mathrm{Ker}(\varphi_2)$. This means that $M_2'$ is finitely presented. □

***Corollary 3.3.3.*** — *The subcategory of coherent $A$-modules is stable under kernels, cokernels and extensions.*

***Proposition 3.3.4.*** — *If $A$ is a coherent ring, then every $A$-module of finite presentation is coherent.*

*Proof.* — Let $M$ be an $A$-module of finite presentation, and $\varphi : A^r \to M$ be a morphism of $A$-modules. We have to show that $\mathrm{Ker}(\varphi)$ is finitely generated. Choose a surjection $\psi : A^s \to M$. There exists a morphism $\theta : A^r \to A^s$ such that the following diagram is commutative:

$$\begin{array}{ccc} A^r & \xrightarrow{\varphi} & M \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\psi} & \\ & A^s & \end{array}$$

Then one has an exact sequence

$$0 \to \mathrm{Ker}(\theta) \to \mathrm{Ker}(\varphi) \to \mathrm{Ker}(\psi) \to \mathrm{Coker}(\theta),$$

where $\mathrm{Ker}(\psi)$ is finitely generated since $M$ is of finite presentation, and $\mathrm{Coker}(\theta)$ is clearly of finite presentation. The assumption implies that $A^s$ is coherent by the previous Lemma. Hence, $\mathrm{Ker}(\theta)$ is finitely generated, so is $\mathrm{Ker}(\psi)$. □

***Remark 3.3.5.*** — Note that if $A$ is a coherent ring and if $I \subseteq A$ is a finitely generated ideal, then $A/I$ is also coherent.

***Proposition 3.3.6.*** — *Let $A$ be an $\mathcal{O}_K$-algebra of topologically finite presentation. Then $A$ is a coherent ring. In particular, an $A$-module is coherent if and only if it is of finite presentation.*

*Proof.* — One has to show that every finitely generated ideal $I$ of $A$ is of finite presentation. Consider first the case $A = \mathcal{O}_K\langle T_1, \cdots, T_n\rangle$. Then the ideal $I$ is flat over $\mathcal{O}_K$. The Proposition follows from Theorem 3.2.3.

Assume now $A$ is general. One writes $A$ as $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle/\mathfrak{a}$ for some finitely generated ideal $\mathfrak{a}$. In particular, $A$ is a coherent $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$-module, hence $I \subseteq A$ is a finitely presented $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$-module. By assumption, one has an exact sequence $0 \to J \to A^r \to I \to 0$. It follows that $J$ is $\mathcal{O}_K\langle T_1, \cdots, T_n\rangle$-module of finite type.

□

**3.3.7.** — Consider a formal $\mathcal{O}_K$-scheme $X = \mathrm{Spf}(A)$ such that $A$ is of topologically finite type. Let $M$ be an $A$-module. One can associate to $M$ an $\mathcal{O}_X$-module $M^\Delta$ as follows. For any $f \in A$, we define

$$M^\Delta(D_f) = \varprojlim_n M \otimes_A A/\pi^n[f^{-1}].$$

Since $\varprojlim_n$ is left exact, we see that the presheaf $M^\Delta$ is indeed a sheaf on $X$. In other words, if we view $X$ as the inductive limit of schemes $X_n = \mathrm{Spec}(A_n)$ with $A_n = A/(\pi^{n+1})$, then $\mathcal{F}$ is the projective limit of the quasi-coherent sheaf $\mathcal{F}_n$ associated to the $A_n$-module $M \otimes_A A_n$.

***Lemma 3.3.8.*** — *If $M$ is finitely generated over $A$, then $M^\Delta(D_f) = M \otimes_A A\langle f^{-1}\rangle$ for any $f \in A$.*

*Proof.* — By definition, $M^\Delta(D_f)$ is the $\pi$-adic completion of $M \otimes_A A[f^{-1}]$. On the other hand, $M \otimes_A A\langle f^{-1}\rangle$ is $\pi$-adically separated and complete. Since $M \otimes_A A[f^{-1}]$ is dense in $M \otimes_A A\langle f^{-1}\rangle$, we see that $M^\Delta(D_f) = M \otimes_A A\langle f^{-1}\rangle$. □

***Proposition 3.3.9.*** — 1. *The functor $M \mapsto M^\Delta$ is exact from the category of finite $A$-modules to the category of $\mathcal{O}_X$-modules is fully faithful and exact.*
2. *Assume that $X$ is of topologically finite presentation. Then, on the category of coherent $A$-modules, the functor $M \mapsto M^\Delta$ commutes with kernels, images, cokernels and tensor product. Moreover, a sequence of coherent $A$-modules*
$$0 \to M' \to M \to M'' \to 0$$
*is exact if and only if the associated sequence of $\mathcal{O}_X$-modules is exact.*

*Proof.* — Exercise.

□

***Definition 3.3.10.*** — Let $X$ be a formal $\mathcal{O}_K$-scheme locally of topologically finite presentation. We say an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent if there exists an affine covering $X = \bigcup_{i\in I} U_i$ such that $\mathcal{F}|_{U_i}$ is the $\mathcal{O}_{U_i}$-module associated to a coherent $\mathcal{O}_X(U_i)$-module.

***Proposition 3.3.11.*** — *Let $X = \mathrm{Spf}(A)$ be an affine formal $\mathcal{O}_K$-scheme of topologically finite presentation, and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module if and only if $\mathcal{F}$ is associated to a coherent $A$-module.*

*Proof.* — Suppose that $(D_{f_i})_{i\in I}$ be a standard finite affine covering such that, for each $i \in I$, $\mathcal{F}|_{D_{f_i}}$ is associated to a coherent $A_i = A\langle f_i^{-1}\rangle$-module $M_i$. Put $M_{i,j} := \Gamma(D_{f_i} \cap D_{f_j}, \mathcal{F}) = M_i \otimes_{A_i} A\langle (f_i f_j)^{-1}\rangle$ for any $i, j \in I$.

Consider the sheaf $\mathcal{F}_n := \mathcal{F} \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi^{n+1}$ scheme $X_n = \mathrm{Spec}(A/\pi^{n+1}A)$ for $n \geq 0$. Then $\mathcal{F}_n|_{D_{f_i}}$ is a quasi-coherent coherent sheaf associated to the $A_i/(\pi^{n+1}) = A/(\pi^{n+1})[f_i^{-1}]$-module $M_{i,n} := M_i/\pi^{n+1}M_i$. By the theory of quasi-coherent sheaves on schemes, $\mathcal{F}_n$ is a quasi-coherent $\mathcal{O}_{X_n}$-module associated to $M_n := \Gamma(X_n, \mathcal{F}_n)$, so that $M_{i,n} = M_n \otimes_{A/(\pi^{n+1})} A/(\pi^{n+1})[f_i^{-1}]$. We have $M_n = M_{n+1}/\pi^{n+1}M_{n+1}$. Put $M = \varprojlim_n M_n$. Then we claim that $M \cong \Gamma(X, \mathcal{F})$. Indeed, we have, for each $n \in \mathbb{N}$, an exact sequence
$$0 \to M_n \to \prod_{i\in I} M_{i,n} \to \prod_{i,j} M_{i,j,n},$$
where $M_{i,j,n} := \Gamma(D_{f_i f_j}, \mathcal{F}_n)$. Since $\varprojlim_n$ is left exact, one gets an exact sequence
$$0 \to M \to \prod_{i\in I} M_i \to \prod_{i,j} M_{i,j}.$$
This means that $M = \Gamma(X, \mathcal{F})$.

We now prove that $M$ is a coherent $A$-module, or equivalently $M$ is an $A$-module of finite presentation by Proposition 3.3.6. Since each $M_{i,n}$ is finitely presented over $A_i/\pi^{n+1}A_i$, we see that $M_n = M/\pi^{n+1}M$ is finitely presented over $A/\pi^{n+1}A$ by Lemma 3.3.12 below.

On the other hand, note that $M$ is separated for $\pi$-adic topology. By Lemma 1.2.6, it follows that $M$ is a finitely generated $A$-module. There exists thus a short exact sequence

$$0 \to N \to A^r \xrightarrow{\theta} M \to 0.$$

We have to show that $N$ is finitely generated. Let $M^{\mathrm{tor}} \subseteq M$ the submodule consisting of $\pi$-power torsion elements. Then $M/M^{\mathrm{tor}}$ is finitely generated $A$-module which is flat over $\mathcal{O}_K$. By Theorem 3.2.3, $M/M^{\mathrm{tor}}$ is finitely presented. Hence, $M^{\mathrm{tor}}$ is finitely generated, and there exists an integer $m \in \mathbb{N}$ such that $\pi^m \cdot M^{\mathrm{tor}} = 0$. Tensoring the exact sequence above with $\mathcal{O}_K/\pi^n$, we get

$$0 \to \mathrm{Tor}_1^A(A/(\pi^m), M) \to N/\pi^m N \xrightarrow{\theta_m} A/(\pi^m)^r \to M/\pi^m M \to 0$$

Note that $\mathrm{Tor}_1^A(A/(\pi^m), M) = M^{\mathrm{tor}}$, hence it is finitely generated. But Lemma 3.3.12 implies that $M_{m-1} = M/\pi^m M$ is an $A/\pi^m A$-module of finite presentation. Hence, $\mathrm{Ker}(\theta_m)$ is finitely generated. It follows that $N/\pi^m N$ is finitely generated. Note that $N$ is separated for the $\pi$-adic topology, since it is a submodule of the separated $A$-module $A^r$. We conclude by Lemma 1.2.6 that $N$ is finitely generated over $A$. This finishes the proof. □

***Lemma 3.3.12.*** — *Let $R$ be a ring, and $M$ be an $R$-module. Let $f_1, \cdots, f_s \in R$ be elements that generate the unit ideal. Suppose that $M[f_i^{-1}] := M \otimes_R R[f_i^{-1}]$ is a finitely presented $R[f_i^{-1}]$-module for each $i$. Then $M$ is a finite presented $R$-module.*

*Proof.* — We prove first that $M$ is finitely generated over $R$. Suppose $M[f_i^{-1}]$ is generated by $\{x_{i,j} : j = 1, \cdots, n_i\}$ for each $i$. Then for a sufficiently integer $N$, we have $f_i^N x_{i,j} \in M$ for all $i$ and $j$. Then show that $\{f_i^N x_{i,j} : i = 1, \cdots, s, j = 1, \cdots, n_i\}$ generate $M$.

Choose now a surjection $\varphi : R^n \to M$, and denote by $N$ its kernel. Since $M[f_i^{-1}]$ is an $R[f_i^{-1}]$-module of finite presentation, $N[f_i^{-1}]$ finitely generated over $R[f_i^{-1}]$. The discussion above applied to $N$ shows that $N$ is finitely generated over $R$. □

***Corollary 3.3.13.*** — *Let $X$ be a formal $\mathcal{O}_K$-scheme locally of topologically finite presentation, and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if for any affine open formal subscheme $U \subseteq X$, $\mathcal{F}|_U$ is associated to a coherent $\mathcal{O}_X(U)$-module.*

## 3.4. Admissible formal blowing-up

Let $X$ be a formal $\mathcal{O}_K$-scheme locally of topologically finite presentation. A coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ is called *open*, if, locally on $X$, $\mathcal{I}$ contains some power of $\pi$.

***Definition 3.4.1.*** — Let $X$ be a formal $\mathcal{O}_K$-scheme locally of topologically finite presentation, and $\mathcal{I} \subseteq \mathcal{O}_X$ be an open coherent ideal sheaf. We define the blowing-up of $X$ along $\mathcal{I}$ as

$$\mathrm{Bl}_{\mathcal{I}}(X) = \varinjlim_n \mathrm{Proj}(\bigoplus_{d=0}^{\infty} \mathcal{I}^d \otimes_{\mathcal{O}_X} \mathcal{O}_X/(\pi^n)).$$

Such a blowing-up is called an *admissible formal blowing-up* of $X$.

***Proposition 3.4.2.*** — *Let $f : X' \to X$ be a flat morphism formal $\mathcal{O}_K$-scheme locally of topologically finite presentation, and $\mathcal{I} \subseteq \mathcal{O}_X$ be an open coherent ideal sheaf. Then we have a canonical isomorphism*

$$\mathrm{Bl}_{\mathcal{I}'}(X') \cong \mathrm{Bl}_{\mathcal{I}}(X) \times_X X',$$

*where $\mathcal{I}' = f^{-1}(\mathcal{I})\mathcal{O}_{X'}$.*

*Proof.* — We may assume that $X = \mathrm{Spf}(A)$ and $X' = \mathrm{Spf}(A')$ are both affine, and let $I = \mathcal{I}(X)$. By definition, we have

$$\mathrm{Bl}_{\mathcal{I}}(X) \times_X X' = \varinjlim_n \mathrm{Proj}\,\big(\bigoplus_{d\geq 0} I^d \otimes_A A' \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi^n\big).$$

However, since $A'$ is flat over $A$, $I^d \otimes_A A'$ is identified with its image $IA' = \mathcal{I}'(X')$. □

Consider the case that $X = \mathrm{Spf}(A)$ is an affine formal $\mathcal{O}_K$-scheme of topologically finite presentation, and $\mathcal{I} = I^{\Delta}$ is associated to an open coherent ideal $I$. Then one has

$$\mathrm{Bl}_{\mathcal{I}}(X) = \varinjlim_n \mathrm{Proj}\,\big(\bigoplus_{d=0}^{\infty} I^d \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi^n\big) = \varinjlim_n \mathrm{Proj}\,\big(\bigoplus_{d=0}^{\infty} I^d\big) \otimes_{\mathcal{O}_K} \mathcal{O}_K/(\pi^n),$$

which is nothing but the $\pi$-adic completion of the blowing-up of $\mathrm{Spec}(A)$ along the ideal $I$.

***Example 3.4.3.*** — Let $X = \mathrm{Spf}(A)$ with $A = \mathcal{O}_K\langle x, y\rangle/(xy - \pi^2)$. We consider the admissible formal blowing-up of $X$ along the open ideal $I = (x, y, \pi)$. The corresponding algebraic blowing-up is given by

$$\mathrm{Bl}_I(\mathrm{Spec}(A)) = \mathrm{Proj}(\bigoplus_{d=0}^{\infty} I^d),$$

which can be viewed as a closed subscheme of $\mathbb{P}^2_A = \mathrm{Proj}(A[T_0, T_1, T_2])$ by sending $T_0 \mapsto \pi$, $T_1 \mapsto x$, and $T_2 \mapsto y$. Here, $x, y, \pi \in I$ are considered as homogeneous elements of $\bigoplus_{d=0}^{\infty} I^d$ of degree 1. We have an affine covering $X = U_0 \cup U_1 \cup U_2$, where $U_i$ is the inverse image of the open locus of $\mathbb{P}^2_A$ where $T_i \neq 0$. Explicitly, one has

$$\begin{aligned} U_0 &= \mathrm{Spec}(A[t_1, t_2]/(t_1t_2 - 1, x - \pi t_1, y - \pi t_2)),\\ U_1 &= \mathrm{Spec}(A[t_0, t_2]/(y - xt_2, \pi - xt_0, t_2 - t_0^2)),\\ U_2 &= \mathrm{Spec}(A[t_0, t_1]/(x - yt_1, \pi - yt_0, t_1 - t_0^2)). \end{aligned}$$

Therefore, the admissible formal blowing-up has a covering

$$\mathrm{Bl}_I(X) = \hat{U}_0 \cup \hat{U}_1 \cup \hat{U}_2,$$

where $\hat{U}_i$ is the $\pi$-adic completions of $U_i$. Geometrically, the formal blowing-up $\mathrm{Bl}_I(X)$ has the same topological space as $\mathrm{Bl}_I(\mathrm{Spec}(A)) \otimes_{\mathcal{O}_K} k$, where $k$ is the residue field of $\mathcal{O}_K$.

Explicitly, one has

$$U_0 \otimes_{\mathcal{O}_K} k = \mathrm{Spec}(k[t_1, t_2]/(1 - t_1 t_2)),$$
$$U_1 \otimes_{\mathcal{O}_K} k = \mathrm{Spec}(k[t_0, x]/(t_0 x)),$$
$$U_2 \otimes_{\mathcal{O}_K} k = \mathrm{Spec}(k[s, y]/(sy)).$$

***Proposition 3.4.4.*** — *Let $X = \mathrm{Spf}(A)$ be an affine admissible formal $\mathcal{O}_K$-scheme, and $\mathcal{I} = I^{\Delta}$ be an open coherent ideal of $\mathcal{O}_X$. Assume $I = (f_0, \cdots, f_r) \subseteq A$. Then*

1. *The ideal $\mathcal{I}\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}}(X)}$ is invertible.*
2. *Let $U_i \subseteq \mathrm{Bl}_{\mathcal{I}}(X)$ be the locus where $\mathcal{I}\mathcal{O}_{\mathrm{Bl}_{\mathcal{I}}(X)}$ is generated by $f_i$. Then $(U_i)_{i=0,\cdots,r}$ is an affine covering of $X$, and $U_i = \mathrm{Spf}(A_i)$, where $A_i$ is the maximal $f_i$-torsion free quotient of*

$$\hat{C}_i = A\langle \frac{f_j}{f_i} : j \neq i \rangle,$$

*i.e. $A_i = \hat{C}_i/(\pi - torsion)$.*

For the proof of this Proposition, we will need the following Lemma due to Gabber:

***Lemma 3.4.5*** **([Bo08] §2.6 Lemma 2).** — *Let $A$ be an $\mathcal{O}_K$-algebra of topologically finite type and $C$ be an $A$-algebra of finite type. Then the $\pi$-adic completion $\hat{C}$ of $C$ is flat over $C$.*

*Proof.* — We give a sketch of the proof of Gabber's Lemma. The strategy is to show first the following

***Sublemma 3.4.5.1.*** — *Let $M$ be a module over some ring $A$, and let $\pi$ be an element, which is not a zero divisor in $A$. Then the following are equivalent:*

(i) *$M$ is flat over $A$.*
(ii) *$M$ has no $\pi$-torsion elements, $M/\pi M$ is flat over $A/\pi A$, and $M[\frac{1}{\pi}]$ is flat over $A[\frac{1}{\pi}]$.*

Granting this sublemma, the proof of Gabber's Lemma proceeds as follows. By the sublemma, since $\hat{C}/\pi\hat{C} \cong C/\pi C$, it suffices to show that $\hat{C}[1/\pi]$ is flat over $C[1/\pi]$.

One consider first the case $C = A[T_1, \cdots, T_n]$ for some integer $n$. We proceed similarly as for the flatness of the rigid analytification over the an algebraic $K$-scheme. It suffices to show that for every maximal ideal $\mathfrak{m}$ of $\hat{C}[1/\pi] = A_K\langle T_1, \cdots, T_n\rangle$ and its restriction $\mathfrak{n}$ to $C[1/\pi]$, the canonical map

$$A_K[T_1, \cdots, T_n]_{\mathfrak{n}} \to A_K\langle T_1, \cdots, T_n\rangle_{\mathfrak{m}}$$

is flat. Since $A_K$ is a noetherian ring, it is enough to show that the above morphism of local rings induces an isomorphism between their completions. It thus suffices to show that for every integer $n \geq 1$, the map $A_K[T_1, \cdots, T_n] \to A\langle T_1, \cdots, T_n\rangle$ induces isomorphisms

$$A_K[T_1, \cdots, T_n]/\mathfrak{n}^n \cong A\langle T_1, \cdots, T_n\rangle/\mathfrak{m}^n$$

for all integer $n \geq 0$.

Then the rest of the proof proceeds exactly as Corollary 2.3.2 □

*Proof of 3.4.4.* — Put $S = \bigoplus_{d\geq 0} I^d$. Then the algebraic blowing-up of $\tilde{X} = \mathrm{Spec}(A)$ along $I$ is given by

$$\tilde{X}' = \mathrm{Proj}\,(S).$$

Since $I$ is generated by $(f_0, \cdots, f_d)$, $S$ is a quotient of $A[T_0, \cdots, T_r]$ with each $T_i$ homogeneous of degree 1, or equivalently $\tilde{X}'$ is a closed subscheme of $\mathbb{P}^r_A$. Then $\tilde{X}'$ admits an affine covering $\tilde{X}' = \bigcup_{i=0}^r D(f_i)$, where $D(f_i)$ is the inverse image of the open subset of $\mathbb{P}^r_A$ where the $i$-th homogeneous coordinate does not vanish. Then $D(f_i) = \mathrm{Spec}(S_{(f_i)})$, where $S_{(f_i)}$ is the degree 0 part of the usual localization of $S$ by $f_i$. The formal blowing-up is then covering by the $\pi$-adic completions $\hat{D}(f_i) = \mathrm{Spf}(\hat{S}_{(f_i)})$.

Explicitly, $S_{(f_i)}$ is the quotient of

$$C_i = A[\frac{f_j}{f_i} : j \neq i] = A[t_j : j \neq i]/(f_j - t_j f_i)$$

by its $f_i$-torsion subgroup, i.e. $S_{(f_i)} \cong C_i/(f_i - \mathrm{torsion})_{C_i}$. Since some power of $\pi$ lies in the ideal $I$ and $f_i$ generates $I$ in $S_{(f_i)}$, the $f_i$-torsion submodule of $C_i$ is contained in the $\pi$-torsion submodule. But $X$ is supposed to be admissible, there is no $\pi$-torsion elements in $A$, and hence in $S$ and $S_{(f_i)}$. Therefore, we have

$$(f_i - \mathrm{torsion})_{C_i} = (\pi - \mathrm{torsion})_{C_i},$$

and $S_{(f_i)}$ is the quotient $C_i/(\pi - \mathrm{torsion})_{C_i}$.

Now to get the results for the formal admissible blowing-up $\mathrm{Bl}_{\mathcal{I}}(X)$, we pass to $\pi$-adic completions. Let

$$\hat{C}_i = A\langle \frac{f_j}{f_i} : j \neq i\rangle$$

be the $\pi$-adic completion of $C_i$. By Gabber's Lemma, $\hat{C}_i$ is flat over $C_i$. Therefore, one has

$$(\pi - \mathrm{torsion})_{\hat{C}_i} = (\pi - \mathrm{torsion})_{C_i} \otimes_{C_i} \hat{C}_i$$

and $\hat{S}_{(f_i)}$ is thus the quotient of $\hat{C}_i$ by its $\pi$-adic torsion submodules.

□

***Corollary 3.4.6.*** — *If $X$ is an admissible formal $\mathcal{O}_K$-scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent open ideal, then $\mathrm{Bl}_{\mathcal{I}}(X)$ is also an admissible $\mathcal{O}_K$-scheme.*

Recall that the algebraic blowing-up of a scheme $X$ along a coherent ideal $\mathcal{I}$ can be characterized by a certain universal property. Similar property holds for the admissible formal blowing-up.

***Proposition 3.4.7.*** — *Let $X$ be an admissible formal $\mathcal{O}_K$-scheme, and $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent open ideal. Then the formal admissible blowing-up $\mathrm{Bl}_{\mathcal{I}}(X)$ satisfies the following universal property: any morphism of admissible formal $\mathcal{O}_K$-schemes $\varphi : Y \to X$ such that $\varphi^{-1}(\mathcal{I})\mathcal{O}_Y$ is invertible factors through uniquely through $\mathrm{Bl}_{\mathcal{I}}(X) \to X$.*

*Proof.* — We may assume that $X = \mathrm{Spf}(A)$ is affine, and $\mathcal{I}$ is the ideal sheaf associated to a coherent ideal $I = (f_0, \cdots, f_r) \subseteq A$. Let $\varphi : Y \to X$ be a morphism of formal $\mathcal{O}_K$-schemes such that $\varphi^{-1}(\mathcal{I})$ is invertible. We may assume that $Y = \mathrm{Spf}(B)$ is affine and that $IB$ is generated by some $\varphi^*(f_i)$ and $\varphi(f_i)$ is not a zero divisor in $B$. Therefore, $\varphi^*(f_j)$ is uniquely divisible by $\varphi^*(f_i)$ in $B$ for any $j$, and there exists a unique morphism

$$A\langle \frac{f_j}{f_i} : 0 \leq j \leq r\rangle/(\pi - torsion) \to B$$

extending $\varphi^*$.

□

***Lemma 3.4.8.*** — *Let $X$ be an admissible formal $\mathcal{O}_K$-scheme, and $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$ be two coherent open ideals. Let $X_{\mathcal{I}} = \mathrm{Bl}_{\mathcal{I}}(X)$, $\mathcal{J}' = \mathcal{J}\mathcal{O}_{X_{\mathcal{I}}}$, and $(X_{\mathcal{I}})_{\mathcal{J}'} = \mathrm{Bl}_{\mathcal{J}'}(X_{\mathcal{I}})$. Then the composition*

$$(X_{\mathcal{I}})_{\mathcal{J}'} \to X_{\mathcal{I}} \to X$$

*is identified with the formal admissible blowing-up of $X$ at $\mathcal{I}\mathcal{J}$.*

*Proof.* — This Lemma follows from the universal property of the formal admissible blowing-up. Let $X_{\mathcal{I}\mathcal{J}}$ denote the formal admissible blowing-up of $X$ at $\mathcal{I}\mathcal{J}$. We may assume that $X = \mathrm{Spf}(A)$ is affine, and $\mathcal{I}$ is associated to $I = (f_0, \cdots, f_r)$ and $\mathcal{J}$ associated to $J = (g_0, \cdots, g_s)$.

First, we check that $I\mathcal{O}_{X_{\mathcal{I}\mathcal{J}}}$ is invertible. For $0 \leq i \leq r$ and $0 \leq j \leq s$, let $U_{i,j}$ be the open subset of $X_{\mathcal{I}\mathcal{J}}$ such that $IJ\mathcal{O}_{X_{\mathcal{I}\mathcal{J}}}$ is generated by $f_i g_j$. Then $U_{i,j}$ form an affine covering of $X_{\mathcal{I}\mathcal{J}}$, and $I\mathcal{O}_{X_{\mathcal{I}\mathcal{J}}}$ is generated by $f_i$ over $U_{i,j}$. This shows that $I\mathcal{O}_{\mathcal{I}\mathcal{J}}$ is invertible. By the universal property of the formal admissible blowing-up, the projection [$X_{\mathcal{I}\mathcal{J}} \to X$ factors through uniquely as $X_{\mathcal{I}\mathcal{J}} \to X_{\mathcal{I}} \to X$. Similarly, one sees that $\mathcal{J}'\mathcal{O}_{\mathcal{I}\mathcal{J}} = J\mathcal{O}_{\mathcal{I}\mathcal{J}}$ is invertible. Therefore, $X_{\mathcal{I}\mathcal{J}} \to X_{\mathcal{I}}$ factors through uniquely as $X_{\mathcal{I}\mathcal{J}} \to (X_{\mathcal{I}})_{\mathcal{J}'} \to X_{\mathcal{I}}$.

Conversely, one can check that $IJ\mathcal{O}_{(X_{\mathcal{I}})_{\mathcal{J}}}$ is invertible. Therefore, $(X_{\mathcal{I}})_{\mathcal{J}'} \to X$ factors uniquely through $(X_{\mathcal{I}})_{\mathcal{J}} \to X_{\mathcal{I}\mathcal{J}}$. By the uniqueness of these morphisms, this morphism is the inverse of $X_{\mathcal{I}\mathcal{J}} \to (X_{\mathcal{I}})_{\mathcal{J}'}$ constructed above.

□

***Proposition 3.4.9.*** — *Let $X' \to X$ and $X'' \to X'$ be two formal admissible blowing-ups of admissible formal $\mathcal{O}_K$-schemes. Then the composite $X'' \to X' \to X$ is also a formal admissible blowing-up.*

*Proof.* — Let $\mathcal{I} \subseteq \mathcal{O}_X$ and $\mathcal{I}' \subseteq \mathcal{O}_{X'}$ be coherent open ideals such that $X' = \mathrm{Bl}_{\mathcal{I}}(X)$ and $X'' = \mathrm{Bl}_{\mathcal{I}'}(X')$.

We only do the proof when $X = \mathrm{Spf}(A)$ is affine. For the general case, see [**Bo08**, §2.6 Prop. 11]. Write $\mathcal{I} = I^{\Delta}$ for some coherent ideal $I \subseteq A$. Let $\tilde{X}'$ denote the algebraic blowing-up of $\tilde{X} = \mathrm{Spec}(A)$ along the ideal $I$. Then $X'$ is identified with the $\pi$-adic completion of $\tilde{X}'$. Since $\mathcal{I}'$ is open, there exists an integer $n \geq 1$ such that $\pi^n \in \mathcal{I}'$. But note that $\mathcal{O}_{\tilde{X}'}/\pi^n \cong \mathcal{O}_{X'}/\pi^n$, there exists thus a coherent open ideal sheaf $\tilde{\mathcal{I}}' \subseteq \mathcal{O}_{\tilde{X}'}$ such that $\mathcal{I}' = \tilde{\mathcal{I}}'\mathcal{O}_{X'}$.

Let $\tilde{X}''$ denote the algebraic blowing-up of $\tilde{X}'$ along the ideal $\tilde{\mathcal{I}}'$. Then $X''$ is the $\pi$-adic completion of $\tilde{X}''$. To finish the proof, it suffices to show that the composed map

$$\tilde{X}'' \to \tilde{X}' \to \tilde{X}$$

is the algebraic blowing-up of $\tilde{X}$ along some coherent open ideal $\tilde{J} \subseteq A$. Since then $X'' \to X$ will be the formal admissible blowing-up of $X$ at the ideal $\tilde{\mathcal{J}}\mathcal{O}_X$. Denote $\mathcal{L} = \tilde{\mathcal{I}}\mathcal{O}_{\tilde{X}'}$, which is an invertible sheaf on $\tilde{X}'$, relative ample with respect to the projective morphism $\tilde{X}' \to \tilde{X}$. Then there exists an integer $n_0$ such that for all $n \geq n_0$, $\tilde{\mathcal{I}}' \otimes \mathcal{L}^n$ is generated by global sections.

Consider the blowing-up $\tilde{\varphi} : \tilde{X}' \to X$, which is an isomorphism over the complement of $V(I)$ in $X$. Write $I = (f_0, \cdots, f_n)$. Let $\tilde{X}'_i = \operatorname{Spec}(\tilde{A}'_i)$ be the open subset of $\tilde{X}'$ where $I\mathcal{O}_{\tilde{X}'}$ is generated by $f_i$. Then the natural map $A \to \tilde{A}'_i$ induces an isomorphism $A_{f_i} \xrightarrow{\sim} A'_{i,f_i}$. Therefore, any section of $A_{i,f_i}$ is induced by elements of $A$ up to multiplying a sufficiently large power of $f_i$. Let $n_0 \geq 0$ be as above, and choose $n \geq n_0$. Then there exists a finite set of generators of $\tilde{\mathcal{I}}' \otimes \mathcal{L}^n$, which come from $A$. Define an ideal $\mathcal{I}'' \subseteq \mathcal{O}_{\tilde{X}}$ by the following exact sequence

$$0 \to \tilde{\mathcal{I}}'' \to \mathcal{O}_{\tilde{X}} \to \varphi_*\mathcal{O}_{\tilde{X}'}/\tilde{\varphi}_*(\tilde{\mathcal{I}}' \otimes \mathcal{L}^n).$$

Then $\tilde{\mathcal{I}}''$ is quasi-coherent, and thus associated to an ideal $\tilde{I}''$ By construction, the inverse image $\varphi^{-1}(I'')$ generate $\tilde{\mathcal{I}}' \otimes \mathcal{L}^n$. Since $\tilde{\mathcal{I}}' \otimes \mathcal{L}^n$ is generated by a finite set of generators, there exists a finitely generated ideal $\tilde{J}'' \subseteq \tilde{I}''$ such that $\tilde{J}''\mathcal{O}_{\tilde{X}'} = \tilde{\mathcal{I}}' \otimes \mathcal{L}^n$. Since $A$ is coherent, $\tilde{J}''$ is coherent. By Lemma 3.4.8, $X'' \to X$ is the blowing-up of $X$ along the coherent ideal $\tilde{J} = \tilde{J}''\tilde{I}$.

□

***Proposition 3.4.10.*** — *Let $f : X \to Y$ be a morphism of admissible formal $\mathcal{O}_K$-schemes, and $Y' \to Y$ be a formal admissible blowing-up. Then there exists a formal admissible blowing-up $X' \to X$ and a morphism $f' : X' \to Y'$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

*Proof.* — Assume that $Y' = \mathrm{Bl}_{\mathcal{J}}(Y)$ for some coherent open ideal $\mathcal{J} \subseteq \mathcal{O}_Y$. Put $\mathcal{I} = \mathcal{J}\mathcal{O}_X$, and $X' = \mathrm{Bl}_{\mathcal{I}}(X)$. Then the composition $X' \to X \xrightarrow{f} Y$ factors through uniquely by $Y' \to Y$ according to the universal property of the formal admissible blowing-up.

□

## 3.5. Rigid geometry via formal schemes

Recall first the definition of the localization of a category.

***Definition 3.5.1.*** — Let $\mathfrak{C}$ be a category, and $S$ be a class of morphisms in $\mathfrak{C}$. Then a localization of $\mathfrak{C}$ at $S$ is a category $\mathfrak{C}_S$ together with a functor $Q : \mathfrak{C} \to \mathfrak{C}_S$ such that

1. $Q(s)$ is an isomorphism for all $s \in S$, and
2. if $F : \mathfrak{C} \to \mathfrak{D}$ is any functor such that $F(s)$ is an isomorphism for all $s \in S$, then there exists a unique functor $G : \mathfrak{C}_S \to \mathfrak{D}$ such that $F = G \circ Q$.

***Lemma 3.5.2.*** — *Assume that the class of morphisms $S$ satisfies the following properties:*

1. $\mathrm{id}_X \in S$ *for every object $X$ of $\mathfrak{C}$.*
2. *$S$ is stable under composition, i.e. if $s : X \to Y$ and $t : Y \to Z$ belong to $S$, then so does $t \circ s$.*
3. *If $s : X' \to X$ and $t : X'' \to X$ are two morphisms in $S$ with the same target, then there exists a third morphism $u : \tilde{X} \to X$ in $S$ dominant both $s, t$, i.e. $u$ factors through both $s$ and $t$.*
4. *If $f : X \to Y$ is a morphism in $\mathfrak{C}$ and $t : Y' \to Y$ is a morphism in $S$, then there exists a morphism $s : X' \to X$ in $S$ and a morphism $f' : X' \to Y'$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

*Then the localization of $\mathfrak{C}$ with respect to $S$ exists.*

*Proof.* — We consider the category $\mathfrak{C}'_S$ defined as follows:

1. the set of objects of $\mathfrak{C}'_S$ is the same as $\mathfrak{C}$, and
2. for any two objects $X, Y \in \mathfrak{C}'_S$, we define

$$\mathrm{Hom}_{\mathfrak{C}'_S}(X, Y) = \varinjlim_{(X' \xrightarrow{s} X) \in S_X} \mathrm{Hom}_{\mathfrak{C}}(X', Y),$$

where $S_X$ is the set of morphisms $s : X' \to X$ in $S$ with target $X$, and the transition map

$$\mathrm{Hom}_{\mathfrak{C}}(X', Y) \to \mathrm{Hom}_{\mathfrak{C}}(X'', Y)$$

is defined by the composition with $f$, whenever $t : X'' \to X$ factors as $X'' \xrightarrow{f} X' \xrightarrow{s} X$.

Then one checks easily that $\mathfrak{C}'_S$ satisfies the universal properties for the localization of $\mathfrak{C}$ with respect to $S$. Therefore, it is exactly the localization of $\mathfrak{C}$ with respect to $S$. □

We consider now the induced morphism on rigid generic fibers by a formal admissible blow-up.

***Proposition 3.5.3.*** — *Let $X$ be an admissible formal $\mathcal{O}_K$-scheme, and $X_{\mathcal{A}} \to X$ be a formal admissible blowing-up of $X$ at a coherent open ideal $\mathcal{I}$. Then the natural projection $X_{\mathcal{I}} \to X$ induces an isomorphism of rigid spaces*

$$X_{\mathcal{I}}^{\mathrm{rig}} \cong X^{\mathrm{rig}}.$$

*In other words, the functor* rig *from the category of admissible formal $\mathcal{O}_K$-schemes to the category of rigid $K$-spaces factors through the localization of the category of formal $\mathcal{O}_K$-schemes with respect to formal admissible blowing-ups.*

*Proof.* — We only prove here when $X = \mathrm{Spf}(A)$ is affine, and the general case will follow by gluing. Assume that $\mathcal{I} = (f_0, \cdots, f_r)$, and let $X_{\mathcal{I},i}$ be the open subset of $X_{\mathcal{I}}$ where $\mathcal{I}\mathcal{O}_{X_{\mathcal{I}}}$ is generated by $f_i$ for each $0 \leq i \leq r$. Then one has $X_{\mathcal{I},i} = \mathrm{Spf}(A_i)$ with

$$A_i = A\langle \frac{f_j}{f_i} : 0 \leq j \leq r\rangle/(\pi - torsion).$$

Therefore, $X_{\mathcal{I},i}^{\mathrm{rig}} = \mathrm{Sp}(A_K\langle \frac{f_j}{f_i}\rangle : 0 \leq j \leq i)$ is identified with the affinoid subdomain

$$\{x \in X^{\mathrm{rig}} \mid |f_j(x)| \leq |f_i(x)|\}$$

of $X^{\mathrm{rig}}$, and $X_{\mathcal{I}}^{\mathrm{rig}} = \bigcup_i X_{\mathcal{I},i}$ is identified with the rational covering of $X^{\mathrm{rig}}$ defined by $(f_0, \cdots, f_r)$. □

If we restrict to certain subcategory of admissible formal $\mathcal{O}_K$-schemes, we will actually get an equivalence of categories. For this, let us recall some topological conditions.

***Definition 3.5.4.*** — A topological space $X$ is called quasi-paracompact if there exists an open covering $X = \bigcup_{\in I} X_i$ such that:

1. $X_i$ is quasi-compact for each $i \in I$,
2. The covering is locally finite; i.e., for any index $i \in I$, the intersection $X_i \cap X_j$ is non-empty for at most finitely many indices $j \in I$.

Recall that a rigid $K$-space $X$ is called quasi-separated, if the diagonal morphism $X \to X \times_K X$ is quasi-compact, i.e. the intersection of two quasi-compact admissible open subset of $X$ is again quasi-compact.

***Theorem 3.5.5* (Raynaud).** — *The functor* rig *induces an equivalence of categories between*

1. *the localizaiton of the category of quasi-paracompact admissible formal $\mathcal{O}_K$-schemes with respect to the class of formal admissible blowing-ups, and*
2. *the category of quasi-paracompact and quasi-separated rigid $K$-spaces.*

Before giving the proof, we first note that all admissible formal $\mathcal{O}_K$-scheme is automatically quasi-separated, hence so is its associated rigid space. However, there exist indeed examples of non-quasi-separated rigid spaces, which do not come from any quasi-paracompact admissible formal $\mathcal{O}_K$-scheme. For instance, the rigid space obtained by glueing two copies of the unit disk $\mathbb{B}_K^1 = \mathrm{Sp}(K\langle T\rangle)$ along the open unit ball $\mathbb{B}_-^1 = \{x \in \mathbb{B}_K^1 \mid |x| < 1\}$ is not quasi-separated.

The rest of this section is devoted to the proof of this Theorem. We will prove the following assertions.

(a) The functor rig transforms formal admissible blowing-ups to isomorphisms.
(b) Two morphisms of admissible formal $\mathcal{O}_K$-schemes $\varphi, \psi : X \to Y$ coincide if $\varphi^{\mathrm{rig}}, \psi^{\mathrm{rig}} : X^{\mathrm{rig}} \to Y^{\mathrm{rig}}$ coincide.
(c) If $X, Y$ are two quasi-paracompact admissible formal $\mathcal{O}_K$-schemes and $\varphi_K : X^{\mathrm{rig}} \to Y^{\mathrm{rig}}$ is a morphism of their associated rigid spaces, then there exists a formal admissible blowing-up $\tau : X' \to X$ and a morphism $\varphi' : X' \to Y$ such that $\varphi'^{\mathrm{rig}} = \varphi_K \circ \tau^{\mathrm{rig}}$.
(d) For rigid $K$-space $X$, which is quasi-paracompact and quasi-separated, then there exists a quasi-paracompact admssible formal $\mathcal{O}_K$-scheme $\mathcal{X}$ such that $\mathcal{X}^{\mathrm{rig}} \cong X$.

Before giving the proof of these assertions, let us see why these assertions imply Raynaud's theorem.

Let $\mathbf{FS}(\mathcal{O}_K)$ denote the category of quasi-paracompact admissible formal $\mathcal{O}_K$-schemes,let $S$ be the class of and let $\mathbf{Rig}(K)$ be the category of quasi-paracompact quasi-separated rigid spaces over $K$. Assertion (a) implies that the functor rig : $\mathbf{FS}(\mathcal{O}_K) \to \mathbf{Rig}(K)$, induces a functor

$$\mathrm{rig}_S : \mathbf{FS}(\mathcal{O}_K)_S \to \mathbf{Rig}(K).$$

Assertion (d) implies that the functor $\mathrm{rig}_S$ is essentially surjective. It remains to see that it is fully faithful. The faithfulness follows from assertion (b). Indeed, if

$$X \xleftarrow{\tau_1} X_1' \xrightarrow{\varphi_1} Y, \quad X \xleftarrow{\tau_2} X_2' \xrightarrow{\varphi_2} Y$$

are two morphisms in $\mathbf{FS}(\mathcal{O}_K)_S$ giving rise to the same morphism of associated spaces $\varphi_K : X^{\mathrm{rig}} \to Y^{\mathrm{rig}}$, we can find an admissible formal blowing-up $\tau : X' \to X$ dominant both $\tau_1$ and $\tau_2$. Then the two composite morphisms

$$\psi_i : X' \to X_i \xrightarrow{\varphi_i} Y$$

will induce the same morphism of rigid spaces $\varphi_K \circ \tau^{\mathrm{rig}} : X'^{\mathrm{rig}} \to Y^{\mathrm{rig}}$. Therefore, by assertion (b), there exists a further admissible blowing-up $X'' \to X'$ whose composite with $\psi_1$ and $\psi_2$ coincide. By definition of morphisms in the localization $\mathbf{FS}(\mathcal{O}_K)_S$, this implies that the original two morphisms in $\mathbf{FS}(\mathcal{O}_K)_S$ are the same.

Finally, assertion (c) guarantees that every morphism in $\mathbf{Rig}(K)$ comes from some morphism in $\mathbf{FS}(\mathcal{O}_K)_S$ via the functor $\mathrm{rig}_S$. This finishes the proof of fully faithfulness of $\mathrm{rig}_S$, and thus the proof of Theorem 3.5.5.

Now we turn to the proof of the assertions (a)-(d). Assertion (a) has been proved in Proposition 3.5.3. For assertion (b), the statement is local in $X$, thus we may assume that $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(B)$ are affine. Since $X$ and $Y$ are admissible, there exists natural inclusions $A \subseteq A_K$ and $B \subseteq B_K$. Therefore, if $\varphi^*, \psi^*$ agree on $A_K$, they have to agree on $A$.

To justify assertion (c), we need some preliminaries.

***Lemma 3.5.6.*** — *Let $X$ be an admissible formal $\mathcal{O}_K$-scheme which is quasi-paracompact, and let $U$ be an open formal subscheme which is quasi-compact. Then any coherent open* *ideal* $\mathcal{I}_U \subseteq \mathcal{O}_U$ *extends to a coherent open ideal* $\mathcal{I} \subseteq \mathcal{O}_X$ *such that* $\mathcal{I}|_V = \mathcal{O}_V$ *for any formal open subscheme disjoint from* $u$*.*

*Proof.* — Since $\mathcal{I}_U$ is open and $U$ quasi-compact, there is an integer $n \in \mathbb{N}$ such that $\pi^n \in \mathcal{I}_U$. Then one can view $\mathcal{I}_U/\pi^n\mathcal{O}_U$ as a coherent ideal of $\mathcal{O}_U/\pi^n\mathcal{O}_U$. By [EGA, I, 6.9.6], one can extend $\mathcal{I}_U/\pi^n\mathcal{O}_U$ to a coherent ideal $\mathcal{I}_n \subseteq \mathcal{O}_X/\pi^n\mathcal{O}_X$ such that $\mathcal{I}_n|_V = \mathcal{O}_V/\pi^n$ for any open formal subscheme $V$ disjoint from $U$. Then one can take $\mathcal{I}$ as the inverse image of $\mathcal{I}_n$ in $\mathcal{O}_X$. □

***Lemma 3.5.7.*** — *Let* $X$ *be an admissible quasi-paracompact formal* $\mathcal{O}_K$*-scheme, and let* $\mathfrak{U}_K$ *be an admissible covering of associated rigid* $K$*-space* $X^{\text{rig}}$*, consisting of quasi-compact open subspaces of* $X^{\text{rig}}$*, such that each object* $U \in \mathfrak{U}_K$ *intersect only with finitely many other objects in* $\mathfrak{U}_K$*. Then there exists an admissible formal blowing-up* $\tau : X' \to X$ *and a covering* $\mathfrak{U}'$ *of* $X$ *such that the associated family* $\mathfrak{U}'^{\text{rig}}$ *of rigid* $K$*-spaces of* $X^{\text{rig}}$ *coincides with* $\mathfrak{U}_K$*.*

*Proof.* — We consider first the case when $X = \mathrm{Spf}(A)$ is affine. In this case, $\mathfrak{U}_K$ is a finite covering. By Gerritzen-Grauert's theorem [**Bo08**, §1.6, Theorem 20], each $U \in \mathfrak{U}_K$ is a finite union of rational subdomain. Hence, we may assume that $\mathfrak{U}_K$ is a rational covering of $X$ generated by $f_0, \cdots, f_r \in A_K$, which generate the unit ideal of $A$. Up to multiplying by a scalar in $K$, we may assume that $f_i \in A$. Consider the admissible formal blowing-up $\tau : X' \to X$ along the open ideal $I = (f_0, \cdots, f_r)$. Let $U_i'$ be the open subset of $X'$ where $I\mathcal{O}_{X'}$ is generated by $f_i$. Then we have seen, in the proof of Proposition 3.5.3, that $U_i'^{\text{rig}}$ is the rational subdomain

$$X^{\text{rig}}\Big(\frac{f_j}{f_i}; 0 \le j \le r\Big).$$

Consider now the general case. We take an affine covering $X = \bigcup_{j\in J} X_j$ be an affine covering by admissible formal schemes. For each object $X_j$, the restriction of $\mathfrak{U}_K$ to $X_j^{\text{rig}}$ yields a finite covering of $X_j^{\text{rig}}$ by quasi-compact admissible open subsets (note that this uses the fact that $X$ is quasi-separated). By the discussion above, there exists an admissible formal blowing-up $X_{j,\mathcal{I}_j} \to X_j$ along some ideal $\mathcal{I}_j \subseteq \mathcal{O}_{X_j}$ such that the finite covering $\mathfrak{U}_K|_{X_j}$ is induced by a covering of $X_{j,\mathcal{I}_j}$ by quasi-compact open subsets. Let $\overline{\mathcal{I}}_j \subseteq \mathcal{O}_X$ be an extension of $\mathcal{I}_j$ given by Lemma 3.5.6. Put $\overline{\mathcal{I}} = \prod_j \overline{\mathcal{I}}_j$. Consider the admissible blowing-up $\tau : X_{\overline{\mathcal{I}}} = \mathrm{Bl}_{\overline{\mathcal{I}}}(X) \to X$. Then the restriction $\tau^{-1}(X_j) \to X_j$ factors through $X_{j,\mathcal{I}_j}$ for each $j \in J$, and every element of the covering $\mathfrak{U}_K$ can be induced by some quasi-compact open subse of $X_{\overline{\mathcal{I}}}$. We obtain thus a family of quasi-compact open subsets $\mathfrak{U}$ of $X_{\overline{\mathcal{I}}}$. It remains to see that $\mathfrak{U}$ form a covering of $X_{\overline{\mathcal{I}}}$. By construction, the restriction of the family $\mathfrak{U}$ to $\tau_j^{-1}(X_j)$ comes from a covering of $X_{j,\mathcal{I}_j}$. Therefore, $\mathfrak{U}$ is indeed a covering of $X$. □

***Lemma 3.5.8.*** — *Let* $A$ *be an admissible* $\mathcal{O}_K$*-algebra so that* $A$ *can be viewed as a subalgebra of* $A_K = A \otimes_{\mathcal{O}_K} K$*. Let* $f_1, \cdots, f_n \in A_K$ *be elements such that* $\|f_i\|_{\sup} \le 1$ *for all* $i$*. Then* $A' = A[f_1, \cdots, f_n]$ *is an admissible* $\mathcal{O}_K$*-algebra which is finite over* $A$ *as an* $A$*-module. Moreover, if* $c \in \mathcal{O}_K\backslash\{0\}$ *is an element such that* $cf_1, \cdots, cf_n \in A$*, then* *the canonical morphism* $\tau : \mathrm{Spf}(A') \to \mathrm{Spf}(A)$ *is identified with the formal blowing-up of* $\mathrm{Spf}(A)$ *along the ideal* $I = (c, cf_1, \cdots, cf_n)$.

*Proof.* — For an affinoid algebra $B$ over $K$, denote by $B^\circ$ the subring of power bounded elements. By Proposition 1.4.13, we have

$$B^\circ = \{f \in B \mid \|f\|_{\sup} \le 1\}.$$

Choose a surjection $\phi : \mathcal{O}_K\langle T_1, \cdots, T_r\rangle \to A$. Proposition 1.4.13 implies that $f_i$ is integral over $A$, hence $A' = A[f_1, \cdots, f_n]$ is a finite $A$-module, in particular an $\mathcal{O}_K$-algebra of topologically finite type. Since $A'$ is flat over $\mathcal{O}_K$, it is an admissible $\mathcal{O}_K$-algebra by Corollary 3.2.4.

To see that $\tau : \mathrm{Spf}(A') \to \mathrm{Spf}(A)$ is the admissible blowing-up along $I$, we just need to check that $\tau$ satisfies the universal property of the admissible formal blowing-up. Note first that $IA'$ is generated by $c$, hence invertible. Consider now a homomorphism of admissible $\mathcal{O}_K$-algebras $A \to B$ such that the ideal $IB$ is invertible. If $IB$ is generated by $c$, then $cf_i \in cB$, and hence $f_i \in B$ since $c$ is not a zero-divisor in $B$. If $IB$ is generated by $cf_i$, then $c = cf_i g_i$ for some $g_i \in B$. We have $f_i^{-1} = g_i \in B$. As $f_i$ is integral over $A$, we take an monic equation

$$f_i^s + b_1 f_i^{s-1} + \cdots + b_s = 0,$$

and one gets

$$f_i = -b_1 - \cdots - b_s f_i^{1-s} \in B.$$

Thus $f_i$ is a unit in $B$, and $IB$ is generated by $c$. □

We come back to the proof of assertion (c). Consider two admissible quasi-paracompact formal $\mathcal{O}_K$-schemes $X, Y$, and a morphism $\varphi_K : X^{\mathrm{rig}} \to Y^{\mathrm{rig}}$ between the associated rigid $K$-spaces. We want to show that there exists an admissible formal blowing-up $\tau : X' \to X$ and a morphism $\varphi : X' \to Y$ such that $\varphi^{\mathrm{rig}} = \varphi_K \circ \tau^{\mathrm{rig}}$. Consider first the case $X = \mathrm{Spf}(A)$ and $Y = \mathrm{Spf}(B)$ are both affine. The morphism $\varphi_K$ corresponds to a morphism of affinoid $K$-algebras $\varphi_K^* : B_K \to A_K$. We can view $B$ as a subalgebra of $B_K$, and we have

$$B \subseteq B_K^\circ = \{f \in B_K \mid \|f\|_{\sup} \le 1\}$$

since all elements of $B$ are clearly power bounded.

Choose $g_1, \cdots, g_s \in B$ such that $\alpha : \mathcal{O}_K\langle T_1, \cdots, T_s\rangle \to B$ sending $T_i \mapsto f_i$ is surjective. Let $f_i = \varphi_K^*(g_i)$. Then $f_i$ is also power bounded, and hence $\|f_i\|_{\sup} \le 1$. Then $A' = A[f_1, \cdots, f_s]$ is an admissible $\mathcal{O}_K$-algebra and $X' = \mathrm{Spf}(A') \to X$ is an admissible formal blowing-up. On the other hand, the morphism $\varphi_K^*$ restricts to a well defined homomorphism of admissible $\mathcal{O}_K$-algebras $\varphi^* : B \to A'$, which corresponds to a morphism $\varphi : X' \to Y$ as desired.

Now consider the general case. One chooses affine open coverings $\mathfrak{U}$ of $X$ and $\mathfrak{V}$ of $Y$ which are locally finite. Denote by $\mathfrak{U}^{\mathrm{rig}}$ and $\mathfrak{V}^{\mathrm{rig}}$ the derived coverings of the rigid spaces. Restricting the pull-back $\varphi_K^{-1}(\mathfrak{V}^{\mathrm{rig}})$ to each member $U^{\mathrm{rig}}$ of $\mathfrak{U}^{\mathrm{rig}}$, we get a refinement $\mathfrak{U}_K$ of $\mathfrak{U}^{\mathrm{rig}}$, which is an admissible affinoid covering of $X^{\mathrm{rig}}$ which is locally finite. By Lemma 3.5.7, we may assume that $\mathfrak{U}_K$ is induced by some open affine covering of the formal scheme $X$. Up to replacing $\mathfrak{U}$ by this new covering, we may assume that $\varphi_K$ sends each member $U^{\mathrm{rig}}$ of $\mathfrak{U}^{\mathrm{rig}}$ to some member $V^{\mathrm{rig}}$ of $\mathfrak{V}^{\mathrm{rig}}$. By the discussion in the affine case, there exists an admissible formal blowing-up $\tau_U : U' \to U$ and a morphism $\varphi_U : U' \to V$ such that $\varphi_U^{\mathrm{rig}} = \varphi_K|_U \circ \tau_U^{\mathrm{rig}}$. By Lemma 3.5.6, there exists an admissible formal blowing-up $X' \to X$ which, when restricted to each $U$ in $\mathfrak{U}$, dominates the admissible blowing-up $\tau_U : U' \to U$. Then the composition $\tau^{-1}(U) \to U' \xrightarrow{\varphi_U} Y$ glue together to get a well defined morphism $\varphi : X' \to Y$ such that $\varphi^{\mathrm{rig}} = \varphi_K \circ \tau^{\mathrm{rig}}$.

Finally, we turn to the proof of assertion (d). Given an object $X_K \in \mathbf{Rig}(K)$, we need to construct an admissible formal $\mathcal{O}_K$-scheme $X \in \mathbf{FS}(\mathcal{O}_K)$ such that $X^{\mathrm{rig}} = X_K$. Since $X_K$ is quasi-paracompact, there exists an admissible locally finite covering of $X$ by affinoid open subsets $X_K = \bigcup_{j\in J} X_{K,j}$.

We consider first the case when $J$ is a finite set. We proceed by induction on $|J| \geq 1$. When $|J| = 1$, $X_K = \mathrm{Sp}(A_K)$ is affinoid, we choose a surjection

$$\phi : K\langle T_1, \cdots , T_n\rangle \to A_K,$$

and take $A = \phi(\mathcal{O}\langle T_1, \cdots , T_n\rangle$. Then $X = \mathrm{Spf}(A)$ is an admissible formal scheme with $X^{\mathrm{rig}} = X_K$. Note that finite union of quasi-compact open subsets of $X_K$ is still quasi-compact. Assume now that $X_K = U_{1,K} \cup U_{2,K}$ where $U_{1,K}$ and $U_{2,K}$ are both quasi-compact open subsets of $X_K$, and admit formal integral model $U_1$ and $U_2$ respectively. Let $W_K = U_{1,K} \cap U_{2,K}$. Then $W_K$ is quasi-compact since $X_K$ is quasi-separated. Choose a finite admissible covering of $U_{1,K}$ by quasi-compact open subsets which contains $W_K$ as a member. Then by Lemma 3.5.7, there exists a formal admissible blowing-up $U_1' \to U_1$ such that the covering chosen above is induced from a covering of $U_1'$ by quasi-compact open formal subschemes by taking rigid generic fibres. In particular, there exists a quasi-compact open subset $W_1' \subseteq U_1'$ such that the open immersion $W_K \subseteq U_{1,K}$ is identified with $W_1'^{\mathrm{rig}} \hookrightarrow U_1'^{\mathrm{rig}}$. Similarly, we have an admissible formal blowing-up $U_2' \to U_2$ and an open quasi-compact subset $W_2' \subseteq U_2'$ such that the open immersion $W_K \hookrightarrow U_{2,K}$ is identified with $W_2'^{\mathrm{rig}} \hookrightarrow U_2'^{\mathrm{rig}}$. By assertion (c), there exists admissible formal blowing-ups $W'' \to W_1'$ and $W'' \to W_2'$ such that the canonical isomorphism

$$W_1'^{\mathrm{rig}} \cong W_K \cong W_2'^{\mathrm{rig}}$$

is induced by

$$W_1' \leftarrow W'' \to W_2'.$$

By Lemma 3.5.6, the blowing-ups $W'' \to W_1'$ and $W'' \to W_2'$ can be extended to admissible formal blowing-ups $U_i'' \to U_i'$ for $i = 1, 2$. Then one can glue $U_1''$ and $U_2''$ along $W''$ to get an integral formal model of $X_K$.

Consider the case when $|J|$ is infinite. We may assume that $X_K$ and all $X_{K,i}$ are all connected. Then $J$ is countable. We fix $U_{K,0} = X_{K,i_0}$ for some $i_0 \in J$, and define inductively $U_{K,u+1}$ as the union of $X_{K,i}$ which are not contained in the union

$$V_{K,n} := U_{K,0} \cup \cdots \cup U_{K,n}$$

but which meet $U_{K,n}$. Since the covering $X_K = \bigcup_{i\in J} X_{K,i}$ is locally finite, $U_{n+1}$ consists of only finitely many $X_{K,i}$'s, hence it is quasi-compact. It is easy to see that $\bigcup_{n\geq 0} U_{K,n}$ equals a connected component of $X_K$, hence equals to $X_K$. Moreover, by construction, it is also easy to check that $U_{K,n} \cap U_{K,m} = \emptyset$ for all $m < n - 1$.

Then we proceed by induction to show that each $V_{K,n}$ admits an integral formal model $V_n$ together with open immersions $U_i \hookrightarrow V_n$ which induces $U_{K,i} \hookrightarrow V_{K,n}$ by taking rigid generic fibers.

### 3.6. Rigid points of admissible formal schemes

***Definition 3.6.1.*** — Let $X$ be an admissible formal $\mathcal{O}_K$-scheme. A rigid point of $X$ is a morphism $u : T \to X$ of admissible formal $\mathcal{O}_K$-schemes such that

1. $u$ is a closed immersion,
2. $T = \mathrm{Spf}(B)$, with $B$ a local integral domain of dimension 1.

In this setup, the fraction field of $B$ is called the residue field of $u$.

***Lemma 3.6.2.*** — *Let $T = \mathrm{Spf}(B)$ be an admissible formal $\mathcal{O}_K$-scheme, where $B$ is a local integral domain of dimension 1. Then $B$ is a finite $\mathcal{O}_K$-module, and the integral closure of $B$ in its fraction field is the valuation ring in a finite extension of $K$.*

*Proof.* — Since every element of $\mathfrak{m}_K$ is topologically nilpotent in $B$, the maximal ideal of $B$ contains $\mathfrak{m}_K$ and hence $B \otimes_{\mathcal{O}_K} k$ is a local $k$-algebra of finite type. It follows by Noether's normallization that $B \otimes_{\mathcal{O}_K} k$ is finite dimensional over $k$. Choose a surjection $\phi : \mathcal{O}_K\langle T_1, \cdots, T_n\rangle \to B$ such that the residue classes of $x_i = \phi(T_i)$ generate $B \otimes_{\mathcal{O}_K} k$ as a $k$-vector space. Then there exists $\pi \in \mathfrak{m}_K$ such that

$$\phi(T_iT_j) \in \sum_{i=1}^{n} \mathcal{O}_K x_i + \pi B.$$

It follows by Nakayama's Lemma that $B$ is generated by $x_i$'s as an $\mathcal{O}_K$-module. □

***Lemma 3.6.3.*** — *Let $X = \mathrm{Spf}(A)$ be an affine admissible formal $\mathcal{O}_K$-scheme, and $f \in A$. Let $u : T = \mathrm{Spf}(B) \to X_f = \mathrm{Spf}(A\langle 1/f\rangle)$ be a rigid point. Then the natural composed map*

$$v : T \xrightarrow{u} X_f \to X$$

*is also a rigid point.*

*Proof.* — It suffices to show that the composed morphism $v$ is a closed immersion, i.e. the homomorphism of admissible $\mathcal{O}_K$-algebras

$$v^* : A \to A\langle 1/f\rangle \xrightarrow{u^*} B$$

is surjective. By assumption, $u^*$ is known to be surjective. Let $B'$ be the image of $v^*$. By Lemma 3.6.2, $B$ is finite over $\mathcal{O}_K$, hence finite over $B'$. Let $\bar{f} \in B$ be the image of $f \in A$. Then $\bar{f}^{-1} \in B$ is integral over $B'$. Taking a monic equation of $\bar{f}^{-1}$ over $B'$, we see easily that $\bar{f}^{-1} \in B'$. Since $A[f^{-1}]$ is dense in $A\langle f^{-1}\rangle$, so is $B'$ in $B$. Since both $B$ and $B'$ are complete for the $\pi$-adic topology, it follows that $B = B'$.

□

Two rigid points $u : T \to X$ and $u' : T' \to X'$ are called equivalent, if there exists an isomorphism $\sigma : T \to T'$ such that $u = u' \circ \sigma$. Denote by $\mathrm{rig} - \mathrm{pts}(X)$ the set of isomorphism classes of rigid points of $X$.

***Lemma 3.6.4.*** — *Let $X = \mathrm{Spf}(A)$ be an affine admissible formal $\mathcal{O}_K$-scheme. Then there exist canonical bijections between the following sets of points:*

(a) *Isomorphism classes of rigid points of $X$.*
(b) *Non-open prime ideals $\mathfrak{p} \subseteq A$ such that $\dim(A/\mathfrak{p}) = 1$.*
(c) *Maximal ideals in $A \otimes_{\mathcal{O}_K} K$.*

*Moreover, the bijections between the three sets of points are given as follows*

1. *Given a rigid point $u : T = \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ defined by a surjection $u^* : A \to B$, we associate to $u$ the prime ideal $\mathfrak{p} := \mathrm{Ker}(u^*)$, which is a point of type (b).*
2. *Given a point of type (b), i.e. a non-open prime ideal $\mathfrak{p} \subseteq A$, we associate to it the prime ideal $\mathfrak{p}_K := \mathfrak{p} \otimes_{\mathcal{O}_K} K \subset A \otimes_{\mathcal{O}_K} K$, which is a maximal ideal of $A \otimes_{\mathcal{O}_K} K$.*
3. *Given a maximal ideal $\mathfrak{m} \subseteq A \otimes_{\mathcal{O}_K} K$, let $\mathfrak{p} = \mathfrak{m} \cap A$, we associate to it the canonical morphism $\mathrm{Spf}(A/\mathfrak{p}) \to \mathrm{Spf}(A)$.*

*Proof.* — First, we show that the morphisms described above are well defined. It is clear that given a rigid point $u : \mathrm{Spf}(B) \to \mathrm{Spf}(A)$, the kernel $\mathfrak{p} := \mathrm{Ker}(u^*)$ is a prime ideal of $A$ with $\dim(A/\mathfrak{p}) = 1$. Since $B$ is flat over $\mathcal{O}_K$, we see that $\mathfrak{p}$ is not open.

Now given such a non-open prime ideal $\mathfrak{p}$ with $\dim(A/\mathfrak{p}) = 1$, we want to show that $\mathfrak{p} \otimes_{\mathcal{O}_K} K$ is a maximal ideal of $A_K$. It suffices to see that $(A/\mathfrak{p}) \otimes_{\mathcal{O}_K} K = (A/\mathfrak{p})[1/\pi]$ is a field, i.e. 0 is the only prime ideal of $(A/\mathfrak{p})[1/\pi]$. But the prime ideals of $(A/\mathfrak{p})[1/\pi]$ corresponds to the prime ideals of $A/\mathfrak{p}$ which do not contain $\pi$. It suffices to see that all non-zero prime ideals of $A/\mathfrak{p}$ contains $\pi$. But since $\dim(A/\mathfrak{p}) = 1$, all non-zero ideals are maximal. On the other hand, since $\pi$ is topologically nilpotent in $A/\mathfrak{p}$, all maximal ideals of $A/\mathfrak{p}$ contains $\pi$.

Finally, given a maximal ideal $\mathfrak{m} \subseteq A_K$, we want to show that $\mathfrak{p} := \mathfrak{m} \otimes A$ is a non-open ideal with $\dim(A/\mathfrak{p}) = 1$. We have a natural inclusion

$$B := A/\mathfrak{p} \hookrightarrow K' := A_K/\mathfrak{m},$$

where $K'$ is a finite extension of $K$. Note that every element of $A$ is power bounded in $A_K$, it follows that every element of $B$ is power bounded in $K'$. Thus, if $\mathcal{O}_{K'}$ denotes the valuation ring of $K'$, we have $B \subseteq \mathcal{O}_{K'}$. In particular, $B$ is integral over $\mathcal{O}_K$. It follows that

$$\dim(B) = \dim(\mathcal{O}_K) = 1.$$

By construction, it is easy to see that all these maps between the the three sets of points are injective. Hence, all the three sets of points are in natural bijection.

□

***Corollary 3.6.5.*** — *Let $X$ be an admissible formal $\mathcal{O}_K$-scheme. Then there is a canonical bijection of sets:*

$$\mathrm{rig} - \mathrm{pts}(X) \xrightarrow{\sim} X^{\mathrm{rig}}.$$

Let $u : T = \mathrm{Spf}(R) \to X$ be a rigid point. It induces by reduction a natural closed immersion

$$u_k : \mathrm{Spec}(R \otimes_{\mathcal{O}_K} k) \to X_k = X \otimes_{\mathcal{O}_K} k.$$

Since $R$ is a local integral domain admissible as an $\mathcal{O}_K$-algebra, $\mathrm{Spec}(R \otimes_{\mathcal{O}_K} k)$ consists only of a single point. Thus, the image $u_k$ is a well-defined closed point on the special fiber $X_k$, called usually the *specialization of* $u$. Using Corollary 3.6.5, we see that there exists a canonical specialization map

$$sp : X^{\mathrm{rig}} \to X_k.$$

***Example 3.6.6.*** — Let $X = \mathrm{Spf}(\mathcal{O}_K\langle T\rangle)$, and $\tau : X' \to X$ be the formal admissible blowing-up along the ideal $I = (\pi, T)$. Then $X'$ is covered by two affine formal pieces $X' = U_1 \cup U_2$, where

$$U_1 = \mathrm{Spf}(\mathcal{O}_K\langle \frac{T}{\pi}\rangle), \quad U_2 = \mathrm{Spf}\left(\mathcal{O}_K\langle T, S\rangle/(\pi - TS)\right)$$

Consider the specialization map

$$sp : X'^{\mathrm{rig}} \cong X^{\mathrm{rig}} \to X'_k = U_{1,k} \cup U_{2,k}$$

where $U_{1,k}^{red} \cong \mathbb{A}^1_k$ and $U_{2,k}^{red} = \mathrm{Spec}(k[T,S]/(ST))$. The inverse image $sp^{-1}(x)$ of a closed point of $X'_k$ is a open ball; in particular, it is is not quasi-compact. Note that inverse image of the singular point $\bar{x} \in U_{2,k}$ is given by

$$sp^{-1}(\bar{x}) = \{x \in X^{\mathrm{rig}} : |\pi| < |T(x)| < 1\}.$$

On the other hand, if $U_k \subseteq X'_k$ is a Zariski open subset, then we have

$$sp^{-1}(U_k) = U^{\mathrm{rig}},$$

where $U \subseteq X$ is the open formal subscheme corresponding to $U_k$.

In the sequel, let $X$ be an admissible formal $\mathcal{O}_K$-scheme with special fiber $X_k = X \otimes_{\mathcal{O}_K} k$.

***Lemma 3.6.7.*** — *Let $U \subseteq X$ be an open formal subscheme with reduction $U_0 \subseteq X_k$. Then we have*

$$sp^{-1}(U_0) = U^{\mathrm{rig}}.$$

*Proof.* — Note that if the statement is true for $U_i$ with $i \in I$, then so is for $\bigcup_{i\in I} U_i$. We may assume thus that $X = \mathrm{Spf}(A)$ is affine and $U = D(f) = \mathrm{Spf}(A\langle f\rangle)$ for some $f \in A$. Let $u : T = \mathrm{Spf}(B) \to X$ be a rigid point. Then $sp(x) \in U_0$ if and only if $|f(x)| \geq 1$ (actually, the equality holds here). Therefore, $sp^{-1}(U_0) = U^{\mathrm{rig}} = \mathrm{Sp}(A_K\langle \frac{1}{f}\rangle)$. □

***Lemma 3.6.8.*** — *Let $Y_0 \subseteq X_0$ be a locally closed subscheme. Assume that there exist $f_1, \cdots, f_r, g_1, \cdots, g_s \in \Gamma(X, \mathcal{O}_X)$ of reductions $\bar{f}_1, \cdots, \bar{f}_r, \bar{g}_1, \cdots, \bar{g}_s \in \Gamma(X_0, \mathcal{O}_{X_0})$ such that $Y_0 = Z_0 \cap U_0$ with $Y_0 = V(\bar{f}_1, \cdots, \bar{f}_r)$ and $U_0 = D(\bar{g}_1) \cup \cdots \cup D(\bar{g}_s)$. Then $sp^{-1}(Y_0)$ is the admissible open subset of $X^{\mathrm{rig}}$ defined by conditions:*

$$|f_i(x)| < 1, \forall i; \quad \exists j, |g_j(x)| = 1.$$

*Proof.* — We may assume that $X = \mathrm{Spf}(A)$ is affine. Since $sp^{-1}(Y_0) = sp^{-1}(Z_0) \cap sp^{-1}(U_0)$, we may assume that $Y_0 = Z_0$ or $Y_0 = U_0$. We have that $sp^{-1}(D(\bar{g}_i)) = D(g_i)^{\mathrm{rig}}$, and the case of $Y_0 = U_0$ follows. Since $V(\bar{f}_i)$ is the complement of $D(\bar{f}_i)$, we have

$$sp^{-1}(V(\bar{f}_i)) = \{x \in X^{\mathrm{rig}} \mid |f_i(x)| < 1\}.$$

The case of $Y_0 = Z_0$ also follows immediately. □

***Remark 3.6.9.*** — In the literature, $sp^{-1}(Y_0)$ is also denoted by $]Y_0[$, and called the tube of $Y_0$ in $X^{\mathrm{rig}}$. From the discussion above, we see that if $Y_0 \subseteq X_0$ is open, then $]Y_0[$ is quasi-compact.

***Proposition 3.6.10.*** — *The specialization map of topological spaces $sp : X^{\mathrm{rig}} \to X_0$ is surjective onto the set of closed ponts of $X_0$.*

*Proof.* — By Lemma 3.6.7, the problem is local for $X$. We may assume thus that $X = \mathrm{Spf}(A)$ is affine. When $A = \mathcal{O}_K\langle T_1, \cdots, T_n\rangle$, the statement follows by a direct computation.

Consider now the general case. By Noether's normalization theorem, there exists an injective homomorphism

$$R = \mathcal{O}_K\langle T_1, \cdots, T_d\rangle \to A$$

such that $A$ is integral over $\mathcal{O}_K\langle T_1, \cdots, T_d\rangle$. Then $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is surjective.

Consider first the case when $A$ is an integral domain. Recall that the going-down theorem in commutative algebra says that if $R \to S$ is an extension of rings such that $S$ is an integral domain and $R$ is integrally closed, then $R \to S$ satisfies the going-down property, namely if $\mathfrak{p}_1 \supset \mathfrak{p}_2$ is a chain of prime ideals in $R$ and $\mathfrak{q}_2$ is a prime ideal of $S$ lying above $\mathfrak{p}_2$, then there exists a prime ideal $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ which lies above $\mathfrak{p}_1$. In our case, we claim that the ring $R$ is indeed integrally closed. Assuming the claim, we prove the proposition as follows. Let $\mathfrak{m}$ be a maximal ideal of $A$, and $\mathfrak{n} = R \cap \mathfrak{m}$ be the image in $\mathrm{Spec}(R)$. By the discussion above, $\mathfrak{n}$ lies in the image of the specialization map, i.e. there exists a non-open prime ideal $\mathfrak{p} \subseteq \mathfrak{n}$ with $\dim(R/\mathfrak{p}) = 1$. Then the going-down property for $R \to A$ implies the existence of a prime ideal $\mathfrak{q} \subseteq \mathfrak{m}$ lying above $\mathfrak{p}$. Clearly, $\mathfrak{q}$ is non-open and $\dim(A/\mathfrak{q}) = 1$.

Now we prove that $R$ is integrally close. If $f$ is an element in the fractional field of $R$ integral over $R$, then $f \in K\langle T_1, \cdots, T_d\rangle$ since we have seen that Tate algebras are integrally closed. Let

$$f^r + a_1 f^{r-1} + \cdots + a_r = 0$$

be a monic equation with $a_i \in R$. Then we see that $\|f\|_{\sup} \leq \max\{\|a_i\|_{\sup}^{1/i}\} \leq 1$. Hence $f$ has coefficients in $\mathcal{O}_K$, i.e. $f \in R$.

Finally, we consider the general case. Let $\mathfrak{p}_{K,s}, \cdots, \mathfrak{p}_{K,s}$ be the minimal prime ideals of $A_K$, and $\mathfrak{p}_i = \mathfrak{p}_{K,i} \cap A$. For any maximal ideal $\mathfrak{m}$ of $A$, there exists $\mathfrak{p}_i$ such that $\mathfrak{m} \supset \mathfrak{p}_i$.

We consider the diagram

$$\begin{CD} A @>>> A_K \\ @VVV @VVV \\ A/\mathfrak{p}_i @>>> A_K/\mathfrak{p}_{K,i} \end{CD}$$

Note that $A/\mathfrak{p}_i$ is an admissible $\mathcal{O}_K$-algebra, since it is of topologically finite type and flat over $\mathcal{O}_K$. By the integral case, the surjection $A/\mathfrak{p}_i \to A/\mathfrak{m}$ lift to a surjection $A/\mathfrak{p}_i \to B$ corresponding to a rigid point $x : \mathrm{Spf}(B) \to \mathrm{Spf}(A/\mathfrak{p}_i)$. Then composition $\mathrm{Spf}(B) \to \mathrm{Spf}(A/\mathfrak{p}_i) \to \mathrm{Spf}(A)$ gives a rigid point with specialization $\mathfrak{m}$.

□

## 3.7. Rigid generic fiber of a not necessarily admissible formal $\mathcal{O}_K$-scheme

To avoid technical difficulties, we assume in this section that $\mathcal{O}_K$ is a discrete valuation ring, or equivalently $\mathcal{O}_K$ is noetherian. We fix a uniformizer $\pi \in \mathcal{O}_K$.

Let $X$ be a locally noetherian formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$, and $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal of definition. We suppose that the reduced closed subscheme $X_0$ defined by $\mathcal{I}$ is an $\mathcal{O}_K$-scheme locally of finite type. Note that this condition is independent of the choice of the definition ideal. Under this assumption, we can generalize the construction of rigid generic fiber defined in section 3.2.13 to this case.

Assume first that $X = \mathrm{Spf}(A)$ is affine, with definition ideal $I = (f_1, \cdots, f_r)$. Then for all $n \geq 1$, we put

$$B_n = A\langle T_1, \cdots, T_r\rangle/(\pi T_i - f_i^n).$$

The hypothesis on $A$ implies that

$$B_n/\pi B_n \cong A/(\pi, f_1, \cdots, f_r)[T_1, \cdots, T_r],$$

is a $k = \mathcal{O}_K/(\pi)$-algebra of finite type. Therefore, $B_n$ is an $\mathcal{O}_K$-algebra of topologically finite type. According to Section 3.2.13, one has the rigid generic fiber attached to $\mathrm{Spf}(B_n)$. For $m \geq n$, there is a canonical homomorphism $B_m \to B_n$ sending $T_i$ to $f_i^{m-n}T_i$ and an induced morphism of rigid spaces

$$\mathrm{Spf}(B_n)^{\mathrm{rig}} \to \mathrm{Spf}(B_m)^{\mathrm{rig}}$$

which identifies $\mathrm{Spf}(B_n)^{\mathrm{rig}}$ as the affinoid subdomain of $\mathrm{Spf}(B_m)^{\mathrm{rig}}$ defined by $|f_i(x)| \leq |\pi|^{1/n}$. We define the rigid generic fiber of $X$ as

$$X^{\mathrm{rig}} := \bigcup_{n\geq 1} \mathrm{Spf}(B_n)^{\mathrm{rig}}.$$

***Lemma 3.7.1.*** — *Under the above notation, the rigid space $X^{\mathrm{rig}}$ does not depend on the choice of the definition ideal $I$ nor the generators $f_1, \cdots, f_r$.*

*Proof.* — Let $J = (g_1, \cdots, g_s)$ be another definition ideal of the adic ring $A$. Then there exists an integers $a, b \geq 1$ such that $J^a \subseteq I^b$, and hence $g_i^a = \sum_j h_{i,j} f_j^b$ with $h_{i,j} \in A$. Let $C_n$ be the $\mathcal{O}_K$-algebra constructed in the same way as $B_n$ with $f_i$'s replaced by $g_i$'s. For any integer $n \geq 1$, we have $g_i^{ans}/\pi \in B_{nb}$, and thus morphisms of $\mathcal{O}_K$-algeras $C_{ans} \to B_{nb}$, which induces morphisms of rigid spaces

$$\mathrm{Spf}(B_{nb})^{\mathrm{rig}} \to \mathrm{Spf}(C_{na})^{\mathrm{rig}}.$$

Letting $n$ vary, this induces an isomorphism of rigid spaces

$$\bigcup_n \mathrm{Spf}(B_{nb})^{\mathrm{rig}} \cong \bigcup_{n \geq 1} \mathrm{Spf}(C_{na})^{\mathrm{rig}}.$$

□

***Remark 3.7.2.*** — Note that if $X$ is not admissible, $X^{\mathrm{rig}}$ is not quasi-compact in general. For instance, if $X = \mathrm{Spf}(\mathcal{O}_K[[T]])$ with definition ideal $I = (\pi, T)$, then

$$B_n = \mathcal{O}_K[[T]]\langle S \rangle / (\pi S - T^n)$$

and $\mathrm{Spf}(B_n)^{\mathrm{rig}} = \mathrm{Sp}(B_n \otimes_{\mathcal{O}_K} K) = \mathrm{Spf}(K\langle T, \frac{T^n}{\pi}\rangle)$ is the closed disk with center 0 and radius $|\pi|^{1/n}$. Taking union in $n$, we see that $X^{\mathrm{rig}}$ is the open unit disk.

Let $X_0$ be the reduced closed subscheme of $X$, i.e. $X_0 = \mathrm{Spec}(\mathcal{O}_X/\mathcal{I})^{red}$ for some definition ideal. We have also a morphism of specialization $sp : X^{\mathrm{rig}} \to X_0$ defined as follows. Assume that $X = \mathrm{Spf}(A)$ is affine, and let $B_n$ be the algebras in the construction of $X^{\mathrm{rig}}$ for all $n \geq 1$. Then for $m \geq n$, we have a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Sp}(B_n \otimes_{\mathcal{O}_K} K) & \xrightarrow{sp} & \mathrm{Spec}(B_n \otimes k)^{red} & \longrightarrow & X_0 \\
\downarrow & & \downarrow & & \| \\
\mathrm{Sp}(B_m \otimes_{\mathcal{O}_K} K) & \xrightarrow{sp} & \mathrm{Spec}(B_m \otimes k)^{red} & \longrightarrow & X_0.
\end{array}$$

Taking union in $n$, we get a well-defined morphism:

$$sp : X^{\mathrm{rig}} \to X_0$$

***Proposition 3.7.3.*** — *Let $X$ be a locally noetherian formal $\mathcal{O}_K$-scheme with a definition ideal $\mathcal{I}$. Let $X_0 = V(\mathcal{I})$ and $Y_0 \subseteq X_0$ be a closed subscheme. Let $\hat{X}$ be the completion of $X$ along $Y_0$. Then $sp^{-1}(Y_0)$ is an admissible open subset of $X^{\mathrm{rig}}$, and we have a canonical isomorphism*

$$\hat{X}^{\mathrm{rig}} = sp^{-1}(Y_0) = ]Y_0[.$$

*Proof.* — Let $I = \Gamma(X, \mathcal{I}) = (f_1, \cdots, f_r)$, and $J \supset I$ be the ideal defining $Y_0$. Write $J = (g_1, \cdots, g_s)$,

$$B_n = A\langle T_1, \cdots, T_r \rangle / (T_i \pi - f_i^n)$$

and $U_n = \mathrm{Sp}(B_n \otimes K)$. By Lemma 3.6.8, we have

$$sp^{-1}(Y_0) \cap U_n = \{x \in U_n \mid |g_i(x)| < 1, \forall i\},$$

which is admissible open in $U_n$. Since $U_n$ form an admissible covering of $X^{\mathrm{rig}}$, $sp^{-1}(Y_0) = \bigcup_n sp^{-1}(Y_0) \cap U_n$ is thus admissible open in $X$.

It remains to prove $sp^{-1}(Y_0) = \hat{X}^{\mathrm{rig}}$. Put

$$V_n = \mathrm{Sp}(B'_n \otimes K), \quad B'_n = B_n\langle T_1, \cdots, T_s\rangle/(g_j^n - \pi T_j).$$

Then we have $sp^{-1}(Y_0) = \bigcup_n V_n$. On the other hand, if $\hat{A}$ is the completion of $A$ with respect to $J$, and we put

$$C_n = \hat{A}\langle T'_1, \cdots, T'_s\rangle/(g_j^n - \pi T'_j).$$

Then $\hat{X}^{\mathrm{rig}}$ is by construction the union of $W_n = \mathrm{Sp}(C_n \otimes K)$. Since there exists $h_{i,j} \in A$ such that $f_i = \sum_j h_{i,j} g_j$, the elements $f_i^{ns}/\pi$ are power bounded in $C_n \otimes K$, so that the homomorphism $A \to C_n \otimes K$ factors through $B_{ns} \otimes K$ and then through $B'_{ns} \otimes K$ as well. Hence, the morphism $\varphi : \hat{X}^{\mathrm{rig}} \to X^{\mathrm{rig}}$ $W_n$ into $V_{ns}$. On the other hand, the algebra $B'_{ns}$ is by definition complete for the $\pi$-adic topology, and also for the $JB'_{ns}$-adic topology thanks to the relation $g_j^n = \pi T_j$ in $B'_{ns}$. The map $A \to B'_{ns}$ factors through $\hat{A}$. We deduce thus a morphism $C_{ns} \otimes K \to B'_{ns} \otimes K$, which corresponds to a map of rigid spaces $V_{ns} \to W_{ns}$. Therefore, we see that $\varphi$ induces an isomorphism between $\hat{X}^{\mathrm{rig}}$ and $sp^{-1}(Y_0)$. □

## 3.8. Permanence of good properties by formal integral models

In this section, we suppose all rigid spaces are quasi-separated and quasi-compact. Let $f : X \to Y$ be a morphism of quasi-compact admissible formal $\mathcal{O}_K$-schemes, with induced morphism $f^{\mathrm{rig}} : X^{\mathrm{rig}} \to Y^{\mathrm{rig}}$ on the rigid generic fibers. Suppose that $f^{\mathrm{rig}}$ has a good property (such as flat, quasi-finite, finite étale, ...). In general, $f$ does not satisfy the same property. However, one can achieve this by performing admissible formal blowing-ups on $Y$ and on $X$.

***Theorem 3.8.1*** **([BL93] Thm. 5.2).** — *Let $f : X \to Y$ be a morphism of quasi-compact admissible formal $\mathcal{O}_K$-schemes. Suppose that $f^{\mathrm{rig}}$ is flat. Then there exists a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y, \end{array}$$

*where $f'$ is flat, $Y' \to Y$ is an admissible formal blowing-up at some ideal $\mathcal{I} \subseteq Y$, and $X' \to X$ is the formal blowing-up at $\mathcal{I}\mathcal{O}_X$.*

Here, $X'$ can be also described as the strict transform of $X$ with respect to the admissible formal blowing-up $Y' \to Y$, which is obtained from $X \times_Y Y'$ by quotient out the structure sheaf by the all torsion with respect to the pullback of the ideal $\mathcal{I}$.

***Corollary 3.8.2.*** — *Let $f_K : X_K \to Y_K$ be a morphism of quasi-compact and quasi-separated rigid spaces. Then the image $f_K(X_K)$ is an admissible open subset of $Y_K$.*

*Proof.* — By Theorem 3.8.1, there exists a flat morphism $f : X \to Y$ of quasi-compact admissible formal $\mathcal{O}_K$-schemes such that $f^{\mathrm{rig}} = f_K$. Let $f_k : X_k \to Y_k$ be the induced morphism of the special fibers, which is flat by base change. Then it is well known that the image of $f_k$ is an open compact subset $V_k \subseteq Y_k$. Let $V \subseteq Y$ be the corresponding open formal subscheme. Then $f$ induces a faithfully flat map $X \to V$. Hence, $f_K : X_K \to Y_K$ factors through $V^{\mathrm{rig}}$. We claim that $f_K$ is surjection onto $V^{\mathrm{rig}}$. It suffices to show that $f_K$ is surjective onto every rigid point of $V^{\mathrm{rig}}$. Indeed, let $u : \mathrm{Spf}(B) \to V$ be a rigid point, then the base change map

$$X \times_V \mathrm{Spf}(B) \to \mathrm{Spf}(B)$$

is faithfully flat. In particular, $X \times_V \mathrm{Spf}(B)$ is a non-empty admissible formal $\mathcal{O}_K$-scheme. Then every rigid point of $X \times_V \mathrm{Spf}(B)$ induces a rigid point of $X_K$ mapping to $u$.

□

***Example 3.8.3.*** — Let $f : X = \mathrm{Spf}(\mathcal{O}_K\langle x, y\rangle/(xy^p - \pi)) \to Y = \mathrm{Spf}(\mathcal{O}_K\langle x\rangle)$ be the natural morphism, where $p$ is the characteristic of the residue field of $\mathcal{O}_K$. Then $f^{\mathrm{rig}}$ is a degree $p$ finite étale cover over the open subdomain defined by $|\pi|^{1/p} \leq |x| \leq 1$. However, $f$ is not flat. Take $Y' \to Y$ to be the blowing-up at $I = (x, \pi)$. Then the induced morphism $f : X' \to Y'$ is a finite flat map of degree $p$.

***Example 3.8.4.*** — A main obstruction for a morphism to be flat is that the dimension of fibers is not locally constant. The following example shows that how admissible blowing-up can be used to get local constancy of the dimension of the fibers in the case of quasi-finite morphisms. Assume that $Y = \mathrm{Spf}(A)$ is affine, and $X$ is the closed formal subscheme of $\mathrm{Spf}(A\langle x\rangle)$ defined by the equation:

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with $a_i \in A$. Assume that the ideal $I = (a_1, \cdots, a_n) \subseteq A$ is open. Then the morphism $f_{\mathrm{rig}} : X \to Y$ is quasi-finite, i.e. the fibres of $f_{\mathrm{rig}}$ are of dimension 0. If $I$ is a principal ideal, say generated by $c \in A$, then the equation writes $P = cQ$ for some $Q \in A[x]$. Note that the openness assumption on $I$ implies that $c$ divides certain power of $\pi$. The formal closed subscheme of $\mathrm{Spf}(A\langle x\rangle)$ defined by the equation $Q = 0$ has the same rigid generic fiber as $X$, and the morphism $\mathrm{Spf}(A\langle x\rangle/(Q))$ is quasi-finite everywhere over $Y$. If $I$ is not principal, we consider the admissible blowing-up $Y' \to Y$ at $I$. Then the ideal $I\mathcal{O}_{Y'}$ is principal, and the discussion above shows that the strict transform of $X' \to Y'$ is quasi-finite everywhere.

The following theorem is a corollary of Theorem 3.8.1.

***Theorem 3.8.5*** (**[BL93] Cor. 5.4**). — *Let $f : X \to Y$ be a morphism of admissible formal schemes. Assume $f^{\mathrm{rig}}$ satisfies one of the following properties:*

1. *$f^{\mathrm{rig}}$ is quasi-finite.*
2. *$f^{\mathrm{rig}}$ is an open immersion.*
3. *$f^{\mathrm{rig}}$ is a closed immersion.*

*Then there exists an admissible formal blowing-up $Y' \to Y$ such that the induced morphism $f' : X' \to Y'$ statisfies the same property.*

*Proof.* — (1) Assume that $f^{\mathrm{rig}}$ is quasi-finite. Up to replacing $Y$ by an admissible blowing-up $Y' \to Y$ and $X$ by its strict transform, we may assume that $f : X \to Y$ is flat. Let $\bar{y} \in Y$ be a closed point. Since the specialization map is always surjective, there exists a rigid point $y : \mathrm{Spf}(B) \to Y$ lifting $\bar{y}$. The base change $f_y : X \times_Y y \to y$ is flat. Since for flat morphisms, the dimension of fibers are locally constant, we see that $f^{-1}(\bar{y})$ has the same dimension as $f^{-1}(y_K)$ which is 0.

(2) As above, we may assume that $f : X \to Y$ is flat. Consider the diagonal map

$$X \hookrightarrow X \times_Y X.$$

Let $\mathcal{I}$ be the kernel of $\mathcal{O}_X \hat{\otimes}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{O}_X$. Then $\mathcal{I}$ is flat over $\mathcal{O}_X$ hence over $\mathcal{O}_K$, since $\mathcal{O}_X$ is flat over $\mathcal{O}_Y$. As $f^{\mathrm{rig}}$ is an open immersion, the diagonal map

$$X^{\mathrm{rig}} \hookrightarrow X^{\mathrm{rig}} \times_{Y^{\mathrm{rig}}} X^{\mathrm{rig}}$$

is an isomorphism, $\mathcal{I}_K = 0$. It follows that $\mathcal{I} = 0$, i.e. $X \xrightarrow{\sim} X \times_Y X$ is an isomorphism, this shows that $f$ is a flat monomorphism. By faithfully flat descent, the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism for all $x \in X$. Hence, $X \to Y$ is an open immersion. Note also that if $: X \to Y$ is a flat morphism such that $f^{\mathrm{rig}}$ is an isomorphism, then $f$ itself is an isomorphism.

(3) Let $Z \subseteq Y$ be the closed formal subscheme defined by the kernel $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Then $f$ necessarily factors through $Z$, and the induced morphism $g : X \to Z$ is an isomorphism over the rigid generic fibers. By assumption, $g^{\mathrm{rig}}$ is an isomorphism. By (2), there exists an admissible blowing-up of $Z' \to Z$ such that the induced morphism $f : X' \to Z'$ is isomorphism. Since every admissible blowing-up of $Z$ is induced by an admissible blowing-up of $Y$, the assertion follows.

□

**3.8.6. Permanence of properties of rigid varieties.** — In applications, we often start with a rigid analytic variety $X_K$ over $K$, which has some good properties such as smoothness, étaleness, or with geometrically reduced fibers. We want to find out some formal integral models $X$ of $X_K$, which have the same good properties as $X_K$.

***Example 3.8.7.*** — Let $X_K = \mathrm{Sp}(A_K)$ with $A_K = K\langle x, y\rangle/(xy - \pi))$. We claim that $X_K$ does not admit any smooth integral fomral models. In fact, note that

$$A := A_K^\circ = \mathcal{O}_K\langle x, y\rangle/(xy - \pi)$$

is the subring of power bounded elements of $A_K$. Then $\mathrm{Spf}(A)$ is an formal integral model of $X_K$, and the special fiber of $X_0$ consists of two affine lines, say $V(x)$ and $V(y)$, intersecting at a simple double point. Let $X$ be an admissible formal $\mathcal{O}_K$-scheme with $X^{\mathrm{rig}} = X_K$. For any open affine subset $U = \mathrm{Spf}(B) \subseteq X$, the natural restriction map

$$A_K = \Gamma(X^{\mathrm{rig}}, \mathcal{O}_{X^{\mathrm{rig}}}) \to \Gamma(U^{\mathrm{rig}}, \mathcal{O}_{U^{\mathrm{rig}}})$$

sends $A$ to $B = \Gamma(U, \mathcal{O}_U)$. Therefore, we get a map

$$A \to \Gamma(X, \mathcal{O}_X),$$

which gives rise to a morphism of formal $\mathcal{O}_K$-schemes $X \to \mathrm{Spf}(A)$. Then $X \to \mathrm{Spf}(A)$ has to be an admissible blowing-up at some open ideal of $A$. In particular, $X \to \mathrm{Spf}(A)$ is generically an isomorphism. Assume that $X$ is smooth. Let $Z_1$ (resp. $Z_2$) be an irreducible component of $X_k$ mapping to the irreducible component $V(x)$ of $\mathrm{Spec}(A_k)$. Since $X_k$ is smooth over $k$, irreducible components of $X_k$ coincide with the connected components of $X_k$. Hence, we see that $Z_1$ and $Z_2$ are distinct connected components of $X_k$. In particular $X$ is not connected, which contradicts with the fact that $X^{\mathrm{rig}}$ is connected.

The example above shows that, it is not possible to find proper smooth integral models for a proper smooth rigid analytic variety. However, it is indeed possible to do so for the property of having geometrically reduced fibers. This is given by so-called the reduced geometric fiber theorem.

We start with a finiteness theorem by Grauert and Remmert.

***Theorem 3.8.8.*** — *Let $A_K$ be a geometrically reduced affinoid algebra over $K$. Then there exists a finite extension $L/K$ such that the subring*

$$A_L^\circ := \{f \in A_K \otimes_K L \mid \|f\|_{\sup} \leq 1\}$$

*is of topologically of finite type over $\mathcal{O}_L$, and the special fiber $A_L^\circ \otimes_{\mathcal{O}_L} k_L$ is geometrically reduced. Moreover, the formation of $A_L^\circ$ is stable under base change in the following sense, for any finite extension $L'/L$, we have $A_{L'}^\circ = (A_L^\circ) \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$, and the special fiber $A_{L'}^\circ \otimes_{\mathcal{O}_{L'}} k_{L'}$ is geometrically reduced.*

Note that if $A$ is an admissible $\mathcal{O}_K$-algebra such that $A \otimes_{\mathcal{O}_K} K \cong A_K$, then $A_L^\circ$ is the normalization of $A$ in $A_K \otimes_K L$. In particular, if $K$ is algebraically closed, then the subring $A_K^\circ$ has a reduced special fiber.

***Example 3.8.9.*** — Assume that $K$ is a discrete valuation field with uniformizer $\pi$, and the residue field $k$ is algebraically closed of characteristic 0. Then $\mathcal{O}_K$ contains all roots of unity. Let $A_K = K\langle x, y\rangle/(x^a y^b - \pi)$, which is geometrically reduced. However,

$$A_K^\circ = \mathcal{O}_K\langle x, y\rangle/(x^a y^b - \pi),$$

and the special fiber $A_K^\circ \otimes k \cong k[x,y]/(x^a y^b)$ is not reduced. Let $d = \gcd(a,b)$, $a = d\alpha$, $b = d\beta$, and $n = d\alpha\beta n'$. We consider the extension $L = K[\pi']/(\pi'^n - \pi)$. Then

$$A_L^\circ = \prod_{\zeta \in \mu_d(\mathcal{O}_L)} \{\mathcal{O}_L\langle x, y\rangle/(\pi'^{\alpha\beta} - \zeta x^\alpha y^\beta)\}^{norm},$$

where $\mu_d(\mathcal{O}_L)$ denotes the group of roots of unity in $\mathcal{O}_L$, and $\{\}^{norm}$ means the normalization. For each $\zeta \in \mu_d(\mathcal{O}_L)$, choose $\theta \in \mu_{d\alpha\beta}(\mathcal{O}_L)$ with $\theta^{\alpha\beta} = \zeta$. Consider the map

$$f^* : B = \mathcal{O}_L\langle x, y\rangle/(\pi'^{\alpha\beta} - \zeta x^\alpha y^\beta) \to \mathcal{O}_L\langle u, v\rangle/(\pi' - \theta uv) = B'$$

given by $f^*(x) = u^\beta$ and $f^*(y) = v^\alpha$. Then $f^*$ induces an injection of the fraction field of $B$ into that of $B'$. Since $(\alpha, \beta) = 1$, there exists integers $\mu, \nu \in \mathbb{Z}$ such that $\alpha\mu + \beta\nu = 1$. Then

$$x = x^{\alpha\mu + \beta\mu} = \big(\frac{\pi'^{\alpha\beta}}{\theta y^\beta}\big)^\mu x^{\beta\nu} = u'^\beta, \quad \text{with } u' = \left(\frac{\pi'^{\alpha\mu} x^\nu}{\theta^\mu y^\mu}\right)^\beta.$$

Similarly, there exists $v' \in \mathrm{Frac}(B)$ such that $y = v'^{\alpha}$. Then one has $f^*(u') = u$ and $f^*(v') = v$. We see that $f^*$ induces an isomorphism between $\mathrm{Frac}(B)$ with $\mathrm{Frac}(B')$, and $B'$ is the normalization of $B$. Therefore, we see that $A_L^\circ \otimes_{\mathcal{O}_L} k$ is geometrically reduced.

***Definition 3.8.10.*** — Let $f : S' \to S$ be a morphism of admissible formal $\mathcal{O}_K$-schemes. We say that $f$ is a rig-étale cover, if it is a composition $S' \to S^\dagger \to S$, where $S^\dagger \to S$ is an admissible formal blowing-up and $S' \to S^\dagger$ is quasi-finite, flat, surjective and étale over the rigid generic fibers.

***Theorem 3.8.11* (Reduced fiber theroem).** — *Let $S$ be an admissible formal $\mathcal{O}_K$-scheme, and $f : X \to S$ be a flat morphism of admissible $\mathcal{O}_K$-schemes such that $f^{\mathrm{rig}} : X^{\mathrm{rig}} \to S^{\mathrm{rig}}$ has reduced geometric fibers.*

*Then there exists a commutative diagram of formal $\mathcal{O}_K$-schemes*

$$\begin{array}{ccc} Y' & & \\ \downarrow & & \\ X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & X, \end{array}$$

*such that*

1. *$S' \to S$ is a rig-étale cover, and $X' = X \times_S S'$,*
2. *$Y' \to X'$ is a finite morphism such that $Y'^{\mathrm{rig}} \to X'^{\mathrm{rig}}$ is an isomorphism,*
3. *$Y' \to S'$ is flat and has reduced geometric fibers.*

***Remark 3.8.12.*** — There exists a similar version of the reduced fiber theorem for schemes. See [**BLR**, Theorem 2.1'].

We describe roughly the ideas behind the proof of Theorem 3.8.11. Let $X \to S$ be a morphism of admissible formal schemes. For a closed point $s \in S$, denote by $X(s)$ the fiber over $s$, and by $X(\bar{s})$ the geometric fiber over a geometric point $\bar{s}$ above $s$. Then $X(\bar{s})$ is reduced if and only if

1. $X(\bar{s})$ is reduced at its generic points, and
2. $X(\bar{s})$ has no embedded components.

So the proof of Theorem 3.8.11 consists of two parts: one being killing the geometric multiplicities of irreducible components of $X(s)$, and the second being killing embedded components.

# CHAPTER 4

# UNIFORMIZATION OF RIGID ANALYTIC CURVES

## 4.1. Riemann-Roch Theorem for proper rigid curves

**4.1.1. Meromorphic functions.** — A good reference for this section is [**Bo83**].

For any ring $R$, let $Q(R)$ denote the total ring of fractions, i.e. $Q(R)$ is the localization of $R$ with respect to the set of non-zero divisors. Let $X$ be a rigid analytic variety over $K$, and $U \subseteq X$ be an affinoid subdomain. The restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is flat, and it induces thus a map $Q(\mathcal{O}_X(X)) \to Q(\mathcal{O}_X(U))$. Let $\mathcal{M}_X$ be the presheaf on $X$ whose value on an affinoid subdomain $U$ is

$$\mathcal{M}_X(U) = Q(\mathcal{O}_X(U)).$$

***Lemma 4.1.2.*** — *The presheaf $\mathcal{M}_X$ is a sheaf.*

*Proof.* — Let $f = \frac{g}{h}$ be an element of $Q(\mathcal{O}_X(X))$ where $g, h \in \mathcal{O}_X$ and $h$ is not a zero divisor. Let $(\mathcal{O}_X, f)$ be the kernel of the map

$$\psi : \mathcal{O}_X \to \mathcal{O}_X/h\mathcal{O}_X$$

given by the multiplication by $g$. Then $(\mathcal{O}_X : f)$ is a coherent ideal of $\mathcal{O}_X$ whose value at an affinoid subdomain $U$ consisting of $a \in \mathcal{O}_X(U)$ such that $af \in \mathcal{O}_X(U)$.

Consider now an admissible affinoid covering $\{U_1, \cdots, U_n\}$ of $X$, and functions $f_i \in \mathcal{M}_X(U_i)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j$. Consider the sheaves $(\mathcal{O}_{U_i} : f_i)$ for $i = 1, \cdots, n$. For any $i, j$, $(\mathcal{O}_{U_i} : f_i)$ coincides with $(\mathcal{O}_{U_j} : f_j)$ over $U_i \cap U_j$. Hence, they define an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$. We claim that there exists a non-zero divisor in $\mathcal{J}(X) \subseteq \mathcal{O}_X(X)$. Otherwise, $\mathcal{J}(X)$ is contained in the union of associated primes of $\mathcal{O}_X(X)$, and hence in one associated prime ideal of $\mathcal{O}_X(X)$. In particular, $\mathcal{J}(X)$ is annihilated by a non-zero element $a \in \mathcal{O}_X(X)$. However, since not all elements of $\mathcal{J}(U_i)$ is not a zero divisor, it follows that $a|_{U_i} = 0$, and hence $a = 0$. Contradiction. Choose now a non-zero element $h \in \mathcal{J}(X)$. They the elements $(h|_{U_i})f_i \in \mathcal{O}_X(U_i)$ coincide with each other over the intersections $U_i \cap U_j$, hence they define a function $g \in \mathcal{O}_X(X)$. We have thus $f = \frac{g}{h}$. □

The sheaf $\mathcal{M}_X$ can be generalized to a general rigid analytic variety $X$ in an evident way. Note that, in general, $\mathcal{M}_X(X)$ is not the ring of fractions of $\mathcal{O}_X(X)$.

**4.1.3. Curves.** — Let $X$ be a rigid analytic curve over $K$, i.e. a separated and reduced rigid $K$-space over $K$ of pure dimension 1. A regular divisor on $X$ is a formal finite sum of regular points. The degree of a regular divisor $D = \sum_P n_P P$ is

$$\deg(D) = \sum_{P \in X} n_P [K(P) : K].$$

Note that if $P$ is a regular point of $X$, the local ring $\mathcal{O}_{X,P}$ is a discrete valuation ring. In particular, $P$ can be defined locally by a single equation. In other words, every regular divisor on $X$ is a Cartier divisor.

Let $D$ be a regular divisor on $X$. As in algebraic geometry, one can attach to $D$ a line bundle $\mathcal{O}_Z(D)$ on $X$ as follows. For any admissible open subset $U \subseteq X$, we put

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{M}_X(U) \mid \operatorname{div}(f) + D \geq 0.\}$$

Then $\mathcal{L}(D)$ is an invertible sheaf on $X$.

Assume now $X$ is proper. By Kiehl's finiteness theorem, the cohomology groups $H^i(X, \mathcal{L}(D))$ are finite dimensional over $K$ for $i \geq 0$, and we denote its dimension by $h^i(\mathcal{O}_X(D))$, and we put

$$\chi(\mathcal{O}_X(D)) = \sum_{i=0}^{\infty} (-1)^i h^i(\mathcal{O}_X(D)).$$

***Lemma 4.1.4.*** — *Let $P \in X$ be a point on a proper curve $X$. Then for any regular divisor $D$ on $X$, we have*

1. $0 \leq h^0(\mathcal{O}_X(D+P)) - h^0(\mathcal{O}_X(D)) \leq \dim_K K(P)$.
2. $0 \leq h^1(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D+P)) \leq \dim_K K(P)$.
3. $\chi(\mathcal{O}_X(D+P)) = \chi(\mathcal{O}_X(D)) + \dim_K K(P)$.

*Proof.* — The proof proceeds exactly as in algebraic geometry. Consider the exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to i_{P,*}K(P) \to 0.$$

Tensoring with $\mathcal{O}_X(D)$, we get an exact sequence sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+P) \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a coherent sheaf concentrated on $P$ with $h^0(\mathcal{F}) = \dim_K K(P)$. The statement follows immediately by taking the long exact sequence. □

***Theorem 4.1.5* (Riemann-Roch theorem).** — *For any regular divisor $D$ on a proper curve $X$, the equality*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg(D)$$

*holds.*

*Proof.* — Write $D = \sum_P n_P P$. The theorem follows by induction on $\sum_P |n_P|$ from the previous Lemma. □

***Theorem 4.1.6.*** — *Every proper rigid analytic curve is projective, hence comes from the analytification of proper algebraic curves over $K$.*

*Sketch.* — Let $X$ be a proper rigid analytic curve with irreducible components $X_i$ for $i = 1, \cdots, n$. Choose a regular point $P_i \in X_i$. By Lemma 4.1.4, there exists an integer $n_i > 0$ such that $h^0(\mathcal{O}_X(n_i P_i)) > h^0(\mathcal{O}_X((n_i - 1)P_i))$. This gives rise to a meromorphic function $f_i$ on $X$ which is analytic everywhere except for a pole of order $n_i$ at $P_i$. Let $f = \sum_i f_i$. Then $f$ defines a morphism $\phi : X \to \mathbb{P}^{1,\mathrm{an}}_K$, which is non-constant at every irreducible component.

Since both $X$ and $\mathbb{P}^{1,\mathrm{an}}_K$ have dimension 1, the morphism $\phi$ is quasi-finite. Since $X$ is proper and $\mathbb{P}^{1,\mathrm{an}}_K$ is separated, $\phi$ is proper as well. Using Stein factorization, one sees easily as in algebraic geometry that proper and quasi-finite morphisms are always finite. Hence, $\phi : X \to \mathbb{P}^{1,\mathrm{an}}_K$ is finite. By rigid GAGA, any finite analytic variety over $\mathbb{P}^{1,\mathrm{an}}_K$ is the analytification of a finite algebraic variety over $\mathbb{P}^1_K$. It follows that $X$ is projective. □

## 4.2. Tate's uniformization of elliptic curves

Recall the classical uniformization of elliptic curves over $\mathbb{C}$: for every elliptic curve $E$ over $\mathbb{C}$, there exists a lattice $\Lambda \subseteq \mathbb{C}$ such that

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda.$$

Obviously, $\Lambda$ and $c\Lambda$ for any $c \in \mathbb{C}^\times$ will define isomorphic elliptic curves. We may assume that $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ with $\mathrm{Im}(\tau) > 0$.

We may ask whether this uniformization can be generalized to a non-archimedean field $K$. A naive idea is to consider the quotient $K/\Lambda$ for some finite free $\mathbb{Z}$-submodule $\Lambda \subseteq K$. However, this quotient is quite problematic, because the natural translation action of $\Lambda$ on $K$ is not discrete.

We consider a slightly different formulation of the complex uniformization. Consider the exponential map

$$\exp : \mathbb{C} \to \mathbb{C}^\times, \quad z \mapsto e^{2\pi i z}.$$

The image of the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ is $q^{\mathbb{Z}} \subseteq \mathbb{C}^\times$ with $q = e^{2\pi i \tau}$ satisfying $0 < |q| < 1$. We have thus

$$E(\mathbb{C}) \cong \mathbb{C}^\times / q^{\mathbb{Z}}.$$

This is this formulation that has a generalization to non-archimedean fields.

Let $K$ be a complete non-archimedean field. The analogue of $\mathbb{C}^\times$ is the rigid space

$$\mathbb{G}_m^{\mathrm{an}} := \bigcup_{n \geq 0} \mathrm{Sp}\big(K\langle \pi^n T, \frac{\pi^n}{T}\rangle\big)$$

such that $\mathbb{G}_m^{\mathrm{an}}(L) = L^\times$ for any extension $L/K$. Let $q \in K$ with $0 < |q| < 1$. We want to consider the quotient of $\mathbb{G}_m^{\mathrm{an}}$ by the subgroup $q^{\mathbb{Z}}$.

***Definition 4.2.1.*** — Let $Y$ be a rigid analytic space, and $\Gamma$ be a group of automorphisms of $Y$. We say that the action of $\Gamma$ is discontinuous, if there exists an admissible affinoid covering $\{Y_i\}$ of $Y$ such that

$$\{\gamma \in \Gamma : \gamma Y_i \cap Y_i \neq \emptyset\}$$

is finite for each $i$.

***Proposition 4.2.2.*** — *Let $Y$ be a rigid analytic space and $\Gamma$ be a group of automorphisms acting discontinuously on $Y$. Then there exists a rigid analytic space $Y/\Gamma$ and a morphism $p : Y \to Y/\Gamma$ such that for any admissible open $\Gamma$-invariant $U \subseteq Y$ and any $\Gamma$-invariant morphism $f : U \to X$, the set $p(U)$ is an admissible open and $f$ factors uniquely through $p(U)$.*

*Proof.* — The underlying set of $Y/\Gamma$ is the set-theoretic quotient. As a $G$-topological space, a set $U \subseteq Y/\Gamma$ is admissible if and only if $p^{-1}(U)$ is, and a covering $\{U_i\}$ of $U$ is admissible if $\{p^{-1}(U_i)\}$ is an admissible covering of $p^{-1}(U)$. Finally, we define the structure sheaf as $\mathcal{O}_{Y/\Gamma}(U) = \mathcal{O}_Y(p^{-1}(U))^{\Gamma}$. □

It is easy to see that the action of $q^{\mathbb{Z}}$ on $\mathbb{G}_m^{\mathrm{an}}$ is discontinuous. We define the Tate curve

$$\mathrm{Tate}_q = \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}.$$

***Proposition 4.2.3.*** — *The Tate curve* $\mathrm{Tate}_q$ *is a connected, regular, separated and proper rigid space of dimension* 1.

*Proof.* — We give a more explicit description of $\mathrm{Tate}_q$. Let $p : \mathbb{G}_m^{\mathrm{an}} \to \mathrm{Tate}_q$ be the canonical projection,

$$U_0 = p\Big(\{x \in \mathbb{G}_m^{\mathrm{an}} : |q|^{1/2} \leq |z| \leq 1\}\Big)$$
$$U_1 = p\Big(\{x \in \mathbb{G}_m^{\mathrm{an}} : |q| \leq |z| \leq |q|^{1/2}\}\Big)$$

Then $U_0$ and $U_1$ give an admissible covering of $\mathrm{Tate}_q$, and we have

$$U_0 \cap U_1 = U_- \coprod U_+,$$

where $U_-$ (resp. $U_+$) is the image of the thin annulus at $|q|^{1/2}$ and (resp. at 1).

Properness of $\mathrm{Tate}_q$ is easy. We put

$$U_0' = p\Big(\{x \in \mathbb{G}_m^{\mathrm{an}} : |q|^{1/2+\epsilon} \leq |z| \leq |q|^{-\epsilon}\}\Big)$$
$$U_1' = p\Big(\{x \in \mathbb{G}_m^{\mathrm{an}} : |q|^{1+\epsilon} \leq |z| \leq |q|^{1/2-\epsilon}\}\Big)$$

for some small rational $\epsilon < 1/4$. Then $\{U_0', U_1'\}$ is also an admissible covering of $\mathrm{Tate}_q$, and we have $U_0 \Subset U_0'$ and $U_1 \Subset U_1'$.

Regularity of $\mathrm{Tate}_q$ is also clear, since every point of $\mathrm{Tate}_q$ has an affinoid neighborhood isomorphic to $\{x \in \mathbb{G}_m^{\mathrm{an}} : |q|^r \leq |z| \leq |q|^s\}$ which is clearly smooth over $K$.

It remains to prove the separateness of $\mathrm{Tate}_q$. We have to show that the diagonal morphism

$$\Delta : \mathrm{Tate}_q \to \mathrm{Tate}_q \times \mathrm{Tate}_q$$

is a closed immersion. We consider the admissible covering $\{U_0 \times U_0, U_0 \times U_1, U_1 \times U_1\}$ of $\mathrm{Tate}_q \times \mathrm{Tate}_q$, and it suffices to show that the restriction of $\Delta$ to each member of this covering is a closed immersion. Since $U_0, U_1$ are separated, it is enough to show that

$$U_0 \cap U_1 = U_- \coprod U_+ \to U_0 \times U_1$$

is a closed immersion, i.e. the homomorphism

$$\mathcal{O}_{U_0}(U_0) \hat{\otimes}_K \mathcal{O}_{U_1}(U_1) \to \mathcal{O}_{U_-}(U_-) \oplus \mathcal{O}_{U_+}(U_+)$$

is surjective. Up to making an extension of the base field, we may assume that there exists an $\pi \in K$ with $|\pi|^2 = |q|$. Then we have

$$\mathcal{O}_{U_0}(U_0) = K\langle \frac{\pi}{T}, T \rangle, \quad \mathcal{O}_{U_1}(U_1) = K\langle \frac{q}{T}, \frac{T}{\pi} \rangle$$
$$\mathcal{O}_{U_-}(U_-) = K\langle \frac{T}{\pi}, \frac{\pi}{T} \rangle, \quad \mathcal{O}_{U_+}(U_+) = K\langle T, \frac{1}{T} \rangle.$$

Note that the morphism $\mathcal{O}_{U_1}(U_1) \to \mathcal{O}_{U_+}(U_+)$ is given by $T \mapsto qT$. It is direct to check that the desired map is surjective.

□

***Corollary 4.2.4.*** — $\mathrm{Tate}_q$ *is a projective elliptic curve over $K$.*

*Proof.* — This follows directly from Proposition 4.2.3 and Theorem 4.1.6.

Using formula forms, one can also give an explicit equation for $\mathrm{Tate}_q$:

$$y^2 + xy = x^3 + a_4(q)x + a_6(q), \tag{4.2.4.1}$$

where

$$a_4(q) = -5\sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = -(5q + 45q^2 + 140q^3 + \cdots)$$
$$a_6(q) = -\sum_{n=1}^{\infty} \frac{7n^5 + 5n^3}{12} \times \frac{q^n}{1-q^n} = -(q + 23q^2 + 154q^3 + \cdots).$$

From this equation, one can compute easily that the genus of $\mathrm{Tate}_q$ is 1. □

Using the equation 4.2.4.1, it is easy to see that the $j$-invariant of $\mathrm{Tate}_q$ is given by the classical formula

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$

which has absolute value $|j(q)| > 1$. The equation (4.2.4.1) actually defines an algebraic curve over $\mathcal{O}_K$. Its completion along the special fiber gives a formal model, denoted by $\mathfrak{X}_q$, for the rigid analytic curve $\mathrm{Tate}_q$. Note that the special fiber of $\mathfrak{X}_q$ has affine equation

$$y^2 + xy = x^3$$

which is isomorphic to $\mathbb{P}^1$ cutting itself at a simple double point $P = (0,0)$. Note that the two tangent vectors of $\mathfrak{X}_q \otimes k$ at $P$ are all defined over $K$; in this case, we say that $\mathfrak{X}_q$ has split multiplicative reduction.

These properties actually characterize the Tate curve $\mathrm{Tate}_q$.

***Theorem 4.2.5.*** — 1. *Two Tate curves* $\mathrm{Tate}_{q_1}$ *and* $\mathrm{Tate}_{q_2}$ *are isomorphic if and only if* $q_1 = q_2$.

2. *Assume that* $K$ *has characteristic* 0. *If* $E$ *is an elliptic curve over* $K$ *with* $|j(E)| > 1$, *then there exists a Galois extension* $L/K$ *of degree* $\leq 2$ *such that* $E_L^{\mathrm{an}}$ *is isomorphic to a Tate curve. Moreover, one can take* $L = K$ *if the minimal Weierstrass equation for* $E$ *has an integral model over* $k$ *with split multiplicative reduction.*

*Proof.* — Assertion (1) follows from the fact that the equation $j(q) = j$ has a unique solution if $|j| > 1$. Indeed, one writes

$$q^{-1} = j - \sum_{n=0}^{\infty} c_n q^n = j - c_0 - \sum_{n=1}^{\infty} c_n q^n \quad \text{with } c_n \in \mathbb{Z},$$

and hence

$$\begin{aligned} q &= 1/(j - c_0 - \sum_{n=1}^{\infty} c_n q^n) = (j - c_0)^{-1}(1 - (j - c_0)^{-1} \sum_{n=1}^{\infty} c_n q^n)^{-1} \\ &= (j - c_0)^{-1} \sum_{m=0}^{\infty} (j - c_0)^{-m} (\sum_{n=1}^{\infty} c_n q^n)^m. \end{aligned}$$

We denote by $F_j(q)$ the expression on the right hand side, viewed as a function of $q$ parametrized by $j$. Note that $|(j - c_0)^{-1}| = |j|^{-1} < 1$ and $|c_n| \leq 1$ for all $n \geq 1$. Then we have

$$|F_j(q_1) - F_j(q_2)| < |q_1 - q_2|,$$

i.e. $q \mapsto F_j(q)$ is a contracting map on the complete metric space $K$. Therefore, the squence $q, F_j(q), F_j \circ F_j(q), \cdots, F_j^{(n)}(q), \cdots$, is a Cauchy sequence in $K$, whose limit gives a unique solution to $j(q) = j$.

(2) Let $q \in K$ be the unique solution to $j(q) = j(E)$. Then $E$ and $\mathrm{Tate}_q$ have the same $j$-invariant, hence isomorphic to an algebraic closure of $K$, denoted by $\overline{K}$. Such elliptic curves are classified by the Galois cohomology group

$$H^1(\mathrm{Gal}(\overline{K}/K), \mathrm{Aut}(\mathrm{Tate}_q \otimes_K \overline{K})))$$

But $\mathrm{Tate}_q$ does not have complex multiplication (otherwise, the $j$-invariant of $\mathrm{Tate}_q$ would be an algebraic integer), we have $\mathrm{Aut}(\mathrm{Tate}_q \otimes_K \overline{K}) = \{\pm 1\}$, and

$$H^1(\mathrm{Gal}(\overline{K}/K), \mathrm{Aut}(\mathrm{Tate}_q \otimes_K \overline{K})) \cong \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \{\pm 1\}).$$

Then every class of $h \in \mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K), \{\pm 1\})$ is split over a quadratic extension of $K$, namely the subfield of $\overline{K}$ fixed by $\mathrm{Ker}(h)$.

□

## 4.3. Mumford's analytic uniformization of curves of genus $g \geq 2$

In this section, we assume that $K$ is a discrete valuation field with uniformizer $\pi$. Tate's uniformization of elliptic curves admits a generalization to higher genus case, due to Mumford [**Mu**]. We first state the main theorem of Mumford's theory.

***Lemma 4.3.1.*** — *Let $\gamma \in \mathrm{PGL}_2(K)$ be the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$. The following conditions are equivalent:*

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has two eigenvalues in $K$ with distinct valuations.*

(b) *$\gamma$ is conjugate to the image of $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ with $q \in K$ and $0 < |q| < 1$.*

(c) *$a + d \neq 0$ and $(ad - bc)/(a + d)^2 \in \mathfrak{m}_K$.*

*Proof.* — $(a) \Leftrightarrow (b)$ and (b)⇒(c) are trivial. Suppose now (c) holds. Then the characteristic polynomial of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$X^2 - (a + d)X + (ad - bc) = 0$$

By Hensel's Lemma, this equation has two roots of the form

$$\lambda_1 = u \cdot (a + d), \quad \lambda_2 = v \cdot u^{-1}(a + d),$$

where $u \in \mathcal{O}_K^\times$ and $v = (ad - bc)/(a + d)^2 \in \mathfrak{m}_K$. Then $|\lambda_2| < |\lambda_1|$. This shows $(c) \Rightarrow (a)$. □

***Definition 4.3.2.*** — (1) An element $\gamma \in \mathrm{PGL}_2(K)$ is called hyperbolic if it satisfies the equivalent conditions in Lemma 4.3.1.

(2) A Schottky group $\Gamma \subset \mathrm{PGL}_2(K)$ is a finitely generated subgroup such that every $\gamma \in \Gamma$ with $\gamma \neq 1$ is hyperbolic.

We fix a Schottky subgroup $\Gamma \subseteq \mathrm{PGL}_2(K)$. Recall that one has a natural action of $\mathrm{PGL}_2(K)$ on $\mathbb{P}^1_K$.

***Definition 4.3.3.*** — A point $P \in \mathbb{P}^1_K(\overline{K})$ is called a limit point of $\Gamma$ if there exists $Q \in \mathbb{P}^1(\overline{K})$ and distinct $\gamma_n \in \Gamma$ such that $P = \lim_{n\to\infty} \gamma_n(Q)$. We denote by $\mathcal{L}(\Gamma)$ the set of limit points of $\Gamma$.

***Example 4.3.4.*** — Let $q \in K$ with $0 < |q| < 1$. Then $\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ is hyperbolic, and $\Gamma = \gamma^{\mathbb{Z}}$ is a Schottky subgroup with limit set $\mathcal{L}(\Gamma) = \{0, \infty\}$.

***Definition 4.3.5.*** — Let $C$ be a connected proper curve over $k$. Then we say $X$ is $k$-split degenerate if the following conditions hold:

1. $C$ is reduced.
2. The normalization of each irreducible component of $C$ is isomorphic to $\mathbb{P}^1_k$.
3. A singular point of $C$ is rational over $k$ and its completed local ring is isomorphic to $k[[x, y]]/(xy)$, i.e. $C$ has only $k$-rational ordinary double points as its singuarity.

The main theorem of Mumford is the following

***Theorem 4.3.6* (Mumford).** — *Let $\Gamma \subseteq \mathrm{PGL}_2(K)$ be a Schottky subgroup with limit set $\mathcal{L}(\Gamma)$, and $\Omega_\Gamma = \mathbb{P}^{1,\mathrm{an}}_K - \mathcal{L}(\Gamma)$. Then the following holds*

1. *The quotient* $X_\Gamma := \Omega_\Gamma/\Gamma$ *is a connected proper and smooth curve over* $K$.
2. $X_\Gamma$ *admits a canonical formal model* $\mathfrak{X}_\Gamma$*, whose special fiber* $\mathfrak{X}_{\Gamma,k}$ *is* $k$*-split degenerate.*

***Remark 4.3.7.*** — Recall that a $k$-split degenerate curve is completely described in terms of its dual graph. We will give later an explicit description of the dual graph of $\mathfrak{X}_{\Gamma,k}$ in terms of $\Gamma$.

We have also an inverse to Mumford's construction.

***Theorem 4.3.8.*** — *Let* $\mathfrak{X}$ *be a projective flat curve over* $\mathrm{Spf}(\mathcal{O}_K)$ *(or* $\mathrm{Spec}(\mathcal{O}_K)$*) such that the generic fiber is smooth of genus* $g \geq 2$*, and the special fiber is* $k$*-split degenerated and stable in the sense that each irreducible component of meets other components at least* 3 *points. Then there exists a Schottky group* $\Gamma \subseteq \mathrm{PGL}_2(K)$ *such that* $\mathfrak{X}$ *is isomorphic to* $\mathfrak{X}_\Gamma$.

The rest of this chapter will be devoted to the proof of Mumford's Theorems 4.3.6 and 4.3.8.

## 4.4. Trees and $p$-adic Schottky groups

In this section, we discuss some preliminaries on graph theory for the proof of Theorem 4.3.6.

**4.4.1. The Bruhat-Tits tree of** $\mathrm{PGL}_2(K)$**.** — Let $V$ be a $K$-vector space of dimension 2. We identify $\mathrm{GL}(V) \cong \mathrm{GL}_2(K)$ by fixing a standard basis $(e_1, e_2)$ of $V$ over $K$. A lattice $M$ of $V$ is a free $\mathcal{O}_K$-submodule of $V$ of rank 2. Two lattices $M$ and $M'$ are called equivalent if there exists $c \in K^*$ such that $M' = cM$. Let $\mathcal{S}$ denote the set of equivalence classes of lattices in $V$.

If $M$ and $M'$ are two lattices in $V$. By theorem of elementary divisors, we can find a basis $v_1, v_2$ of $M$ and integers $n_1, n_2$ such that $(\pi^{n_1}v_1, \pi^{n_2}v_2)$ is a basis of $M'$. The integer $d(M, M') := |n_2 - n_1|$ depends only on the equivalent classes of $M$ and $M'$, and it is called the distance of $M$ and $M'$. Up to multiplying $M'$ by a scalar, we may assume that $n_1 = 0$ and $n_2 \geq 0$; in this case, we say that $M \supset M'$ are in standard position.

We construct a graph $\mathcal{T}$ with the set of vertices given by $\mathcal{S}$ as follows: Two vertices are connected by an edge if their distance is 1.

***Proposition 4.4.2.*** — *(1) The graph* $\mathcal{T}$ *is a tree, i.e.* $\mathcal{T}$ *is connected and there is no loops in* $\mathcal{T}$.

*(2) Let* $[M] \in \mathcal{S}$ *be a vertex. The set of edges connecting to* $[M]$ *is in natural bijection with* $\mathbb{P}(M \otimes_{\mathcal{O}_K} k)$.

*Proof.* — To see that $\mathcal{T}$ is connected, consider two lattices $M, M'$ in $V$. Up to rescaling, there exists basis $v_1, v_2$ of $M$ such that $M' = \mathcal{O}_K \cdot v \oplus \pi^n \mathcal{O}_K v_2$. For $r = 0, \cdots, n$, put $M_r = \mathcal{O}_K \cdot v_1 \oplus \pi^r \mathcal{O}_K \cdot v_2$. Then we have $d(M_r, M_{r+1}) = 1$ for all $0 \leq r \leq n-1$. Therefore, $M = M_0$ and $M' = M_n$ is connected by the chain $M_0M_1, M_1M_2, \cdots, M_{n-1}M_n$.

To see $\mathcal{T}$ contains no loops, we prove by induction on $r \geq 1$ that if $[M], [M'] \in \mathcal{S}$ are two vertices connected by a chain of edges of length $r$, then $d(M, M') = r$. The case $r = 1$

follows from definition. Assume now that $[M], [M']$ are connected by a chain of edges of length $r+1$. Let $[M_r]$ be the lattice connecting to $[M]$ of length $r$, and to $[M']$ of length 1. By induction hypothesis, one has $d(M, M_r) = 4$. Thus, up to rescaling $M_r$, we may assume that there exist a basis $(v_1, v_2)$ of $M$ such that $(v_1, \pi^r v_2)$ is a basis of $M_r$. Up to rescaling $M'$, we may assume that

$$M_r \supsetneq M' \supsetneq \pi M_r.$$

Since the chain connecting $[M]$ and $[M']$ has length $r$, we have $M' \neq \pi M_{r-1} := \pi \mathcal{O}_K v_1 \oplus \pi^r \mathcal{O}_K v_2$. Note that $M_r/\pi M_r$ is 2-dimensional vector space over $k$ with basis $\bar{v}_1, \overline{\pi^r v_2}$. The inequality $M' \neq \pi M_{r-1}$ implies that the image of $M'/\pi M_r$ is generated by a vector of the form $\bar{v}_1 + \lambda \cdot \overline{\pi^r v_2}$ for some $\lambda \in k$. Therefore, the image of $M'/\pi M_r$ in the quotient

$$M/\pi M_r = \mathcal{O}_K/\pi \cdot v_1 \oplus \mathcal{O}_K/\pi^{r+1}\mathcal{O}_K \cdot v_2$$

is generated by $\bar{v}_1 + \lambda \cdot \overline{\pi^r v_2}$. It follows that

$$M/M' \cong (M/\pi M_r)/(M'/\pi M_r).$$

(2) Indeed, the set edges connecting to $[M]$ is given by the equivalent classes of lattices $M'$ with $M \supsetneq M' \supsetneq \pi M$. Such $M'$'s are in natural bijection with the one dimensional quotients of $M/\pi M$. □

***Corollary 4.4.3.*** — *For any two vertices $[M], [M'] \in \mathcal{T}$, there is a unique segment $[MM']$ of length $d(M, M')$ in $\mathcal{T}$ connecting $[M]$ and $[M']$.*

*Proof.* — The existence of the segment $[MM']$ follows from the connectness of the graph $\mathcal{T}$, and the uniqueness follows from the fact that $\mathcal{T}$ is a tree. The fact that the segment $[MM']$ has length $d(M, M')$ follows from the proof of the previous proposition. □

The tree $\mathcal{T}$ is usually called the Bruhat-Tits building of the group $\mathrm{PGL}_2(K)$. It has a natural transitive and isometric action by $\mathrm{PGL}_2(K)$. The stabilizer of a lattice $M$ is the image of $\mathrm{GL}(M)$, and conjugate to $\mathrm{PGL}_2(\mathcal{O}_K)$. The strict stabilizer of an edge is conjugate to the image of the Iwahoric subgroup

$$\mathrm{Iw} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K) \,|\, c \equiv 0 \mod \pi.\}$$

***Corollary 4.4.4.*** — *Let $\Gamma \subseteq \mathrm{PGL}_2(K)$ be a Schottky subgroup. Then the action of $\Gamma$ on $\mathcal{T}$ is free, and $\Gamma$ is a free group.*

*Proof.* — Indeed, if $\gamma \in \Gamma$ with $\gamma \neq e$ has a fixed point $P$ on $\mathcal{T}$. Then $P$ is either a vertex or the midpoint of an edge. In the second case, $\gamma^2$ fixes the 2 endpoints of the edge. Therefore, $\gamma^2$ belongs to the stabilizer of a vertex, which is necessarily of the form

$$g\mathrm{PGL}_2(\mathcal{O}_K)g^{-1} \subseteq \mathrm{PGL}_2(K).$$

However, since $\mathrm{PGL}_2(\mathcal{O}_K)$ contains no parabolic elements, it follows that $\Gamma$ acts freely on $\mathcal{T}$. Now it is a well known fact that a group acting freely on a tree is a free group (See [**Se03**, §3 Theorem 4]). □

**4.4.5. Geometry of the tree $\mathcal{T}$.** — An infinite path starting from a fixed vertex $P \in \mathcal{T}$ without backtracking is called a semi-line. Two such semi-lines define the same end if they coincide except a finite number of vertices. A semi-line of $\mathcal{T}$ is represented by a chain of lattices:

$$M_0 \supset M_1 \supset \cdots$$

such that $M_0/M_n \cong \mathcal{O}_K/(\pi^n)$. Then $D = \cap M_n$ is a direct factor of each $M_i$ and $D \otimes_{\mathcal{O}_K} K$ is a line in $V$. In this way, one establishes $\mathrm{PGL}_2(K)$-equivariant bijections among the three sets of

- the set of ends of $\mathcal{T}$,
- lines $L \subseteq V$, and
- $K$-rational points of $\mathbb{P}(V)$.

Conversely, given a vertex $[M] \in \mathcal{S}$ and an end $L \subseteq V$, there exists a unique semi-line starting from $[M]$ with end $L$, namely the chain of lattices

$$M_0 = M \supset M_1 \supset \cdots \supset M_r \supset \cdots$$

with $M_r = M \cap L + \pi^n M$ for $n \geq 0$.

Given two distinct ends $x_1, x_2 \in \mathbb{P}(V)$, one can construct a straight line $[x_1x_2] \subset \mathcal{T}$. Indeed, if $v_1, v_2$ are generators of the lines given by $x_1, x_2$, then we have an unbounded infinite chain of lattices:

$$\cdots \supset M_{-1} \supset M_0 \supset M_1 \supset \cdots$$

where $M_r = \mathcal{O}_K v_1 + \pi^r \mathcal{O}_K v_2$.

***Example 4.4.6.*** — Let $\gamma \in \mathrm{PGL}_2(K)$ be a hyperbolic element. Then there exists $g \in \mathrm{PGL}_2(K)$ such that $\gamma = g \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} g^{-1}$ for some $q \in K$ with $0 < |q| < 1$. Therefore, $\gamma$ has two fixed points on $\mathbb{P}^1_K$, namely $\tilde{g}(e_1), \tilde{g}(e_2) \in \mathbb{P}^1(K)$, where $\tilde{g} \in \mathrm{GL}_2(K)$ is any lift of $g$. Thus, $\gamma$ determines a straight line in $\mathcal{T}$, called the axis of $\gamma$, which is given by the chain of lattices

$$\cdots \supset M_r \supset M_{r+1} \supset \cdots$$

with $M_r = \mathcal{O}_K \tilde{g}(e_1) + \mathcal{O}_K \pi^r \tilde{g}(e_2)$ for $r \in \mathbb{Z}$. The axis of $\gamma$ is stable under the action of $\gamma$, which sends $[M_r] \mapsto [M_{r+v_\pi(q)}]$.

Let $x_1, x_2, x_3 \in \mathbb{P}(V)$ be three distinct ends in $\mathbb{P}(V)$. Let $v_i$ be a generators of the line in $V$ corresponding to $x_i$. Then there exists a linear equation

$$a_1v_1 + a_2v_2 + a_3v_3 = 0,$$

where $(a_1, a_2, a_3) \in (K^*)^3$ is unique up to scalar. Then $\sum_{i=1}^3 \mathcal{O}_K \cdot a_i v_i$ is a lattice in $V$, and we denote its equivalent class by $\sigma(x_1, x_2, x_3)$. Alternatively, one can also define $\sigma(x_1, x_2, x_3)$ as the unique intersection of the three straight lines $[x_1x_2]$, $[x_2x_3]$ and $[x_1x_3]$.

***Example 4.4.7.*** — Let $x_1 = (1:0), x_2 = (0:1), x_3 = (1:1) \in \mathbb{P}^1(K)$. Then we have $\sigma(x_1, x_2, x_3)$ is the vertex given by the lattice $\mathcal{O}_K e_1 + \mathcal{O}_K e_2$. In general, for $x_3' = (a:b) \in \mathbb{P}^1(K)$, we have $\sigma(x_1, x_2, x_3') = [a\mathcal{O}_K e_1 + b\mathcal{O}_K e_2]$.

***Lemma 4.4.8.*** — *Let $x_1, x_2, x_3 \in \mathbb{P}(V)$ be three distinct ends, and $P = \sigma(x_1, x_2, x_3)$. Then the semi-lines starting from $P$ to the 3 ends $x_1, x_2, x_3$ all start off on different edges.*

*Proof.* — It suffices to show that any two directions, say to $x_1$ and $x_2$, are different. Choosing a suitable basis for $V$, we may assume that $x_1 = 0, x_2 = \infty$ as points of $\mathbb{P}(V) \cong \mathbb{P}^1_K$. So we are in the situation of previous example. The semi-lines starting from $P$ to $x_1$ and $x_2$ are the two directions of the axis of $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$. □

**4.4.9. Tree associated to a Schottky group.** — Let $\Gamma$ be a non-abelian Schottky subgroup of $\mathrm{PGL}_2(K)$. Note that each $\gamma \in \Gamma$ has two fixed points in $\mathbb{P}(V)$. Let $\Sigma \subseteq \mathbb{P}(V)$ be the set of fixed points of some elements in $\Gamma$. Then $\Sigma$ is stable under the natural action of $\Gamma$ on $\mathbb{P}(V)$. Let $\mathcal{S}_\Gamma$ be the subset of $\mathcal{T}$ consisting of vertices of the form $\sigma(x, y, z)$ for $x, y, z \in \Sigma$. Since $\Gamma$ is non-abelian, $\mathcal{S}_\Gamma$ is not empty.

***Proposition 4.4.10.*** — *The set $\mathcal{S}_\Gamma$ is the set of vertices of a unique subtree $\mathcal{T}_\Gamma \subseteq \mathcal{T}$.*

For the proof this propoistion, we need a Lemma.

***Lemma 4.4.11.*** — *Let $[M_1], [M_2], [M_3] \in \mathcal{S}$ be distinct vertices. Let $m = d(M_1, M_2)$ and $n = d(M_1, M_3)$, and*

$$M_1 \supset M_2 \supset \pi^n M_1, \quad M_1 \supset M_3 \supset \pi^m M_1$$

*be representatitves in standard position. Put $N = M_2 + M_3$, and $r = d(M_1, N)$. Then there exist $u, v \in M_1$ such that*

$$M_1 =< u, v >, \quad N =< u, \pi^r v >$$
$$M_2 =< u, \pi^m v >, \quad M_3 =< u + \pi^r v, \pi^n v > .$$

*Proof.* — Since $M_1 \supset M_2$ are in standard position, we may choose first $u_1, u_2 \in M_1$ such that $M =< u_1, u_2 >$ and $M_2 =< u_1, \pi^m u_2 >$. Consider the exact sequence

$$0 \to N/M_2 \to M_1/M_2 \to M_1/N \to 0.$$

Since $M_1/M_2 \cong \mathcal{O}_K/(\pi^m)\bar{u}_2$ and $M_1/N \cong \mathcal{O}_K/(\pi^r)$, $N/M_2$ must the submodule $\pi^r \mathcal{O}_K/(\pi^n)\bar{u}_2$, and hence $N =< u_1, \pi^r u_2 >$. Consider now the exact sequence

$$0 \to N/M_3 \to M_1/M_3 \to M_1/N \to 0.$$

Since $M_1/M_3 \cong \mathcal{O}_K/\pi^n$ and $M_1/N \cong \mathcal{O}_K/\pi^r$ with a generator given by the image of $u_2$, $u_2$ also gives a generator of $M_1/M_3$. Thus, there exists $a \in \mathcal{O}_K$ such that $u_1 + au_2 \in M_3$. Since $N/M_3$ is the submodule $\pi^r \mathcal{O}_K/(\pi^n)\bar{u}_2$ and $N = M_2 + M_3$, we have $v_\pi(a) = r$. Write thus $a = \pi^r \lambda$ with $\lambda \in \mathcal{O}_K^\times$. We have $M_3 = (u_1 + \pi^r \lambda u_2)\mathcal{O}_K + \pi^n u_2 \mathcal{O}_K$. The Lemma holds with $u = u_1$ and $v = \lambda u_2$.

□

*Proof of Proposition 4.4.10.* — We define the graph $\mathcal{T}_\Gamma$ as the union of $\mathcal{S}_\Gamma$ together with the segments connecting any two vertices in $\mathcal{S}_\Gamma$. Let $[M_j] = \sigma(x_j, y_j, z_j)$ for $j = 1, 2, 3$ be three distinct vertices in $\mathcal{S}_\Gamma$. We choose representatives $M_1, M_2, M_3$ so that

$$M_1 \supset M_2 \supset \pi^m M_1, \quad M_1 \supset M_3 \supset \pi^n M_1$$

are in standard position. Let $N = M_2 + M_3$. Then $[N]$ is the unique common vertex lying on the three segments $[M_1M_2], [M_2M_3], [M_3M_1]$. To show that $\mathcal{T}_\Gamma$ is a tree, it is enough to show that $[N] \in \mathcal{S}_\Gamma$.

If $N = M_2$ or $M_3$, the statement is trivial. Assume thus $N \neq M_2, M_3$. Choose generators $u_j, v_j, w_j \in V$ for $x_j, y_j, z_j \in \mathbb{P}(V)$ so that one has linear equations $a_ju_j + b_jv_j + c_jw_j = 0$ with $a_j, b_j, c_j \in \mathcal{O}_K^\times$. Hence, we have $M_j =< u_j, v_j >$ for $j = 1, 2, 3$. Since $M_2 \not\subset \mathfrak{m}_K M_1$, one of $u_2, v_2, w_2$ must have non-zero image in $M_1/\mathfrak{m}_K M_1$. Renaming, we may assume that $u_2 \notin \mathfrak{m}_K M_1$. Similarly, we may assume that $u_3 \notin \mathfrak{m}_K M_1$. Since the images $\bar{u}_1, \bar{v}_1, \bar{w}_1 \in M_1/\mathfrak{m}_K M_1$ of $u_1, v_1, w_1$ generate different lines in $M_1/\mathfrak{m}_K M_1$, one of them is linearly independent from $\bar{u}_2$ or $\bar{u}_3$. Renaming, we may assume that the line generated by $\bar{u}_1$ is different from those by $\bar{u}_2$ and $\bar{u}_3$. Hence, $\bar{u}_1, \bar{u}_2$ form a basis of $M_1/\mathfrak{m}_K M_1$, and we have a linear relation

$$\bar{\alpha}\bar{u}_1 + \bar{\beta}\bar{u}_2 + \bar{u}_3 = 0,$$

where $\bar{\alpha}, \bar{\beta} \in k$ and $\bar{\beta} \neq 0$. It follows that $u_1, u_2, u_3$ satisfy a relation

$$\alpha u_1 + \beta u_2 + u_3 = 0, \tag{4.4.11.1}$$

where $\alpha, \beta \in \mathcal{O}_K$ and $\beta \notin \mathfrak{m}_K$. Choose generators $u, v$ of $M_1$ as in the previous Lemma. One has

$$\begin{cases} u_2 = \lambda_2 u + \mu_2 \pi^m v, & \lambda_2, \mu_2 \in \mathcal{O}_K, \lambda_2 \notin \mathfrak{m}_K, \\ u_3 = \lambda_3(u + \pi^r v) + \mu_3 \pi^n v, & \lambda_3, \mu_3 \in \mathcal{O}_K, \lambda_3 \notin \mathfrak{m}_K. \end{cases}$$

Since $N \neq M_2, M_3$, we have $r > m, n$ and $u_2$ and $u_3$ are linearly independent over $K$. Therefore, we have $\alpha \neq 0$ in (4.4.11.1) and $x_1, x_2, x_3$ are three distinct ends of $\mathcal{T}$. The vertex $\sigma(x_1, x_2, x_3)$ is represented by the lattice $\mathcal{O}_K u_2 + \mathcal{O}_K u_3$, which coincides with $N = \mathcal{O}_K u + \pi^r \mathcal{O}_K v$. This finishes the proof.

□

The notions of semi-lines and ends apply to the tree $\mathcal{T}_\Gamma$ as well.

***Lemma 4.4.12.*** — *Let $\partial(\mathcal{T}_\Gamma)$ denote the set of ends of $\mathcal{T}_\Gamma$. Then we have $\Sigma \subseteq \partial(\mathcal{T}_\Gamma) \subseteq \mathbb{P}^1(K)$.*

*Proof.* — Let $\gamma \in \Gamma$ with fixed points $x_0, x_\infty \in \Sigma \subseteq \mathbb{P}^1(K)$. Let $y \in \Sigma$ be a third element distinct from $x_0, x_\infty$, whose existence is guaranteed by the fact that $\Gamma$ is non-abelian. Let $P = \sigma(x_0, x_\infty, y) \in \mathcal{S}_\Gamma$. Then $P$ is a vertex of $\mathcal{T}_\Gamma$ lying on the axis of $\gamma$ in $\mathcal{T}$, and $\gamma^n(P) = \sigma(x_0, x_\infty, \gamma^n(y))$ for $n \in \mathbb{Z}$ are distinct vertices in $\mathcal{T}_\Gamma$ on the axis of $\gamma$. It follows that the axis of $\gamma$ is contained in $\mathcal{T}_\Gamma$ (but note that not all vertices lying on the axis of $\gamma$ are contained in $\mathcal{T}_\Gamma$), and hence $x_0, x_\infty \in \partial(\mathcal{T}_\Gamma)$. □

The group $\Gamma$ acts on $\mathcal{S}_\Gamma$, and thus acts (freely) on the tree $\mathcal{T}_\Gamma$.

***Proposition 4.4.13.*** — *The quotient graph $\mathcal{T}_\Gamma/\Gamma$ is finite.*

*Proof.* — Let $\gamma_1, \cdots, \gamma_n \in \Gamma$ be free generators of $\Gamma$ and let $v \in \mathcal{T}_\Gamma$ be any vertex of $\mathcal{T}_\Gamma$. Put

$$\mathcal{D} = [v\gamma_1(v)] \cup \cdots \cup [v\gamma_n(v)],$$

where $[v\gamma_i(v)]$ is the segment connecting $v$ and $\gamma_i(v)$. Then $\mathcal{D}$ is a finite subtree of $\mathcal{T}_\Gamma$, and we claim that $\mathcal{D}$ maps onto $\mathcal{T}_\Gamma/\Gamma$. This is equivalent to saying that

$$\tilde{\mathcal{D}} = \bigcup_{\gamma\in\Gamma} \gamma(\mathcal{D}) = \bigcup_{\gamma\in\Gamma} \bigcup_{i=1}^{n} [\gamma(v), \gamma\gamma_i(v)]$$

is equal to $\mathcal{T}_\Gamma$.

Note first that $\tilde{\mathcal{D}}$ is connected. Secondly, note that every $x \in \Sigma$ is also an end of the tree $\tilde{D}$. In fact, suppose that $x$ is a fixed point of $\gamma \in \Gamma$. If $u \in \tilde{\mathcal{D}}$ is a vertex such that $d(u, \gamma(u))$ is minimal, then the union of the segment $[\gamma^i(u), \gamma^{i+1}(u)]$ is the axis of $\gamma$. Thus, $x$, which is one of the two ends of the axis of $\gamma$, is an end of $\tilde{\mathcal{D}}$. Finally, we prove that all vertices in $\mathcal{T}_\Gamma$ are contained in $\tilde{\mathcal{D}}$. Let $w = \sigma(x, y, z)$ be a vertex of $\mathcal{T}_\Gamma$. If $w \notin \tilde{\mathcal{D}}$, then $w$ is connected to $\tilde{\mathcal{D}}$ by a unique path $\tau$ (if there were two such paths, then there would exist a circuit in $\mathcal{T}_\Gamma$). Thus all the paths from $w$ to all ends of $\mathcal{D}$ start with the same edge. Since $x, y, z \in \Sigma$ are all ends of $\tilde{\mathcal{D}}$, this contradicts with Lemma 4.4.8.

□

***Remark 4.4.14.*** — If $\mathfrak{X}_\Gamma$ is the projective flat curve over $\mathcal{O}_K$ associated to $\Gamma$ given by Theorem 4.3.6, we will see later that

$$\mathfrak{X}_{\Gamma,K}^{\mathrm{an}} \cong (\mathbb{P}_K^{1,\mathrm{an}} - \partial(\mathcal{T}_\Gamma))/\Gamma.$$

and $\mathcal{T}_\Gamma/\Gamma$ is the dual graph of the special fiber of $\mathfrak{X}_\Gamma$.

***Corollary 4.4.15.*** — *The tree $\mathcal{T}_\Gamma$ is locally finite.*

*Proof.* — Indeed, $\mathcal{T}_\Gamma$ is locally isomorphic to the finite graph $\mathcal{T}_\Gamma/\Gamma$. □

## 4.5. From trees to curves

We keep the notation of the previous section, and follow the discussion of [**Ra72**] Consider the projective line $\mathbb{P}(V) \cong \mathbb{P}_K^1$. For each lattice $M \subseteq V$, let

$$\mathbb{P}(M) = \mathrm{Proj}\,\big(\mathrm{Sym} M\big).$$

Then $\mathbb{P}(M)$ is a projective line over $\mathcal{O}_K$ with a canonical isomorphism $\mathbb{P}(M)_K \cong \mathbb{P}(V)$. For two lattices $M$ and $M'$ of $V$, the identity isomorphism on the generic fibers extends to an isomorphism $\mathbb{P}(M) \cong \mathbb{P}(M')$ if and only if $M$ and $M'$ are homothetic.

Let $M \supset M' \supset \pi^n M$ be two lattices with standard position with disctance $n$. Then $M'/\pi^n M$ is a direct factor of $M/\pi^n M$, and it defines thus a section of $\mathbb{P}(M)$ over $\mathcal{O}_K/\pi^n$. Alternatively, $M'$ generates an homogeneous ideal $I(M') \subseteq \mathrm{Sym}(M)$, which corresponds to an ideal sheaf $\mathcal{I}(M') \subseteq \mathcal{O}_{\mathbb{P}(M)}$ defining the section given by $M'$. In this way, segments of the tree $\mathcal{T}$ of length $n$ with origin $M$ is in natural bijection with the sections of $\mathbb{P}(M)$ over $\mathcal{O}_K/\pi^n$.

Define the joint of $\mathbb{P}(M)$ and $\mathbb{P}(M')$ as the closure in $\mathbb{P}(M)\times\mathbb{P}(M')$ of the graph of the identity morphism on the generic fibers. We have thus a commutative diagram

$$\begin{array}{ccc} & \mathbb{P}(MM') & \\ {\scriptstyle\rho}\swarrow & & \searrow{\scriptstyle\rho'} \\ \mathbb{P}(M) & & \mathbb{P}(M') \end{array}$$

***Lemma 4.5.1.*** — *(1) Assume that $M \supseteq M' \supseteq \pi^n M$ are in standard position. Then $\rho$ is the blowing-up of $\mathbb{P}(M)$ at the ideal $\mathcal{I}(M')$ and $\rho'$ is the blowing-up of $\mathbb{P}(M')$.*

*(2) The special fiber of $\mathbb{P}(MM')$ consists of two $\mathbb{P}^1_k$ cutting each other transversally at one point $P$, and the complete local ring of $\mathbb{P}(MM')$ at $P$ is isomorphic to $\mathcal{O}_K[[x,y]]/(xy-\pi^n)$.*

*Proof.* — Statement (2) is an immediate consequence of (1). To prove (1), we write $M = \mathcal{O}_K e_1 + \mathcal{O}_K e_2$ and $M' = \mathcal{O}_K e_1 + \mathcal{O}_K \pi^n e_2$. Let $X_1, Y_1$ be the corresponding homogeneous coordinates of $\mathbb{P}(M)$, and $X_2, Y_2$ be those of $\mathbb{P}(M')$. Then $\mathbb{P}(MM')$ is the closed subscheme of $\mathbb{P}(M)\times\mathbb{P}(M')$ defined by the homogeneous equations

$$X_1 = X_2, \quad \pi^n Y_1 = Y_2.$$

Consider the affine covering $\mathbb{P}(M) = U_1 \cup V_1$, where $U_1$ is the open affine defined by $Y_1 \neq 0$ with coordinate $x_1 = X_1/Y_1$, and $V_1$ defined by $X_1 \neq 0$ with coordinate $y_1 = 1/x_1$. Similarly, we have affine coordinates $x_2$ and $y_2 = 1/x_2$ on $\mathbb{P}(M')$. Then the equation above writes as

$$x_1 = \pi^n x_2, \quad x_1 y_2 = \pi^n.$$

Then we see that $\mathbb{P}(MM')$ is the blowing-up of $\mathbb{P}(M)$ at the ideal $\mathcal{I}(M') = (x_1, \pi^n)$.

□

Now let $\Lambda \subseteq \mathcal{T}$ be a finite subtree with vertex points $[M_1], \cdots, [M_n]$, we define the joint $\mathbb{P}(\Lambda)$ as the closure in the product

$$\mathbb{P}(M_1)\times\mathbb{P}(M_2)\times\cdots\times\mathbb{P}(M_n)$$

of the diagonal embedding $\mathbb{P}(V) \hookrightarrow \prod_{i=1}^n \mathbb{P}(V)$. We can also think of $\mathbb{P}(\Lambda)$ as a successive blowing-up of $\mathbb{P}(M_1)$ according to Lemma 4.5.1. The special fiber of $\mathbb{P}(\Lambda)$ is a stable curve whose each irreducible component is isomorphic to $\mathbb{P}^1$, and whose dual graph is given by $\Lambda$. More precisely, for all $i$, there exists a unique irreducible component $C_i$ of $\overline{\mathbb{P}(\Lambda)}$, which maps isomorphically to $\mathbb{P}(M_i)$ via the canonical projection $\mathbb{P}(\Lambda) \to \mathbb{P}(M_i)$. An edge of length $n$ $[M_iM_j]$ in $\Lambda$ correspond to an intersection point of $C_i$ with $C_j$ and the complete local ring of $\mathbb{P}(\Lambda)$ at this point is isomorphic to $\mathcal{O}_K[[x,y]]/(xy-\pi^n)$. We note also that $\mathbb{P}(\Lambda)$ is proper, flat over $\mathrm{Spec}(\mathcal{O}_K)$ and normal.

Let $\Lambda$ be a locally finite subtree of $\mathcal{T}$ with vertices points $[M_i]$ for $i = 0, 1, \cdots$. We choose a vertex $[M]$ and consider $\Lambda$ as the union of the subtree $\Lambda_n$ where $\Lambda_n$ is the union of the segments of $\Lambda$, with origin $M$, containing at most $n+1$ vertices. Then the schemes $\mathbb{P}(\Lambda_n)$ form a projective system. The edges of $\Lambda_{n+1} - \Lambda_n$ define a finite number of rational non-singular points of $\mathbb{P}(\Lambda_n)$. Let $U_n$ be the complement of those non-singular points in $\mathbb{P}(\Lambda_n)$. Then for all integer $m \geq 0$, the canonical morphisms $\mathbb{P}(\Lambda_{n+m}) \to \mathbb{P}(\Lambda_n)$ are isomorphisms over $U_n$ so that we get open immersions $U_0 \hookrightarrow U_1 \hookrightarrow \cdots$. We denote by $\mathbb{P}(\Lambda)$ the union of $U_n$.

Then the resulting scheme $\mathbb{P}(\Lambda)$ is a flat normal $\mathcal{O}_K$-scheme of generic fiber $\mathbb{P}(V)$, and the special fiber $\overline{\mathbb{P}}(\Lambda)$ is a stable curve with each irreducible component isomorphic to $\mathbb{P}^1$, and the dual graph of $\overline{\mathbb{P}}(\Lambda)$ is isomorphic to $\Lambda$.

Now consider a closed point $x \in \mathbb{P}(V)$. Since each $\mathbb{P}(\Lambda_n)$ is proper over $\mathcal{O}_K$, $x$ specializes to a closed point of the closed fiber $\overline{\mathbb{P}}(\Lambda_n)$. Passing to the limit, it is easy to see that $x$ specializes to a point of $\overline{\mathbb{P}}(\Lambda)$ if and only if $x$ is not a rational point of $\mathbb{P}(V)$ corresponding to the end of $\Lambda$. Let $\mathcal{P}(\Lambda)$ be the completion of $\mathbb{P}(\Lambda)$ along its special fiber. Therefore, the rigid generic fiber of $\mathcal{P}(\Lambda)$ is

$$\mathcal{P}(\Lambda) = \mathbb{P}^{1,\mathrm{an}}_K - \partial(\Lambda)$$

where $\partial(\Lambda) \subseteq \mathbb{P}^1(K)$ is the set of ends of $\Lambda$, and the special fiber is identified with $\overline{\mathbb{P}}(\Lambda)$.

**4.5.2. Passing to the quotient by $\Gamma$.** — Let $\Gamma \subseteq \mathrm{PGL}_2(K)$ be a non-abelian Schottky subgroup and $\mathcal{T}_\Gamma$ be the associated tree. Applying the previous section to $\mathcal{T}_\Gamma$, we get a formal $\mathcal{O}_K$-scheme $\mathcal{P}(\mathcal{T}_\Gamma)$ equipped with an action by $\Gamma$. Recall that $\Gamma$ acts freely on the tree $\mathcal{T}_\Gamma$, and hence freely also on the special fibre $\overline{\mathcal{P}}(\mathcal{T}_\Gamma)$.

***Theorem 4.5.3.*** — *There exists a unique pair $(\mathfrak{X}_\Gamma, p)$ consisting of an admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X}_\Gamma$ and a surjective étale $\mathcal{O}_K$-morphism*

$$p : \mathcal{P}(\mathcal{T}_\Gamma) \to \mathfrak{X}_\Gamma$$

*such that*

1. *for all $\gamma \in \Gamma$, if $[\gamma]$ represents the induced automorphism of $\mathcal{P}(\mathcal{T}_\Gamma)$, then $p \circ [\gamma] = p$.*
2. *for $x, y \in \mathcal{P}(\mathcal{T}_\Gamma)$, $p(x) = p(y)$ if and only if $x = [\gamma]y$ for some $\gamma \in \Gamma$.*

*Moreover, $\mathfrak{X}_\Gamma$ is normal, flat $\mathcal{O}_K$, and there exists a unique projective flat curve $\mathfrak{X}^{\mathrm{alg}}_\Gamma$ over* $\mathrm{Spec}(\mathcal{O}_K)$*, whose formal completion along the special fiber gives $\mathfrak{X}_\Gamma$.*

*Proof.* — We will construct the quotient $\mathcal{P}(\mathcal{T}_\Gamma)/\Gamma$ in two steps. First, let $\Gamma_0 \subseteq \Gamma$ be a normal subgroup of finite index such that a non-neutral element does not send a vertex of $\mathcal{T}_\Gamma$ to a adjacent vertex. Therefore, no $\gamma \in \Gamma_0$ takes a component of $\overline{\mathcal{P}}(\mathcal{T}_\Gamma)$ into itself or into another component meeting the first one. It follows that the action of $\Gamma_0$ on $\overline{\mathcal{P}}(\mathcal{T}_\Gamma)$ is discontinuous i.e. one can cover $\overline{\mathbb{P}}(\mathcal{T}_\Gamma)$ by affine open subsets $\overline{U}_i$ such that $\gamma(\overline{U}_i) \cap \overline{U}_i = \emptyset$ for all $i$ and $\gamma \neq 1$. Let $U_i \subseteq \mathcal{P}(\mathcal{T}_\Gamma)$ be the affine open subset corresponding to $\overline{U}_i$. Then the affine open subsets $\{U_i\}$ cover $\mathcal{P}(\mathcal{T}_\Gamma)$ and we construct the formal scheme $\mathfrak{X}_{\Gamma_0} := \mathcal{P}(\mathcal{T}_\Gamma)/\Gamma_0$ by simply glueing the $U_i$. More precisely, for every $i, j$, there is at most one element $\gamma_{i,j} \in \Gamma_0$ such that

$$\gamma_{i,j}(U_i) \cap U_j \neq \emptyset.$$

Then one glues $U_i$ to $U_j$ on this overlap via the map $[\gamma_{i,j}]$. This gives a formal $\mathcal{O}_K$-scheme $\mathfrak{X}_{\Gamma_0}$ and a morphism

$$p_0 : \mathcal{P}(\mathcal{T}_\Gamma) \to \mathfrak{X}_{\Gamma_0}$$

which is surjective and locally an isomorphism such that

– $p_0 \circ [\gamma] = p_0$ for all $\gamma \in \Gamma_0$ and

– $p_0(x) = p_0(y)$ implies that $x = [\gamma]y$ for some $\gamma \in \Gamma_0$.

Note that the graph of the special fiber $\overline{\mathfrak{X}}_{\Gamma_0}$ is the quotient graph $\mathcal{T}_\Gamma/\Gamma_0$. Since $\mathcal{T}_\Gamma/\Gamma$ is finite, so is $\mathcal{T}_\Gamma/\Gamma_0$. It follows that $\mathfrak{X}_{\Gamma_0}$ is of finite type, and hence proper and flat over $\mathcal{O}_K$ and normal. For each irreducible component $E$ of $\overline{\mathfrak{X}}_{\Gamma_0}$, we choose a relatively ample divisor $D_E$ supported in the non-singular locus of $E$, and put $D = \sum_E D_E$. Then $D$ is a relatively ample divisor on $\mathfrak{X}_{\Gamma_0}$, hence it follows that $\mathfrak{X}_{\Gamma_0}$ is projective. By Grothendieck's algebraization theorem [EGA III, 5], we conclude that $\mathfrak{X}_{\Gamma_0}$ is the formal completion of a unique projective scheme $\mathfrak{X}_{\Gamma_0}^{\mathrm{alg}}$ over $\mathcal{O}_K$ along its special fiber.

The finite group $\Gamma/\Gamma_0$ acts freely on $Y$. By [SGA I, VIII, Corollaire 7.6], the quotient of $\mathfrak{X}_{\Gamma_0}^{\mathrm{alg}}$ by $\Gamma/\Gamma_0$ exists. Let $\mathfrak{X}_\Gamma^{\mathrm{alg}}$ denote the quotient $\mathfrak{X}_\Gamma^{\mathrm{alg}}/(\Gamma/\Gamma_0)$, and $\mathfrak{X}_\Gamma$ be the formal completion of $\mathfrak{X}_\Gamma^{\mathrm{alg}}$ along its special fiber. Then the composite

$$p : \mathcal{P}(\mathcal{T}_\Gamma) \to \mathfrak{X}_{\Gamma_0} \to \mathfrak{X}_\Gamma$$

gives the desired quotient of $\mathcal{P}(\mathcal{T}_\Gamma)$ by $\Gamma$. □

***Proposition 4.5.4.*** — *If $g$ is the number of generators of the free group $\Gamma$, then $\mathfrak{X}_\Gamma$ is a stable curve over $\mathcal{O}_K$ of genus $g$, whose generic fiber is smooth over $K$ and whose special fiber is $k$-split degenerate with dual graph $\mathcal{T}_\Gamma/\Gamma$.*

*Proof.* — By definition of $\mathfrak{X}_\Gamma$, its special fiber $\overline{\mathfrak{X}}_\Gamma$ is isomorphic to $\mathcal{P}(\mathcal{T}_\Gamma)/\Gamma$, hence the normalization of each irreducible component of $\overline{\mathfrak{X}}_\Gamma$ is isomorphic to $\mathbb{P}^1$, and the dual graph of $\overline{\mathfrak{X}}_\Gamma$ is the quotient $\mathcal{T}_\Gamma/\Gamma$. Recall that every vertex of $\mathcal{T}_\Gamma$ has at least 3 edges starting with it, hence each $\mathbb{P}^1$ in $\overline{\mathfrak{X}}_\Gamma$ has at least 3 double points. Finally, from the explicit description of complete local rings, we see that the generic fiber of $\mathfrak{X}_\Gamma$ is regular, hence smooth over $K$.

To see that $\mathfrak{X}_\Gamma$ has genus $g$, we consider the special fiber $\overline{\mathfrak{X}}_\Gamma$. We have

$$g(\mathfrak{X}_{\Gamma,K}) = g(\overline{\mathfrak{X}}_\Gamma) = h^1(\mathcal{T}_\Gamma/\Gamma) = e - v + 1,$$

where $g(\overline{\mathfrak{X}}_\Gamma)$ denotes the arithmetic genus of $\overline{\mathfrak{X}}_\Gamma$, $e$ and $v$ denote respectively the number of edges and vertices of the graph $\mathcal{T}_\Gamma/\Gamma$. By the construction of $\mathcal{T}_\Gamma/\Gamma$, it is obtained by gluing an arbitrary vertex $P$ with its translates $\gamma_i(P)$ for $i = 1, \cdots, g$, where $\{\gamma_i : 1 \leq i \leq g\}$ is a set of generators of $\Gamma$. Then it is easy to see that $h^1(\mathcal{T}_\Gamma)/\Gamma = g$. □

## 4.6. Uniqueness and Existence

In this section, we study whether the curve $\mathfrak{X}_\Gamma$ determines the group $\Gamma$, and conversely if all stable curves with $k$-split degenerate reduction come from Mumford's construction.

For two schemes $X, Y$ over $S = \mathrm{Spec}(\mathcal{O}_K)$, we denote by $\underline{\mathrm{Isom}}_S(X, Y)$ the functor which associates to each $S$-scheme $T$ the set of isomorphisms between $X$ and $Y$. If both $X$ and $Y$ are stable curves, a general result of Deligne-Mumford [**DM**, Theorem 1.11] says that $\underline{\mathrm{Isom}}_S(X, Y)$ is representable by a scheme finite and unramified over $S$.

***Proposition 4.6.1.*** — *If $C_1, C_2$ are 2 stable curves over $S = \mathrm{Spec}(\mathcal{O}_K)$ with $k$-split degenerate special fibers, then the scheme $\underline{\mathrm{Isom}}_S(C_1, C_2)$ exists and is isomorphic to a disjoint union $\coprod_{i=1}^N S_i$ of closed subschemes of $S$.*

*Proof.* — By the result of Deligne-Mumford mentioned above, $\underline{\mathrm{Isom}}_S(C_1, C_2)$ is finite and unramified over $S$. Since $S$ is henselian, it is a disjoint union of schemes $S_i$ with only a single point over the closed fiber of $S$. Because the special fibers $\overline{C}_1$ and $\overline{C}_2$ are $k$-split degenerate, every isomorphism of $\overline{C}_1$ and $\overline{C}_2$ is uniquely determined by its induced morphism from the dual graph of $\overline{C}_1$ to that of $\overline{C}_2$ (Note that any automorphism of $\mathbb{P}^1$ fixing three distinct points is trivial). Moreover, any isomorphism between $\overline{C}_1$ and $\overline{C}_2$ must be rational over $k$. Hence, the closed points $s_i \in S_i$ are all $k$-rational. Since $S_i$ is finite and unramified over $S$, $S_i \to S$ is in fact a closed immersion.

□

***Corollary 4.6.2.*** — *Let $C_1, C_2$ be two stable curves as in the proposition. Every isomorphism of the generic geometric fibers between $(C_1)_{\bar\eta}$ and $(C_2)_{\bar\eta}$ extends uniquely to an isomorphism of $C_1$ and $C_2$.*

*Proof.* — Indeed, let $S_i \subseteq \underline{\mathrm{Isom}}_S(C_1, C_2)$ be the component whose generic geometric fiber is non-empty. Then $S_i \cong S$ by the Proposition, and it defines hence an isomorphism $C_1 \cong C_2$. □

***Proposition 4.6.3.*** — *Let $\Gamma$ be a non-abelian Schottky group. Then the projection*

$$\mathcal{P}(\mathcal{T}_\Gamma) \to \mathfrak{X}_\Gamma$$

*makes $\mathcal{P}(\mathcal{T}_\Gamma)$ the universal covering of $\mathfrak{X}_\Gamma$.*

*Proof.* — The category of étale coverings of $\mathfrak{X}_\Gamma$ is equivalent to that of the special fiber $\overline{\mathfrak{X}}_\Gamma$. Since the closed fiber $\overline{\mathcal{P}}(\mathcal{T}_\Gamma)$ is connected and is tree-like union of copies of $\mathbb{P}^1_k$, it is simply connected and must be the universal covering space of $\overline{\mathfrak{X}}_\Gamma$. □

For a scheme $X$ over $S$, we denote by $X_{\bar\eta}$ the geometric generic fiber $X \times_S \mathrm{Spec}(\overline{K})$.

***Corollary 4.6.4.*** — *Let $\Gamma_1, \Gamma_2$ be two Schottky groups. Every isomorphism of the geometric generic fibers $\varphi_{\bar\eta} : \mathfrak{X}^{\mathrm{alg}}_{\Gamma_1,\bar\eta} \to \mathfrak{X}^{\mathrm{alg}}_{\Gamma_2,\bar\eta}$ extends uniquely to an isomorphism*

$$\varphi : \mathfrak{X}^{\mathrm{alg}}_{\Gamma_1} \to \mathfrak{X}^{\mathrm{alg}}_{\Gamma_2},$$

*hence induces an isomorphism $\hat\varphi : \mathfrak{X}_{\Gamma_1} \to \mathfrak{X}_{\Gamma_2}$, and hence a pair $(\tilde\varphi, \alpha)$ consisting of an isomorphism*

$$\alpha : \Gamma_1 \cong \Gamma_2$$

*and an $\alpha$-equivariant isomorphism of formal schemes*

$$\tilde\varphi : \mathcal{P}(\mathcal{T}_{\Gamma_1}) \cong \mathcal{P}(\mathcal{T}_{\Gamma_2}).$$

To prove Theorem 4.6.8, we need to show that an isomorphism $\tilde\varphi : \mathcal{P}(\mathcal{T}_{\Gamma_1}) \cong \mathcal{P}(\mathcal{T}_{\Gamma_2})$ comes from an element of $\mathrm{PGL}_2(K)$. For this, note that $\tilde\varphi$ induces an isomorphism of trees $\mathcal{T}_{\Gamma_1} \cong \mathcal{T}_{\Gamma_2}$. We choose a vertex point $v_0 = [M_0] \in \mathcal{T}_{\Gamma_1}$, which corresponds to an irreducible component $E$ of the special fiber $\overline{\mathcal{P}}(\mathcal{T}_{\Gamma_1})$. Let $x_0 \in E$ be a non-singular point of $E$, and

$D_1$ be a relative Cartier divisor of $\mathcal{P}(\mathcal{T}_{\Gamma_1})$ over $\mathcal{O}_K$ such that its restriction to the special fiber is given by $\overline{D}_1 = [x_0]$. Consider the line bundle $\mathcal{L}_1 = \mathcal{O}_{\mathcal{P}(\mathcal{T}_{\Gamma_1})}(D_1)$, and let $\overline{\mathcal{L}}_1$ be the restriction of $\mathcal{L}_1$ to the special fiber. Let

$$A = \bigoplus_{n=0}^{\infty} H^0(\mathcal{P}(\mathcal{T}_{\Gamma_1}), \mathcal{L}_1^{\otimes n}),$$

and

$$\overline{A} = \bigoplus_{n=0}^{\infty} H^0(\overline{\mathcal{P}}(\mathcal{T}_{\Gamma_1}), \overline{\mathcal{L}}_1^{\otimes n}).$$

***Lemma 4.6.5.*** — *We have $H^1(\overline{\mathcal{P}}(\mathcal{T}_{\Gamma_1}), \overline{\mathcal{L}}^{\otimes n}) = 0$ for all $n \geq 0$.*

*Proof.* — The set of edges of $\mathcal{T}_\Gamma$, denoted by $e(\mathcal{T}_\Gamma)$, is in bijection with the set of singular points of $\overline{\mathcal{P}}(\mathcal{T}_\Gamma)$, and the set of vertices, denoted by $v(\mathcal{T}_\Gamma)$, is in bijection with that of irreducible components. Let $p : \overline{\mathcal{P}}(\mathcal{T}_\Gamma)^{\mathrm{norm}} \to \overline{\mathcal{P}}(\mathcal{T}_\Gamma)$ denote the canonical map. We have an exact sequence

$$0 \to \mathcal{L}^{\otimes n} \to p_* p^* \mathcal{L}^{\otimes n} \to \bigoplus_{e \in e(\mathcal{T}_\Gamma)} i_{e,*} k \to 0,$$

where $i_{e,*} : e \hookrightarrow \overline{\mathcal{P}}(\mathcal{T}_\Gamma)$ is the canonical closed immersion. Taking cohomology groups, one gets

$$0 \to H^0(\overline{\mathcal{P}}(\mathcal{T}_\Gamma), \mathcal{L}^{\otimes n}) \to \bigoplus_{v \in v(\mathcal{T}_\Gamma)} H^0(\mathbb{P}^1_{k,v}, \mathcal{L}^{\otimes n}|_{\mathbb{P}^1_{k,v}}) \xrightarrow{\partial} \bigoplus_{e \in e(\mathcal{T}_\Gamma)} k \to H^1(\overline{\mathcal{P}}(\mathcal{T}_\Gamma), \mathcal{L}^{\otimes n})$$
$$\to \bigoplus_{v \in v(\mathcal{T}_\Gamma)} H^1(\mathbb{P}^1_{k,v}, \mathcal{L}^{\otimes n}|_{\mathbb{P}^1_{k,v}}).$$

Note that $\mathcal{L}|_{\mathbb{P}^1_{k,v}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{k,v}}(1)$ if $v = v_0$, and to $\mathcal{O}_{\mathbb{P}^1_{k,v}}$ if $v \neq v_0$. It follows that $H^1(\mathbb{P}^1_{k,v}, \mathcal{L}^{\otimes n}|_{\mathbb{P}^1 k,v}) = 0$ for all $n \geq 0$. Note that $\mathcal{O}_{\mathcal{P}(\mathcal{T}_\Gamma)}$ is a subsheaf of $\mathcal{L}$, and the restriction of $\partial$ to $\bigoplus_{v \in v(\mathcal{T}_\Gamma)} H^0(\mathbb{P}^1_{v,k}, \mathcal{O}_{\mathbb{P}^1_{v,k}})$ is the morphism

$$\partial : k^{v(\mathcal{T}_\Gamma)} \to k^{e(\mathcal{T}_\Gamma)}$$

defining the cohomology groups of the graph $\mathcal{T}_\Gamma$. Note that $H^1(\mathcal{T}_\Gamma, k) = 0$, as $\mathcal{T}_\Gamma$ is a tree. It follows that $\partial$ is surjective, and hence $H^1(\overline{\mathcal{P}}(\mathcal{T}_\Gamma), \mathcal{L}^{\otimes n}) = 0$ for all $n \geq 0$. □

***Proposition 4.6.6.*** —
1. *The $\mathcal{O}_K$-algebra $A$ is flat and of finite type, and the canonical map $A \otimes k \to \overline{A}$ is an isomorphism.*
2. *The canonical map*
$$\mathcal{P}(\mathcal{T}_{\Gamma_1}) \to \mathrm{Proj}(A)$$
*is identified with the morphism $\mathcal{P}(\mathcal{T}_{\Gamma_1}) \to \mathbb{P}(M_0)$.*

*Proof.* — (1) Consider the natural projection

$$\bar{q} : \overline{\mathcal{P}}(\mathcal{T}_\Gamma) \to \overline{\mathbb{P}}(M_0).$$

Since $\mathcal{L}|_{\mathbb{P}^1_{k,v}} \cong \mathcal{O}_{\mathbb{P}^1_{k,v}}$ for all vertices $v \neq v_0$ and $\mathcal{L}|_{\mathbb{P}^1_{k,v_0}} \cong \mathcal{O}_{\mathbb{P}^1_{k,v_0}}(1)$, we have $\bar{q}_*(\mathcal{L}^{\otimes n}) \cong \mathcal{O}_{\mathbb{P}^1_{v_0,k}}(n)$. We have

$$\overline{A} \cong \bigoplus_{n\geq 0} H^0(\mathbb{P}^1_{v_0,k}, \mathcal{O}(n)) \cong k[x,y],$$

which is clearly a finitely generated $k$-algebra. By Lemma 4.6.5, we have

$$H^0(\mathcal{P}(\mathcal{T}_\Gamma), \mathcal{L}^{\otimes n}) \otimes_{\mathcal{O}_K} k \cong H^0(\overline{\mathcal{P}}(\mathcal{T}_\Gamma), \mathcal{L}^{\otimes n})$$

for all $n \geq 0$. It follows that $\overline{A} = A \otimes_{\mathcal{O}_K} k$, and $A$ is a finitely generated $\mathcal{O}_K$-algebra.

Statement (2) follows from the proof of (1) above. □

***Corollary 4.6.7.*** — *Let $(\alpha, \tilde{\varphi})$ be a pair consisting of an isomorphism $\alpha : \Gamma_1 \cong \Gamma_2$ and an $\alpha$-equivariant isomorphism $\tilde{\varphi} : \mathcal{P}(\mathcal{T}_{\Gamma_1}) \cong \mathcal{P}(\mathcal{T}_{\Gamma_2})$. Then there exists a $g \in \mathrm{PGL}(V)$ such that $\alpha$ is given by the conjugation $\gamma \mapsto g\gamma g^{-1}$, and $\tilde{\varphi}$ is induced by the isomorphism $[g] : \mathbb{P}(V) \cong \mathbb{P}(V)$.*

*Proof.* — Choose a vertex $i_0 = [M_0] \in \mathcal{T}_{\Gamma_1}$ as above. Let $i'_0 = [M'_0] \in \mathcal{T}_{\Gamma_2}$ be the image of $i_0$ via the isomorphism of graphs induced by $\tilde{\varphi}$. Then one has a commutative diagram of isomorphisms.

$$\begin{array}{ccc} \mathcal{P}(\mathcal{T}_{\Gamma_1}) & \xrightarrow{\tilde{\varphi}} & \mathcal{P}(\mathcal{T}_{\Gamma_2}) \\ \downarrow & & \downarrow \\ \mathbb{P}(M_0) & \longrightarrow & \mathbb{P}(M'_0) \end{array}$$

If we identify the generic fibers of $\mathbb{P}(M_0)$ and $\mathbb{P}(M'_0)$ with $\mathbb{P}(V)$, we see that that isomorphism $\tilde{\varphi}$ is induced by an isomorphism $[g] : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V)$ for some $g \in \mathrm{PGL}(V)$. Since $\tilde{\varphi}$ is $\alpha$-equivariant, we have

$$[\alpha(\gamma)] \circ [g] = [g] \circ [\gamma].$$

It follows that $\alpha(\gamma) = g\gamma g^{-1}$. □

***Theorem 4.6.8.*** — *Let $\Gamma_1$ and $\Gamma_2$ be two Schottky subgroups of $\mathrm{PGL}_2(K)$.*

1. *The geometric generic fibers of $\mathfrak{X}^{\mathrm{alg}}_{\Gamma_1}$ and $\mathfrak{X}^{\mathrm{alg}}_{\Gamma_2}$ are isomorphic if and only if $\Gamma_1$ and $\Gamma_2$ are conjugate in $\mathrm{PGL}_2(K)$.*
2. *We have*

$$\mathrm{Aut}(\mathfrak{X}^{\mathrm{alg}}_\Gamma) \cong N(\Gamma)/\Gamma$$

*where $N(\Gamma)$ is the normalizer of $\Gamma$ in $\mathrm{PGL}_2(K)$.*

*Proof.* — This follows immediately from Corollary 4.6.4 and 4.6.7.

□

**4.6.9. Existence theorem.** — We turn to the proof of Theorem 4.3.8. Let $\mathfrak{X}$ be a projective stable curve over $\mathcal{O}_K$ such that its generic fiber is smooth of genus $g \geq 2$ and its special fiber $\overline{\mathfrak{X}}$ is $k$-split degenerate. Then $\overline{\mathfrak{X}}$ admits a universal covering $\overline{p} : \overline{P} \to \overline{\mathfrak{X}}$ of Galois group $\Gamma$ which is free of $g$ generators. Then the graph $\Lambda$ of $\overline{P}$ is locally finite such that each vertex has at least 3 edges starting from it. Since the category of étale coverings of $\overline{\mathfrak{X}}$ is equivalent to that of étale coverings of $\mathfrak{X}$, there exists a unique étale covering $p : \mathfrak{P} \to \mathfrak{X}$ that lifts $\overline{p}$. Since $\mathfrak{X}$ is flat over $\mathcal{O}_K$ and normal, it is the same as $\mathfrak{P}$.

We need to realize $\Lambda$ as a subtree of the Bruhat-Tits tree $\mathcal{T}$ and $\Gamma$ as a Schottky subgroup of $\mathrm{PGL}_2(K)$. Let $I$ be the set of vertices of $\Lambda$. For all $i \in I$, let $D_i$ be a positive relative Cartier divisor of $\mathfrak{P}$ whose support meets only the irreducible component $C_i$ of $\overline{\mathfrak{P}}$ corresponding to the index $i$. Let $\Lambda'$ be a finite subtree of $\Lambda$ and $D' = \sum_{i \in \Lambda'} D_i$. Let $\mathcal{L}'$ be the formal invertible sheaf on $\mathfrak{P}$ given by $D'$, and let $\overline{\mathcal{L}}'$ be the restriction to the special fiber $\overline{\mathfrak{P}}$.

Then the same arguments as in Proposition 4.6.6 show that the $k$-algebra

$$\bigoplus_{n \geq 0} H^0(\overline{\mathfrak{P}}, \mathcal{L}'^{\otimes n})$$

is of finite type and the canonical morphism

$$\overline{q}_{\Lambda'} : \overline{\mathfrak{P}} \to \operatorname{Proj}\big(\bigoplus_{n \geq 0} H^0(\overline{\mathfrak{X}}, \mathcal{L}'^{\otimes n})\big)$$

is identified with the canonical projection $\overline{P} \to C_{\Lambda'}$ where $C_{\Lambda'} \subseteq \overline{\mathfrak{P}}$ is the union of the irreducible components $C_i$ with $i \in \Lambda'$.

Similarly to Lemma 4.6.5, we have

$$H^1(\overline{\mathfrak{P}}, \overline{\mathcal{L}}'^{\otimes n}) = 0, \quad \forall n \geq 0.$$

By the flatness of $\mathfrak{P}$, we see that the canonical morphism

$$H^0(\mathfrak{P}, \mathcal{L}'^{\otimes n}) \otimes k \xrightarrow{\sim} H^0(\overline{\mathfrak{P}}, \mathcal{L}'^{\otimes n})$$

is an isomorphism. It follows that $H^0(\mathfrak{P}, \mathcal{L}'^{\otimes n})$ is a flat $\mathcal{O}_K$-module of finite type, and $\operatorname{Proj}(\bigoplus_{n \geq 0} H^0(\mathfrak{P}, \mathcal{L}'^{\otimes n}))$ is a proper and normal $\mathcal{O}_K$-scheme $\mathbb{P}(\Lambda')$ flat over $\mathcal{O}_K$. We have a morphism of formal schemes

$$q_{\Lambda'} : \mathfrak{P} \to \hat{\mathbb{P}}(\Lambda')$$

which lifts $\overline{q}_{\Lambda'}$. Since the special fiber of $\mathbb{P}(\Lambda')$ is isomorphic to $C_{\Lambda'}$, which has arithmetic genus 0, it follows that the generic fiber of $\mathbb{P}(\Lambda')$ is isomorphic to the projective line over $K$. We choose a vertex $i_0 \in \Lambda'$. Then we have a canonical morphism

$$r_{i_0} : \mathbb{P}(\Lambda') \to \mathbb{P}(\Lambda_0) \cong \mathbb{P}^1_{\mathcal{O}_K},$$

where $\Lambda_0$ is the subtree of $\Lambda'$ consisting only of the vertex $i_0$, and $r_{i_0}$ is an isomorphism on generic fibers. Using the identification

$$\mathbb{P}(\Lambda')_K \cong \mathbb{P}(\Lambda_0)_K \cong \mathbb{P}^1_K,$$

we can realize $\Lambda'$ as a subtree of the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(K)$. Passing to the limit, one can realize $\Lambda$ as a subtree of $\mathcal{T}$. Moreover, the action of $\Gamma$ on $\Lambda$, and hence on $\mathfrak{P}$, induces an action on $\mathbb{P}(\Lambda_0)_K \cong \mathbb{P}^1_K$. This gives an embedding $\Gamma \hookrightarrow \mathrm{PGL}_2(K)$.

Let $\gamma \in \Gamma$ with $\gamma \neq 1$. Then $\Gamma$ on $\Lambda$ acts without fixed points, since $\mathbb{P}^1_k$ is simply connected. If $v_0 \in \Lambda$ is a vertex such that $d(v_0, \gamma(v_0))$ is minimal, then $\bigcup_{n\in\mathbb{Z}}[\gamma^n(v_0), \gamma^{n+1}(v_0)]$ is a straight line on which $\gamma$ acts by translation. Therefore, the two ends of this straight line correspond to the two fixed points of $\gamma$ in $\mathbb{P}^1_K$. Therefore, $\gamma$ is hyperbolic, and $\Gamma \subseteq \mathrm{PGL}_2(K)$ is a Schottky subgroup. Moreover, this argument also shows that the set of fixed points of $\Gamma$ is contained in the set of ends of $\Gamma$. To finish the proof, we have to show that $\Lambda$ is the canonical graph associated to $\Gamma$. Let $v$ be a vertex of $\Lambda$ and $v_0$ be its image in $\Lambda_\Gamma := \Lambda/\Gamma$. Note that $\Lambda_\Gamma$ is the dual graph of $\overline{\mathfrak{X}}$. Since $\overline{\mathfrak{X}}$ is stable, it follows that there exist 3 oriented circuits in $\Lambda_\Gamma$, which depart from $v_0$ along tree different oriented edges. These three circuits correspond to 3 different elements, say $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. The union of segments

$$L_i := \bigcup_{n\geq 0}[v, \gamma_i^n(v)]$$

for $i = 1, 2, 3$ is a semi-line with origin $v$, and they start with 3 different edges. The end of $L_i$ is one of the fixed points of $\Gamma_i$, say $x_i$, for $i = 1, 2, 3$. Then we have $\sigma(x, y, z) = v$. This shows that $\Lambda$ is the canonical tree $\mathcal{T}_\Gamma$ associated to $\Gamma$.

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