Exc. 1: i) Let R be a comm. ring, p_i, $i\le m$ prime ideals and I an ideal.

Let $I \subseteq \bigcup_{i=1}^{m} p_i$ and wlog let $I \nsubseteq \bigcup_{i\neq j} p_i$ for all $1\le j\le n$.

(otherwise reduce the number of p_i and repeat).

So take for each j an $x_j \in I \setminus \bigcup_{i\neq j} p_i$.

Consider $x_1 + \prod_{j=2}^{m} x_j \in I$. If $x_1 + \prod_{j=2}^{m} x_j \in p_1 \underset{x_1\in p_1}{\Rightarrow} \prod_{j=2}^{m} x_j \in p_1 \Rightarrow \exists 2\le j\le m: x_j \in p_1$ ↯.

$\Rightarrow \exists 2\le k\le m: x_1 + \underbrace{\prod_{j=2}^{m} x_j}_{\in p_k} \in p_k \Rightarrow x_1 \in p_k$ ↯. $\Rightarrow$ Claim. and $\exists k\le m: I\subseteq p_k$.

ii) Let I_i, $i\le m$ be any ideals in R and p a prime ideal.

Let $\bigcap_{i=1}^{m} I_i \subseteq p$, but $I_i \nsubseteq p$, $\forall i$. Let $x_i \in I_i \setminus p$, $\forall i$.

$\underset{\Rightarrow}{p \text{ prime}} \prod_{i=1}^{n} x_i \notin p$, but $\prod_{i=1}^{n} x_i \in \bigcap_{i=1}^{m} I_i \subseteq p$ ↯. $\Rightarrow \exists k\le m: I_k \subseteq p$.

iii) Let I_i, $i\le m$ any ideals in R, p prime.

Let $\bigcap_{i=1}^{m} I_i = p \underset{ii)}{\Rightarrow} \exists k\le m: I_k \subseteq p$. If $I_k \subsetneq p \Rightarrow \bigcap_{i=1}^{m} I_i \subsetneq p$ ↯.

$\Rightarrow I_k = p$.

iv) $R = k[X,Y]$ for a field k, $I=(X,Y)$, $J = \{(f): f \text{ irreducible}\} \subseteq \operatorname{Spec} R$.

Let $p\in I \Rightarrow p$ does not have a constant term, so it is nonconstant and not a unit in $k[X,Y] \underset{R \text{ a UFD}}{\Rightarrow} \exists \varepsilon \in k[X,Y]^\times, f_1,\dots,f_n$ irred. with $p = \varepsilon \prod_{i=1}^{n} f_i$.

In particular $p\in (f_1)$, so $I \subseteq \bigcup_{q\in J} q$, but if $I\subseteq (f)$ for f irred, so $x = af$, $y = bf$ for some $a,b\in k[X,Y]$.

$\underset{\text{irred.}}{\overset{X,Y}{\Rightarrow}} a,b\in \underset{k}{k[X,Y]^\times}$, so $a^{-1}x = f = b^{-1}y$ ↯ to R a UFD.

Exc. 2: i) $\Rightarrow$ ii) R a Jacobson ring and B a field, finitely gen. as an R-algebra.

Thus we have a ring morphism of finite type $R \xrightarrow{f} B$.

$\underset{\text{HNS}}{\overset{\text{Generalized}}{\Rightarrow}}$ The preimage of the maximal ideal $(0)\subset B$, which is the kernel of f is again maximal and $R/\ker f \hookrightarrow B$ is a finite extension of fields.

In particular B is a finitely gen. $R/\ker f$-module, so it is a finitely gen. R-module.

ii) ⇒ iii) R a ring satisfying property ii), let $p \in \operatorname{Spec} R \setminus \operatorname{Specm} R$ with $p \subsetneq \bigcap_{\substack{q \in \operatorname{Spec} R \\ p \subsetneq q}} q$.

By quotienting out p, we obtain an integral domain R/p with

$(0) \subsetneq \bigcap_{\substack{q \in \operatorname{Spec} R/p \\ (0) \subsetneq q}} q$ which is not a field.

Let $a \in \bigcap_{\substack{q \in \operatorname{Spec} R/p \\ (0) \subsetneq q}} q \setminus \{0\}$. Then $(R/p)_a \neq 0$ since $a \neq 0$ in an integral domain,

so it is a nonzerodivisor. Also the prime ideals in $(R/p)_a$ correspond to

those in R/p which do not contain a, which is only one, the zero ideal by construction of a.

Also $(R/p)_a$ is a domain, as a nonzero localization of a domain.

Thus it is a field, since $|\operatorname{Spec}(R/p)_a| = 1$ and (0) is prime.

So it is (by ii)) (since it is isomorphic to $(R/p)[t]/(ta-1)$, in part. fin. gen. as an ~~[illegible]~~ R-algebra)

a finitely generated R-module, so it also is a fin. gen. R/p-module.

But we also have $\underbrace{\operatorname{Quot}((R/p)_a)}_{\substack{(R/p)_a \\ \text{is a field}}} \cong \operatorname{Quot}(R/p)$, a fin. gen. R/p-module.

$\overset{\text{sheet 3}}{\underset{\text{Ex 2}}{\Longrightarrow}}$ R/p ~~is~~ is a field. $\Rightarrow p \in \operatorname{Specm} R$ ↯.

iii) ⇒ iv) Let R be a ring fulfilling property iii).

Since for each ring homomorphism $f: R \to R'$ the image $f(R)$ is isomorphic to $R/\ker f$,

it is enough to show iv) for all quotients of R. ~~By further quotienting out the nilradical of R/I, we can assume R/I to be reduced. So let R/I be reduced with $\operatorname{Jac}(R/I) \neq 0$. Let $f \in \operatorname{Jac}(R/I) \setminus \{0\}$, since R/I is reduced, f~~

I first wanted to reduce to R/I reduced but this is not needed, proceed as follows:

Let $I \subset R$ an ideal with $\operatorname{Nil}(R/I) \subsetneq \operatorname{Jac}(R/I)$, let $\bar{x} \in \operatorname{Jac}(R/I) \setminus \operatorname{Nil}(R/I)$ for some $x \in R$.

Since $\bar{x} \notin \operatorname{Nil}(R/I)$, $(R/I)_{\bar{x}} \neq 0$. Let $m \in \operatorname{Specm}(R/I)_{\bar{x}}$.

The pullback $m \cap R/I$ is a prime ideal in R/I which is maximal under those <u>not</u>

containing $\bar{x}$. (by the prime correspondence for localisations), so it is not maximal, since $\bar{x} \in \operatorname{Jac}(R/I)$.

Ex 2: iii) ⇒ iv) The pullback $m \cap R/I \cap I = p$ is a prime ideal in R maximal under those
<u>Cont.</u>

lying over I and not containing x. Also $p \notin \operatorname{Specm} R$, since $m \cap R/I \notin \operatorname{Specm} R/I$.

Now we have $p = \bigcap_{\substack{p \subsetneq q \\ q \in \operatorname{Spec} R}} q$, the left side does not contain x.

So there is a $q \in \operatorname{Spec} R$, $p \subsetneq q$ also not containing x ↯ to the maximality of p.

$\Rightarrow \operatorname{Nil}(R/I) = \operatorname{Jac}(R/I)$.

iv) ⇒ i) Let $I \subseteq R$ be an ~~[illegible]~~ ideal, R a ring satisfying iv), $R \twoheadrightarrow R/I$.

Then $\operatorname{Nil}(R/I) = \operatorname{Jac}(R/I)$, so $\underbrace{\bigcap_{\substack{q \in \operatorname{Spec} R \\ I \subseteq q}} q}_{= \sqrt{I}} = R \cap \operatorname{Nil}(R/I) = R \cap \operatorname{Jac}(R/I) = \bigcap_{\substack{q \in \operatorname{Max} R \\ I \subseteq q}} q$

<u>Solution 1:</u> B a field which is a fin. gen. $\mathbb{Z}$-algebra $\overset{\mathbb{Z}\text{ Jac.}}{\underset{\text{by characterisation iii)}}{\Longrightarrow}}$ ~~[illegible]~~.

and by Generalised HNS the pullback of (0) under $\mathbb{Z} \to B$ is a maximal ideal $p\mathbb{Z}$,

and $\underbrace{\mathbb{Z}/p\mathbb{Z}}_{= \mathbb{F}_p} \hookrightarrow B$ is a finite field extension, so B is a finite field.

<u>Solution 2:</u> B a field which is a fin. gen. $\mathbb{Z}$-algebra $\overset{\mathbb{Z}\text{ Jac}}{\Longrightarrow}$ B is a fin. gen. $\mathbb{Z}$-module.

B contains $\mathbb{F}_p$ for some p or $\mathbb{Q}$. Since submodules of fin. gen. $\mathbb{Z}$-modules are fin. gen.

($\mathbb{Z}$ is Noetherian), but $\mathbb{Q}$ is not a fin. gen. $\mathbb{Z}$-mod. as it is the quotient field of $\mathbb{Z}$, but

$\mathbb{Z}$ is not a field. Thus B has characteristic p for $p > 0$ and $\mathbb{Z} \to B$ factors through $\mathbb{F}_p$

as the inclusion of $\mathbb{F}_p$ into B.

Now as B is a fin. gen. $\mathbb{Z}$-module and $\exists\ \mathbb{Z}^n \twoheadrightarrow B$ surjective map for some n, which

again factors surjectively as $\mathbb{F}_p^n \twoheadrightarrow B$, so $\mathbb{F}_p \hookrightarrow B$ is a finite field extension and B is

a finite field.

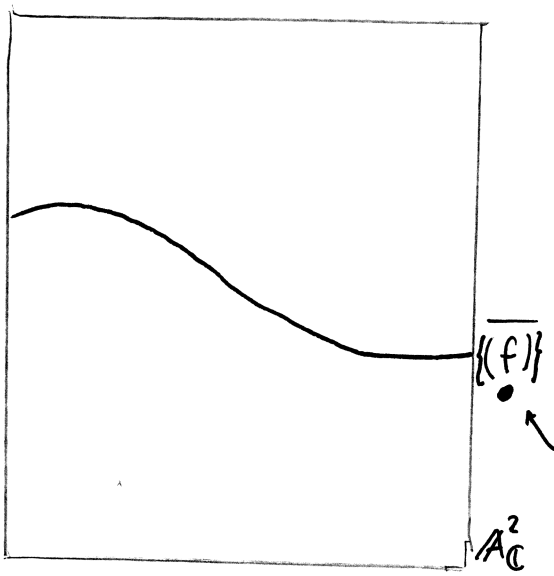
Exc. 3: We have three kinds of prime ideals in $\mathbb{C}[X,Y]$:

Those of height zero, height one and height two.

Height zero prime ideals are minimal prime ideals which are only (0) in an integral domain.

Prime ideals of height two are exactly maximal ideals which are by HNS ($\mathbb{C}=\overline{\mathbb{C}}$!) just ideals of the form $(X-a, Y-b)$, $a,b\in\mathbb{C}$.

Height one prime ideals ~~[illegible]~~ are the principal ideals which are generated by an irreducible polynomial in two variables.



The closure of (f) can be identified by identifying $\operatorname{Max}\mathbb{C}[X,Y]\subseteq\operatorname{Spec}\mathbb{C}[X,Y]$ with $\mathbb{A}^2_{\mathbb{C}}$ with an irreducible variety $V(f)$ of dimension one in $\mathbb{A}^2_{\mathbb{C}}$ (a "curve").

This is the additional point associated to (f).

Rmk: This is not entirely true, as not all of $\overline{\{(f)\}}$ lies in $\operatorname{Max}\mathbb{C}[X,Y]$; everything does except for (f) itself. So additionally to the closed points of the closure we get an additional nonclosed point in (f), which's closure is (of course) $\overline{\{(f)\}}$.

Exc. 4: These varieties are irreducible iff the polynomial p is irreducible. (using $k[V(p)]\cong k[x,y]/p$ and $V(p)$ being irred. $\Leftrightarrow$ $k[V(p)]$ is an integral domain)
($V(p)$, $p\in k[x,y]$)

i) $xy^3 = x\cdot y^3$ not irreducible.

ii) We consider $x^2+xy+y^3\in(k[y])[x]$ and factor it into two non-unit polynomials p,q (Want to get a ↯).

Case 1: $\deg p=2$, $\deg q=0$, so $p=p_2x^2+p_1x+p_0$, $q=q_0$ with $q_0,p_0,p_1,p_2\in k[y]$, $q_0\notin k[y]^\times$

$\Rightarrow pq=x^2+xy+y^3$, so $p_2q_0=1$ in $k[y]$ by comparing coefficients. $\Rightarrow q_0\in k[y]^\times$ ↯.

Case 2: $\deg p=\deg q=1$, $p=p_1x+p_0$, $q=q_1x+q_0$, $q_0,q_1,p_0,p_1\in k[y]$.

$\Rightarrow pq=x^2+xy+y^3$, so $p_0q_0=y^3 \underset{\text{UFD}}{\overset{k[y]}{\Longrightarrow}} p_0=\varepsilon y^{\ell}$, $q_0=\varepsilon' y^{\ell'}$ for $\varepsilon,\varepsilon'\in k[y]^\times=k$, $\ell,\ell'\in\mathbb{N}$.

So $\underbrace{p_0q_0}_{=y^3}=\varepsilon\varepsilon' y^{\ell+\ell'}$ $\Rightarrow \varepsilon'=\varepsilon^{-1}$, $\ell'=3-\ell$ $\rightsquigarrow p_0=\varepsilon y^{\ell}$, $q_0=\varepsilon^{-1}y^{3-\ell}$, also $p_1q_1=1\Rightarrow p_1=\alpha$, $q_1=\alpha^{-1}$, $\alpha\in \underbrace{k[y]^\times}_{=k}$

$xy=x(p_1q_0+p_0q_1)=x(\alpha\varepsilon^{-1}y^{3-\ell}+\alpha^{-1}\varepsilon y^{\ell}) \underset{\text{UFD}}{\overset{k[x,y]}{\Longrightarrow}} y=\alpha\varepsilon^{-1}y^{3-\ell}+\alpha^{-1}\varepsilon y^{\ell}$, $3-\ell\neq\ell$ so ~~[illegible]~~ $\{y^{\ell},y^{3-\ell},y\}$ lin. indep. ↯.

k alg. closed

Exc. 5: First we need the following fact: $m \geq n$: $\mathbb{A}^n_k \xrightarrow{\varphi} \mathbb{A}^m_k$ induced by $k[t_1,\dots,t_m] \xrightarrow{\varphi^\#} k[x_1,\dots,x_n]$
$(a_1,\dots,a_n) \longmapsto (a_1,\dots,a_n,\lambda_{n+1},\dots,\lambda_m)$, $t_i \longmapsto x_i$, $i \leq n$

is continuous (in the Zariski top.) for fixed $\lambda_{n+1},\dots,\lambda_m \in k$, $t_i \longmapsto \lambda_i$, $i > n$.

(This holds more generally for any two varieties $X \subseteq \mathbb{A}^k$, $Y \subseteq \mathbb{A}^j$ and for a regular map $f: X \to Y$ between them.

$\ulcorner$ Let $I \subseteq k[t_1,\dots,t_m]$ an ideal. Then $\varphi^{-1}(V(I)) = V(\varphi^\#(I))$, where $\varphi^\#(I)$ is an ideal in $k[x_1,\dots,x_n]$ by surjectivity of $\varphi^\#$. ($\varphi^\#(f)(a_1,\dots,a_n) = f(\varphi(a_1,\dots,a_n))$ for $f \in k[t_1,\dots,t_m]$.)
Thus preimages of closed sets are closed and φ is continuous. $\lrcorner$

i) Suppose $\Gamma_{\sin} = \{(x,\sin(x)) : x \in \mathbb{C}\} \subseteq \mathbb{A}^2_\mathbb{C}$ closed and consider $\varphi: \mathbb{A}^1_\mathbb{C} \to \mathbb{A}^2_\mathbb{C}$, $x \longmapsto (x,0)$ as above.

$\Rightarrow \varphi^{-1}(\Gamma_{\sin}) = \{x \in \mathbb{C} : \sin x = 0\} \subseteq \mathbb{A}^1_\mathbb{C}$ is closed.
$\{x \in \mathbb{C} : \sin x = 0\} = 2\pi\mathbb{Z}$

But this is an infinite closed set in $\mathbb{A}^1_\mathbb{C}$, $\mathbb{C}$ alg. closed ↯.

(You can also do this directly (it is essentially the same argument) by letting $\Gamma_{\sin} = V(I)$, taking $f \in I \subseteq \mathbb{C}[X,Y]$ and considering $f(x,0) \in \mathbb{C}[X]$ with infinite zeroes).

ii) Similarly to i) suppose the set Z of $n \times n$-matrices with rank k for $0 < k \leq n$ were closed in $\mathbb{A}^{n^2}_k$. The following set is clearly closed in $\mathbb{A}^{n^2}_k$ (identifying it with $n \times n$-matrices)

$A = \left\{ \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & \\ 0 & & & & 0 \end{pmatrix} : \lambda \in k \text{ with } k-1 \leq n \text{ ones on the diagonal} \right\}$ as it is the zero set of

$M = \{ t_{ij} : i \neq j,\ t_{ii} : i > k,\ t_{ii} - 1 : i < k \} \subseteq k[t_{11},\dots,t_{nn}]$.

Thus $A \cap Z \subseteq \mathbb{A}^{n^2}_k$ is closed, but these are just all matrices in A with $\lambda \neq 0$ as

$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & \\ 0 & & & & 0 \end{pmatrix}$ has rank $k \Leftrightarrow \lambda \in k^\times = k \setminus \{0\}$.

Now embed $\mathbb{A}^1_k \xrightarrow{\varphi} \mathbb{A}^{n^2}_k$, $x \longmapsto \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & x & \\ 0 & & & & 0 \end{pmatrix}$, this is a map as in the fact, so $\varphi^{-1}(A \cap Z) \subseteq \mathbb{A}^1_k$ is closed.
$\varphi^{-1}(A \cap Z) = \mathbb{A}^1_k \setminus \{0\}$.

This is infinite, since $k = \bar{k}$ so it cannot be closed ↯.

iii) We know from linear algebra that an $n\times n$-matrix A is nilpotent iff $A^n=0$.

But we can, by expanding $A^n=(b_{ij})_{i,j\leq n}$ for each b_{ij} in the entries a_{ij} of A polynomially, express each b_{ij} as a polynomial $p_{ij}(a_{11},\dots,a_{nn})$.

Thus the set of nilpotent matrices is exactly $V(p_{11},\dots,p_{nn}) \subseteq \mathbb{A}_k^{n^2}$.