Graduate Seminar on Representation Theory
Hopf algebras and tensor categories

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THORSTEN HEIDERSDORF

Part 1. Tannaka categories

The main sources for this part are the book of Milne [Mil17] and the article by Deligne-Milne [DM82]. Further material on monoidal categories can be found in [EGNO15].

Talk 1: Monoidal categories

One should introduce the notion of a monoidal category following [DM82] and explain some standard defintions: k-linearity, tensor functors,... An additional reference is [EGNO15]. Note: Not every commutative diagram needs to be written down... One should then introduce the notion of a a dual object, the evaluation and coevaluation, rigidity, the internal Hom and the symmetry isomorphism. The categories Vec_k and $sVec_k$ (s = super) should be explained. One should also introduce the notion of a tensor category as a rigid symmetric monoidal abelian category with $End(\mathbf{1}) \cong k$, and observe that such categories are automatically k-linear. The main source is [DM82], but [EGNO15] might be helpful as well.

Talk 2: Algebraic groups and their representations

Explain the definition of an affine algebraic groups along with some examples as in [Mil17, Chapter 1, a-d, Chapter 2 a]. Representations should be defined. The category of finite dimensional representations is an abelian symmetric rigid k-linear category with $End(\mathbf{1}) = k$. This talk should be in close alignment with talk 3; and the two speakers should adjust the contents so that both talks are of equal length.

Talk 3: Hopf algebras and comodules

The speaker should introduce Hopf algebras and some standard notions about them. One should explain the correspondence between affine algebraic groups and commutative Hopf algebras as in [Mil17, Chapter 3 a-c], in particular [Mil17, Corollary 3.7]. The notion of a comodule needs to be introduced. Then one should explain how to pass between representations of G and comodules of $k[G]$ (see also [DM82, Proposition 2.2]). One should prove that every representation of G is a union of finite-dimensional ones.

An alternative source for talk 2 and 3 is the book [Jan03].

Talk 4: Automorphisms of the fiber functor

Given $Rep(G)$, one can consider the forgetful functor $Rep(G) \to Vec$. Its group of tensor automorphisms can be identified with G. This is on page 19, 20 in [DM82] and is deceptively short, but one needs to prove some results used in the proof of this theorem.

Talk 5: Tannakian categories 1

The general concept of a tannakian category should be introduced. Then one should quickly pass to neutral tannakian categories [DM82, Definition 2.19] and announce the main theorem [DM82, Theorem 2.11]. The rest of talk 5 and 6 are devoted to a proof of this theorem. An alternative source for a proof of the main theorem is in [Mil17].

Talk 6: Tannakian categories 2

See talk 5. If time permits, one could choose an application or an example from [DM82, Page 25 -30].

Talk 7: Internal characterization of tannakian categories 1

It is often not clear whether a given tensor category admits a fiber functor. If $char(k) = 0$ and k is algebraically closed (or if one is willing to use a finite field extension k'/k) there is a very convenient criterion due to Deligne [Del90]. Given a pre-tannakian category $\mathcal{C}$, it admits a fiber functor iff the categorial dimension of each object is a natural number iff each object is annihilated by a suitable exterior power. The main goal of the talk is to explain Deligne's criterion [13, th. 3.1] precisely. For this one should introduce the notion of a dimension and an exterior power in a symmetric monoidal category along with some examples and standard properties.

Talk 8: Internal characterization of tannakian categories 2

We now prove Deligne's theorem. For this one needs to introduce algebras and modules in a symmetric monoidal category and study the splitting of objects and morphisms.

Part 2. Reshitikhin-Turaev invariants

The main references for this are [JM19], [Oht02] and [Kas95]. In this part we will occasionally have to skip some proofs.

Talk 9: Knots, links and tangles

The speaker should introduce knots and links (and tangles), their framed versions and explain that one would like construct invariants up to isotopy. Then the associated diagrams should be explained and how isotopy invariance leads to the considerations of Reidemeister and Turaev moves. This is a summary talk without complete proofs as in [JM19, Sections 1-3] or [Oht02] or [Kas95, Chapter 10].

Talk 10: Quasitriangular Hopf algebras

The general formalism of Reshetikhin-Turaev takes objects in a ribbon category and colors link or tangle diagrams with these objects to get an invariant. However it is not so easy to construct reasonable ribbon categories that lead to interesting invariants. The main examples all arise from Hopf algebras. However the category of finite dimensional modules of a Hopf algebra is in general not braided (or even a ribbon category). This is however true if the Hopf algebra is quasitriangular. One should define quasitriangular Hopf algebras, why this extra structures gives rise to a braiding (the universal R-matrix), the Yang-Baxter equation etc. (see [JM19, Section 8.2] and [Oht02, Section 4.1].

Talk 11: Ribbon Hopf algebras

The speaker should discuss ribbon Hopf algebras and their modules as in [JM19, Section 8.3] and [Oht02, Section 4.1]. Basically one should obtain that the module category of a ribbon Hopf algebra is a ribbon category.

Talk 12: The Reshitikhin-Turaev functor 1

Talk 12 and 13 should define the RT functor and check that this yields an invariants of framed links/tangles as in [Oht02, Chapters 4.2, 4.3] and [JM19, Chapter 7, Chapter 9.1 -9.3]. The two speakers should discuss the two talks together. One should explain chapter 7.1 and chapter 7.2 from [JM19] and progress towards theorem 7.17, but avoid to check all the Turaev move conditions.... Chapter 7.5 needs to be discussed. The main part of talk 12 and 13 should be devoted to [JM19, Chapter 9.1 -9.3]. Occasionally some proofs will need to be skipped. Alternative sources are the books [Kas95], [KRT97] [Tur16].

Talk 13: The Reshitikhin-Turaev functor 2

See talk 12.

Talk 14: The quantum group $U_v(\mathfrak{sl}_2)$

As indicated before, it is not easy to construct ribbon categories with non-symmetric braiding. The two most famous constructions are the Drinfeld double of a Hopf algebra and the quantum deformations of the univeral enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra. We will study the latter example in the special case $\mathfrak{sl}_2$. In this talk we introduce the quantum group $U_v(\mathfrak{sl}_2)$ by introducing a v-deformation of the generators of $U(\mathfrak{sl}_2)$ where v is an indeterminate. We need the construction, the notion of a $U_v(\mathfrak{sl}_2)$-module (of type 1), that $U_v(\mathfrak{sl}_2)$ is a Hopf algebra and a description of $Rep(U_v(\mathfrak{sl}_2))$ (basically saying it looks almost like the usual $Rep(U(\mathfrak{sl}_2))$. All of this is in [JM19], [Oht02] and [Kas95, Section 6,7]. The purpose of this talk to give an overview, and some results about modules cannot be proven due to time constraints.

Talk 15: The Jones polynomial as quantum invariant

In this talk the universal R-matrix of $U_v(\mathfrak{sl}_2)$ should be discussed. One should discuss the braiding and that one almost gets a ribbon Hopf algebra in this way. Actually one needs to pass to a completion to make this work for $Rep(U_v(\mathfrak{sl}_2)$. The braiding should be written down explicitely, but presumably without proof. In the rest of the talk the quantum invariant of a knot should be computed if we take as a color of the knot the standard representation of $U_v(\mathfrak{sl}_2)$ as in [JM19] [Oht02]. This is done by computing it explicitely on the circle, the crossings etc. and determine the skein relations. The content of this talk is mostly contained in [JM19, Section 9.4] or [Oht02, Section 4.4].

References

[Del90] P. Deligne. Catégories tannakiennes. (Tannaka categories). The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. II, Prog. Math. 87, 111-195 (1990)., 1990.

[DM82] Pierre Deligne and J. S. Milne. Tannakian categories. Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 101-228 (1982)., 1982.

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories.*, volume 205. Providence, RI: American Mathematical Society (AMS), 2015.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups. 2nd ed.*, volume 107. Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2003.

[JM19] David M. Jackson and Iain Moffatt. *An introduction to quantum and Vassiliev knot invariants.* Cham: Springer, 2019.

[Kas95] Christian Kassel. *Quantum groups.*, volume 155. New York, NY: Springer-Verlag, 1995.

[KRT97] Christian Kassel, Marc Rosso, and Vladimir Turaev. *Quantum groups and knot invariants.*, volume 5. Paris: Société Mathématique de France, 1997.

[Mil17] J. S. Milne. *Algebraic groups. The theory of group schemes of finite type over a field.*, volume 170. Cambridge: Cambridge University Press, 2017.

[Oht02] Tomotada Ohtsuki. *Quantum invariants. A study of knots, 3-manifolds, and their sets.*, volume 29. Singapore: World Scientific, 2002.

[Tur16] Vladimir G. Turaev. *Quantum invariants of knots and 3-manifolds. 3rd edition.*, volume 18. Berlin: Walter de Gruyter, 3rd edition edition, 2016.