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Exercises for the lecture Algebra 1
—Exercise sheet 9—

Exercise 1 (15 points). (Regularity) Let R be a local noetherian ring with $\dim R = d$. Let m be the maximal ideal of R and let $k = R/m$. Show:

(i) $d \leq \dim_k m/m^2$.

(ii) We have $d = \dim_k m/m^2$ if and only if m is generated by d elements. In this case we call R regular.

Let k be algebraically closed and $f \in k[x_1, \ldots, x_n]$ a product of pairwise distinct irreducible polynomials. A point $P \in V(f)$ is called non-singular if there exists $i \in \{1, \ldots, n\}$ with $\frac{\partial f}{\partial x_i}(P) \neq 0$ where $\frac{\partial f}{\partial x_i}$ is the formal derivative of f with respect to x_i. Show:

(iii) $P \in Z(f)$ is non-singular if and only if the local ring $(k[x_1, \ldots, x_n]/(f))_m$ is regular where m is the maximal ideal corresponding to P.

Exercise 2 (10 points). (Height and small dimension) Let R be a ring and $p \in \operatorname{Spec} R$. Recall: The height of p is given by $\operatorname{ht}(p) = \dim R_p$. Show:

(i) If R is noetherian and $q \in \operatorname{Spec} R$ mit $\operatorname{ht}(q) \geq 2$, then q is a union of infinitely many distinct prime ideals $p \in \operatorname{Spec} R$ with $\operatorname{ht}(p) = 1$.

(ii) If $\operatorname{Spec}(R)$ is finite and R noetherian, then $\dim R \leq 1$.

(iii) There exists a ring R with $\dim(R) = 0$ and a noetherian ring R' with $\dim(R') = 1$ such that $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R')$ are infinite.

Due data: Thursday, 06.06.2019, around 2pm before the lecture.