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COURSE NOTES ON INTERSECTION THEORY

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CONTENTS

CHAPTER 1

CHOW GROUPS

1.0.1. Notation. — Let k be a field. All the schemes in this course are supposed to be separate of finite type over k. A *variety* is a separated, reduced and irreducible scheme over k. For a variety X, we denote by $k(X)$ the field of rational functions on X. Let $V \subseteq X$ be a subvariety in a scheme X, we denote by $\mathcal{O}_{X,V}$ the local ring of X at the generic point of V.

1.1. Cycles and rational equivalence

Let X be a variety or a scheme (over k). For an integer d, a d-cycle on X is a finite formal sum

$$\sum_i n_i[V_i]$$

where $n_i \in \mathbb{Z}$ and $V_i \subseteq X$ are subvarieties of dimension d. The set of d-cycles on X form a natural abelian group, which we denote by $Z_d(X)$. We put

$$Z(X) \colon = \bigoplus_{d=0}^{\infty} Z_d(X),$$

and call it the cycle group of X. A cycle $\sum_i n_i[V_i]$ is called *effective* if $n_i \geq 0$ for all i.

Let $Y \subseteq X$ be a subscheme. Since Y is of finite type over k hence Noetherian, Y has only finitely many irreducible components, say $Y_1, \cdots, Y_s$. Then each local ring $\mathcal{O}_{Y,Y_i}$ is Artinian. We define

$$[Y] \colon = \sum_{i=1}^{s} \operatorname{leng}(\mathcal{O}_{Y,\eta_i})[Y_i],$$

and call it the cycle associated to Y.

1.1.1. Order function. — Let A be a one-dimensional Noetherian domain with fraction field K. One can define an order function

$$\operatorname{ord}_A : K^\times \to \mathbb{Z}$$

as follows. For $a \in A\backslash\{0\}$, $A/(a)$ is a finite length A-module, and we put

$$\operatorname{ord}_A(a) = \operatorname{leng}(A/(a)).$$

Then it is easy to see that

$$\operatorname{leng}(A/(ab)) = \operatorname{leng}(A/(a)) + \operatorname{leng}(A/(b)), \quad \forall a, b \in A.$$

For $x = \frac{a}{b} \in K^\times$, we put

$$\operatorname{ord}_A(x) \colon = \operatorname{ord}_A(a) - \operatorname{ord}_A(b),$$

which is independent of the writing $x = \frac{a}{b}$. It is easily seen that ord_A is multiplicative:

$$\operatorname{ord}_A(xy) = \operatorname{ord}_A(x) + \operatorname{ord}_A(y), \quad \forall x, y \in K^\times.$$

Note that we do not require that A to be neither local nor normal. If this is the case, A is a discrete valuation ring, and there exists a uniformizer $\pi \in A$ such that any element $f \in k(X)^\times$ writes uniquely as $f = \pi^n u$ with $u \in A^\times$ and $\operatorname{ord}_A(f) = n$.

Let X be a variety of dimension n, and $V \subseteq X$ be a codimension 1 subvariety. The ring $\mathcal{O}_{X,V}$ is a one-dimensional local domain. For any $f \in k(X)^\times$, we put

$$\operatorname{ord}_V(f) = \operatorname{ord}_{\mathcal{O}_{X,V}}(f),$$

and define the divisor associated to f as

$$\operatorname{div}(f) = \sum_{V \subseteq X} \operatorname{ord}_V(X)[V],$$

where the sum runs through the closed subvarieties of codimension 1 of X. If $\operatorname{ord}_V(f) = n > 0$ (resp. $\operatorname{ord}_V(f) = n < 0$), then we say that f has a zero (resp. pole) of order $|n|$ at V. By the additivity of the functions ord_V, the map $f \mapsto \operatorname{div}(f)$ defines a homomorphism of groups

$$\operatorname{div} \colon k(X)^* \to Z_{n-1}(X).$$

If there is risk of confusion, we use the notation div_X to emphasize the dependence on X.

Example 1.1.2. — Let C be the projective plane curve with homogeneous equation $Y^2 Z = X^3$. Let $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$. Then one has $k(C) = k(x, y)/(y^2 - x^3)$, and $\operatorname{div}_C(x) = 2[P] - 2[\infty]$ and $\operatorname{div}_C(y) = 3[P] - 3[\infty]$, where P is the point $(0, 0, 1) \in C$ and $\infty = (0, 1, 0)$. If U denotes the affine curve $C\backslash\{\infty\}$, then $\operatorname{div}_U(x) = 2[P]$ and $\operatorname{div}_U(y) = 3[P]$.

Definition 1.1.3. — Let X be a scheme over k. A d-cycle $\alpha \in Z_d(X)$ is *rationally equivalent to* 0, denoted as $\alpha \sim 0$, if there exist a finite number of $(d+1)$-dimensional closed subvarieties $W_i \subseteq X$ and $f_i \in k(W_i)^\times$ such that

$$\alpha = \sum_i \operatorname{div}(f_i).$$

Two d-cycles $\alpha, \beta \in A_d(X)$ are called *rationally equivalent*, written as $\alpha \sim \beta$, if $\alpha - \beta$ is rationally equivalent to 0. The d-cycles rationally equivalent to 0 form a subgroup $\operatorname{Rat}_d(X) \subseteq Z_d(X)$. We put

$$A_d(X) \colon = Z_d(X) / \operatorname{Rat}_d(X),$$

and call it the Chow group of d-dimensional cycles on X. We put also $A(X) = \oplus_d A_d(X)$.

Example 1.1.4. — (a) One has $A_0(\mathbb{P}^1) \cong \mathbb{Z}[\infty]$ and $A_1(\mathbb{P}^1) = \mathbb{Z}[\mathbb{P}^1]$.
(b) One has $A_0(\mathbb{A}^1) = 0$ and $A_1(\mathbb{A}^1) = \mathbb{Z}[\mathbb{A}^1]$.

We have the following basic properties of Chow groups:

Lemma 1.1.5. — *Let X be a scheme over k.*
(a) If $X_{\mathrm{red}} \subset X$ is the closed reduced subscheme, then $A_d(X) = A_d(X_{\mathrm{red}})$.
(b) If X is a disjoint union of schemes $X_1, \cdots, X_s$, then $Z_d(X) = \oplus Z_d(X_i)$ and $A_d(X) = \oplus A_d(X_i)$.
(c) If X_1, X_2 are closed subschemes of X, then there are exact sequences

$$A_d(X_1 \cap X_2) \to A_d(X_1) \oplus A_d(X_2) \to A_d(X_1 \cup X_2) \to 0,$$

where the first arrow is $\alpha \mapsto (\alpha, \alpha)$ and the second is $(\beta, \gamma) \mapsto \beta - \gamma$.
(d) Let $Y \subseteq X$ a closed subscheme, and $U = X \backslash Y$. Then one has an exact sequence

$$A_d(Y) \xrightarrow{\alpha} A_d(X) \xrightarrow{\beta} A_d(U) \to 0,$$

where α is the natural map, and β is induced by $\eta = \sum_i n_i[V_i] \mapsto \eta|_U = \sum_i n_i[V_i \cap U]$.

Proof. — The statements (a) and (b) are evident. To prove (c), note that the natural map$Z_d(X_1) \oplus Z_d(X_2) \to Z_d(X_1 \cup X_2)$ is surjective, hence so is $A_d(X_1) \oplus A_d(X_2) \to A_d(X_1 \cup X_2)$. To show that the exactness in the middle, let $\alpha \in Z_d(X_1)$ and $\beta \in Z_d(X_2)$ be such that $\alpha - \beta \in \mathrm{Rat}_d(X_1 \cup X_2)$. Then by definition, there exists a finite number of $(d+1)$-dimensional subvarieties $W_i \subseteq X_1 \cup X_2$ and $f_i \in k(W_i)^\times$ such that

$$\alpha - \beta = \sum_{i \in I} \mathrm{div}(f_i).$$

Since each W_i is irreducible and reduced, one has either $W_i \subseteq X_1$ or $W_i \subseteq X_2$. There exists thus a partition $I = I_1 \sqcup I_2$ such that $i \in I_j$ if and only if $W_i \subseteq X_j$. Note that $\sum_{i \in I_j} \mathrm{div}(f_i) \in \mathrm{Rat}_d(X_j)$. Therefore, up to replacing α by $\alpha - \sum_{i \in I_1} \mathrm{div}(f_i)$ and β by $\beta - \sum_{i \in I_2} \mathrm{div}(f_i)$, we may assume that $\alpha = \beta$. This means that (α, β) lies in the image of $Z_d(X_1 \cap X_2) \to Z_d(X_1) \oplus Z_d(X_2)$.

For (d), one has obviously an exact sequence

$$0 \to Z_d(Y) \to Z_d(X) \xrightarrow{|_U} Z_d(U) \to 0, \tag{1.1.5.1}$$

from which the surjectivity of $\beta : A_d(X) \to A_d(U)$ follows. Now let $\eta \in Z_d(X)$ be a d-cycle such that $\eta|_U \sim 0$. One has to show that η is rationally equivalent to d-cycle in Y. Indeed, the assumption on η implies that there exist $(d+1)$-dimensional closed subvarieties $W_i \subseteq U$ and $f_i \in k(W_i)^\times$ such that

$$\eta|_U = \sum_i \mathrm{div}_{W_i}(f_i)$$

Let $\bar{W}_i$ denote the Zariski closure of W_i in X. Then one has $\eta|_U = \sum_i \mathrm{div}_{\bar{W}_i}(f_i)|_U$. By the exact sequence (1.1.5.1), it follows that

$$\eta - \sum_i \mathrm{div}_{\bar{W}_i}(f_i) \in Z_d(Y).$$

□

Remark 1.1.6. — In general, the morphism α in Lemma 1.1.5(d) is not injective.

1.2. Proper push-forward of cycles

Let $f : X \to Y$ be a *proper* morphism of schemes. If $V \subseteq X$ is a closed subvariety of dimension d, its image $W = f(V)$ is closed in Y. There is an induced embedding $k(W) \subseteq k(V)$. Note that $k(V)/k(W)$ must be a finite extension of field, since they are both finitely generated extensions over k of the same transcendence degree. We put

$$f_*[V] = \begin{cases} 0 & \text{if } \dim(W) < \dim(V); \\ [k(V) : k(W)][W] & \text{if } \dim(W) = \dim(V). \end{cases}$$

This extends by linearity to a homomorphism of d-cycles

$$f_* : Z_d(X) \to Z_d(Y).$$

From this definition, it is clear that $(gf)_* = g_* f_*$ for any proper morphism $g : Y \to Z$.

Theorem 1.2.1. — *The map $f_* : Z_d(X) \to Z_d(Y)$ preserves linear equivalence, and it induces thus a map $f_* : A_d(X) \to A_d(Y)$.*

Proof. — If $\alpha = \mathrm{div}(r)$ is a d-cycle given by a rational function r defined over a closed subvariety $V \subseteq X$. One has to show that $f_*(\mathrm{div}(r))$ is also rationally equivalent to 0. Up to replacing X by V and Y by $f(V)$, one reduces to show the following □

Proposition 1.2.2. — *Let $f : X \to Y$ be a proper, surjective morphism of varieties, and $r \in k(X)^*$. Then*

$$f_*(\mathrm{div}(r)) = \begin{cases} 0 & \textit{if } \dim(Y) < \dim(X); \\ \mathrm{div}(N(r)) & \textit{if } \dim(Y) = \dim(X), \end{cases}$$

where $N(r) \in k(Y)^\times$ is the norm of r.

Before proving Proposition 1.2.2, one deduces some corollary from it. Let X be a *proper* scheme, i.e. the structural morphism $p : X \to S = \mathrm{Spec}(k)$ is proper. For any zero-cycle $\alpha = \sum_P n_P[P] \in Z_0(X)$, we define the degree of α as

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P[k(P) : k];$$

or equivalently, if we identify $A_0(S) = \mathbb{Z}[S]$ with $\mathbb{Z}$ via the basis $[P]$, we have

$$\deg(\alpha) = p_*(\alpha),$$

called the degree of α.

Corollary 1.2.3. — *(a) If X is a proper scheme, then the degree map induces a homomorphism*

$$\int_X : A_0(X) \to \mathbb{Z}.$$

(b) If $f : X \to Y$ is a morphism of proper schemes, then $\int_Y f_*(\alpha) = \int_X \alpha$ for any $\alpha \in A_0(X)$.

Proof. — (a) By Theorem 1.2.1, $p_*\colon Z_0(X) \to Z_0(S)$ induces a map $p_* : A_0(X) \to A_0(S) \cong \mathbb{Z}$. (b) It follows from $p_* = q_* f_*$, where $q : Y \to S$ is the structural map of Y. □

We now turn to the proof of Proposition 1.2.2.

Let R be a domain with fractional field K. If M is a finitely generated torsion free R-module and $\varphi : M \to M$ is a R-endomorphism, then $\varphi \otimes_R K$ is an endomorphism of the finite dimensional K-vector space $M \otimes_R K$. We define $\det(\varphi) \in K$ as the determinant of $\varphi \otimes_R K$.

Lemma 1.2.4. — *Let R be a 1-dimensional local Noetherian domain which is the localization of a finitely generated k-algebra, and $\varphi : M \to M$ be an endomorphism of a finitely generated torsion free R-module. Assume that* $\det(\varphi) \neq 0$. *Then φ is injective and we have*

$$\mathrm{leng}(M/\varphi(M)) = \mathrm{ord}_R(\det(\varphi)),$$

where $\mathrm{ord}_R : K^\times \to \mathbb{Z}$ *is the order function defined in Subsection 1.1.1. In particular, if M has rank n, then one has, for any $r \in R$,*

$$\mathrm{leng}(M/rM) = n\mathrm{leng}(R/rR) = n \cdot \mathrm{ord}_R(r).$$

Proof. — The endomorphism $\varphi \otimes_R K$ is an isomorphism if and only if $\det(\varphi) \neq 0$. The injectivity of φ follows immediately from the fact that M is torsion free.

First, we reduce the problem to the case when R is a discrete valuation ring. Let R' be the integral closure of R in K. Then R' is a Dedekind domain, and the quotient R'/R is a R-module of finite length. Let M' be the quotient of $M \otimes_R R'$ by its torsion submodule. Then one has a natural inclusion $M \subseteq M'$, and the quotient M'/M is a R-module of finite length. Moreover, φ induces an endomorphism $\varphi' : M' \to M'$ of R'-modules, and one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & M'/M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\bar\varphi'} & & \\ 0 & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & M'/M & \longrightarrow & 0. \end{array}$$

By Snake Lemma, one obtains

$$0 \to \mathrm{Ker}(\bar\varphi') \to \mathrm{Coker}(\varphi) \to \mathrm{Coker}(\varphi') \to \mathrm{Coker}(\bar\varphi') \to 0.$$

By the additivity of the length function, one gets

$$\mathrm{leng}(\mathrm{Coker}(\varphi)) - \mathrm{leng}(\mathrm{Coker}(\varphi')) = \mathrm{leng}(\mathrm{Ker}(\bar\varphi')) - \mathrm{leng}(\mathrm{Coker}(\bar\varphi')).$$

On the other hand, it follows from the exact sequence

$$0 \to \mathrm{Ker}(\bar{\varphi}') \to M'/M \to M'/M \to \mathrm{Coker}(\bar{\varphi}') \to 0$$

that

$$\mathrm{leng}(\mathrm{Ker}(\bar{\varphi}')) - \mathrm{leng}(\mathrm{Coker}(\bar{\varphi}')) = \mathrm{leng}(M'/M) - \mathrm{leng}(M'/M) = 0.$$

Therefore, one gets $\mathrm{leng}(\mathrm{Coker}(\varphi)) = \mathrm{leng}(\mathrm{Coker}(\varphi'))$. Note that $\varphi \otimes_R K = \varphi' \otimes_{R'} K$. Hence, one is reduced to proving that $\det(\varphi') = \mathrm{leng}(\mathrm{Coker}(\varphi'))$.

Up to replacing R by R' and M by M', one may assume that R is a Dedekind domain with finitely many maximal ideals, hence R is a principal ideal domain. By the structural theorem of finitely generated modules over a PID, there exists a basis $(e_1, \cdots, e_n)$ of M over R and $a_1, \cdots, a_n \in R$ such that $(a_1e_1, \cdots, a_ne_n)$ is a basis of $\varphi(M)$. Therefore, one has

$$\mathrm{leng}(M/\varphi(M)) = \sum_{i=1}^{n} \mathrm{leng}(R/a_iR).$$

On the other hand, since $(\varphi(e_1), \cdots, \varphi(e_n))$ is another basis of $\varphi(M)$, there exists a matrix $B \in \mathrm{GL}_n(R)$ such that

$$(\varphi(e_1), \cdots, \varphi(e_n)) = (a_1e_1, \cdots, a_ne_n)B,$$

i.e., φ has matrix $\mathrm{diag}(a_1, \cdots, a_n)B$ under the basis $(e_1, \cdots, e_n)$. Hence, one gets $\det(\varphi) = a_1 \cdots a_n \cdot u$ with $u = \det(B) \in R^\times$, and

$$\mathrm{ord}_R(\det(\varphi)) = \mathrm{ord}_R(\prod_{i=1}^{n} a_i) = \sum_{i=1}^{n} \mathrm{leng}(R/a_iR) = \mathrm{leng}(M/\varphi(M)).$$

□

Proof of Proposition 1.2.2. — (a) Assume $\dim(Y) = \dim(X)$. Let

$$X \to X' = \mathrm{Spec}(f_*(\mathcal{O}_X)) \to Y$$

denote the Stein factorization of f. We fix a subvariety W of Y of codimension 1. Put $A = \mathcal{O}_{Y,W}$, and $B = f_*(\mathcal{O}_X)_{Y,W}$. Then A is a Noetherian local domain of dimension 1, and B is a finite A-algebra such that the maximal ideals $\mathfrak{m}_i$ of B correspond to the subvarieties $V_i \subseteq X$ mapping onto W. Then to prove the Proposition in this case, it suffices to show that

$$\sum_i \mathrm{ord}_{V_i}(r)[k(V_i) : k(W)] = \mathrm{ord}_W(N(r)).$$

Since both sides above are multiplicative in r, it is enough to prove this when $r \in B\backslash\{0\}$. Consider the homomorphism $B \xrightarrow{\times r} B$. By Lemma 1.2.4, we have

$$\mathrm{ord}_W(N(r)) = \mathrm{leng}_A(B/rB),$$

where leng_A means the length as A-module. But B/rB is a Artinian ring, and we have

$$\begin{aligned}\mathrm{leng}(B/rB) &= \sum_i \mathrm{leng}_A(B_{\mathfrak{m}_i}/rB_{\mathfrak{m}_i}) = \mathrm{leng}_{B_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}/rB_{\mathfrak{m}_i})[k(\mathfrak{m}_i) : k(\mathfrak{m}_A)] \\ &= \sum_i \mathrm{ord}_{V_i}(r)[k(V_i) : k(W)].\end{aligned}$$

This finishes the proof of Proposition 1.2.2 when $\dim(X) = \dim(Y)$.

(b) Assume $\dim(Y) < \dim(X)$. Consider first the case $Y = \mathrm{Spec}(K)$ is the spectrum of a field K, and $X = \mathbb{P}^1_K$. The field of rational functions on X is $K(t)$. We may assume that r is an irreducible polynomial in t of degree d. Then ideal $(r) \subseteq K[t]$ corresponds to a point P of $\mathbb{P}^1_K$ of degree d, and we have

$$\mathrm{div}(r) = [P] - d[\infty].$$

Note that $K(P)$ is a finite extension of K of degree d. One obtains $f_*(\mathrm{div}(r)) = d[Y] - d[Y] = 0$.

In the general case, one may assume that $\dim(Y) = \dim(X) - 1$, otherwise the statement is trivial. $f_*(\mathrm{div}(r))$ is a multiple of $[Y]$ with coefficients

$$\sum_V \mathrm{ord}_V(r)[k(V) : K],$$

with $K = k(Y)$, where V runs through all the codimension 1 subvarieties of X mapping surjectively to Y. Put $X_K = X \times_Y \mathrm{Spec}(K)$. So one may assume that X is a curve over $Y = \mathrm{Spec}(K)$. Let $g : \tilde{X} \to X$ denote the normalization. Then the rational function r defines a finite dominant map $h : \tilde{X} \to \mathbb{P}^1_K$. One has thus a commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{P}^1_K & \xrightarrow{p} & \mathrm{Spec}(K) \end{array}$$

If $\tilde{r}$ denotes the rational function of $\tilde{X}$ defined by the isomorphism $K(\tilde{X}) \cong K(X)$, one has

$$f_*(\mathrm{div}(r)) = f_*g_*(\mathrm{div}(\tilde{r})) = p_*h_*\,\mathrm{div}(\tilde{r})$$

However, one has $h_*(\mathrm{div}(\tilde{r})) = \mathrm{div}(N(r))$ by part (a). Hence, Proposition 1.2.2 when $\dim(Y) < \dim(X)$ follows from the case of $\mathbb{P}^1$.

□

1.3. Geometric definition of rational equivalence

Let X be a scheme. We introduce a "boundary" map $\partial : Z(X \times \mathbb{P}^1) \to Z(X)$ as follows: Let $W \subseteq X \times \mathbb{P}^1$ be a closed subvariety. If $\mathrm{pr}_2 : W \to \mathbb{P}^1$ is not dominant, then $\mathrm{pr}_2(W)$ must be a closed point of $\mathbb{P}^1$ and we put $\partial([W]) = 0$. If $\mathrm{pr}_2 : W \to \mathbb{P}^1$ is dominant, then put $W_0 = \mathrm{pr}_1(\mathrm{pr}_2^{-1}(\{0\}))$ and $W_\infty = \mathrm{pr}_1(\mathrm{pr}_2^{-1}(\{\infty\}))$ (scheme theoretic preimage). We define

$$\partial([W]) = [W_0] - [W_\infty].$$

Proposition 1.3.1. — *A cycle $\alpha \in Z_d(X)$ is rationally equivalent to* 0 *if and only if it is a linear combination of cycles of the form $\partial(W)$, for some subvariety $W \subseteq X \times \mathbb{P}^1_k$ of dimension $d+1$ dominating over $\mathbb{P}^1$.*

Proof. — First, we prove that $\partial(W)$ is rationally equivalent to 0. Indeed, the second projection $\mathrm{pr}_2 : W \to \mathbb{P}^1$ gives rise to a rational function $r \in k(W)^\times$. Then we have $\mathrm{div}(r) = \mathrm{pr}_2^{-1}(W_0) - \mathrm{pr}_2^{-1}(W_\infty)$, and

$$\partial(W) = \mathrm{pr}_{1,*}(\mathrm{div}(r)).$$

It follows from Proposition 1.2.2 that $\partial(W)$ is rationally equivalent to 0. Conversely, if V is a subvariety of X of dimension $d+1$ and $r \in k(V)^\times$, we prove that $\mathrm{div}(r)$ is has the form $\partial(W)$. Indeed, such a rational function gives a rational morphism from V to $\mathbb{P}^1$, i.e. there exists an open dense subset $U \subseteq V$ such that $r : U \to \mathbb{P}^1$ is dominant. We take W to be the closure of the graph of r in $X \times \mathbb{P}^1$. Then the projection to X maps W birationally and properly onto V. Then we have $\mathrm{div}(r) = \partial(W)$.

□

Example 1.3.2. — Let $C, D \subseteq \mathbb{P}^N$ be two distinct hypersurfaces defined by two homogeneous polynomials $F(x), G(x) \in k[x_0, x_1, \cdots, x_N]$ of the same degree. We define

$$W = \{(x,t) \in \mathbb{P}^N \times \mathbb{P}^1 \mid t_0 F(x) + t_1 G(x) = 0\},$$

where $t = (t_0, t_1)$ is the homogeneous coordinate on $\mathbb{P}^1$. Then $W \to \mathbb{P}^1$ is dominant it is clear that $W_0 = C$ and $W_\infty = D$, and hence $C \sim D$. In particular, if $S_d \subseteq \mathbb{P}^N$ is any hypersurface of degree d, then $[S_d] = d[H]$ with $H \subseteq \mathbb{P}^N$ a hyperplane.

1.3.3. Hilbert polynomial. — For any coherent sheaf $\mathcal{F}$ on a proper scheme Y, recall that the Euler-Poincaré characteristic of $\mathcal{F}$ is defined as

$$\chi(\mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(Y, \mathcal{F}).$$

Assume $Y \subseteq \mathbb{P}^N$ is projective. Let $\mathcal{O}_Y(1)$ be the canonical ample line bundle, and $\mathcal{O}_Y(n)$ the n-th twist. We put $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n)$. The function $n \mapsto \chi(\mathcal{F}(n))$ is a polynomial in n, and call it the Hilbert polynomial of $\mathcal{F}$. If $\mathcal{F} = \mathcal{O}_Z$ for some closed subvariety $Z \subset Y$, we also call $\chi(\mathcal{O}_Z(n))$ the Hilbert polynomial of Z.

Proposition 1.3.4. — *Assume X is proper. Let $W \subseteq X \times \mathbb{P}^1$ be a closed subvariety dominating $\mathbb{P}^1$, and $p, q \in \mathbb{P}^1(k)$ be two points lying in the image of W. Let W_p and W_q denote respectively the fibres of W at p and q. Then we have $\chi(\mathcal{O}_{W_p}) = \chi(\mathcal{O}_{W_q})$. Moreover, if X is projective, then the Hilbert polynomial of W_p and W_q are the same, i.e. we have*

$$\chi(\mathcal{O}_{W_p}(n)) = \chi(\mathcal{O}_{W_q}(n)), \quad \forall n \in \mathbb{Z}$$

Proof. — For any some variety $Z \subseteq X \times \mathbb{P}^1$, denote by $\mathcal{O}_Z(0,b)$ the pull-back to Z of $\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(b)$. Let (t_0,t_1) denote coordinates on $\mathbb{P}^1$ such that p (resp. q) is defined as $t_0 = 0$ (resp. $t_1 = 0$). Since W is a variety dominating $\mathbb{P}^1$, it is flat over $\mathbb{P}^1$. One has two sequences

$$0 \to \mathcal{O}_W(0,-1) \xrightarrow{t_0} \mathcal{O}_W \to \mathcal{O}_{W_p} \to 0$$
$$0 \to \mathcal{O}_W(0,-1) \xrightarrow{t_1} \mathcal{O}_W \to \mathcal{O}_{W_q} \to 0,$$

and these sequences are exact because W is a variety dominating over $\mathbb{P}^1$, hence flat over $\mathbb{P}^1$ around p,q. Now the equality follows from the additivity of χ.

Now assume that X is projective. Twisting the sequences by $\mathrm{pr}_1^*\mathcal{O}_X(n)$ and using the additivity of Euler-Poincaré characteristics, one gets

$$\chi(\mathcal{O}_{W_p}(n)) = \chi(\mathcal{O}_{W_q}(n)), \quad \forall n \in \mathbb{Z}.$$

□

Remark 1.3.5. — The previous Proposition is a special case of the following general fact: Let $f : X \to S$ be a proper morphism of schemes, and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-sheaf flat over S. For any $s \in S$, put $\mathcal{F}_s = \mathcal{F} \otimes_{f^{-1}\mathcal{O}_S} k(s)$. Then $\chi(\mathcal{F}_s)$ is a locally constant function in $s \in S$.

Assume X is proper. By the additivity of Euler-Poincaré characteristic, one can extend χ by linearity to $Z(X)$. By Proposition 1.3.4, χ induces a function

$$\chi : A(X) \to \mathbb{Z}.$$

If X is projective, one can even attach a Hilbert polynomial $P_\alpha(x)$ to a cycle class $\alpha \in A(X)$: if $\alpha = \sum_V n_V[V]$, $P_\alpha(x) \in \mathbb{Q}[x]$ is the polynomial such that

$$P_\alpha(x) := \sum_V n_V \cdot \chi(\mathcal{O}_V(x)), \quad \forall x \in \mathbb{Z}.$$

1.4. Flat pull-back

Let $f : X \to Y$ be a flat morphism of schemes. Assume that f has relative dimension n, i.e. for any subvariety V of Y and all irreducible components V' of $f^{-1}(V)$, one has $\dim(V') = \dim(V) + n$. For any closed subvariety $V \subseteq Y$ of dimension d, set

$$f^*[V] = [f^{-1}(V)].$$

This extends by linearity to a pull-back morphism

$$f^* : Z_d(Y) \to Z_{d+n}(X).$$

Lemma 1.4.1. — *For any closed subscheme $Z \subseteq Y$, one has*

$$f^*[Z] = [f^{-1}(Z)].$$

Proof. — By definition, one has

$$f^*[Z] = f^* \sum_{V \subseteq Z} \mathrm{leng}(\mathcal{O}_{Z,V}) f^*[V] = \sum_{V} \mathrm{leng}(\mathcal{O}_{Z,V})[f^{-1}(V)]$$

and

$$[f^{-1}(Z)] = \sum_{W \subseteq f^{-1}(Z)} \mathrm{leng}(\mathcal{O}_{f^{-1}(Z),W})[W],$$

where V (resp. W) runs through the irreducible (reduced) components of Z (resp. of $f^{-1}Z$). For each irreducible component W of $f^{-1}(Z)$, the closure of the image $f(W)$, say V, must be an irreducible component of Z. Let $A = \mathcal{O}_{Z,V}$ and $B = \mathcal{O}_{f^{-1}(Z),W}$. Then both A and B are local Artinian rings, so that A admits a decreasing filtration by A-submodules

$$M_0 = A \supset M_1 \supset \cdots \supset M_n = 0$$

such that each M_i/M_{i+1} is isomorphic to $A/\mathfrak{m}_A$. By the flatness of B, one has

$$\mathrm{leng}_B(B) = n \cdot \mathrm{leng}_B(B/\mathfrak{m}_A B) = \mathrm{leng}_A(A)\mathrm{leng}_B(B/\mathfrak{m}_A B).$$

But the left hand side is the coefficient of $[W]$ in $[f^{-1}(Z)]$, and the right hand side is that of $[W]$ in $f^*[Z]$. The Lemma follows immediately.

□

Proposition 1.4.2. — *Given a cartesian diagram of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f proper and g flat of relative dimension n, then, for any $\alpha \in Z_d(X)$, the equality

$$g^* f_*(\alpha) = f'_* g'^*(\alpha)$$

in $Z_{d+n}(Y)$.

Proof. — We may assume that $\alpha = [V]$ for some closed subvariety of X of dimension d. Put $W = f(V)$, $V' = g'^{-1}(V)$ and $W' = g^{-1}(W)$. Then every irreducible component of V' (resp. of W') has dimension $d + n$ (resp. $\dim(W) + n$). So if $\dim(W) < \dim(V)$, then the desired equality is trivial.

Assume thus $\dim(W) = d$. Up to replacing Y by W and X by V, one may assume that $f : X \to Y$ is a proper surjective morphism of varieties and $\alpha = [X]$. Since the problem is local for Y, after taking the base change $\mathrm{Spec}(k(Y)) \to Y$, one may assume that $Y = \mathrm{Spec}(K)$ and $X = \mathrm{Spec}(L)$ with L/K finite. Then one has

$$g^* f_*([X]) = [L : K] g^*([Y]) = [L : K][Y'].$$

On the other hand, we has

$$f'_* g'^*([X]) = f'_*([Y' \otimes_K L]) = [L : K][Y'].$$

□

Theorem 1.4.3. — *Let $f : X \to Y$ be a flat morphism of relative dimension n. Then f^* preserves rational equivalence, hence it induces a homomorphism of Chow groups:*

$$f^* \colon A_d(Y) \to A_{d+n}(X)$$

Proof. — Let $W \subseteq Y$ be a closed subvariety of dimension $d+1$, and $r \in k(W)^\times$. We have to show that $f^* \operatorname{div}(r)$ is rationally equivalent to 0. Up to replacing Y by $W \cap f(X)$, we may assume that $Y = W$ is a variety and f is surjective. Let X_i be the irreducible components of X, and

$$[X] = \sum_i n_i [X_i], \quad \text{with } n_i = \operatorname{leng}(\mathcal{O}_{X,X_i}).$$

Then for each i, the restriction $f|_{X_i}$ induces an injection $k(Y) \hookrightarrow k(X_i)$. Let $r_i \in k(X_i)$ denote the image of r. We claim that

$$f^* \operatorname{div}(r) = \sum_i n_i [\operatorname{div}(r_i)].$$

By definition, we have

$$f^* \operatorname{div}(r) = \sum_V \operatorname{ord}_V(r) f^*[V] = \sum_V \operatorname{ord}_V(r)[f^{-1}(V)].$$

Fix a subvariety $Z \subseteq X$ of codimension 1. Let V be the closure of $f(Z)$. Put $A = \mathcal{O}_{Y,V}$ and $B = \mathcal{O}_{X,Z}$. Then A is a Noetherian local domain of dimension 1 with fraction field $k(Y)$, and B is a local ring of dimension 1 flat over A. The minimal primes $\mathfrak{p}_i$ of B are in bijection with the irreducible components X_i such that $Z \subseteq X_i$. Then the coefficient of $[Z]$ in $f^* \operatorname{div}(r)$ is $\operatorname{ord}_V(r) \operatorname{leng}(B/\mathfrak{m}_A B)$, and that in $\sum_i n_i [\operatorname{div}(r_i)]$ is $\sum_{\mathfrak{p}_i} \operatorname{leng}(B_{\mathfrak{p}_i}) \operatorname{ord}_Z(r_i)$. We may assume that $r \in \mathfrak{m}_A$. To finish the proof, it is enough to show that

$$\operatorname{leng}(A/rA)\operatorname{leng}(B/\mathfrak{m}_A B) = \sum_{\mathfrak{p}_i} \operatorname{leng}(B_{\mathfrak{p}_i}) \operatorname{leng}(B/\mathfrak{p}_i + rB).$$

Since B is flat over A, one sees easily that $\operatorname{leng}(A/rA)\operatorname{leng}(B/\mathfrak{m}_A B) = \operatorname{leng}(B/rB)$. Now the Theorem follows immediately from the following Lemma.

□

Lemma 1.4.4. — *Let B be a Noetherian local ring of dimension 1 with minimal ideals $\mathfrak{p}_i$, and M is a finitely generated B-module. Let $x \in \mathfrak{m}_B$ which is M-regular, i.e. $M \xrightarrow{\times x} M$ is injective. Then one has*

$$\operatorname{leng}(M/xM) = \sum_{\mathfrak{p}_i} \operatorname{leng}(M_{\mathfrak{p}_i}) \operatorname{leng}(B/(\mathfrak{p}_i + xB)).$$

Proof. — See [**Fu98**, Lemma A.2.7.] □

1.5. Affine bundles

A scheme $p : E \to X$ over X, is called an affine bundle of rank n, if there exists an open covering U_α of X such that $p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{A}^n$.

Proposition 1.5.1. — *Let $p : E \to X$ be an affine bundle of rank n. Then the pull-back morphism*

$$p^* : A_d(X) \to A_{d+n}(E)$$

is surjective.

Proof. — By Noetherian induction on X, we may assume that the affine bundle is trivial, i.e. $E \cong X \times \mathbb{A}^n$ and X is an affine variety. By induction on n, we reduce to the case $n = 1$. One has to prove that every $d+1$-dimensional subvariety $V \subseteq X \times \mathbb{A}^1$ is rationally equivalent to a cycle of the form $p^*\alpha$. We may assume that V dominates X. Let A be the coordinate ring of X with fraction field K. Let $\mathfrak{q} \subseteq A[t]$ be the prime ideal corresponding to the subvariety $V \subseteq X \times \mathbb{A}^1$. If $\dim(X) = d$, then $V = E$ and hence $[V] = p^*[X]$. Otherwise, one has $\dim(X) = d + 1$. Since V dominates X, the prime ideal $\mathfrak{q}K[t] \subseteq K[t]$ is generated by a polynomial $r \in K[t]$. Then

$$[V] - \mathrm{div}(r) = \sum_i n_i[V_i]$$

for some subvarieties $V_i \subseteq X \times \mathbb{A}^1$ of dimension $d + 1$ not dominating X. Putting $W_i = p(V_i)$, then one has $V_i = p^{-1}(W_i)$. The Proposition follows immediately. □

Corollary 1.5.2. — *Let U be an open dense subset of $\mathbb{A}^n$. Then $A_d(U) = 0$ for any $d \neq n$, and $A_n(U)$ is a free abelian group generated by $[U]$.*

Proof. — For $U = \mathbb{A}^n$, this follows immediately from Proposition 1.5.1. The general case follows from Lemma 1.1.5(d). □

Corollary 1.5.3. — *Let X be a scheme with a cellular decomposition:*

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subschemes, such that each $X_i - X_{i-1}$ is a disjoint union of schemes $U_{i,j}$ isomorphic to $\mathbb{A}^{n_{i,j}}$. Let $V_{i,j}$ denote the closure in X of $U_{i,j}$. Then the Chow group $A(X)$ is generated by $[V_{i,j}]$.

Proof. — We proceed by induction on m with $m \leq n$ to show that $A(X_m)$ is generated by $[V_{i,j}]$ for $i \leq m$. This follows immediately from Corollary 1.5.2 and the exact sequence:

$$A(X_{m-1}) \to A(X_m) \to A(X_m - X_{m-1}) \to 0.$$

□

Example 1.5.4. — Consider the projective space $\mathbb{P}^n$. One has a cellular decomposition:

$$\mathbb{P}^n \supset \mathbb{P}^{n-1} \supset \cdots \mathbb{P}^1 \supset \mathbb{P}^0 = \{(0, \cdots, 0, 1)\},$$

where $\mathbb{P}^{n-r}$ is the closed subvariety of $\mathbb{P}^n$ defined by vanishing of the first r-coordinates. Denote by $L^r \in A_{n-r}(\mathbb{P}^n)$ the class of $\mathbb{P}^{n-r}$. By Corollary 1.5.3, $A_{n-r}(\mathbb{P}^n)$ is generated by

L^r. We claim that $A_{n-r}(\mathbb{P}^n)$ is actually a free abelian group with generator L^r. Indeed, it suffices to see that L^r is not a torsion class in $A_{n-r}(X)$. Recall that the Hilbert polynomial of a class of $A(\mathbb{P}^n)$ is well defined, and that of L^r is

$$P_{L^r}(x) = \chi(\mathcal{O}_{\mathbb{P}^{n-r}}(x)) = \binom{x+n-r}{n-r}.$$

Hence, L^r is not torsion, and we conclude that $A_{n-r}(\mathbb{P}^n) = \mathbb{Z} \cdot L^r$.

Now let $X \subseteq \mathbb{P}^n$ be an arbitrary closed subscheme of pure dimension $n-r$. The Hilbert polynomial of X has the form

$$P_X(x) = c_0 \frac{x^{n-r}}{(n-r)!} + \text{lower terms}, \quad \text{with } c_0 \in \mathbb{Z}\backslash\{0\}.$$

We put $\deg(X) = c_0$ and call it the degree of X. Then it is easy to see that $[X] = \deg(X)L^d$ in $A_{n-d}(\mathbb{P}^n)$.

Exercise 1.5.5. — Let $S \subseteq \mathbb{P}^N$ be a hypersurface defined by a homogeneous equation of degree d. Then the degree of S in the sense above is d.

Example 1.5.6. — Let G be a reductive group over k, $T \subset B \subset G$ be a maximal torus and Boreal subgroup defined over k. Let $W = N_G(T)/T$ be the Weyl group. Consider the flag variety. One has the Bruhat decomposition:

$$G/B = \coprod_{w \in W} B\dot{w}B/B,$$

where $\dot{w}$ is a representative of $w \in W$. Note that, for each $w \in W$, we have

$$B\dot{w}B/B \cong B/B \cap \dot{w}B\dot{w}^{-1} \cong \mathbb{A}^{\ell(w)},$$

where $\ell(w)$ is the length of $w \in W$. Denote by $\overline{B\dot{w}B}/B$ the closure of $B\dot{w}B/B$ in G/B. Then we have

$$\overline{B\dot{w}B}/B - B\dot{w}B/B = \coprod_{w'<w} B\dot{w}'B/B.$$

In particular, $G/B = \coprod_{w \in W} B\dot{w}B/B$ is a cellular decomposition of G/B. Let σ_w denote the class of $\overline{B\dot{w}B}/B$ in $A(G/B)$. Then by Corollary 1.5.3, $A(G/B)$ is generated by $\{\sigma_w : w \in G/B\}$.

CHAPTER 2

INTERSECTION PRODUCT ON NON-SINGULAR VARIETIES

We are going to define the intersection product on the Chow group of a variety X over k. There are two approaches to the definition of intersection product. The first approach uses moving lemma, and it applies only to smooth quasi-projective varieties over k. The second approach uses the technique of deformation to the normal cone, and this is the approach adapted in [**Fu98**]. In this chapter, we will discuss the first approach via moving Lemma by following [**EH13**].

In this chapter, we assume k is algebraically closed.

2.1. Moving Lemma and intersection products

Definition 2.1.1. — Let X be a variety.

(a) Let A, B be subschemes of X. We say A and B are *dimensionally transverse* if for every irreducible component C of $A \cap B$, one has

$$\operatorname{codim}_X(C) = \operatorname{codim}_X(A) + \operatorname{codim}_X(B).$$

(b) Subvarieties A and B are transverse at a point $p \in X$ if X, A and B are smooth at p and their tangent spaces satisfy $T_p(X) = T_p(A) + T_p(B)$.

(c) Subvarieties A, B are generically transverse if every irreducible component of $A \cap B$ contains a point p at which A and B are transverse.

Lemma 2.1.2. — *Subvarieties A, B of X are generically transverse if and only if they are dimensionally transverse and each irreducible component C of $A \cap B$ is reduced and contains a closed point of p such that C and X are smooth at p.*

Proof. — First, we suppose that A and B are generically transverse. By definition, every irreducible component C of $A \cap B$ contains a point $p \in X$ such that A, B and X are all smooth at p and $T_pX = T_pA + T_pB$. One has to show that C is smooth at p and $\operatorname{codim}(C) = \operatorname{codim}(A) + \operatorname{codim}(B)$. Indeed, Let $R = \mathcal{O}_{X,p}$ and $I \subseteq R$ and $J \subseteq R$ be the ideals given by A and B. By assumption, R, R/I and R/J are all regular local rings of dimension $n = \dim(X)$, $a = \dim(A)$ and $b = \dim(B)$. Let $\mathfrak{m}$ denote the maximal ideal of R. The condition $T_pX = T_pA + T_pB$ is equivalent to saying that the natural map

$$\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/(I + \mathfrak{m}^2) \oplus \mathfrak{m}/(J + \mathfrak{m}^2)$$

is injective, i.e. if $\bar{I}$ and $\bar{J}$ denote respectively the image of I and J in $\mathfrak{m}/\mathfrak{m}^2$, then one has $\bar{I} \cap \bar{J} = 0$. It follows that the image of $I + J$ in $\mathfrak{m}/\mathfrak{m}^2$ has dimension $n - a + n - b$, and $R/(I + J) = \mathcal{O}_{C,p}$ is regular of dimension $n - (2n - a - b) = a + b - n$.

Let C be an irreducible component of $A \cap B$ such that $\operatorname{codim}(C) = \operatorname{codim}(A) + \operatorname{codim}(B)$ and C contains a smooth point $p \in C$. Let R, I, J be as above. Then one has

$$\dim(A) = \dim(R/I) \le \dim_k \mathfrak{m}/(I + \mathfrak{m}^2) = n - \dim_k(\bar{I}),$$

i.e. $\dim_k \bar{I} \le \operatorname{codim}(A)$, and the equality holds if and only if A is smooth at p. Similarly, we have $\dim_k \bar{J} \le \operatorname{codim}(B)$. But C is smooth at p, it follows that

$$\operatorname{codim}(C) = \dim_k(\overline{I + J}) \le \dim_k(\bar{I}) + \dim_k(\bar{J}) \le \operatorname{codim}(A) + \operatorname{codim}(B).$$

As $\operatorname{codim}(C) = \operatorname{codim}(A) + \operatorname{codim}(B)$, hence all the equalities above hold, that is, A and B are smooth at p, and $\dim_k(\overline{I + J}) = \dim_k(\bar{I}) + \dim_k(\bar{J})$, which is equivalent to saying that $T_pX = T_pA + T_pB$.

□

It is natural to generalize the notion of *dimensional transverseness* from subvarieties to cycles. Two cycles $\alpha = \sum_i m_i[A_i]$ and $\beta = \sum_j n_j[B_j]$ are called *dimensionally* (resp. *generically*) transverse if A_i, B_j are dimensionally (resp. generically) transverse. If $A = \sum_i m_i[A_i]$ and $B = \sum_j n_j[B_j]$ are two cycles on X intersecting generically transversely, then we put

$$A \cap B = \sum_{i,j} m_i n_j [A_i \cap B_j].$$

***Theorem 2.1.3* (Moving Lemma).** — *Let X be a smooth quasi-projective variety.*

(a) *Let $\alpha \in A(X)$ and $B \in Z(X)$. Then there exists a cycle $A \in Z(X)$ which represents α and intersects B generically transversely.*
(b) *If A and B are cycles intersecting generically transversely, then the class $[A \cap B] \in A(X)$ depends only on the class $[A], [B] \in A(X)$.*

Remark 2.1.4. — Even if α and β are classes of effective cycles, it is not always possible to find effective representatives A, B in the Moving Lemma intersecting transversally.

Given the moving Lemma, one can define the intersection product on the Chow group of a smooth quasi-projective variety.

Theorem 2.1.5. — *Let X be a smooth quasi-projective variety. There exists a unique bilinear product on $A(X)$ such that*

$$[A] \cdot [B] = [A \cap B]$$

whenever A and B are subvarieties that are generically transverse. Under this product structure, $A(X)$ becomes an associative, commutative ring.

Assuming the Moving Lemma, the proof of this Theorem is straightforward, and we will leave it as an exercise.

Let $n = \dim(X)$. For each integer $i \geq 0$, we put $A^i(X) = A_{n-i}(X)$. Then it is easy to see that

$$A^i(X) \cdot A^j(X) \subseteq A^{i+j}(X).$$

Therefore, $A(X) = \bigoplus_{i\in\mathbb{Z}} A^i(X)$ is a commutative graded ring.

Example 2.1.6. — Let $\xi \in A^1(\mathbb{P}^n)$ denote the class of a hyperplane. Then ξ^d is the class of a $(n-d)$-dimensional linear subspace. We have seen that $A^d(\mathbb{P}^n)$ is a free abelian group generated by ξ^d. Thus, we have an isomorphism of commutative graded rings:

$$A(\mathbb{P}^n) = \mathbb{Z}[\xi]/\xi^{n+1}.$$

Recall that for each subvariety $X \subset \mathbb{P}^n$ of codimension d (or more generally, for a cycle $\alpha \in A^d(\mathbb{P}^n)$), we have defined a degree $\deg(X)$. It has the following geometric interpretation: $\deg(X)$ is the number of intersection points of X with a general linear subspace in $\mathbb{P}^n$ of dimension d.

Let $H_1, \cdots, H_n$ be n hypersurfaces of degree $d_1, \cdots, d_n$ respectively. If $H_1 \cap \cdots \cap H_n$ consists of some reduce points, then

$$\#(H_1 \cap \cdots \cap H_n) = \deg([H_1] \cdots [H_n]) = \prod_i d_i.$$

Example 2.1.7. — Using the Chow ring of $\mathbb{P}^n$, one can show the following statement: *Any morphism from $\mathbb{P}^n$ to a quasi-projective variety of dimension strictly less than n is constant.*

Indeed, suppose $\varphi : \mathbb{P}^n \to X$ is such a map. We may assume that φ is surjective. Suppose that φ is not constant, i.e. $\dim(X) > 0$. Then the preimage of a general hyperplane section of X will be disjoint from the preimage of a general point of X. The preimage of a general hyperplane section of X has dimension $n-1$ and that of a general point on X has dimension > 0. But two such subvarieties of $\mathbb{P}^n$ must meet, which is a contradiction.

Example 2.1.8. — Consider the Veronese embedding

$$v_{n,d} : \mathbb{P}^n \to \mathbb{P}^N, \quad [x_0, \cdots, x_n] \mapsto [\cdots, X^I, \cdots],$$

where $N = \binom{n+d}{d} - 1$ and X^I runs over all monomials of degree d in $n+1$ variables. Denote by $\Phi_{n,d}$ its image. Then $\deg(\Phi_{n,d})$ is the number of points of the intersection of $\Phi_{n,d}$ with n general hyperplanes. Since the inverse image of a hyperplane in $\mathbb{P}^N$ via $v_{n,d}$ is a hypersurface of degree d, thus $\deg(\Phi_{n,d})$ equals also to the number of n general hypersurfaces of degree d, which is d^n by Bezout.

Example 2.1.9. — A product of projective spaces has also an evident cellular decomposition. It follows that

$$A(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}) = \mathbb{Z}[t_1, \cdots, t_r]/(t_1^{n_1+1}, \cdots, t_r^{n_r+1}).$$

This can be viewed as a Kunneth formula for Chows groups of projective spaces. However, note that Kunneth formula is in general FALSE for Chow groups.

We focus on the case of $r = 2$.

As shown by the following examples, the smoothness of X is essential for the moving Lemma to hold.

Example 2.1.10. — Let $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ be a smooth conic and $p \notin \mathbb{P}^2$. Let $X = \overline{pC} \subset \mathbb{P}^3$ be the cone with vertex p. Let $L \subset X$ be a line passing through p. We claim that every cycle on X rationally equivalent to L has support containing the singular point p, and hence the conclusion of the first part of the Moving Lemma fails for X.

Via the natural embedding $X \hookrightarrow \mathbb{P}^3$, one can define the degree for cycles on X. It is clear that $\deg(L) = 1$ and $\deg(C) = 2$. On the other hand, if $D \subseteq X$ is a curve not containing p, then the natural projection from p defines a finite flat map $D \to C$. We claim that $\deg(D) = \deg(f)\deg(C)$. Indeed, choose a line $H' \subseteq \mathbb{P}^2$ cutting transversally with C, and let $H = \overline{pH'}$ be the unique plane containing p and H'. Then we have

$$\deg(D) = \#(H \cap D) = \deg(f)\#(H \cap C) = \deg(f)\deg(C).$$

Hence, any cycle of dimension 1 on X with support disjoint from p has even degree. Since the degree is invariant under rational equivalence, the claim follows.

Example 2.1.11. — Let $Q \subseteq \mathbb{P}^3$ be a smooth quadratic surface and let $X = \overline{pQ}$ be the cone in $\mathbb{P}^4$ with vertex point $p \notin \mathbb{P}^3$. It is well known that the quadratic surface Q contains two rulings of lines $\{M_t\}$ and $\{N_t\}$ parametrized by $t \in \mathbb{P}^1$, and they are the only lines contained in Q. For instance, if Q is the surface with affine equation $x^2 + y^2 - z^2 = 1$, then the lines are parametrized by the circle $x^2 + y^2 = 1$ with equations

$$\begin{cases} x = \pm az + b \\ y = \mp bz + a \end{cases} \qquad \text{with } a^2 + b^2 = 1.$$

Correspondingly, the cone has two families of planes $\{\Lambda_t = \overline{pM_t}\}$ and $\{\Gamma_t = \overline{pN_t}\}$.

Let $L \subset X$ be a line not passing through p. Under the projection from p, L maps a line on Q. Hence, its image must lies in a plane Λ_t or Γ_t. Note that lines M_t and $M_{t'}$ on Q in the same ruling are disjoint if $t \neq t'$, and lines M_t and $N_{t'}$ from different ruling intersect at one point. It follows that, if M is a general line lying on Λ_t, then M is disjoint from $\Lambda_{t'}$ for $t \neq t'$ and cuts with $\Gamma_{t'}$ at one point. Therefore, if there exists a good theory of intersection product on X satisfying the same condition as in Theorem 2.1.5, one would have

$$[M] \cdot [\Lambda_t] = 0, \quad [M] \cdot [\Gamma_t] = [q],$$

for some point $q \in X$. Similarly, if N is a general line on Γ_t, one would have

$$[N] \cdot [\Lambda_t] = [r], \quad [N] \cdot [\Gamma_t] = 0.$$

Note also that $[q], [r] \in A_0(X)$ are necessarily non-zero, since their degrees are non-zero. However, note that all lines on X are rationally equivalent, because one can move a general line M on Λ_t to a line passing through p, and them move it to a general line on the other ruling $\Gamma_{t'}$.

2.2. Intersection Multiplicities

Let X be a smooth quasi-projective variety. We have defined the intersection product on $A(X)$ using Moving Lemma, so that if A, B are two subvarieties are generically transversal, then

$$[A][B] = [A \cap B].$$

We want to extend this formula to the case when A, B are just dimensionally transversal.

Definition 2.2.1. — Let A and B be two subschemes of X which are dimensionally transverse, and let Z be an irreducible component of $A \cap B$. We define the intersection multiplicity of A, B at Z as

$$m_Z(A,B) := \sum_{i=0}^{\dim(X)} (-1)^i \mathrm{leng}_{\mathcal{O}_{A\cap B,Z}} \mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z}).$$

Recall the following definition of Cohen-Macaulay rings. Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. A sequence $(x_1, \cdots, x_r)$ in $\mathfrak{m}$ is called R-regular (or just regular), if the map

$$R/(x_1, \cdots, x_{i-1}) \xrightarrow{\times x_i} R/(x_1, \cdots, x_{i-1})$$

is injective. The maximal length of R-regular sequences in $\mathfrak{m}$ is called the depth of R, denoted by $\mathrm{depth}(R)$. We say R is Cohen-Macaulay if we have $\mathrm{depth}(R) = \dim(R)$. A noetherian local ring A is called a complete intersection ring, if $\widehat{A}$ can be written as $\widehat{A} = R/(x_1, \cdots, x_r)$, where R is a local regular ring and $(x_1, \cdots, x_r)$ is a R-regular sequence. A local complete intersection ring is always Cohen-Macaulay.

Lemma 2.2.2. — *In the situation above, if $\mathcal{O}_{A,Z}$ and $\mathcal{O}_{B,Z}$ are Cohen-Macaulay, then* $\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z}) = 0$ *for $i > 0$, and hence*

$$m_Z(A,B) = \mathrm{leng}(\mathcal{O}_{A\cap B,Z}).$$

Proof. — Since $\widehat{\mathcal{O}}_{X,Z}$ is faithfully flat over $\mathcal{O}_{X,Z}$, we have

$$\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(M,N) \otimes_{\mathcal{O}_{X,Z}} \widehat{\mathcal{O}}_{X,Z} = \mathrm{Tor}_i^{\widehat{\mathcal{O}}_{X,Z}}(\hat{M}, \hat{N}),$$

for any finitely generated $\mathcal{O}_{X,Z}$-module M, N. Since we are in the equal characteristic case, the complete regular local ring $\widehat{\mathcal{O}}_{X,Z} \cong \kappa(Z)[[x_1, \cdots, x_n]]$ where $\kappa(Z)$ is the residue field of $\mathcal{O}_{X,Z}$. We reduce to showing the following statement: *Let $R = k[[x_1, \cdots, x_n]]$, and $R/I, R/J$ be two quotients of R of dimension $n-r$ and r respectively. If R/I and R/J are both Cohen-Macaulay and* $\dim(R/(I+J)) = 0$*, then*

$$\mathrm{Tor}_i^R(R/I, R/J) = 0, \quad \forall i > 0.$$

Suppose first that R/I is a local complete intersection, i.e. I is generated by a regular sequence $(x_1, \cdots, x_r)$. Let $\mathrm{Kos}^\bullet(\underline{x})$ be the Koszul complex associated to the sequence $(x_1, \cdots, x_r)$. Then $\mathrm{Kos}^\bullet(\underline{x})$ is a free resolution of R/I. Hence, we have

$$\mathrm{Tor}_i^R(R/I, R/J) = H^{-i}(\mathrm{Kos}^\bullet(\underline{x}) \otimes_R R/J)$$

The condition that
$$\dim(R/(I+J)) = \dim(R/J) - r$$
implies that the image of $(x_1, \cdots, x_r)$ in R/J is also a regular sequence. Therefore, we have $H^{-i}(\mathrm{Kos}^\bullet(\underline{x}) \otimes_R R/J) = 0$.

The general case can be reduced to the previous case by "diagonal trick". We consider R as a quotient of $R\widehat{\otimes}_k R = k[[1 \otimes x_i, x_i \otimes 1 : 1 \leq i \leq n]]$ modulo the ideal $(1 \otimes x_i - x_i \otimes 1, 1 \leq i \leq n)$. Then one has
$$\mathrm{Tor}_i^R(R/I, R/J) = \mathrm{Tor}_i^{R\widehat{\otimes}_k R}(R, R/I\widehat{\otimes}_k R/J)$$
Note that R is a quotient of $R\widehat{\otimes} R$ by an ideal generated by a regular sequence, and $R/I\widehat{\otimes}_k R/J$ Cohen-Macaulay. Hence, we are reduced to the previous case.

□

Theorem 2.2.3 **(Serre)**. — *Let X be a smooth quasi-projective variety, and $A, B \subset X$ be two dimensionally transversal subschemes. Then we have*
$$[A][B] = \sum_Z m_Z(A,B)[Z].$$
When A and B are both Cohen-Macaulay at the generic point of each irreducible component of Z, then one has
$$[A][B] = [A \cap B].$$

Remark 2.2.4. — Without the Cohen-Macaulay assumption, the equality $[A][B] = [A \cap B]$ is false in general.

Example 2.2.5 **(Bézout)**. — Let $X = \mathbb{P}^n$, and $H_1, \cdots, H_n$ be hypersurfaces of degrees $d_1, \cdots, d_n$ respectively. Suppose that $H_1 \cap \cdots \cap H_n$ contain only finitely many points, say $P_1, \cdots, P_m$. In this case, $H_1 \cap \cdots \cap H_i$ is a complete intersection for each i with $1 \leq i \leq n$, hence Cohen-Macaulay. Then we have
$$\sum_{i=1}^m \mathrm{leng}(\mathcal{O}_{H_1 \cap \cdots \cap H_n, P_i}) = d_1 \cdots d_n.$$

Example 2.2.6. — Let $X = \mathbb{P}^4$, and V_1, V_2 be two general planes meeting at a unique point p and $A = V_1 \cup V_2$. Let B_1 be a plane in $\mathbb{P}^4$ not passing through p. Then B_1 intersects with each of V_1, V_2 at a unique reduced point. Hence, B_1 intersects with A generically transversally, and we have
$$\deg([A][B_1]) = \deg([A \cap B_1]) = 2.$$
Consider another plane B_2 which passes p and does not meet A elsewhere. Since B_1 is rationally equivalent to B_2, we have $m_p(A, B_2) = 2$. However, one will see below that $\mathrm{leng}(\mathcal{O}_{A \cap B_2, p}) = 3$, and hence $[A][B_2] \neq [A \cap B_2]$. The problem comes from the fact that $\mathcal{O}_{A,p}$ is not Cohen-Macaulay.

To compute $\operatorname{leng}(\mathcal{O}_{A\cap B_2,p})$, we choose local coordinates so that $\widehat{\mathcal{O}}_{X,p} = k[[x_1, x_2, x_3, x_4]]$, and

$$I(V_1) = (x_1, x_2), \quad I(V_2) = (x_3, x_4),$$
$$I(A) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4),\ I(B_2) = (x_1 - x_3, x_2 - x_4).$$

Hence, we get

$$\mathcal{O}_{A\cap B_2,p} \cong k[[x_1, \cdots, x_4]]/I(A) + I(B_2) \cong k[x_1, x_2]/(x_1^2, x_2^2, x_1x_2),$$

which clearly has length 3. The difference $3 - 2 = 1$ comes from the length of $\operatorname{Tor}_1^{\mathcal{O}_{X,p}}(\mathcal{O}_{A,p}, \mathcal{O}_{B_2,p})$.

2.3. Pullbacks for generically separable formulas

We have defined the pullback map $f^* : A(X) \to A(Y)$ for a flat morphism $f : Y \to X$. Using the same ideas as Moving Lemma, one can extend the definition of f^* to more general morphisms of quasi-projective varieties.

Definition 2.3.1. — Let $f : Y \to X$ be a projective map of quasi-projective varieties. A subvariety $V \subseteq X$ of codimension c is called "generically transverse" to f if

1. $f^{-1}(V)$ is generically reduced of codimension c.
2. Y is smooth at a general point q of each irreducible component of $f^{-1}(V)$ and X and V are smooth at $f(q)$.

Note that when $f : Y \to X$ is a closed immersion, then a subvariety $V \subseteq X$ generically transverse to f is equivalent to saying that V, Y are generically transverse in the sense of Definition 2.1.1.

A cycle $A = \sum_i m_i[A_i]$ on X is said to be *generically transverse* to f if every component A_i is. For such a cycle, we define

$$f^*(A) = \sum_i m_i[f^{-1}(A_i)].$$

Recall that $f : Y \to X$ is generically separable if the field extension $k(Y)/k(X)$ is separable, that is, $k(Y)$ is a finite separable extension of purely transcendental extension of $k(X)$.

Theorem 2.3.2 **(Moving Lemma for morphisms)**. — *Let $f : X \to Y$ be a generically separable morphism of smooth quasi-projective varieties.*

(a) *For any Chow class $\alpha \in A(Y)$, there exists a cycle $A \in Z(X)$ such that $[A] = \alpha$ and A is generically transverse to f.*
(b) *If $A \in Z(Y)$ is a cycle generically transverse to f, the class $f^{-1}(A)$ depends only on $[A] \in A(Y)$.*

Using Theorem 2.3.2, one can define the pullback $f^* : A(X) \to A(Y)$ for more general morphisms as follows.

Theorem 2.3.3. — *Let $f : Y \to X$ be a generically separable map of smooth projective varieties.*

(a) *There exists a unique ring homomorphism $f^* : A(X) \to A(Y)$ such that if A is a subvariety of X generically transverse to f then*

$$f^*[A] = [f^{-1}(A)].$$

(b) *(Projection formula) Let $f_* : A(Y) \to A(X)$ be the push-forward map of Chow groups. Then we have, for any $\alpha \in A(X)$ and $\beta \in A(Y)$,*

$$f_*(f^*(\alpha)\beta) = \alpha f_*(\beta).$$

Proof. — We just prove (b), since part (a) follows straightly from Theorem 2.3.2. The formula being linear in β, we may assume that $\beta = [B]$ for some irreducible some variety $B \subset Y$. By Moving Lemma, we may assume that B is generically transverse to the generic fibre of f, and hence generically separable to over its image $f(B)$. We may assume $\alpha = [A]$, where $A \subset X$ is generically transverse to $f|_B$ and to $f(B)$. Note that this implies that $f^{-1}(A)$ is generically transverse to B.

Suppose that B is generically finite of degree d over $f(B)$. Since A is transverse to $f|_B$, the cycle $f^{-1}(A) \cap B$ is also generically finite of the same degree d. In this case, Theorem follows as

$$\begin{aligned} f_*(f^*(\alpha)\beta) &= f_*([f^{-1}(A)]\beta) \qquad (A \text{ is generically transverse to } f) \\ &= f_*([f^{-1}(A) \cap B]) \qquad f^{-1}(A) \text{ is generically transverse to } B \\ &= d[f(f^{-1}(A) \cap B)] \qquad \text{definition of } f_* \\ &= d[A \cap f(B)] \qquad \text{set theoretic equality} \\ &= [A \cap f_*(B)] \qquad \text{definition of } f_* \\ &= \alpha f_*(\beta). \end{aligned}$$

□

The projection formula is extremely useful when computing the degree of 0-cycles. Actually, in the setup of Theorem 2.3.3, if $f^*(\alpha)\beta$ and $\alpha f_*(\beta)$ are 0-cycles, then one have

$$\int_Y f^*(\alpha)\beta = \int_X \alpha f_*(\beta). \tag{2.3.3.1}$$

Example 2.3.4. — Consider the Segre embedding

$$\sigma_{r,s} \colon \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$$

defined by

$$([x_0, \cdots, x_r], [y_0, \cdots, y_s]) \mapsto [\cdots, x_i y_j, \cdots].$$

Denote by $\Sigma_{r,s}$ the image of $\sigma_{r,s}$, and we want to find $\deg(\Sigma_{r,s})$. Recall that

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1})$$

and $A(\mathbb{P}^{(r+1)(s+1)-1}) \cong \mathbb{Z}[\xi]/\xi^{(r+1)(s+1)}$. By the geometric interpretation of degree, we have

$$\deg(\Sigma_{r,s}) = \int_{\mathbb{P}^{(r+1)(s+1)-1}} [\sigma_{r,s,*}(\mathbb{P}^r \times \mathbb{P}^s)] \cdot \xi^{r+s}.$$

Note that $\sigma_{r,s}^*(\xi) = \alpha + \beta$. By (2.3.3.1), we get

$$\deg(\Sigma_{r,s}) = \int_{\mathbb{P}^r \times \mathbb{P}^s} \sigma_{r,s}^*(\xi)^{r+s} = \int_{\mathbb{P}^r \times \mathbb{P}^s} \binom{r+s}{r} \alpha^r \beta^s = \binom{r+s}{r}.$$

Example 2.3.5. — Recall that if X is a scheme, and Y is a closed subscheme of X with ideal sheaf $\mathcal{I}$, the blow-up of X along Y is defined as

$$\mathrm{Bl}_Y(X) := \mathrm{Proj}\,\Big(\bigoplus_{n\geq 0} \mathcal{I}^n\Big)$$

We have a natural projection $\pi : \mathrm{Bl}_Y(X) \to Y$ which induces an isomorphism $\pi^{-1}(X - Y) \xrightarrow{\sim} X - Y$. Let B denote the blow-up of $\mathbb{P}^n$ at a point $p = (1, 0, \ldots, 0)$. Then B is the closed subscheme $\mathbb{P}^n \times \mathbb{P}^{n-1}$ defined as

$$B = \{(x_0, \ldots, x_n), (y_1, \ldots, y_n) | x_i y_j = x_j y_i, \forall i, j \geq 1\}.$$

We have thus natural projections:

$$\mathbb{P}^n \xleftarrow{\ \pi\ } B \xrightarrow{\ \alpha\ } \mathbb{P}^{n-1},$$

where π is an isomorphism over $\mathbb{P}^n - p$ and $E = \pi^{-1}(p) \cong \mathbb{P}^{n-1}$. The projection α is a $\mathbb{P}^1$-fibration. It induces an isomorphism on E and its restriction on $\pi^{-1}(\mathbb{P}^n - p) \cong \mathbb{P}^n - p$ is the projection from p:

$$\alpha \colon (x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n).$$

To compute the Chow ring of B, we first construct a cellular decomposition of B. Let $\Gamma'_{k-1} \subseteq \mathbb{P}^{n-1}$ be the $(k-1)$-dimensional linear subspace given by $y_1 = \cdots = y_{n-k} = 0$. Then we get a flag:

$$\Gamma'_0 \subset \Gamma'_1 \subset \cdots \subset \Gamma'_{n-2} \subset \Gamma'_{n-1} = \mathbb{P}^{n-1}.$$

Put $\Gamma_k = \alpha^{-1}(\Gamma'_{k-1}) \subset B$ for $k = 1, \ldots, n$. Let $\Lambda' \subseteq \mathbb{P}^n$ be the hyperplane given by $x_0 = 0$. Put $\Lambda = \pi^{-1}(\Lambda')$, and

$$\Lambda_k = \Gamma_{k+1} \cap \Lambda \quad \text{for } k = 0, \cdots, n-1.$$

Geometrically, Γ_k is the strict transform of a k-dimensional linear subspace of $\mathbb{P}^n$ containing p. Hence, Λ_k is the preimage of a k-dimensional linear subspace of $\mathbb{P}^n$ which does not contain p. We note that $\{\Lambda_k, \Gamma_k; 0 \leq k \leq n\}$ form a cellular decomposition of B. Indeed, $\Lambda_k - \Lambda_{k-1}$ is clearly isomorphic to $\mathbb{A}^k$, and the open stratum $\tilde{\Gamma}_k := \Gamma_k - \Gamma_{k-1} - \Lambda_{k-1}$ is explicitly given by

$$\tilde{\Gamma}_k = \{(1, 0, \cdots, 0, \lambda, \lambda y_{n-k+2}, \cdots, \lambda y_n), (0, \ldots, 0, 1, y_{n-k+2}, \ldots, y_n) \in \mathbb{P}^n \times \mathbb{P}^{n-1}\}$$

where the free variables $\lambda, y_{n-k+2}, \dots, y_n$ gives an isomorphism $\tilde{\Gamma}_k \cong \mathbb{A}^k$. Hence, it follows that $A(B)$ is generated over $\mathbb{Z}$ by the classes $\lambda_k = [\Lambda_k]$ for $k = 0, \dots, n-1$, and $\gamma_l = [\Gamma_l]$ for $l = 1, \dots, n$.

Consider the homomorphism

$$\alpha^* : A(\mathbb{P}^{n-1}) \cong \mathbb{Z}[\eta]/\eta^n \to A(B)$$

induced by $\alpha : B \to \mathbb{P}^{n-1}$, where η is the class of Γ'_{n-1}. Then $\alpha^*(\eta^l) = \gamma_{n-l}$ for any $0 \leq l \leq n-1$. Similarly, consider the homomorphism

$$\pi^* : A(\mathbb{P}^n) \cong \mathbb{Z}[\xi]/\xi^{n+1} \to A(B).$$

We have $\pi^*(\xi^k) = \lambda_{n-k}$ for $0 \leq k \leq n$. Note that, ξ is also the class of a hyperplane $H \subseteq \mathbb{P}^n$ containing p, and H is generically transverse for π. By Theorem 2.3.3, we have

$$\pi^*(\xi) = [\pi^{-1}(H)] = [E] + [H'].$$

Hence, if $e \in A^1(B)$ denotes the class of the exceptional divisor E, then

$$\pi^*(\xi) = \gamma_{n-1} + e = \lambda_{n-1}$$

Note also that $\lambda_{n-1}e = 0$ because $\Lambda_{n-1} \cap E = \emptyset$. If we put $\lambda = \lambda_{n-1} = \pi^*\xi$, then $\alpha^*(\eta) = \lambda - e$ and thus all the classes γ_k and λ_l can be expressed in terms of polynomials of λ and e, i.e. there exists a natural surjective map of rings

$$\theta : \mathbb{Z}[\lambda, e]/(\lambda e, \lambda^n + (-1)^n e^n) \to A(B),$$

and the degree function $\int_B : A(G) \to A_0(G) \to \mathbb{Z}$ sends λ^n to 1.

We claim that θ is an isomorphism. Indeed, since both the source of θ is a free abelian group of rank $2n$, it suffices to show that $A(B)$ is also a free abelian group of rank $2n$. It is enough to show that, for any integer k with $1 \leq k \leq n-1$, λ_k and γ_k are linearly independent in $A_k(B)$. Consider the degree pairing:

$$A_k(B) \times A_{n-k}(B) \to \mathbb{Z} : \quad (x, y) \mapsto \int_B xy$$

Then we have

$$\begin{cases} \int_B \gamma_k \gamma_{n-k} = 0 \\ \lambda_B \gamma_k \lambda_{n-k} = 1 \end{cases} \qquad \begin{cases} \int_B \lambda_k \gamma_{n-k} = 1 \\ \int_B \lambda_k (\lambda_{n-k} - \gamma_{n-k}) = 0 \end{cases}$$

From this, we see easily that γ_k and λ_k are linearly independent.

CHAPTER 3

LINE BUNDLES AND INTERSECTION THEORY ON SURFACES

3.1. Cartier divisors

Let X be an n-dimensional variety. Let $\mathcal{O}_X^\times$ be the sheaf of invertible functions on X, and $\mathcal{K}(X)^\times$ be the sheaf of non-zero rational functions. Recall that elements of $\Gamma(X, \mathcal{K}(X)^\times/\mathcal{O}_X^\times)$ called Cartier divisors on X. Explicitly, a Cartier divisor on X is given by $D = (U_\alpha, f_\alpha)$ where U_α form an open covering of X, and $f_\alpha \in k(U_\alpha)^\times$ such that f_α/f_β in a unit on $U_\alpha \cap U_\beta$. A Cartier divisor (U_α, f_α) is principal if and only if there exists a global rational function $f \in k(X)^\times$ such that $f_\alpha/(f|_{U_\alpha})$ is a unit on U_α.

Cartier divisors form a natural abelian group, and we denote it by $\mathrm{Div}(X)$. Principal Cartier divisors form a subgroup of $\mathrm{Div}(X)$, which we denote by $\mathrm{Prin}(X)$. We put

$$\mathrm{Pic}(X) := \mathrm{Div}(X)/\mathrm{Prin}(X).$$

Using the exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathcal{K}(X)^\times \to \mathcal{K}(X)^\times/\mathcal{O}_X^\times \to 1,$$

one gets

$$0 \to \mathrm{Pic}(X) \to H^1(X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{K}(X)^\times).$$

However, it is easy to see that $H^1(X, \mathcal{K}(X)^\times) = 0$, and we obtain thus an isomorphism

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times).$$

Recall that $H^1(X, \mathcal{O}_X)$ is also the group of line bundles on X modulo rational equivalence. If $D = (U_\alpha, f_\alpha)$ is a Cartier divisor, then its associated line bundle $\mathcal{O}_X(D)$ is associated to the Cech cocycle $(U_\alpha, f_{\alpha\beta})$ with

$$f_{\alpha,\beta} = \frac{f_\alpha|_{U_\alpha \cap U_\beta}}{f_\beta|_{U_\alpha \cap U_\beta}} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^\times).$$

Let $D = (U_\alpha, f_\alpha)$ be a Cartier divisor on X, and $V \subseteq X$ be a codimension 1 closed subvariety. Then we put

$$\mathrm{ord}_V(D) = \mathrm{ord}_V(f_\alpha) \in \mathbb{Z}$$

for any α such that $V \cap U_\alpha \neq \emptyset$. It is easy to see that this definition does not depend on the choice of α, and that $\mathrm{ord}_V(D)$ is linear in D. Thus we get a homomorphism of groups

$$\mathrm{Div}(X) \to Z_{n-1}(X)$$

given by $D \mapsto [D] = \sum_{V \subseteq X} \mathrm{ord}_X(V)[V]$. Note that if D is a principal Cartier divisor associated to the rational function $r \in k(X)^\times$, then one has $[D] = \mathrm{div}(r)$. Hence, we get a canonical map

$$\mathrm{Pic}(X) \to A_{n-1}(X) = A^1(X).$$

***Proposition 3.1.1* (EGA IV 21.6).** — (1) *If X is normal, then the canonical maps* $\mathrm{Div}(X) \to Z_{n-1}(X)$ *and* $\mathrm{Pic}(X) \to A^1(X)$ *are both injective.*

(2) *If X is non-singular, then* $\mathrm{div}(X) \to Z_{n-1}(X)$ *and* $\mathrm{Pic}(X) \to A^1(X)$ *are both isomorphisms.*

Proof. — (1) We prove first that $\mathrm{Div}(X) \to A_{n-1}(X)$ is injective. Let $D = (U_\alpha, f_\alpha)$ be a Cartier divisor such that $[D] = 0$. Then by definition, this means that $\mathrm{ord}_V(f_\alpha) = 0$ for any α and any closed subvariety $V \subseteq X$ such that $V \cap U_\alpha \neq \emptyset$. We have to show that f_α is invertible on U_α for any α. We may assume that U_α is affine with coordinate ring A. The condition is equivalent saying that $f_\alpha \in A_\mathfrak{p}^\times$ for any prime ideal $\mathfrak{p} \subseteq A$ of height 1. Since X is normal, hence so is A. We have $A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_\mathfrak{p}$. It follows that

$$f_\alpha \in \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_\mathfrak{p}^\times = A^\times.$$

The injectivity of $\mathrm{Pic}(X) \to A_{n-1}(X)$ follows from the fact that $\mathrm{Div}(X) \cap \mathrm{Rat}(X)_{n-1} = \mathrm{Prin}(X)$.

(2) This is related to the fact that, if X is non-singular, every local ring of X is a unique factorial domain. Hence, every codimension 1 closed subvariety can be defined locally by 1 equation. □

Example 3.1.2. — Let X be the planar projective curve with homogeneous equation $Y^2Z = X^3$. Then one has $A_0(X) \cong \mathbb{Z}$, and $\mathrm{Pic}(X) \to A_0(X)$ is surjective with kernel $(k, +)$.

Indeed, let $\pi : \tilde{X} \to X$ be the normalization of X. Then one has $\tilde{X} \cong \mathbb{P}^1$, and π is an isomorphism outside the singular point $p = (0, 0, 1) \in X$. Given a pint $Q \in X$, it is easy to construct an explicit rational function r on X such that $\mathrm{div}(r) = [Q] - [\infty]$, where $\infty = (0, 1, 0) \in X$. Hence, we see that $A_0(X) \cong \mathbb{Z}$ with a generor $[\infty]$. To compute $\mathrm{Pic}(X)$, we look at the exact sequence

$$0 \to \mathcal{O}_X^\times \to \pi_*\mathcal{O}_{\tilde{X}}^\times \to i_{p,*}k \to 0.$$

From this, we deduce that

$$0 \to k \to H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times) \to 0.$$

We conclude by the fact that $A_0(X) \cong A_0(\tilde{X}) = H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times) \cong \mathbb{Z}$ with generator $\mathcal{O}_{\tilde{X}}(\infty)$.

Exercise 3.1.3. — Let X be the planar projective curve with affine equation $y^2 = x^3 + x^2$. Prove that $A_0(X) \cong \mathbb{Z}$, and the canonical map $\mathrm{Pic}(X) \to A_0(X)$ is surjective with kernel $k^\times$.

Example 3.1.4. — Let $X \subseteq \mathbb{P}^3$ be the singular quadratic surface with equation $z^2 = xy$. Let $L = V(x, z) \subseteq X$. Then one claims $A^1(X) \cong \mathbb{Z}$ with generator $[L]$, and $\mathrm{Pic}(X) \cong \mathbb{Z}$ with generator $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^3}(1)|_X$, but the canonical morphism $\mathrm{Pic}(X) \to A^1(X)$ is injective with cokernel $\mathbb{Z}/2\mathbb{Z}$.

To justifty the claim, we consider the stratification

$$X \supset L \supset \{p\}.$$

This is actually a cellular decomposition. Indeed, L is clearly isomorphic to $\mathbb{P}^1$, and $L - \{p\} \cong \mathbb{A}^1$. Moreover, an isomorphism $\mathbb{A}^2 \xrightarrow{\sim} X - L$ can be explicitly given by $(z, w) \mapsto (1, z^2, z, w)$. Hence, $A_1(X)$ is generated by $[L]$. Note that the image of $\mathcal{O}_X(1)$ in $Z_{n-1}(X)$ is $2[L]$, and L does not come from a Cartier divisor, because L can not be defined by one equation around p. The claim now follows immediately.

3.2. Intersection with Cartier divisors

Let D be a Cartier divisor on a variety X, and $\alpha \in Z_m(X)$. We will define an intersection product

$$D \cdot \alpha \in A_{m-1}(|D| \cap |\alpha|),$$

where $|D|$ and $|\alpha|$ are respectively the support of D and α. By linearity, it suffices to define $D \cdot \alpha$ for $\alpha = [V]$ with $V \subseteq X$ a closed subvariety of dimension m. In this case, denote by $i : V \to X$ the canonical embedding, and we put $D \cdot \alpha = [i^*\mathcal{O}_X(D)]$. More precisely, this means:

- Case 1. $V \subsetneq |D|$. If $D = (U_\alpha, f_\alpha)$, we define $D \cdot \alpha = [(U_\alpha \cap V, f_\alpha|_V)]$.
- Case 2. $V \subseteq |D|$. Let C be a Cartier divisor on V such that $\mathcal{O}_V(C) = i^*\mathcal{O}_X(D)$. Then we put $D \cdot \alpha = [C]$.

Proposition 3.2.1. — (1) *If D is a principal Cartier divisor, then $D \cdot \alpha = 0$ in $A_{m-1}(|D| \cap |\alpha|)$.*

(2) *Let $f : Y \to X$ be a flat morphism of varieties of relative dimension r, D be a Cartier divisor on X and $\alpha \in Z_m(X)$. Then*

$$f^*(D \cdot \alpha) = f^*D \cdot f^*\alpha$$

holds in $A_{m-1+r}(Y)$.

(3) *Let f be a proper morphism of varieties, D a Cartier divisor on X, and $\alpha \in Z_m(Y)$. Let $g : f^{-1}(|D| \cap |\alpha|) \to |D| \cap |\alpha|$ denote the morphism induced by f. Then we have*

$$g_*(f^*D \cdot \alpha) = D \cdot f_*\alpha$$

in $A_{m-1}(|D| \cap \alpha)$.

Proof. — □

Theorem 3.2.2. — *Let D and D' be Cartier divisors on X, then we have*

$$D \cdot [D'] = D' \cdot [D].$$

The basic idea of the proof is to reducing to the basic case that D and D' are effective and intersect properly by using blow-up. We need the following

Lemma 3.2.3. — *Let D, D' be Cartier divisors on X, $\pi : \tilde{X} \to X$ be a proper birational morphism of varieties, and $\pi^*D = B \pm C, \pi^*D' = B' \pm C'$ for Cartier divisors B, C, B', C' on $\tilde{X}$ with $|B| \cup |C| \subseteq \pi^{-1}(|D|)$, $|B'| \cup |C'| \subseteq \pi^{-1}(|D'|)$. Assume that the Theorem holds for the pairs $(B, B'), (B, C'), (C, B')$ and (C, C') on $\tilde{X}$. Then the Theorem holds also for the pair (D, D').*

Proof. — By the compatibility of the intersection product with pull-back and push-forward maps, we have $D \cdot \pi_*\pi^*[D'] = \pi_*(\pi^*D \cdot \pi^*[D'])$. As π is a birational map, one has $\pi_*\pi^*[D'] = [D']$, and thus $D \cdot [D'] = \pi_*(\pi^*D \cdot \pi^*[D'])$. Hence, it follows that

$$\begin{aligned} D \cdot [D'] &= \pi_*(B \pm C) \cdot [B' \pm C'] \\ &= \pi_*(B \cdot [B'] \pm B \cdot [C'] \pm C \cdot [B'] + C \cdot [C']) \\ &= \pi_*(B' \cdot [B] \pm C' \cdot [B] \pm B' \cdot [C] + C' \cdot [C]) \\ &= \pi_*((B' \pm C') \cdot [B \pm C]) \\ &= D' \cdot [D]. \end{aligned}$$

□

Now, we can start to prove Theorem 3.2.2. The proof will be divided into several steps.

Step 1: Reduction to the case that both D and D' are effective. Assume first that one of D and D', say D', is effective. Let $\mathcal{I}$ denote the ideal sheaf of denominators of D: if $U = \operatorname{Spec}(A)$ is an affine open subset of X where D is defined by the local equation $d \in k(U)^\times$, then

$$\mathcal{I}(U) = \{a \in A | ad \in A\}.$$

Consider the blow-up $\pi : \tilde{X} \to X$ of X along the subscheme defined by $\mathcal{I}$, and let E be the exceptional divisor. Then one has

$$\pi^*D = C - E$$

for an effective divisor C on $\tilde{X}$. By Lemma 3.2.3, if the Theorem has been verified for the pairs (C, π^*D') and (E, π^*D'), so does it for (D, D'). Similarly, if both D and D' are not effective, one can reduce to the previous case by blow-up. We assume henceforth that both D and D' are effective.

Step 2: Consider the special case that D and D' are dimensionally transversal. Let $W \subseteq X$ be a codimension 2 closed subvariety. One needs to prove that the coefficients of $[W]$ in $D \cdot [D']$ and $D' \cdot [D]$ are equal. This is a local problem. Let $A = \mathcal{O}_{X,W}$ and $a, a' \in A$ be the local equations defining D and D'. The subvarieties $V \subseteq X$ of codimension 1 containing W correspond to height 1 prime ideals $\mathfrak{p} \subseteq A$. The coefficient of $[V]$ in $[D']$ is $\operatorname{leng}_{A_\mathfrak{p}}(A/a'A_\mathfrak{p})$, and the coefficient of $[W]$ in $D \cdot [D']$ is

$$\sum_{\mathfrak{p}} \operatorname{leng}_{A_\mathfrak{p}}(A_\mathfrak{p}/a'A_\mathfrak{p}) \cdot \operatorname{leng}_{A/\mathfrak{p}}(A/\mathfrak{p} + aA).$$

However, by a lemma in commutative algebra, we see that the expression above equals to $\operatorname{leng}_{A/(a,a')}(A/(a,a')) - \operatorname{leng}\operatorname{Tor}_1^A(A/(a), A/(a'))$. Exchanging the roles of a and a', we see that this also equals to the coefficients of $[W]$ in $D' \cdot [D]$.

Step 3: Reduction to the case when D and D' are dimensionally transversal Cartier divisors. Put

$$\epsilon(D, D') = \max\{\operatorname{ord}_V(D) \cdot \operatorname{ord}_V(D') | \operatorname{codim}(V) = 1\}.$$

Then we have $\epsilon(D, D') \geq 0$, and D and D' are dimensionally transverse if and only if $\epsilon(D, D') = 0$.

We may view D and D' as closed subschemes of X. Let $D \cap D'$ be the scheme theoretic intersection of D and D', and

$$\pi : \tilde{X} \to X$$

be the blow-up of X along $D \cap D'$. Put $E = \pi^{-1}(D \cap D')$. Then we have

$$\pi^* D = E + C, \quad \pi^* D' = E + C'$$

for some effective Cartier divisors C, C' on $\tilde{X}$.

We remark that, by Lemma 3.2.3, Theorem 3.2.2 follows immediately from the following Lemma.

Lemma 3.2.4. — *Under the notation above,*

(a) *C and C' are disjoint,*

(b) *If $\epsilon(D, D') > 0$, then $\epsilon(E, C)$ and $\epsilon(E, C')$ are strictly smaller than $\epsilon(D, D')$.*

Proof. — The assertions are local for X, so that we may assume that $X = \operatorname{Spec}(A)$ is affine, and D, D' are defined by local equations $a, a' \in A$ respectively. If $I = (a, a')$, then $\tilde{X} = \operatorname{Proj}(\bigoplus_{n \geq 0} I^n)$. The surjection of the graded algebra

$$A[S, T] \to \bigoplus_{n \geq 0} I^n$$

given by $S \mapsto a$ and $T \mapsto a'$ gives an embedding $\tilde{X} \hookrightarrow X \times \mathbb{P}^1 = \operatorname{Proj}(A[S, T])$. Actually, $\tilde{X}$ is identified with the closed subscheme of $X \times \mathbb{P}^1$ defined by the vanishing of the homoegeneous ideal $a'S - aT$. Let $\mathcal{O}(1)$ be the pull-back of the canonical ample line bundle on $\mathbb{P}^1$ to $\tilde{X}$, and let s, t denote the sections of $\mathcal{O}(1)$ given by S, T respectively. Let $C = V(s)$, and $C' = V(t)$. Then, one has

$$\pi^* D = E + C$$

Indeed, over the open subset $s \neq 0$, one has $a' = ta/s$ and thus $\pi^* D$ agrees with E; over the open subset $t \neq 0$, one has $a = a's/t$ and thus $\pi^* D = E + C$. Similarly, one has $\pi^* D' = E + C'$. Since $V(s) \cap V(t) = \emptyset$, assertion (a) follows.

For (b), note first that π induces an isomorphism $C \cong D$ and $C' \cong D'$. Hence, if $\tilde{V}$ is a codimension one subvariety of $\tilde{X}$ contained in $C \cap E$ or $C' \cap E$, then $V = \pi(\tilde{V}) \cong \tilde{V}$ is also a codimension 1 closed subvariety of X contained in $D \cap D'$. By the projection formula, one has $[D] = \pi_*\pi^*D = \pi_*([E + C])$, and hence

$$\mathrm{ord}_V(D) = \mathrm{ord}_{\tilde{V}}(E) + \mathrm{ord}_{\tilde{V}}(C)$$

and similarly for D'. Suppose that $\epsilon(C, E) \geq \epsilon(D, D') > 0$, and $\tilde{V}$ is chosen so that $\mathrm{ord}_{\tilde{V}}(C) \cdot \mathrm{ord}_V(E) = \epsilon(C, E)$. Then

$$\begin{aligned}\epsilon(D, D') &\geq \mathrm{ord}_V(D) \cdot \mathrm{ord}_V(D') \\ &= (\mathrm{ord}_{\tilde{V}}(E) + \mathrm{ord}_{\tilde{V}}(C))(\mathrm{ord}_{\tilde{V}}(E) + \mathrm{ord}_{\tilde{V}}(C')) \\ &= \mathrm{ord}_{\tilde{V}}(E)^2 + \epsilon(C, E) > \epsilon(C, E).\end{aligned}$$

This is a contradiction. □

Corollary 3.2.5. — *Let D be a Cartier divisor on X and $\alpha \in Z_m(X)$ rationally equivalent to 0. Then one has*

$$D \cdot \alpha = 0$$

in $A_{m-1}(|D|)$.

Proof. — We may assume that $\alpha = \mathrm{div}(r)$ for some $r \in k(W)^\times$, where W is a closed subvariety of dimension $m + 1$ in X. Up to replacing X by W, one may assume that $W = X$, and α is the cycle of a principal Cartier divisor D' on X. It follows from Theorem 3.2.2 and Proposition 3.2.1(a). □

3.2.6. First Chern class. — In summary, we have obtained well-defined map

$$\mathrm{Pic}(X) \times A_m(X) \to A_{m-1}(X) : \quad (\mathcal{O}_X(D), \alpha) \mapsto D \cdot \alpha.$$

For a line bundle L on X, we also write

$$c_1(L) \cap \alpha := D \cdot \alpha$$

for any Cartier divisor D on X such that $L = \mathcal{O}_X(D)$. Here, the map

$$c_1(L) : A_m(X) \to A_{m-1}(X)$$

is called the first Chern class of L.

Remark 3.2.7. — (1) Note that $c_1(L)$ is additive in L in the sense that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

(2) When X is smooth and quasi-projective, $[L] := c_1(L) \cap [X] \in A^1(X)$ is the cycle class associated to any Cartier divisor D on X such that $L = \mathcal{O}_X(D)$, and $c_1(L) \cap \alpha$ coincides with the intersection product of $[L]$ and α defined in Chapter 2.

3.3. Intersection on surfaces

Let X be a proper smooth surface over an algebraically closed field k. We have a natural symmetric bilinear pairing on $A^1(X) \cong \mathrm{Pic}(X)$

$$A^1(X) \times A^1(X) \to \mathbb{Z}$$

given by $(D, D')_X := \int_X (D \cdot D')$. The basic properties of this pairing are given by

Lemma 3.3.1. — *(1) If D, D' are two effective divisors on X such that $D \cap D'$ is 0-dimensional, then $(D, D')_X = \#(D \cap D')$ counted with multiplicity.*

(2) If $C \subseteq X$ is a curve, then $(C, D)_X = \deg(\mathcal{O}_X(D)|_C)$ for any $D \in A^1(X)$.

Proof. — (1) Let p be a point on $D \cap D'$. Then the local rings $\mathcal{O}_{D,p}$ and $\mathcal{O}_{D',p}$ are both local complete intersection rings, and thus Cohen-Macaulay. Hence, assertion (1) follows from Serre's multiplicity formula 2.2.3.

(2) We have $(C, D)_X = \int_X (D \cdot [C])$, where we view D as a Cartier divisor on X. By definition, we have $D \cdot [C] = \mathcal{O}_X(D)|_C$. □

Proposition 3.3.2. — *For any $D, E \in A^1(X)$, we have*

$$(D, E)_X = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(E)) + \chi(\mathcal{O}_X(D + E)). \tag{3.3.2.1}$$

Proof. — By Bertini's Theorem, one may write $E = A - B$ where A and B are both irreducible curves. Then the left hand side of (3.3.2.1) is

$$(D, E)_X = (D, A)_X - (D, B)_X = \deg(\mathcal{O}_X(D)|_A) - \deg(\mathcal{O}_X(D)|_B).$$

On the other hand, tensoring the exact sequence

$$0 \to \mathcal{O}_X(-B) \to \mathcal{O}_X \to i_{B,*}\mathcal{O}_B \to 0$$

by $\mathcal{O}_X(A)$, we deduce that

$$\chi(\mathcal{O}_X(E)) = \chi(\mathcal{O}_X(A)) - \chi(\mathcal{O}_X(A)|_B).$$

Similarly, we have

$$\chi(\mathcal{O}_X(E + D)) = \chi(\mathcal{O}_X(A + D)) - \chi(\mathcal{O}_X(A + D)|_B).$$

Hence, the right hand side of (3.3.2.1) is

$$\big(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(A))\big) + \big(\chi(\mathcal{O}_X(A+D)) - \chi(\mathcal{O}_X(D))\big) + \big(\chi(\mathcal{O}_X(A)|_B) - \chi(\mathcal{O}_X(A+D)|_B)\big).$$

Twisting the exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to i_{A,*}\mathcal{O}_A \to 0$$

by $\mathcal{O}_X(A)$ and $\mathcal{O}_X(A + D)$, we see that the expression above is

$$\big(-\chi(\mathcal{O}_X(A)|_A) + \chi(\mathcal{O}_X(A + D)|_A)\big) + \big(\chi(\mathcal{O}_X(A)|_B) - \chi(\mathcal{O}_X(A + D)|_B)\big).$$

By Riemann-Roch Theorem for curves, the latter is nothing but

$$\begin{aligned} &\deg(\mathcal{O}_X(A + D)|_A) - \deg(\mathcal{O}_X(A)|_A) + \deg(\mathcal{O}_X(A)|_B) - \deg(\mathcal{O}_X(A + D)|_B) \\ &= \deg(\mathcal{O}_X(D)|_A) - \deg(\mathcal{O}_X(D)|_B). \end{aligned}$$

This finishes the proof. □

***Corollary 3.3.3* (Riemann-Roch).** — *Let $K_X \in \mathrm{Pic}(X)$ be the class of the canonical bundle Ω_X^2. For any divisor D on X, we have*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{(D, D-K_X)}{2}.$$

Proof. — By (3.3.2.1), we have

$$(D, K_X - D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(K_X - D)) + \chi(\mathcal{O}_X(K_X)).$$

By Serre duality, the right hand side above is $2(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D)))$. The Corollary follows immediately. □

We fix an ample divisor H on X, i.e. a divisor whose associated line bundle is ample. Note that we have $(H, D)_X > 0$ for any effective divisor D on X. For any Cartier divisor D, put

$$h^i(D) := \dim_k H^i(X, \mathcal{O}_X(D)), \quad \text{for } i = 0, 1, 2.$$

Note that the Serre duality implies that $h^i(D) = h^{2-i}(K_X - D)$, and $h^0(D) > 0$ is equivalent to saying that D is rationally equivalent to an effective divisor.

Lemma 3.3.4. — *Let D be a divisor on X such that $(D, D)_X > 0$. Then exactly one of the following two possibilities will occur:*

1. *We have $(H, D)_X > 0$, $h^0(nD) \to \infty$ as $n \to \infty$ and $h^0(K_X - nD) = 0$ for $n >> 0$.*
2. *We have $(H, D)_X < 0$, $h^0(K_X - nD) \to \infty$ as $n \to \infty$ and $h^0(nD) = 0$ for $n >> 0$.*

Proof. — It is clear that the two cases are exclusive. By Riemann-Roch Theorem 3.3.3, we have

$$h^0(nD) + h^0(K_X - nD) \geq \frac{(nD, nD - K_X)_X}{2} + \chi(\mathcal{O}_X)$$

which goes to ∞ as $n \to \infty$ as $(D, D)_X > 0$. Therefore, either $h^0(nD) \to \infty$ or $h^0(K_X - nD) \to \infty$. In the first case, nD is rationally equivalenet to an effective divisor for $n >> 0$, so that $(H, D)_X > 0$ and $h^0(K_X - nD)$ must be 0 for n large enough (because $(H, K_X - nD)_X < 0$ for large n). Hence, we have $h^0(nD) > 0$. Similar arguments apply to the case that $h^0(K_X - nD) \to \infty$.

□

A divisor D on X is called numerically equivalent to 0 if for any $E \in \mathrm{Pic}(X)$, we have $(D, E)_X = 0$. Let $\mathrm{Pic}^\tau(X)$ be the subgroup of $\mathrm{Pic}(X)$ of elements which are numerically equivalent to 0, and put

$$\mathrm{Num}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X).$$

Then $\mathrm{Num}(X)$ is a free abelian group, and the intersection pairing on $\mathrm{Pic}(X)$ induces a non-degenerate symmetric bilinear form:

$$(\bullet, \bullet)_X \colon \mathrm{Num}(X) \times \mathrm{Num}(X) \to \mathbb{Z}.$$

We consider $\mathrm{Num}(X)_\mathbb{Q} := \mathrm{Num}(X) \otimes_\mathbb{Z} \mathbb{Q}$. One has a decomposition $\mathrm{Num}(X)_\mathbb{Q} = \mathbb{Q} \cdot H \oplus H^\perp$, where $H^\perp$ is the orthogonal complement of H with respect to $(\bullet, \bullet)_X$.

***Theorem 3.3.5* (Hodge Index Theorem).** — *The intersection pairing* $(\bullet, \bullet)_X$ *is negative definite on* $H^{\perp}$, *i.e. if* D *is a non-zero element of* $\mathrm{Num}(X)$ *such that* $(D, H)_X = 0$, *then* $(D, D)_X < 0$.

Proof. — We may assume D is the class of a divisor on X with $(D, H)_X = 0$. If $(D, D)_X > 0$, then Lemma 3.3.4 implies that either $(D, H)_X > 0$ or $(D, H)_X < 0$, which is a contradiction.

□

Remark 3.3.6. — Actually, $\mathrm{Num}(X)$ is a free abelian group of finite rank. If ρ_X denotes the rank of $\mathrm{Num}(X)$, then Theorem 3.3.5 says that the signature of the intersection pairing on $\mathrm{Num}(X)$ is $(1, \rho_X - 1)$.

We look more examples on positivity of the intersection pairing on surfaces.

***Example 3.3.7* (Mumford).** — Let V be a projective surface (not necessarily smooth), and $p \in V$. Let $\pi : X \to V$ be a proper birational morphism such that X is smooth, and π is an isomorphism outside p. Put $E := \pi^{-1}(p)$. Then for any non-zero divisor D of X supported on E, we have $(D, D)_X < 0$. In other words, the restriction of the intersection pairing to the subgroup of divisors supported on E is negative definite.

Choose a rational function f on V which vanishes at p, so that one can write

$$\mathrm{div}(\pi^*(f)) = \sum_i m_i E_i + Z$$

where $E_i \subset E$ are irreducible components of E, $m_i > 0$ and Z is a divisor not containing any E_i. Note that $Z \cdot E_i \geq 0$ since f vanishes at p. Set $D_i = m_i E_i$. Then a general divisor D of X supported on E writes as $D = \sum_i a_i D_i$ with $a_i \in \mathbb{Q}$. Then Mumford's result follows from the following computation:

$$\begin{aligned}
(D, D)_X &= \sum_i (D_i, \sum_j a_j D_j)_X \\
&= \sum_i a_i (D_i, \sum_j a_j D_j - a_i \,\mathrm{div}(\pi^*(f))) \\
&= \sum_{i \neq j} a_i (a_j - a_i)(D_i, D_j)_X - \sum_i a_i^2 (D_i, Z) \\
&= -\sum_{i<j} (a_i - a_j)^2 (D_i, D_j)_X - \sum_i a_i^2 (D_i, Z) < 0.
\end{aligned}$$

Mumford's result is fundament for the study of the singularity on surfaces. For instance, one can classify singular points on surfaces according to the types of the negative definitive quadratic form associated to these points.

Exercise 3.3.8. — Let V be the singular quadratic surface with equation $z^2 = xy$, and $\pi : X \to V$ be its blow-up at the singular point $p = (0, 0, 0, 1)$. Then $E := \pi^{-1}(p)$ is irreducible and isomorphic to $\mathbb{P}^1$, and we have $(E, E)_X = -2$.

***Proposition 3.3.9* (Adjonction formula).** — *If $C \subseteq X$ is a proper smooth curve, then the genus of C is given by*

$$g_C = \frac{(C, C+K_X)_X}{2} + 1.$$

Proof. — Let $\mathcal{I} = \mathcal{O}_X(-C)$ denote the ideal sheaf defining C. We have an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_X|_C \to \Omega^1_C \to 0.$$

It follows that

$$\Omega^2_X|_C \cong (\mathcal{I}/\mathcal{I}^2) \otimes \Omega^1_C,$$

or equivalently

$$\Omega^1_C \cong \Omega^2_X|_C \otimes (\mathcal{I}/\mathcal{I}^2)^\vee = \mathcal{O}_X(K_X + C)|_C.$$

Hence, one gets

$$2g_C - 2 = \deg(\mathcal{O}_X(K_X + C)|_C) = (K_X + C, C)_X.$$

□

Example 3.3.10. — Let $X \subseteq \mathbb{P}^3$ be smooth hypersurface of degree d. If X contains a line L, then what is the self-intersection $(L, L)_X$?

First, we have an exact sequence:

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega^1_{\mathbb{P}^3}|_X \to \Omega^1_X \to 0.$$

As $\deg(X) = d$, it follows that $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-d)$. Hence, we get

$$\Omega^2_X \cong \Omega^3_{\mathbb{P}^3}|_X \otimes (\mathcal{I}_X/\mathcal{I}_X^2)^\vee = \mathcal{O}_{\mathbb{P}^3}(d-4)|_X.$$

On the other hand, it follows from the genus formula that

$$0 = g(L) = \frac{(L, L+K_X)_X}{2} + 1.$$

It is clear that $(L, K_X)_X = d - 4$, hence we get $(L, L)_X = 2 - d$.

CHAPTER 4

VECTOR BUNDLES AND CHERN CLASSES

We fix a field k. Let X be a scheme over k.

4.1. Vector bundles and Chern classes

A vector bundle of rank n over X is a morphism $\pi : E \to X$ such that there exists an open covering U_α of X with $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{A}_k^n$. If E is a vector bundle of rank n over X, then its sheaf of sections of E is a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules of rank n. Conversely, given a locally free coherent sheaf E of rank n over X, we can produce a vector bundle $E = \operatorname{Spec}(\operatorname{Sym}(\mathcal{E}^\vee))$. In fancy languages, the functor that sends a vector bundle to its sheaf of sections is an equivalence of categories.

The main result of the theory of Chern classes is the following

Theorem 4.1.1. — *For any vector bundle E over a scheme X, there exists a unique homomorphism*

$$c_i(E)\cap\colon A_m(X) \to A_{m-i}(X)$$

for every integer i with $0 \leq i \leq n$ and integer m such that the following properties are satisfied:

(a) *(Normalization) We have $c_0(E) = 1$, and if E is a line bundle associated with a Cartier divisor D, then $c_1(E) \cap [X] = [D]$.*
(b) *(Vanishing) For all vector bundle E, we have $c_i(E) = 0$ for all $i > n = \operatorname{rank}(E)$.*
(c) *(Commutativity) For all vector bundles E, F on X, we have*

$$c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha)$$

(d) *(Projection formula) If $f : X' \to X$ a proper morphism and E is a vector bundle on X, then*

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha).$$

(e) *(Pullback) If $f : X' \to X$ is a flat morphism and E is a vector bundle on X, then*

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha).$$

(f) *(Whitney sum) We put* $c_t(E) = \sum_{i=0}^{\infty} c_i(E)t^i$ *for some free variable* t. *Then for any exact sequence* $0 \to E_1 \to E \to E_2 \to 0$ *of vector bundles, we have*

$$c_t(E) = c_t(E_1)c_t(E_2)$$

i.e. $c_\ell(E) = \sum_{i+j=\ell} c_i(E_1)c_j(E_2)$.

Remark 4.1.2. — When X is a quasi-projective smooth variety, then the map $c_i(E)\cap : A_m(X) \to A_{m-1}(X)$ is given by the intersection product with $c_i(E) \cap [X] \in A^i(X)$. In this case, we also use $c_i(E)$ to denote this cycle class, and call it the ith Chern class of E.

The uniqueness of Theorem 4.1.1 follows from the so-called *splitting principle*.

***Proposition 4.1.3* (Splitting principle).** — *Given a finite collection* $\mathscr{S}$ *of vector bundles on a scheme* X, *there is a flat and proper surjective morphism* $f : X' \to X$ *such that*

- $f^* : A_*(X) \to A_*(X')$ *is injective, and*
- *for each* $E \in \mathscr{S}$, f^*E *has a filtration by subbundles*

$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0$$

with line bundle quotients $L_i = E_i/E_{i-1}$.

Assuming this proposition for a moment, we can deduce the uniqueness of Theorem **??** as follows. Note that, for a line bundle L on X, one has $c_t(L) = 1 + c_1(L)t$ by the normalization and vanishing property. Let E be a vector bundle of rank n over X. We choose a flat and proper surjective morphism $f : X' \to X$ as in the splitting principle so that there exists a filtration by subbundles:

$$E_n = f^*E \supset E_{n-1} \supset \cdots \supset E_1 \supset E_0 = 0$$

with $L_i = E_i/E_{i-1}$ a line bundle. Then by Whitney sum, we have

$$f^*c_t(E) = c_t(f^*E) = \prod_{i=1}^{n}(1 + tc_1(L_i)).$$

This determines $c_t(E)$ since $f^* : A_*(X) \to A_*(X')$ is injective.

4.1.4. Segre classes. — Let E be a vector bundle of rank $r = e+1$ over X. We denote by

$$p : P(E) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^\vee)) \to X$$

the projective bundle associated to E. Geometrically, $P(E)$ classifies family of lines passing through the original in E. There exists a universal sub-line bundle $\mathcal{O}_E(-1) \subseteq p^*\mathcal{E}$ on $P(E)$. We define $\mathcal{O}(1) = \mathcal{O}_E(1)$ as the dual of $\mathcal{O}_E(-1)$ so that there exists a canonical projection $p^*\mathcal{E}^\vee \to \mathcal{O}_E(1)$. Define a homomorphism $\alpha \mapsto s_i(E) \cap \alpha$ from $A_k(X) \to A_{k-i}(X)$ y by

$$s_i(E) \cap \alpha := p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha),$$

and we call it the i-th Segre class of E.

Proposition 4.1.5. — (1) *For all* $\alpha \in A_k(X)$, *we have* $s_i(E) \cap \alpha = 0$ *for* $i < 0$ *and* $s_0(E) \cap \alpha = \alpha$.

(2) *If* E, F *are vector bundles on* X, $\alpha \in A_k(X)$, *then*

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha).$$

(3) *If* $f : X' \to X$ *is proper,* E *a vector bundle on* X, *and* $\alpha \in A_*(X')$, *then*

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha).$$

(4) *If* $f : X' \to X$ *is flat,* E *a vector bundle on* X *and* $\alpha \in A_*(X)$, *then*

$$s_i(f^*E) \cap \alpha = f^*(s_i(E) \cap \alpha)$$

(5) *If* E *is a line bundle on* X, *and* $\alpha \in A_*(X)$, *then*

$$s_1(E) \cap \alpha = -c_1(E) \cap \alpha.$$

Proof. — We will prove first (3) and (4). Given a morphism $f : X' \to X$ and a vector bundle E on X, we have a cartesian diagram

$$\begin{array}{ccc} P(f^*E) & \xrightarrow{f'} & P(E) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

and $f^*\mathcal{O}_E(1) = \mathcal{O}_{f^*E}(1)$. If f is proper and $\alpha \in A_*(X')$, then

$$\begin{aligned} f_*(s_i(f^*E) \cap \alpha) &= f_*p'_*(c_1(\mathcal{O}_{f^*E}(1))^{e+i} \cap p'^*\alpha) \\ &= p_*f'_*(c_1(f^*\mathcal{O}_E(1))^{e+i} \cap p'^*\alpha) \\ &= p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap f'_*p'^*\alpha) \\ &= p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^*f_*\alpha) \\ &= s_i(E) \cap f_*(\alpha). \end{aligned}$$

The proof of (4) is similar, and will be left to the readers.

For (1), we may assume that $\alpha = [V]$ is the class of an m-dimensional subvariety of X. By (3), one may reduced to the case $X = V$. Then one has $A_{m-i}(X) = 0$ for $i < 0$, which proves the first part of (1). For the second part, by noetherian induction, we may assume that E is trivial. Hence $P(E) = X \times \mathbb{P}^e$, and $\mathcal{O}(1)$ has sections whose zero scheme is $X \times \mathbb{P}^{e-1}$. Then

$$c_1(\mathcal{O}(1)) \cap [X \times \mathbb{P}^e] = [X \times \mathbb{P}^{e-1}].$$

repeating e times, one gets (1).

To prove (2), we consider the Cartesian diagram

$$\begin{array}{ccc} Q & \xrightarrow{p'} & P(F) \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ P(E) & \xrightarrow{p} & X \end{array}$$

Let $f+1$ be the rank of F. Then

$$\begin{aligned} s_i(E)\cap(s_j(F)\cap\alpha) &= p_*(c_1(\mathcal{O}_E(1))^{e+i}\cap p^*(q_*(c_1(\mathcal{O}_F(1))^{f+j}\cap q^*\alpha))) \\ &= p_*(c_1(\mathcal{O}_E(1))^{e+i}\cap q'_*p'^*(c_1(\mathcal{O}_F(1))^{f+j}\cap q^*\alpha)) \\ &= p_*q'_*(c_1(q'^*\mathcal{O}_E(1)^{e+i})\cap(c_1(p'^*\mathcal{O}_F(1))^{f+j}\cap p'^*q^*\alpha)) \\ &= q_*p'_*(c_1(p'^*\mathcal{O}_F(1))^{f+j}\cap(c_1(q'^*\mathcal{O}_E(1)))^{e+i}\cap q'^*p^*\alpha)). \end{aligned}$$

Now the expression above is symmetric in E and F, hence the statement (5) follows. □

Corollary 4.1.6. — *The flat pull-back map*

$$p^* : A_m(X)\to A_{m+e}(P(E))$$

admits a left inverse $\beta\mapsto p_*(c_1(\mathcal{O}(1))^e\cap\beta)$. In particular, it is injective.

4.1.7. Construction of Chern classes. — Let E be a vector bundle of rank $r=e+1$ on a scheme X. We consider the formal power series

$$s_t(E)=\sum_{n=0}^{\infty}s_i(E)t^i=1+s_1(E)t+s_2(E)t^2+\cdots.$$

We define the Chern polynomial of E as $c_t(E):=s_t(E)^{-1}$. Explicitly, one has

$$\begin{aligned} &c_0(E)=1,\quad c_1(E)=-s_1(E); \\ &c_2(E)=s_1(E)^2-s_2(E),\quad \ldots, \\ &c_n(E)=-s_1(E)c_{n-1}(E)-s_2(E)c_{n-2}(E)-\cdots-s_n(E). \end{aligned}$$

We will show that this definition of Chern classes satisfies Theorem 4.1.1. Statements (a), (c), (d) and (e) follow immediately from Proposition 4.1.5. By splitting principle, we may assume that E admits a filtration statements (b) and (f) are easy consequences of the following proposition.

Proposition 4.1.8. — *Assume that E admits a filtration by subbundles*

$$E=E_r\supset E_{r-1}\supset\cdots\supset E_0=0$$

such that subquotients $L_i=E_i/E_{i-1}$ are line bundles, then one has

$$c_t(E)=\prod_{i=1}^{r}(1+tc_1(L_i)). \tag{4.1.8.1}$$

For the proof of this proposition, we need the following

Lemma 4.1.9. — *Let E be as in the proceeding proposition. Let s be a section of E and $Z\subseteq X$ be the closed subscheme where s vanishes. Then for any m-cycle α on X, there is $(m-r)$-cycle β on Z such that*

$$\prod_{i=1}^{r}c_i(L_i)\cap\alpha=\beta$$

holds in $A_{m-r}(Z)$*. In particular, if* s *is nowhere vanishing, then* $\prod_{i=1}^r c_i(L_i) = 0$.

Proof. — The section s defines a section $\bar{s}$ of the quotient line bundle L_r. Denote by D_r the vanishing locus of $\bar{s}$. Then D_r is a Cartier divisor on X such that $\mathcal{O}_X(D_r) \cong L_r$, and intersecting with D_r gives a class $D_r \cdot \alpha \in A_{m-1}(X)$ such that

$$c_1(L_r) \cap \alpha = j_*(D_r \cdot \alpha)$$

where $j : D_r \hookrightarrow X$ denotes the canonical closed immersion. By the projection formula, one has

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = j_*(\prod_{i=1}^{r-1} c_1(j^* L_i) \cap (D_r \cdot \alpha)).$$

The bundle $j^* E_{r-1}$ has a section induced by s, whose vanishing locus is Z. By induction on r, the right hand side of the proceeding formula is represented by a cycle on Z. This concludes the proof. □

Proof of Proposition 4.1.8. — Let $p : P(E) \to X$ denote the associated projective bundle. The tautological subbundle $\mathcal{O}(-1) \subseteq p^*E$ gives rise to a trivial line subbundle of $p^*E \otimes \mathcal{O}(1)$. Hence, $p^*E \otimes \mathcal{O}(1)$, which admits a filtration by subbundles with subquotients $p^*L_i \otimes \mathcal{O}(1)$, has a nowhere vanishing section s. Then by the proceeding Lemma, one has

$$\prod_{i=1}^{r} c_1(p^* L_i \otimes \mathcal{O}(1)) = \prod_{i=1}^{r} (\xi + c_1(p^* L_i)) = 0.$$

If we put $\xi = c_1(\mathcal{O}_E(1))$ and let $\tilde{\sigma}_i$ denote the i-th elementary symmetric function of $c_1(p^*L_1), \cdots, c_1(p^*L_r)$, the formula above writes as

$$\xi^r + \tilde{\sigma}_1 \xi^{r-1} + \cdots + \tilde{\sigma}_r = 0 \tag{4.1.9.1}$$

So for all $i \geq 1$, we have

$$\xi^{e+i} + \tilde{\sigma}_1 \xi^{e+i-1} + \cdots + \tilde{\sigma}_r \xi^{i-1} = 0.$$

It follows that for all $\alpha \in A_*(X)$, we have

$$p_*(\xi^{e+i} \cap p^*\alpha) + p_*(\tilde{\sigma}_1 \xi^{e+i-1} \cap p^*\alpha) + \cdots + p_*(\tilde{\sigma}_r \xi^{i-1} \cap p^*\alpha) = 0.$$

By definition of Segre classes, this says

$$s_i(E) \cap \alpha + \sigma_1 s_{i-1}(E) \cap \alpha + \cdots + \sigma_r s_{i-r}(E) \cap \alpha = 0.$$

where σ_i is the i-th elementary symmetric function of $c_1(L_i)$'s. This means that

$$(1 + \sigma_1 t + \cdots + \sigma_r t^r) s_t(E) = 1,$$

which is equivalent to (4.1.8.1).

□

Corollary 4.1.10. — *Let* E *be a vector bundle of rank* $r = e+1$ *over* X*. Let* $p : P(E) \to X$ *be the natural projection of the associated projective bundle, and* $\xi = c_1(\mathcal{O}_E(1))$*. Then one has*

$$\xi^r + c_1(p^*E)\xi^{r-1} + \cdots + c_r(p^*E) = 0.$$

Example 4.1.11. — Let $T_{\mathbb{P}^n}$ be the tangent bundle of $\mathbb{P}^n$. Then one has an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0$$

Then we have $c_t(T_{\mathbb{P}^n}) = (1+ht)^{n+1}$ where $h \in A^1(\mathbb{P}^n)$ is the class of the canonical ample line bundle $\mathcal{O}(1)$.

***Example 4.1.12* (Adjonction formula).** — Let $i : Y \hookrightarrow X$ be a closed embedding of codimension d of non-singular varieties. Let $N = i^*T_X/T_Y$ be the normal bundle of i. If Y is the intersection of divisors $D_1, \cdots, D_r$, then we have

$$c(N) = \prod_{i=1}^{r}(1 + c_1(i^*\mathcal{O}_X(D_i))). \tag{4.1.12.1}$$

Indeed, we have a successive closed immersions

$$Y = D_1 \cap \cdots \cap D_d \hookrightarrow D_1 \cap \cdots \cap D_{r-1} \hookrightarrow \cdots \hookrightarrow D_1 \hookrightarrow X$$

and a filtration

$$T_Y \hookrightarrow T_{Y_{r-1}}|_Y \hookrightarrow \cdots T_{Y_1}|_Y \hookrightarrow T_X|_Y$$

where $Y_i = D_1 \cap \cdots \cap D_i$ for $1 \leq i \leq r$. Then the subquotient $(T_{Y_{i-1}}|_Y)/(T_{Y_i}|_Y)$ is the restriction to Y of the normal bundle of the embedding $Y_i \hookrightarrow Y_{i-1}$. Since the intersection of $D_1, \cdots, D_d$ are transversal, we get

$$T_{Y_{i-1}}|_X/T_{Y_i}|_Y \cong i^*\mathcal{O}_X(D_i).$$

Then the formula (4.1.12.1) follows immediately from (4.1.8.1).

In particular, if $X = \mathbb{P}^m$ and Y is a complete intersection of hypersurfaces D_i's with $\deg(D_i) = n_i$. Then $c(N) = \prod_{i=1}^r (1 + n_i h)$, and hence

$$c(T_Y) = \frac{c(i^*T_X)}{c(N)} = \frac{(1+h)^{m+1}}{\prod_{i=1}^r (1+n_i h)}.$$

Example 4.1.13. — Consider the d-fold Veronese embedding $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ with $m = \binom{n+d}{d} - 1$ given by

$$(x_0, \cdots, x_n) \mapsto (x^{\underline{a}})$$

where $x^{\underline{a}}$ runs through the monomials of $x_0, \cdots, x_n$ of degree d. Recall that $i^*\mathcal{O}_{\mathbb{P}^m}(1) = \mathcal{O}_{\mathbb{P}^n}(d)$. Hence,

$$c(N) = \frac{c(i^*T_{\mathbb{P}^m})}{T_{\mathbb{P}^n}} = \frac{i^*(c(T_{\mathbb{P}^m}))}{c(T_{\mathbb{P}^n})} = \frac{(1+dh)^{m+1}}{(1+h)^{n+1}}.$$

For instance, if $n = 2$ and $d = 2$, we have $m = 5$ and

$$c(N) = \frac{(1+2h)^6}{(1+h)^3} = (1 + 12h + 60h^2)(1 - 3h + 6h^2) = 1 + 9h + 30h^2.$$

Example 4.1.14. — Let E, F be vector bundles of rank r and s over X respectively. We try to compute the Chern classes $E \otimes F$.

By splitting principle, we may assume that $E = \oplus_{i=1}^r L_i$ and $F = \oplus_{j=1}^s M_j$ are direct sums of line bundles. Then

$$c(E) = \prod_{i=1}^r (1+\alpha_i), \quad c(F) = \prod_{j=1}^s (1+\beta_j)$$

where $\alpha_i = c_1(L_i)$ and $\beta_j = c_1(M_j)$. Then

$$E \otimes F = \bigoplus_{i,j} L_i \otimes M_j, \quad c(E\otimes F) = \prod_{i,j}(1+\alpha_i+\beta_j).$$

In particular, $c_1(E\otimes F) = \sum_{i,j}(\alpha_i+\beta_j) = sc_1(E) + rc_1(F)$.

4.2. Chow groups of projective bundles

Let E be a vector bundle of rank $r = e+1$ on a scheme X with projection $\pi : E \to X$. Denote by $p : P(E) \to X$ the projection of the associated projective bundle, and let $\mathcal{O}(1)$ be the canonical bundle on $P(E)$.

Theorem 4.2.1. — (a) *The flat pull-back*

$$\pi^* : A_{m-r}X \to A_m E$$

is an isomorphism for all m. In particular, $A_m E = 0$ if $m < r$.

(b) *Put $\xi = c_1(\mathcal{O}(1))$. Then the map $\theta_E : \bigoplus_{i=0}^e A_{m-e+i}(X) \to A_m P(E)$ defined by*

$$\theta_E(\oplus_i \alpha_i) = \sum_{i=0}^e \xi^i \cap p^*\alpha_i$$

is an isomorphism.

(c) *When X is a smooth quasi-projective variety, then the morphism in (b) induces an isomorphism of Chow rings*

$$A(P(E)) \cong A(X)[\xi]/(\xi^r + c_1(E)\xi^{r-1} + \cdots + c_r(E)).$$

Proof. — By Corollary 4.1.10, statement (c) is an easy consequence of (b). We have already seen the surjectivity of π^*, and it remains to prove the injectivity of π^* and statement (b). We prove first the surjectivity of θ_E. By the usual Noetherian induction, we may reduce to the case that E is a trivial bundle. Let $F = E \oplus 1$. By induction on the rank of E, it suffices to show the surjectivity of θ_F under the assumption of the surjectivity of θ_E.

Let $P = P(E)$, $Q = P(F)$, and $q : Q \to X$ be the projection. Then one has

$$P \xrightarrow{\ i\ } Q \xleftarrow{\ j\ } E\ .$$

By Lemma 1.1.5(d), one has a commutative diagram of Chow groups

$$
\begin{array}{ccccccc}
A_m(P) & \xrightarrow{i_*} & A_m(Q) & \xrightarrow{j^*} & A_m(E) & \longrightarrow & 0\ , \\
 & & \uparrow{\scriptstyle q^*} & \nearrow{\scriptstyle \pi^*} & & & \\
 & & A_{m-e}(X) & & & &
\end{array}
\tag{4.2.1.1}
$$

where the upper row is exact.

Lemma 4.2.2. — *For all $\alpha \in A_*(X)$, one has*

$$c_1(\mathcal{O}_F(1)) \cap q^*\alpha = i_*p^*\alpha.$$

Proof. — We may assume that $\alpha = [V]$. Then the required equality is reduced to showing that

$$c_1(\mathcal{O}_F(1)) \cap [q^{-1}V] = [p^{-1}V].$$

But this follows immediately from the fact that $\mathcal{O}_F(1)$ has a section vanishing precisely at P. □

We return to the proof of the surjectivity of θ_F under the assumption that θ_E is surjective. Let $\beta \in A_m(P)$ Since π^* is surjective, we have $j^*\beta = \pi^*\alpha_0$ for some $\alpha_0 \in A_{m-e}(X)$, i.e. $j^*(\beta - q^*\alpha) = 0$. Since the kernel of j^* coincides with the image of i_*, there exists $\gamma \in A_m(Q)$ such that $\beta - q^*\alpha = i_*\gamma$. By induction hypothesis, γ is of the form

$$\gamma = \sum_{j=0}^{e-1} c_1(\mathcal{O}_E(1))^j \cap q^*\alpha_j.$$

However, note that $c_1(\mathcal{O}_E(1)) = i^*c_1(\mathcal{O}_F(1))$. It follows from the projection formula and Lemma 4.2.2 that

$$
\begin{aligned}
i_*\gamma &= \sum_{j=0}^{e-1} c_1(\mathcal{O}_F(1))^j \cap i_*q^*\alpha_j \\
&= \sum_{j=0}^{e-1} c_1(\mathcal{O}_F(1))^{j+1} \cap q^*\alpha_j
\end{aligned}
$$

It follows that $\beta = q^*\alpha + \sum_{i=1}^{e} \xi^i \cap q^*\alpha_{i-1}$. This shows the surjectivity of θ_F.

We show now that θ_E is also injective. Let $\oplus\alpha_i \in \mathrm{Ker}(\theta_E)$. Then one has $\sum_{i=0}^{e} \xi^i \cap p^*\alpha_i = 0$, and

$$0 = p_*(\sum_{i=0}^{e} \xi^i \cap p_*\alpha_i) = \sum_{i=0}^{e} s_{i-e}(E) \cap \alpha_i = \alpha_e.$$

Hence, one gets $\sum_{i=0}^{e-1} \xi^i \cap p^*\alpha_i = 0$. Multiplying ξ and taking p_*, one gets $\alpha_{e-1} = 0$. Inductively, we see that $\alpha_i = 0$ for all $0 \le i \le e$.

Finally, we prove that $\pi^* : A_{m-r}(X) \to A_m(E)$ is injective. Let $\alpha \in A_{m-r}(X)$ be such that $\pi^*\alpha = 0$. Then by (4.2.1.1), $q^*\alpha$ must lie in the image of i_*. One writes $q^*\alpha = i_*\beta$ for some $\beta \in A_m(P(E))$. By (2), β is of the form

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}_E(1))^i \cap p^*\alpha_i$$

It follows by Lemma 4.2.2 that

$$0 = q^*\alpha - i_*\beta = q^*\alpha - \sum_{i=0}^{e} c_1(\mathcal{O}_F(1))^{e+1} \cap q^*(\alpha_i)$$

This contradicts with the injectivity of θ_F.

□

An iterated application of Theorem 4.2.1 allows one to compute the Chow ring of a flag variety.

Example 4.2.3. — Let V be a 4 dimensional vector space over k. A flag of V is a sequence of subspaces

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq V_4 = V$$

with $\dim(V_i) = i$ for $i = 1, 2, 3, 4$. Let $X = \mathcal{F}l(V)$ be the flag variety that classifies the flags of V. Then it is well known that $\mathcal{F}l(V)$ is a projective smooth variety of dimension 6. One can explicitly construct X as follows: Let $X_1 = P(V) \cong \mathbb{P}^3$ be the projective space of V. Then it classifies the lines of V, and there exists a rank 3 universal quotient bundle on X_1:

$$0 \to \mathcal{O}_{X_1}(-1) \to V \otimes \mathcal{O}_{X_1} \to Q_1 \to 0.$$

Let $\pi_1 : X_2 = P(Q_1) \to X_1$ be the projective bundle associated to Q_1. Then X_2 classifies lines of Q_1. Let $\mathcal{O}_{X_2}(-1) \subseteq \pi_1^* Q_1$ be the universal line subbundle, and $Q_2 = \pi^*Q_1/\mathcal{O}_{X_2}(-1)$. This is a rank 2 vector bundle over X_2. Then the projective bundle $\pi_2 : X_3 = P(Q_2) \to X_1$ is isomorphic to $\mathcal{F}l(V)$. In summary, we get a sequence of projective bundles

$$\mathcal{F}l(V) = X_3 \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 = \mathbb{P}^3.$$

One can compute the Chow ring $A(X_i)$ for $i = 1, 2, 3$ successively. We have

$$A(X_1)\mathbb{Z}[\eta_1]/(\eta_1^3),$$

where $\eta_1 = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$. By construction, the Chern polynomial of Q_1 is

$$c(Q_1) = \frac{1}{c(\mathcal{O}_{\mathbb{P}^3}(-1))} = \frac{1}{1 - \eta_1} = 1 + \eta_1 + \eta_1^2 + \eta_1^3.$$

Hence, by Theroem 4.2.1,

$$A(X_2) = A(X_1)[\eta_2]/(\eta_2^3 + \eta_1\eta_2^2 + \eta_1^2\eta_2 + \eta_1^3).$$

Similarly, the Chern polynomial of Q_2 is

$$\begin{aligned} c(Q_2) &= \frac{c(\pi_1^* Q_1)}{c(\mathcal{O}_{X_2}(-1))} = \frac{1}{(1-\eta_1)(1-\eta_2)} \\ &= (1+\eta_1+\eta_1^2)(1+\eta_2+\eta_2^2) \\ &= 1 + (\eta_1+\eta_2) + (\eta_1^2+\eta_1\eta_2+\eta_2^2). \end{aligned}$$

It follows that

$$\begin{aligned} A(\mathcal{F}l(V)) &= A(X_2)[\eta_3]/(\eta_3^2 + (\eta_1+\eta_2)\eta_3 + \eta_1^2+\eta_1\eta_2+\eta_2^2) \\ &= \mathbb{Z}[\eta_1,\eta_2,\eta_3]/(\eta_1^4, \eta_2^3+\eta_1\eta_2^2+\eta_1^2\eta_2+\eta_1^3, \eta_3^2+(\eta_1+\eta_2)\eta_3+\eta_1^2+\eta_1\eta_2+\eta_2^2). \end{aligned}$$

In particular, $A(\mathcal{F}l(V))$ is a free abelian group of rank $4! = 24$.

Recall that we have Bruhat cellular decomposition

$$\mathcal{F}l(V) = \mathrm{GL}_4/B = \coprod_{w\in W} B\dot{w}B/B,$$

where B is the upper triangular Borel subgroup, $W \cong S_4$ is the Weyl group of GL_4, $\dot{w} \in \mathrm{GL}_4$ is representative of w. Each cell $B\dot{w}B/B \cong \mathbb{A}^{\ell(w)}$ is an affine space. If $\sigma_w \in A_{\ell(w)}(\mathcal{F}l(V))$ denote the class of the closure of $B\dot{w}B/B$, we know that $A(\mathcal{F}l(V))$ is generated over $\mathbb{Z}$ by $\{\sigma_w : w \in W\}$. However, since $A(\mathcal{F}l(V))$ is free of rank $|W| = 24$, it follows that $\{\sigma_w : w \in W\}$ is actually a basis of $A(\mathcal{F}l(V))$ over $\mathbb{Z}$. It is a hard problem to express explicitly the product $\sigma_w \cdot \sigma_{w'}$ in terms of the basis σ_w's.

Example 4.2.4. — We explain the following beautiful example in [**EH13**, §11.8]: *Given 8 general lines $L_1, \cdots, L_8 \subseteq \mathbb{P}^3$, how many plane conics in $\mathbb{P}^3$ meet all of them?*

First, we introduce a parameter space for plane conics in $\mathbb{P}^3$. Consider the space

$$\mathcal{M} = \{(C,H) | H \cong \mathbb{P}^2 \subset \mathbb{P}^3, C = V(F)\}$$

where F is any homogeneous quadratic polynomial on H. Note that H is uniquely determined by C.

Let $\mathbb{P}^{3,*}$ be the dual 3-dimensional projective space, i.e. the space that classifies 3-dimensional vector space of k^4, or equivalently it classifies planes $H \cong \mathbb{P}^2$ in $\mathbb{P}^3$. We have naturally a projection morphism

$$\pi : \mathcal{M} \to \mathbb{P}^{3,*}$$

given by $(C,H) \mapsto H$, and the fibre above a point H is the space of conics contained in H, which is isomorphic to $\mathbb{P}^5$ (the projective space of the quadratic forms on H). Actually, one can view $\mathcal{M}$ as a projective bundle associated to a certain vector bundle. Let S be the vector bundle over $\mathbb{P}^{3,*}$ defined by

$$0 \to S \to \mathcal{O}^4_{\mathbb{P}^{3,*}} \to \mathcal{O}_{\mathbb{P}^{3,*}}(1) \to 0. \tag{4.2.4.1}$$

For any point $x \in \mathbb{P}^{3,*}$, $H_x = P(S_x) \subseteq \mathbb{P}^3 = \mathbb{P}(k^4)$ is the plane of $\mathbb{P}^3$ corresponding to x, and linear functions on H_x is identified with S_x^*. Therefore, we have $\mathcal{M} \cong P(\mathrm{Sym}^2 S^*)$, which is a projective smooth variety of dimension 8.

We determine first $A(\mathcal{M})$. We start with computing $c(\operatorname{Sym}^2 S^*)$. Recall that $A(\mathbb{P}^{3,*}) = \mathbb{Z}[h]/(h^4)$, where $h = c_1(\mathcal{O}_{\mathbb{P}^{3,*}}(1))$. By taking dual of the exact sequence (4.2.4.1), we have

$$c(S^*) = \frac{1}{1-h} = 1 + h + h^2 + h^3.$$

If we write $c(S^*) = (1+\alpha)(1+\beta)(1+\gamma)$, then

$$\begin{aligned} c(\operatorname{Sym}^2 S^*) &= (1+2\alpha)(1+2\beta)(1+2\gamma)(1+\alpha+\beta)(1+\beta+\gamma)(1+\alpha+\gamma) \\ &= 1 + 4h + 10h^2 + 20h^3. \end{aligned}$$

Alternatively, one considers the bundle $\operatorname{Sym}^2(\mathcal{O}^4_{\mathbb{P}^{3,*}})$. Note that if E is a vector bundle with line subbundle $L \subseteq E$ with quotient $F = E/L$. Then one has

$$0 \to L \otimes E \to \operatorname{Sym}^2 E \to \operatorname{Sym}^2 F \to 0.$$

Hence, by the dual of (4.2.4.1), we get

$$0 \to \mathcal{O}_{\mathbb{P}^{3,*}}(-1)^{\oplus 4} \to \operatorname{Sym}^2(\mathcal{O}^4_{\mathbb{P}^{3,*}}) \to \operatorname{Sym}^2 S^* \to 0$$

Thus $c(\operatorname{Sym}^2 S^*) = \frac{1}{(1-h)^4} = 1 + 4h + 10h^2 + 20h^3$.

Therefore, by Theorem 4.2.1, we get

$$A[\mathcal{M}] \cong \mathbb{Z}[h,t]/(h^4, t^6 + 4ht^5 + 10h^2t^4 + 20h^3t^3).$$

We fix a general line $L \subseteq \mathbb{P}^3$, and look the locus

$$Z_L = \{(C,H) \in \mathcal{M} \mid C \cap L \neq \emptyset\}$$

Note that a general plane $H \subseteq \mathbb{P}^3$ intersects with L at a unique point p, and the condition $C \cap L \neq \emptyset$ is equivalent to $p \in C$. If C is defined by a quadratic $F = 0$, then $p \in C$ imposes one equation $F(p) = 0$ to the moduli problem. Hence, Z_L is a closed codimension 1 subscheme in $\mathcal{M}$. Denote by $\delta = [Z_L] \in A^1(\mathcal{M})$ its cycle class, which is independent of L.

The number in question is $\deg(\delta^8)$. Write $\delta = ah + bt \in A^1(\mathcal{M})$, where $a, b \in \mathbb{Z}$ are some coefficients to be determined. We consider two curves $\Gamma, \Phi \subseteq \mathcal{M}$: Γ is a general pencil of conics contained in a fixed plane $H \subseteq \mathbb{P}^3$, and Φ is the intersection of a fixed quadratic surface $Q \subseteq \mathbb{P}^3$ with a pencil of planes in $\mathbb{P}^3$. Put $\gamma = [\Gamma], \phi = [\Phi]$, then one checks easily that

$$\begin{cases} \deg(\gamma \cdot h) = 0 \\ \deg(\gamma \cdot t) = 1 \end{cases} \qquad \begin{cases} \deg(\phi \cdot h) = 1 \\ \deg(\phi \cdot t) = 0 \end{cases}$$

and $\deg(\gamma \cdot \delta) = 1$ and $\deg(\phi \cdot \delta) = 2$. It follows that $\gamma = 2h + t$. Hence, the number in question is

$$\deg(\delta^8) = \deg((2h+t)^8) = t^8 + 16ht^7 + 112h^2t^6 + 448h^3t^5.$$

Note that h^3 the class of a fiber of $\pi : \mathcal{M} \to \mathbb{P}^{3,*}$, and the restriction of t to each fiber is the hyperplane class. So $\deg(h^3t^5) = 1$. Using relations in $A(\mathcal{M})$, we get

$$\deg(h^2t^6) = \deg(h^2(-4t^5h)) = -4, \quad \deg(ht^7) = 6, \quad \deg(t^8) = -4.$$

Finally, one conclude that $\deg(\delta^8) = 92$.

CHAPTER 5

GYSIN MAPS AND EXCESS INTERSECTION FORMULA

5.1. Gysin homomorphisms of vector bundles

Let X be a k-scheme, and $\pi : E \to X$ be a vector bundle of rank r. Denote by $s = s_E : X \to E$ the zero section. Recall that $\pi^* : A_{m-r}(X) \to A_m(E)$ is an isomorphism by Theorem 4.2.1(a) for any $m \in \mathbb{Z}$. We define the *Gysin homomorphism*

(5.1.0.1) $$s^* : A_m(E) \to A_{m-r}(X)$$

by $s^*(\beta) = (\pi^*)^{-1}(\beta)$.

Let $F = E \oplus 1$. We have a commutative diagram of morphisms

$$\begin{array}{ccccc} P(E) & \xrightarrow{i} & Q = P(F) & \xleftarrow{j} & E \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle q} & \swarrow{\scriptstyle \pi} & \\ & & X & & \end{array},$$

and an exact sequence of Chow groups

$$A_m(P) \xrightarrow{i_*} A_m(Q) \xrightarrow{j^*} A_m(E) \to 0.$$

Proposition 5.1.1. — *Let $\beta \in A_m(E)$, and $\bar{\beta} \in A_m(Q)$ be any element whose restriction to E gives β. Then one has*

$$s^*(\beta) = q_*(c_r(M) \cap \bar{\beta}),$$

where M is the universal quotient bundle of rank r over Q.

Proof. — One has to show that

(5.1.1.1) $$j^*\bar{\beta} = \pi^* q_*(c_r(M) \cap \bar{\beta}).$$

By definition of M, one has an exact sequence

$$0 \to \mathcal{O}_Q(-1) \to q^*E \oplus \mathcal{O}_Q \to M \to 0.$$

Let $\xi = c_1(\mathcal{O}_Q(1))$. Then one has

$$c(q^*E) = (1 - \xi)c(M),$$

and hence

$$c_r(M)\xi = 0, \quad \text{and} \quad c_r(M) = \xi^r + \xi^{r-1} q^* c_1(E) + \cdots + q^* c_r(E).$$

By Theorem 4.2.1(b), one may write

$$\bar{\beta} = \sum_{i=0}^{r} \xi^i \cap q^* \alpha_i.$$

Then $j^*\bar{\beta} = \pi^*\alpha_0$, because $\mathcal{O}_F(1)$ has a section with vanishing locus P and hence $j^*\xi = 0$. On the other hand, one has

$$\begin{aligned} q_*(c_r(M) \cap \bar{\beta}) &= \sum_{i=0}^{r} q_*(c_r(M) \cap \xi^i \cap q^*\alpha_i) \\ &= q_*(c_r(M) \cap q^*\alpha_0) \\ &= \sum_{i=0}^{r} q^*[\xi^i \cap q^*(c_{r-i}(E) \cap \alpha_0)] \\ &= \alpha_0. \end{aligned}$$

Now (5.1.1.1) follows immediately. □

Corollary 5.1.2. — *One has*

$$s^* s_* \alpha = c_r(E) \cap \alpha.$$

Proof. — Let $\bar{s} : X \to Q$ be the section $j \circ s$. Then

$$s^* s_*(\alpha) = q_*(c_r(M) \cap \bar{s}_*(\alpha)) = \sum_{i=0}^{r} q_*(q^* c_{r-i}(E) \xi^i \cap \bar{s}_* \alpha).$$

However, note that $\xi \cap \bar{s}_*\alpha = 0$, since $\bar{s}_*(\alpha)$ is supported in E and $\mathcal{O}_F(1)$ has a section with vanishing locus P. Hence, the right hand side above is

$$q_*(q^* c_r(E) \cap \bar{s}_*(\alpha)) = c_r(E) \cap \alpha.$$

□

We consider now an important special case of Proposition 5.1.1. Let $i : Y \hookrightarrow X$ be a closed immersion of schemes of pure dimension $m - c$, and C be a vector subbundle of $E|_Y$ of rank c, so that one has a commutative diagram:

$$\begin{array}{ccc} C & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y & \overset{i}{\hookrightarrow} & X \end{array}$$

Then one can view C as an m-cycle on E.

Proposition 5.1.3. — *Under the above notation, let D denote the quotient bundle $E|_Y/C$. Denote by $d = r - c$ the rank of D. Then one has*

$$s^*[C] = i_*(c_d(D) \cap [Y])$$

Proof. — Consider the projective bundle $P(C \oplus 1)$ over Y. Then one has a commutative diagram

$$\begin{array}{ccccc} P(C\oplus 1) & \hookrightarrow & Q|_Y & \xrightarrow{j} & Q = P(E\oplus 1)\\ & \searrow & \downarrow{\scriptstyle q_Y} & & \downarrow{\scriptstyle q}\\ & & Y & \xhookrightarrow{i} & X.\end{array}$$

Note that $j_*[P(C \oplus 1)]$ is a natural extension of $[C]$ to $A_m(Q)$. By Proposition 5.1.1, one has

$$\begin{aligned} s^*[C] &= q_*(c_r(M) \cap j_*[P(C \oplus 1)]) \\ &= q_* j_*(c_r(M|_Y) \cap [P(C \oplus 1)]) \\ &= i_* q_{Y,*}(c_r(M|_Y) \cap [P(C \oplus 1)]), \end{aligned}$$

where the second equality is the projection formula. Hence, up to replacing X by Y, we may assume that $X = Y$. To finishes the proof, it suffices to show that

$$q_*(c_r(M) \cap [P(C \oplus 1)]) = c_d(D) \cap [X].$$

Let $q_C : P(C \oplus 1) \to X$ denote the natural projection, and $i_C : P(C \oplus 1) \hookrightarrow P(E \oplus 1)$ the natural embedding. Note that $i_C^* \mathcal{O}_F(1) = \mathcal{O}_{P(C\oplus 1)}(1)$. Let M_C be the universal quotient bundle of rank c on $P(C \oplus 1)$ Then one has a commutative digram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{P(C\oplus 1)}(-1) & \longrightarrow & q_C^*(C \oplus 1) & \longrightarrow & M_C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{P(C\oplus 1)}(-1) & \longrightarrow & q_C^*(E \oplus 1) & \longrightarrow & i_C^* M & \longrightarrow & 0. \end{array}$$

One deduces the exact sequence $0 \to M_C \to i_C^* M \to q_C^* D \to 0$. It follows that

$$c_r(i_C^* M) = c_c(M_C) c_d(q_C^* D).$$

Hence, one has

$$\begin{aligned} q_*(c_r(M) \cap [P(C \oplus 1)]) &= q_* i_{C,*}(c_r(i_C^* M) \cap q_C^*[X]) \\ &= q_{C,*}(c_c(M_C) \cap q_C^*(c_d(D) \cap [X])) \\ &= s_C^*(\pi_C^*(c_d(D) \cap [X])) = c_d(D) \cap [X]. \end{aligned}$$

□

5.2. Excess intersections with regular embeddings

5.2.1. Regular sequences. — Let A be a ring. A sequence of elements $(a_1, \cdots, a_d)$ is a called a *regular sequence* of A (of length d) if $I = (a_1, \cdots, a_d) \neq A$, and the image of each a_i in $A/(a_1, \cdots, a_{i-1})$ is not a zero-divisor. Basic properties of regular sequences are summarized as follows

Lemma 5.2.2. — *Let $a_1, \cdots, a_d$ be a regular sequence in A, and let $I = (a_1, \cdots, a_d)$ Then*

(a) *I/I^2 is a free A/I-module of rank d, and*
(b) *the ring homomorphism*

$$A/I[X_1, \cdots, X_d] \to \bigoplus_{i=0}^{\infty} I^n/I^{n+1}$$

given by $X_i \mapsto a_i$ is an isomorphism. In particular, we have $I^n/I^{n+1} = \mathrm{Sym}^n(I/I^2)$.

Proof. — We prove here only statement (a), and statement (b) can be proved in the same way, but with more complicated techniques, or one can refer to [**Fu98**, Appendix A.6.1] for another proof. We proceed by induction on d. If $d = 1$, the statement is trivial. Assume now the statement hold for $d-1$, i.e. I_{d-1}/I_{d-1}^2 is a free A/I_{d-1}-module of rank $d-1$, where $I_{d-1} = (a_1, \cdots, a_{d-1})$. We have an exact sequence

$$0 \to A/I_{d-1} \xrightarrow{\times a_d} A/I_{d-1} \to A/I \to 0,$$

and it follows that $a_d I_{d-1} = I_{d-1} \cap (a_d)$. It is clear that I/I^2 is generated by the image of $a_1, \cdots, a_d$, and we have to show that they are linearly independent over A/I, i.e. if there exist $x_1, \cdots, x_d$ such that $\sum_i b_i a_i \in I^2$ then $b_i \in I$ for all i. Note that $I^2 = I_{d-1}^2 + a_d I_{n-1} + (a_d^2)$. So up to modifying b_i by elements of I, we may assume that $\sum_{i=1}^d a_i b_i = 0$. Consider the element

$$a_d b_d = -\sum_{i=1}^{d-1} a_i b_i \in I_{n-1} \cap (a_n) = a_n I_{n-1}.$$

It follows that $b_d \in I_{d-1}$. So write $b_d = \sum_{i=1}^{d-1} c_i a_i$ for some $c_i \in A$. Then

$$0 = \sum_{i=1}^{d} a_i b_i = \sum_{i=1}^{d-1} (b_i + c_i a_n) a_i.$$

By induction hypothesis, one has $b_i + c_i a_n \in I_{n-1}$, and hence $b_i \in I$. ☐

5.2.3. Regular closed embeddings and normal bundles. — Let $i : X \hookrightarrow Y$ be a closed embedding of schemes with ideal sheaf $\mathcal{I}$. We define the normal cone of X in Y as

$$C_X(Y) = \mathrm{Spec}(\bigoplus_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}).$$

In general, the geometry of $C_X(Y)$ is complicated.

We say that i is a *regular embedding of codimension* d, if every point of X has an affine neighborhood $U = \mathrm{Spec}(A)$ in Y such that $\mathcal{I}(U)$ is generated by a regular sequence of A of length d. If i is a regular embedding of codimension d, then $\mathcal{I}/\mathcal{I}^2$ is a vector bundle of rank d. By Lemma 5.2.2(b), $C_X(Y)$ is the vector bundle associated to the locally free sheaf $(\mathcal{I}/\mathcal{I}^2)^\vee$. We also put $N_X(Y) = C_X(Y)$ and call it the normal bundle of X in Y.

Basic properties of regular embeddings are listed as follows:

1. When X is smooth over k, then i is regular if and only if Y is also smooth at some neighborhood of X. In that case, we have an exact sequence

$$0 \to T_X \to i^*T_Y \to N_X(Y) \to 0.$$

2. If $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow Z$ are both regular closed embeddings, then the composition $j \circ i : X \hookrightarrow Z$ is also a regular closed embedding. Moreover, one has

$$0 \to N_X(Y) \to N_X(Z) \to i^*N_Y(Z) \to 0.$$

3. If X is a complete intersection of d Cartier divisors $D_1, \cdots, D_d$ on Y, then $i : X \hookrightarrow Y$ is a regular closed embedding, and $N_X(Y)$ has a filtration by subbundles of length d with quotients given by $\mathcal{O}_Y(D_i)|_X$.

5.2.4. Intersection product with a regular closed embedding. — Let $i : X \hookrightarrow Y$ be a closed regular immersion of codimension d of k-schemes, V be a purely m-dimensional scheme over k and equipped with a morphism $f : V \to Y$. Put $W = f^{-1}(X)$. We have a Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y. \end{array}$$

Let $N = g^*N_X(Y)$ be the pull-back of the normal bundle of X in Y, and denote by $\pi : N \to W$ the projection. Let $\mathcal{I}$ be the ideal sheaf of X in Y, and $\mathcal{J}$ be the ideal sheaf of W in V. Then there is a canonical surjection $\mathcal{I} \to \mathcal{J}$, which induces a surjection of grades algebras:

$$f^*(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}) \to \bigoplus_{n\geq 0} \mathcal{J}^n/\mathcal{J}^{n+1}.$$

This induces a closed embebding $C_W(V) \hookrightarrow N$ of W-schemes. We define the intersection product

$$X \cdot V := s^*[C_W(V)] \in A_{m-d}(W) \qquad (5.2.4.1)$$

where $s : W \to N$ is the zero section of the vector bundle $\pi : N \to W$, and s^* is the Gysin homomorphism defined in (5.1.0.1).

***Theorem 5.2.5* (Excess-intersection formula).** — *In the situation above, assume that $W = \coprod_\alpha W_\alpha$ such that $W_\alpha \hookrightarrow V$ is a regular closed embedding for each α. Let $N_{W_\alpha}(V)$ be the normal bundle of W_α in V, and put $E_\alpha = g^*N_X(Y)/N_{W_\alpha}(V)$. Then one has*

$$X \cdot V = \sum_\alpha c_{e_\alpha}(E_\alpha) \cap [W_\alpha],$$

where e_α is the rank of the vector bundle E_α over W.

Proof. — Note that $C_W(V) = \coprod_\alpha C_{W_\alpha}(V)$. Since $W_\alpha \hookrightarrow V$ is regular, we have $C_{W_\alpha}(V) = N_\alpha(V)$, which is a subbudle of $N_X(Y)$. So one has

$$X \cdot V = \sum_\alpha s^*[N_{W_\alpha}(V)].$$

The Theorem follows from Proposition 5.1.3. □

***Corollary 5.2.6* (Self-intersection).** — *We have*

$$X \cdot X = c_e(N_X(Y)) \cap [X].$$

Corollary 5.2.7. — *If $j : W \hookrightarrow V$ is a regular embedding with the same codimension as $i : X \hookrightarrow Y$, then $X \cdot V = [W] = [f^{-1}(V)]$. In particular, if $f : V \to Y$ is a closed embebding of smooth varieties and the intersection with X is the transversal, then we have $X \cdot V = [X \cap V]$.*

Assume now the case when $f : V \hookrightarrow Y$ is a closed embedding. Then the intersection product with X gives essentially another way to define an intersection theory on Y. We state without proof some important properties of the intersection product in this case. For more details, one may refer to [**Fu98**, Chap. 6].

(1) *The intersection product with X is invariant under rational equivalence: if V_1 and V_2 are two closed subschemes in Y, which are rationally equivalent, then one has $X \cdot V_1 = X \cdot V_2$* [**Fu98**, §6.2]. The main issue is to show that the cone $C_{X \cap V_1}(V_1)$ is rationally equivalent to $C_{X \cap V_2}(V_2)$ as cycles on $N_X(Y)$ [**Fu98**, Proposition 5.2], and this uses the techenique of deforming to the normal cone.
(2) *If $f : V \hookrightarrow X$ is also a regular closed embedding, then $X \cdot V = V \cdot X$.* See [**Fu98**, Example 6.5.1].
(3) *If Y is a quasi-projective smooth variety over k, then for any closed subscheme $V \subseteq Y$ the intersection product $X \cdot V$ coincides with the one defined using the moving Lemma.* This actually follows from the previous two points, Corollary 5.2.7, and the uniqueness of the intersection products.

Example 5.2.8. — Let X be a smooth projective variety, $D \subseteq X$ be a (Cartier) divisor, and $C \subseteq D$ be a curve. Then

$$\deg(C \cdot D) = \deg(\mathcal{O}_X(D)|_C).$$

Example 5.2.9. — Let S, T be two smooth surfaces in $\mathbb{P}^4$ of degrees s, t respectively. Assume that the intersection of $S \cap T$ consists of a reduced curve C of genus g and degree d together with a zero-dimensional scheme Γ. The problem is to find the degree of Γ.

By assumption, one has $S \cap T = C \coprod \Gamma$. Using the excess intersection formula, one gets

$$[S] \cdot [T] = c_1(N_S(\mathbb{P}^4)|_C / N_C(T)) \cap [C] + [\Gamma].$$

However, note that $0 \to N_C(S) \to N_C(\mathbb{P}^4) \to N_S(\mathbb{P}^4)|_C \to 0$. It follows that

$$c_1(N_S(\mathbb{P}^4)|_C / N_C(T)) = c_1(N_C(\mathbb{P}^4)) - c_1(N_C(S)) - c_1(N_C(T)).$$

Hence, we get

$$st = \deg([S] \cdot [T]) = \deg(c_1(N_C(\mathbb{P}^4))) - \deg(N_C(S)) - \deg(N_C(T)) + \deg(\Gamma)$$

On the other hand, one has $(C, C)_S = \deg(N_C(S))$, $(C, C)_T = \deg(N_C(T))$, and $c_1(N_C(\mathbb{P}^4)) = c_1(T_{\mathbb{P}^4}|_C) - \deg(T_C)$. It follows that

$$\deg(\Gamma) = st - \deg(T_{\mathbb{P}^4}|_C) + \deg(T_C) - (C, C)_S - (C, C)_T.$$

Note that, if $\xi \in A^1(\mathbb{P}^4)$ denotes the class of hyperplane in $\mathbb{P}^4$, then$c_1(T_{\mathbb{P}^4}) = 5\xi$ so that $\deg(T_{\mathbb{P}^4}|_C) = 5\deg(\mathcal{O}_{\mathbb{P}^4}(1)|_C) = 5d$. One has eventually

$$\deg(\Gamma) = st - 5d + 2 - 2g - (C, C)_S - (C, C)_T.$$

5.2.10. Relationship with pullbacks. — Let $f : Y \to X$ be a morphism of quasi-projective smooth varieties. Recall that, using the Moving Lemma, one can define a homomorphism of Chow rings

$$f^* : A^*(X) \to A^*(Y).$$

Theorem 5.2.11. — *Let $Z \subseteq X$ be a smooth closed subvariety of codimension r. Assume that $W = f^{-1}(Z) = \coprod_\alpha W_\alpha$, such that each W_α is smooth of codimension d_α in Y. Then one has*

$$f^*[Z] = \sum_\alpha c_{e_\alpha}(E_\alpha) \cap [W_\alpha],$$

where $E_\alpha = g_\alpha^* N_Z(X)/N_{W_\alpha}(Y)$ and $e_\alpha = r - d_\alpha$.

Proof. — One has to show that $f^*([Z]) = [Z \cdot Y]$, where $Z \cdot Y$ is as defined in (5.2.4.1). For this, one has to check that

(1) if Z is transversal to f, i.e. $f^{-1}(Z)$ is smooth of codimension r in Y, we have $[Z \cdot Y] = [f^{-1}(Z)]$;
(2) the class $[Z \cdot Y] \in A_{m-r}(Y)$ is invariant under the rational equivalence in Z. For more details, see [**Fu98**, §6.6, §6.7].

□

Corollary 5.2.12. — *If $i : Y \hookrightarrow X$ is a closed embedding of smooth quasi-projective varieties of codimension r, then*

$$i^* i_*(\alpha) = c_r(N_X(Y)) \cap \alpha.$$

Proof. — We may assume that $\alpha = [Z]$ with $Z \subseteq X$ a smooth subvariety. Then the excess vector bundle

$$E = N_Z(Y)/N_Z(X) = N_X(Y)|_Z.$$

The corollary follows immediately from Theorem 5.2.11. □

5.3. Chow groups of the blow-up

Let X be a smooth variety, and $i : Z \subseteq X$ be a smooth closed subvariety of codimension r. Let $\pi : Y \to X$ be the blow-up of X at Z, and $E = \pi^{-1}(Z)$ be the exceptional divisor. If $\mathcal{I}$ denotes ideal sheaf of Z, then

$$Y = \operatorname{Proj}\big(\bigoplus_{n\geq 0} \mathcal{I}^n\big), \quad \text{and} \quad E = \operatorname{Proj}\big(\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\big).$$

Note that i is automatically a regular embedding so that $E = P(N_Z(X))$ is the projective bundle over Z associated to $N_Z(X)$. Put $U = X\backslash Z$. We have a commutative diagram

$$\begin{array}{ccccc} U & \xhookrightarrow{j'} & Y & \xleftarrow{i'} & E \\ \| & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_E} \\ U & \xhookrightarrow{j} & X & \xleftarrow{i} & Z. \end{array}$$

Lemma 5.3.1. — *(1) We have $N_E(Y) = i'^*\mathcal{O}_Y(E) = \mathcal{O}_E(-1)$, where $\mathcal{O}_E(-1)$ is canonical line subbundle on $E = P(N_Z(X))$.*

(2) If $y \in A(E)$ such that $\pi_{E,*}(y) = 0$ and $i'^*i'_*(y) = 0$, then $y = 0$.

Proof. — (1) It is equivalent to showing that $\pi^*\mathcal{I}|_E = \mathcal{O}_E(1)$, which follows immediately from the definition of E.

(2) By Theorem 4.2.1, we may write uniquely $y = \sum_{l=0}^{r-1} \zeta^l \cap \pi_E^*(\alpha_l)$ with $\zeta = c_1(\mathcal{O}_E(1))$. Then we have $\pi_{E,*}(y) = \alpha_{r-1} = 0$. By Corollary 5.2.12, we have

$$0 = i'^*i'_*(y) = c_1(N_E(Y)) \cap y = -\sum_{l=1}^{r-1} \zeta^l \cap \pi_E^*(\alpha_{l-1}).$$

By Theorem 4.2.1, we have $\pi_E^*(\alpha_{l-1}) = 0$ for all l. Hence, one has $y = 0$. □

Theorem 5.3.2. — *Under the notation above, let $Q = \pi_E^* N_Z(X)/\mathcal{O}_E(-1)$ denote the canonical quotient bundle on E of rank $r-1$. Then there exists a split exact sequence of additive groups, preserving the grading by dimension:*

$$0 \to A(Z) \xrightarrow{(-i_*,h)} A(X) \oplus A(E) \xrightarrow{(\pi^*,i'_*)} A(Y) \to 0,$$

where $h : A(Z) \to A(E)$ is the map given by $h(\alpha) = c_{r-1}(Q) \cap \pi_E^*(\alpha)$, and a left inverse for $(-i_*, h)$ is given by $(x, y) \mapsto \pi_{E,*}(y)$.

Proof. — We prove first the surjectivity of (π^*, i'_*). Indeed, we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} A_m(Z) & \xrightarrow{i_*} & A_m(X) & \xrightarrow{j^*} & A_m(U) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi^*} & & \| & & \\ A_m(E) & \xrightarrow{i'_*} & A_m(Y) & \xrightarrow{j'^*} & A_m(U) & \longrightarrow & 0, \end{array}$$

Then the surjectivity of (π^*, i'_*) follows immediately by an easy diagram chasing.

Second, we prove that $(\pi^*, i'_*) \circ (-i_*, h) = 0$, i.e. for any $\alpha \in A(Z)$, we have $\pi^*i_*(\alpha) = i'_*(c_{r-1}(Q) \cap \pi_E^*(\alpha))$. We may assume that $\alpha = [V]$ with $V \subseteq Z$ a smooth closed subvariety of dimension m. Then one has the Cartesian diagram:

$$\begin{array}{ccccc} E_V & \longrightarrow & E & \xrightarrow{i} & Y \\ \downarrow{\scriptstyle \pi_{E_V}} & & \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi} \\ V & \longrightarrow & Z & \longrightarrow & X \end{array}$$

Note that

$$\pi_{E_V}^* N_V(X)/N_{E_V}(Y) \cong (\pi_E^* N_Z(X)/N_E(Y))|_{E_V} \cong Q|_{E_V}.$$

It follows from Theorem 5.2.11 that $\pi^*[V] = i'_*(c_{r-1}(Q) \cap [E_V]) = i'_*(c_{r-1}(Q) \cap \pi_E^*[V])$.

Prove now that the sequence is exact in the middle. Let $(x, y) \in A(X) \oplus A(E)$ be such that $\pi^*(x) + i'_*(y) = 0$. Applying π_*, one gets

$$\pi_*(\pi^*(x)) = \pi_* i'_*(-y).$$

However, note that $\pi_*\pi^*(x) = x$ and $\pi_* i'_*(y) = i_*\pi_{E,*}(y)$. It follows thus $x = i_*\pi_{E,*}(-y)$. Hence, if $\alpha = \pi_{E,*}(y)$, then $x = -i_*(\alpha)$ and

$$i'_*(y) = -\pi^*(x) = \pi^* i_*(\alpha) = i'_*(c_{r-1}(Q) \cap \pi_E^*(\alpha)).$$

If $y' = y - c_{r-1}(Q) \cap \pi_E^*(\alpha)$, we have $i'_* i'^*(y') = 0$. On the other hand, if $\zeta = c_1(\mathcal{O}_E(1))$ and $N = N_Z(X)$, we have

$$c(Q) = \frac{\pi_E^* c(N)}{1 - \zeta}$$

and $c_{r-1}(Q) = \zeta^{r-1} + \zeta^{r-2}\pi_E^* c_1(N) + \cdots + \pi_E^* c_{r-1}(N)$. It follows that

$$\pi_{E,*}(c_{r-1}(Q) \cap \pi_E^*\alpha) = \sum_{l=0}^{r-1} \pi_{E,*}(\zeta^l \cap \pi_E^*(\alpha c_{r-1-l}(N))) = \alpha. \tag{5.3.2.1}$$

and hence

$$\pi_{E,*}(y') = \alpha - \pi_{E,*}(c_{r-1}(Q) \cap \pi_{E,*}(\alpha)) = 0.$$

It follows from Lemma 5.3.1(2) that $y' = 0$, and hence $(x, y) = (i_*\alpha, h(\alpha))$.

The fact that $\pi_{E,*}$ is a left inverse of h follows from (5.3.2.1).

□

Remark 5.3.3. — The multiplication on elements of $A(Y)$ are given by

$$\pi^*\alpha \cdot \pi^*\beta = \pi^*(\alpha\beta)$$
$$\pi^*\alpha \cdot i'_*\sigma = i'_*(\sigma \cdot \pi_E^* i^*\alpha)$$
$$i'_*\sigma \cdot i'_*\tau = i'_*(-\zeta\sigma\tau)$$

where $\zeta = c_1(\mathcal{O}_E(1))$. For instance, if $e = [E]$, then $e^i = i'_*(-\zeta)^{i-1}$ for $i \geq 1$.

5.4. Application: The problem of five conics

The aim of this section is to explain how to solve the following problem in enumerative geometry: *Given 5 general smooth plane conics $C_1, \cdots, C_5 \subseteq \mathbb{P}^2$, how many smooth conics are tangent to all 5?*

First, the set of all conics in $\mathbb{P}^2$ is naturally parametrized by $X \cong \mathbb{P}^5$. Let C_i be a fixed smooth conic. Let Z_i be the subset of X consisting of conics that are tangent to C_i. Note that Z_i may contain singular conics (such as a union of two lines). Here, we say a curve C is tangent to C_i at a point $p \in C_i$ if $p \in C$ and $m_p(C, C_i) \geq 2$.

Lemma 5.4.1. — *Z_i is hypersurface of $X \cong \mathbb{P}^5$ of degree* 6.

Proof. — Put

$$\Sigma_i := \{(C,p) \in \mathcal{M} \times C_i |\ C \text{ is a conic tangent to } C_i \text{ at } p\ \}.$$

Then there is a natural structure of algebraic variety on Σ_i, and we have a correspondence:

$$\begin{array}{ccccc} & & \Sigma_i & & \\ & {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} & \\ Z_i & & & & C_i. \end{array}$$

It is clear that π_2 is surjective and π_1 is finite. On the other hand, it is easy to see that the fiber of π_2 has codimension 3 in X, hence Z_i is a divisor in X. To finish the proof, it remains to see that Z_i has degree 6.

We consider the intersection of Z_i with a general pencil of conics in X. Explicitly, let $F_1, F_2 \in k[x,y,z]$ be two general homogeneous polynomial of degree 2 such that F_1, F_2 have no common zeros on C_i. Put

$$L = \{F = aF_1 - bF_2 | a, b \in k\} \subset X.$$

We define a map $f : C_i \to \mathbb{P}^1$ by $f(p) = (F_1(p), F_2(p))$. We identify $\mathbb{P}^1$ with the pencil L by sending $(b,a) \in \mathbb{P}^1$ to the conic C_p defined by $aF_1 - bF_2$. Then, for $p \in \mathbb{P}^1$, the inverse image $f^{-1}(p)$ is the intersection of $C_i \cap C_p$. By Bézout's theorem, we have $\#f^{-1}(p) = 4$ counted with multiplicities. Hence, f is a dominant map of curves of degree 4. Note that the conic C_p lies in $Z_i \cap L$ if and only if $f^{-1}(p)$ has double points, i.e. $L \cap Z_i$ corresponds to the branch locus of the map f. By Hurwitz formula, we see that the number of branch points of f is 6. □

Now given 5 general conics $C_1, \cdots, C_5$, then the subset of conics that are tangent to all 5 is therefore $\bigcap_{i=1}^5 Z_i$. If the intersection is transversal and $\bigcap_{i=1}^5 Z_i$ consists only of smooth conics, then the desired number is

$$\#(\bigcap_i Z_i) = \prod_{i=1}^5 \deg(Z_i) = 6^5 = 7776.$$

However, this is not the correct answer, because the intersection of Z_i's is neither not transversal, nor their intersection consists of only smooth conics! Indeed, let $D \subseteq X$ be the subset of double lines. Then one has $D \cong \mathbb{P}^2$ embedded into $X \cong \mathbb{P}^5$ via Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. According to our definition, a double line is tangent to all conics so that $D \subseteq Z_i$ for all i. Actually, the intersection $\bigcap_{i=1}^5 Z_i$ is the disjoint union of D together with a discrete set Γ. The cardinality of Γ is the number that we want to find.

We consider the blow-up $\pi : \tilde{X} \to X$ along the locus of double lines D, and let $E = \pi^{-1}(D)$. Then we have $A(D) = A(\mathbb{P}^2) = \mathbb{Z}[\sigma]/(\sigma^3)$, $A(X) = \mathbb{Z}[\omega]/\omega^6$, and $i^*\omega = 2\sigma$.

Hence, $N_D(X)$ is a vector bundle of rank 3 with

$$c(N_D(X)) = \frac{i^*c(T_X)}{T_D} = \frac{i^*(1+\omega)^6}{(1+\sigma)^3} = \frac{(1+2\sigma)^6}{(1+\sigma)^3} = 1 + 9\sigma + 30\sigma^2.$$

Therefore, by Theorem 4.2.1, one has

$$A(E) = A(D)[\zeta]/(\zeta^3 + 9\sigma\zeta^2 + 30\sigma^2\zeta) = \mathbb{Z}[\sigma,\zeta]/(\sigma^3, \zeta^3 + 9\sigma\zeta^2 + 30\sigma^2\zeta).$$

Note also that $A_0(E)$ is a free abelian group with basis $\{\sigma^2\zeta^2, \sigma\zeta^3, \zeta^4\}$, and

$$\deg(\sigma^2\zeta^2) = 1, \quad \deg(\sigma\zeta^3) = -9, \quad \deg(\zeta^4) = 51.$$

Lemma 5.4.2. — *Let $\tilde{Z}_i$ denote the strict transform of Z_i in $\tilde{X}$. Then one has*

$$\pi^*[Z_i] = [\tilde{Z}_i] + 2[E],$$

and the intersection $\bigcap_i \tilde{D}_i$ is transversal, and the intersection lies in $\tilde{X}\backslash E \cong X\backslash D$.

Proof. — In general, one always has $\pi^*[Z_i] = [\tilde{Z}_i] + a[E]$ for some $a \in \mathbb{Z}$. To find a, we consider a general pencil of conics $L \subseteq X$ which passes through a unique point $q \in Z_i$. Then a can be interpreted as the intersection multiplicity at q of L and Z_i. As above, such a pencil defines a dominant map $f : C_i \to \mathbb{P}^1$ of degree 4. Then $f^{-1}(q)$ contains exactly two distinct points each with multiplicity 2. By Hurwitz's formula, we see that the branch locus of f contains 4 more points other than q. That is, $L \cap Z_i$ intersect at 4 more points except q. Since Z_i is a hypersurface of degree 6, this shows that $a = m_q(L, Z_i) = 2$.

For the transverality of $\tilde{Z}_i$'s see [**GH78**, Chap. 6].

□

Assuming Lemma above, the intersection number desired is nothing but $\deg(\bigcap_i [\tilde{Z}_i])$. We view $A(X)$ as a ring of $A(\tilde{X})$ via π^*, and put $e = [E]$. Then

$$[\tilde{Z}_i] = \pi^*[Z_i] - 2[E] = 6\omega - 2e.$$

Hence,

$$\deg(\bigcap_i [\tilde{Z}_i]) = \deg((6\omega - 2e)^5) = \sum_{l=0}^{5} \binom{6}{l} (-2)^{5-l} 6^l \deg(\omega^l e^{5-l})$$

Using $\pi^*\alpha i'_*(\beta) = i'_*(\beta\pi_E^* i^*\alpha)$, we see that $\deg(\omega^i e^j) = \deg((2\sigma)^i(-\zeta)^{j-1})$ for $j \geq 1$. Hence,

$$\deg(\bigcap_i [\tilde{Z}_i]) = 6^6 + \binom{5}{2} 6^2 (-2)^3 \deg((2\sigma)^2\zeta^2) + \binom{5}{1} 6(-2)^4 \deg((2\sigma)(-\zeta)^3) + (-2)^5 \deg((-\zeta)^4)$$
$$= 7776 - 11520 + 8640 - 1632 = 3264.$$

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