Graduate Seminar on Advanced Algebra
Supermathematics

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Preliminaries: Good knowledge of the structure and representation theory of Lie algebras. For talks 5,6 some knowledge about algebraic groups and Hopf algebras is helpful. Furthermore some background about the language of monoidal/tensor categories.

Literature: The main sources for the first half of the seminar are the books Cheng-Wang [Mus12] (both for structure and representation theory), the book [Sch79] for statements about the classification, the article [Kac77] for the classification theorem and the survey articles [Kac78] [Ser17] for both structure and representation theory. The talks 9 - 13 are mostly based on the articles [CW12b] and [Ger98].

Content: The aim is to give both an introduction to the theory as well as a survey of various results from different parts of the theory. Most of the results should be proven which limits the possible topics severely. At the same time the methods should be of wider interest and so we study e.g. supercommutative Hopf algebras, Deligne categories, techniques from quiver theory, highest weight categories,... The four first talks are basic, and afterwards we start skipping through the theory.

Talk 1: Super structures and Lie superalgebras (M.R.)

The aim of this talk is to introduce the 'super' language and then discuss the simple Lie superalgebras (over $\mathbb{C}$). These get grouped into different families: *classical* (which gets again divided into *basic* and *strange*) and *of Cartan type*.

- Discuss the category of super vector spaces. Tensor products, duals. Endomorphisms. It becomes an abelian category when we only allow parity preserving morphisms.
- Define superalgebras, the example of the exterior algebra.
- Define Lie superalgebras. Its even part is a Lie algebra. Its odd part is a module under the even part. The associated universal enveloping algebra (no PBW theorem required here).

- As an example, endomorphisms of a super vector space with the super commutator.
- Define a representation of a Lie superalgebra (on a super vector space).
- Define simple, semisimple Lie superalgebras.
- The Killing form. Distinguish between simple Lie superalgebras which have a non-degenerate Killing form and the rest (the first ones are called *classical*).
- Write down all simple finite dimensional Lie superalgebras over $\mathbb{C}$. Discuss in particular the ones coming from some supersymmetric even/odd bilinear form. Derive a matrix realization for $\mathfrak{osp}$ and $\mathfrak{q}(n)$. Construct $D(2,1,\alpha)$. The Cartan type algebras can be treated superficially.
- Introduce Kac's terminology: Classical Lie superalgebras, Basic classical Lie superalgebras, strange. Also type 1 Lie superalgebras and type 2 Lie superalgebras.
- Warning: Semisimple $\neq$ direct sum of simple.

Literature: [CW12a, Chapter 1.1], [Mus12, Chapter 1, 2.1 - 2.4, 4], [Sch79, §4, §6, §7], [Kac78], [Ser17, Sections 1 - 3.1].

Talk 2: Simple Lie superalgebras and root systems (K.K.Z.)

We assume now that $\mathfrak{g}$ is classical. The aim is to discuss the root space decomposition and the systems of roots that arise for the simple classical Lie superalgebras. In the classical case the choice of a Borel subalgebra or, equivalent, a choice of simple roots, does not matter a lot. This is false for Lie superalgebras. One can also associate some form of Dynkin diagram to the simple roots, but the classification of the simple Lie superalgebras does not use these diagrams.

- Cartan subalgebras
- Root space decomposition (see [CW12a, Theorem 1.18], [Kac77, Section 2.5], [Ser17]).
- Even and odd roots, simple roots, bases, isotropic roots
- Different types of root spaces $\mathfrak{g}_\alpha$ (leading to black, white or grey dots in the Dynkin-Kac diagram)
- Simple roots and the Dynkin-Kac diagram [Mus12, Section 3.4].
- Diskuss the above roots for $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, $\mathfrak{p}(n)$ (see [CW12a, Chapter 1.2]).
- Warning: Not all Borel subalgebras are conjugate. Discuss this shortly for $\mathfrak{gl}(m|n)$ following [Mus12,]. Show that this leads to different Dynkin-Kac diagrams as e.g. in [Mus12, Table 3.4.2].
- Introduce the distinguished Borel [Mus12, Section 3.1, 3.4.5]. Show the associated simple roots and the Dynkin diagram.

Literature: [Ser17, Section 3.2], [CW12a, Chapter 1.2 - 1.4], [Mus12, Chapter 3.1 - 3.6, 8.1], [Sch79, §3], [Kac78].

Talk 3: Irreducible representations and highest weight theory I (A.O.)

The aim of the next two talks is the discussion of finite dimensional irreducible representations of a classical Lie superalgebra. Basically the highest weight theory carries over and the irreducible representations of $\mathfrak{g}$ can be obtained by from the ones of the even subalgebra $\mathfrak{g}_0$. However not every representation is a direct sum of irreducible representations.

- The universal enveloping algebra and the PBW theorem (shortly - the proof is essentially like for ordinary Lie algebras) (see [Mus12, Chapter 6.1] for details)
- Define induced representations and show some standard properties.
- Define highest weight representations. Introduce Verma modules.
- Prove: Every irreducible representation of a classical Lie superalgebra is a highest weight representation. This is essentially [CW12a, Chapter 1.5], leaving the $\mathfrak{p}(n)$-case open.
- Prove Proposition 2.1 and 2.2 in [CW12a] (the $\mathfrak{gl}$-case).
- Warning: The Kac module is always finite dimensional, but not necessarily irreducible or semisimple.
- Discuss without proofs the $\mathfrak{osp}$-case [CW12a, Chapter 2.1.4, 2.1.5].
- Warning: The given dominant integral weights depend heavily on the chosen Borel subalgebra.

Literature: [Ser17, Section 4], [CW12a, Chapter 1.4, 2.1, 2.2], [Mus12, Chapter 14], [Kac78].

Talk 4: Irreducible representations and highest weight theory II (T.H.)

Discuss atypical weights, Harish-Chandra isomorphism, failure of semisimplicity, character formula for typical modules

- TO DO [I will give this talk, so I might not flesh out its contents before later]

Talk 5: Algebraic supergroups and supercommutative Hopf algebras I (T.B.)

The main aim of the next two talks is to clarify the correspondence between representations of an algebraic supergroup and representations of its Lie superalgebra. We first start with some general background about algebraic groups and supercommutative Hopf algebras. The notions and arguments are almost the same as for algebraic groups and commutative Hopf algebras. Sources for talks 5 and 6 are [Mas13] [Mas12] [MS17] [Wei09] [Wes09].

- Define the notion of an algebraic supergroup scheme and an algebraic supergroup as a representable functor as e.g. in [Wes09] or [Mas13, Section 4]. The associated super Hopf algebra. Explain the latter notion. Give some examples, e.g. $GL(m|n)$ and $OSp(m|2n)$.

- Almost by definition one has a contravariant equivalence between the categories of algebraic supergroup schemes and supercommutative Hopf algebras.
- Introduce the categories $Rep(G)$ and $Rep(G, \epsilon)$, the finite dimensional representations of a supergroup scheme G and its subcategory $Rep(G, \epsilon)$ [Del02].
- These can be alternatively described as categories of comodules over the supercommutative Hopf algebra $k[G]$.
- All this is parallel to the classical case. The definitions and constructions should be precise, but proofs are not necessary, and this part should not take too much time.
- Cite the following theorem [Wei09, Theorem 6].
- Cite [Del02, Proposition 0.5, Theorem 0.6]. For this the notion of a Schur functor and of exponential growth need to be defined in a tensor category.
- We now study super Harish Chandra pairs. An overview of the theory is in [Mas13, Section 6], but the proofs are in [Mas12] [MS17]. Define the notion of a super Harish-Chandra pair and prove Theorem 6.5 in [Mas13] (see also [Ser11, Theorem 3.1]. For the proof see [Mas12, Section 4]. It won't be possible to finish the proof in this talk, so some preparatory lemmas should be proven

Talk 6: Algebraic supergroups and supercommutative Hopf algebras II (M.A.K.)

- Continue with the proof of Theorem 6.5 in [Mas13]. Note that many of the remarks etc in [Mas12, Section 4] are not needed for the proof.
- We then want to prove that there is an equivalence of categories between the categories of $(\mathfrak{g}, G_0)$-modules and G-modules (see for instance [ES17, Section 1.2, Proposition 1.3] for an account without proof). For a proof see [Gav16, Section 7] (or [Wei09, Theorem 4, Corollary 6] for a reduction to the differentiable case).

Talk 7: Schur-Weyl duality for $\mathfrak{gl}(m|n)$ (J.A.)

From now on we restrict to the case $\mathfrak{gl}(m|n)$. One of the basic results on representations of $GL(n)$ is Schur-Weyl duality. Here one considers the natural representation $V = \mathbb{C}^n$ of $GL(n)$ and views its tensor powers $V^{\otimes r}$ simultaneously as a module under the symmetric group. SW duality then consists of the usual double centralizer theorem and a description of the decomposition of $V^{\otimes r}$ as a $GL(n) \times S_r$-bimodule.

The same question can be asked for $\mathfrak{gl}(m|n)$ and its natural representation $V = \mathbb{C}^{m|n}$. The decomposition of $V^{\otimes r}$ as a $GL(m|n) \times S_r$ module was given in [BR87] [Ser85]. Remarkably $V^{\otimes r}$ is still completely reducible. In talks 7 and 8 we describe this decomposition and prove character formulas for the irreducible constituents in terms of supersymmetric polynomials.

The main source for these talks is [CW12a]. Alternative sources are [Mus12, Chapter 11, 12] and the original articles.

- Define the action of S_d [CW12a, Lemma 3.9]
- Prove [CW12a, Theorem 3.10]. You will need a version of Proposition 3.5, but note that all $\Psi_d(S_d)$ modules are of type M and so the proposition can be proven directly without using too much of section 3.1 in loc cit. Section 3.1.4 should be ignored completely.
- Prove [CW12a, Theorem 3.11].
- If time permits, include Remark 3.14 in loc. cit. or include an example decomposition.

Talk 7: Schur-Weyl duality for $\mathfrak{gl}(m|n)$ II (B.L.)

- Prove [CW12a, Theorem 3.15, Theorem 3.16, Corollary 3.17]. This will need a lot of background from appendix A, e.g. A.26, A.37,...., so the main bulk of the talk is contained in the appendix.
- Compute one example for a small weight.

Talk 9: Mixed tensor powers and Deligne's category $Rep(GL_t)$ (Z.W.)

An obvious question is whether one can extend Schur-Weyl duality to the mixed tensor powers $V^{\otimes r} \otimes (V^{\vee})^{\otimes s}$ for $r, s \in \mathbb{N}$. The correct replacement for the group ring of the symmetric group is the *walled Brauer algebra*. The mixed tensor space is however not completely reducible. The best way to understand the decomposition of this mixed tensor space is by building a symmetric monoidal category $Rep(GL_t)$ which incorporates all these walled Brauer algebras as endomorphism spaces. We will study this in talk 9 and 10. Contrary to previous talks some proofs will have to be skipped. All the main results are contained in the article [CW12b].

- Consider the example $V \otimes V^{\vee}$ where V is the standard representation of $GL(n|n)$ or $\mathfrak{gl}(n|n)$. Show that this is indecomposable, but not irreducible. Hence *mixed tensor powers* are not completely reducible.
- Define (walled) Brauer diagrams and the walled Brauer algebra (dependent on a parameter t).
- Explain as in [CW12b, Section 4.2] the connection to the symmetric group algebra.
- Define the skeletal Deligne category $Rep_0(GL_t)$ as in [CW12b, Section 3.2]. It is a monoidal category (maybe a brief reminder about some monoidal terminology is helpful).
- Define idempotent completion and that monoidal structures extend to this completion.
- Define $Rep(GL_t)$.
- Recall the notion of the dimension of an object in a tensor category. Prove [CW12b, Proposition 3.4.1] and [CW12b, Proposition 3.5.1].

- Conclude that there exists a monoidal functor $F_{m|n} : Rep(GL_{m-n}) \to Rep(GL(m|n))$ sending the distinguished object$\bullet$ to the natural representation V of $GL(m|n)$).
- This functor induces a map on morphism spaces. In particular (r, s) maps to $End(V^{\otimes r} \otimes (V^{\vee})^{\otimes s})$. That means one way to understand the latter is to understand (r, s) and then the kernel of the induced map.

Talk 10: Mixed tensor powers and Deligne's category $Rep(GL_t)$ II (M.B.)

- We first classify the indecomposable objects in $Rep(GL_t)$ as in [CW12b, Section 4].
- State the correspondence between conjugacy classes of primitive idempotents in an algebra A and isomorphism classes of simple A-modules
- Use the correspondence to prove [CW12b, Theorem 4.5.1]. The classification of simple modules of the walled Brauer algebra should be used as a blackbox.
- Prove [CW12b, Theorem 4.6.2].
- We now want to understand the functors $F_{m|n}$ from the previous talk. We first work in a general setting. Prove Proposition 2.7.4 and Theorem 4.7.1 in [CW12b]. Then skip forward to section 8.3 and obtain as a corollary that $F_{m|n}$ is full. In particular Theorem 4.7.1 gives the decomposition of mixed tensor powers provided we can a) do this in $Rep(GL_t)$ and b) can work out the kernel of $F_{m|n}$.
- Time permitting an overview of the general case could be given (i.e. Theorem 1.2.3).

Talk 11: Ext-quivers and relations (T.P.)

In the next three talks we follow the article [Ger98] where he obtains a description of the *singly atypical blocks* in the $\mathfrak{gl}$-case. The first talk 11 is purely formal and will be independent of the previous talks. The aim of this talk is to show that any *nice* category in the sense of Germoni [Ger98] can be a described by a quiver with relations.

- We need [Ger98, Section 1], most notably Theorem 1.4.1.

Talk 12: Representations of $\mathfrak{gl}(m|n)$ and highest weight categories (J.S.)

- We first prove some basic properties about $Rep(\mathfrak{g})$ as in [Ser11, Section 9.1]. We don't need Theorem 9.2 and the remark at the end of the section. Lemma 9.1 and 9.4 are essentially also in [Ger98].
- Recall some background about Kac modules from [Ger98, Section 3.3, 3.4], in particular Theorem 3.4.1 and Lemma 3.4.2
- Good filtrations as in [Ger98, Section 3.6]

- The notion of a highest weight category [CPS88]. Give some examples (without proof).
- $Rep(\mathfrak{gl}(m|n))$ is a highest weight category, see [BKN09, Section 3.4, 3.5] in which the Kac modules are the cell or standard modules.

Talk 13: Singly atypical blocks and their indecomposable modules (J.N.)

- The talk is essentially contained in [Ger98, Section 5]. The difficult part is Proposition 5.1.1.
- Start with a short reminder about blocks, what is a singly atypical block etc.
- Proposition 5.1.1 should be (partially) proven [VHKT90, Theorem 4.3]. It rests on properties of Kac modules established in talk 4 and 12. Due to the length and technicality of the arguments, Lemma 4.1 and 4.2 in [VHKT90] have to be probably used as blackboxes.
- Briefly explain how the indecomposable modules look like. This can be found in [GQS07]. Essentially there are the simples, their projective covers of length 4, they have filtrations with 2 Kac modules or 2 dual Kac modules (the twisted dual) and there are the zigzag modules which are attached to intervals on the numberline as well as their twisted duals.
- State as a theorem without proof: If Γ is a block of atypicality 2 (e.g. the principal block of $GL(n|n)$, $n \geq 2$), the representation type of the block is wild.

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