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Exercises for the lecture Algebra 1
—Exercise sheet 8—

Exercise 1 (5 Punkte). (Products) Let k be an algebraically closed field. Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Show:

(i) $X \times Y \subseteq \mathbb{A}^{m+n}$ is an affine variety with the topology restricted from $\mathbb{A}^{m+n}$. We have $\mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$, i.e. there are maps $f : \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^{m+n}$ and $g : \mathbb{A}^{m+n} \to \mathbb{A}^m \times \mathbb{A}^n$ which are invers to each other.

(ii) The Zariski topology on $X \times Y$ is in general different from the product topology on $X \times Y$.

Exercise 2 (10 Punkte). (Noetherian or not) Let R be a noetherian ring. Which of the following rings are noetherian? Prove or disprove.

(i) The ring of power series in n variables $R[\![x_1, \ldots, x_n]\!]$,

(ii) A subring of R,

(iii) A finitely generated R-algebra,

(iv) The integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$,

(v) The ring of $\mathbb{Z}$-valued, rational polynomials

$$\mathrm{Int}(\mathbb{Z}) := \{\ f \in \mathbb{Q}[x] \mid f(a) \in \mathbb{Z} \text{ für alle } a \in \mathbb{Z}\ \}.$$

Exercise 3 (15 Punkte). (Artinian) Let R be a ring. Show:

(i) If R is artinian, then $Specm(R)$ is finite.

(ii) If R is an artinian domain, then R is a field. In particular every prime ideal in R is maximal.

(iii) If R is a ring such that $(0) = \prod_{i=1}^n m_i$ mit $m_i \in Specm(R)$, then R is artinian if and only if R is noetherian. This holds in particular if $(0) = \bigcap_{i=1}^n m_i$.

(iv) A ring R is artinian if and only if it is noetherian and all prime ideals are maximal.

Exercise 4 (10 Punkte). (Finite fibres) Let R be a ring and A a finitely generated R-algebra which is integral over R. Show that for $p \in Spec(R)$ there are only finitely many prime ideals in A which lie over p (this means geometrically that the induced map between the spectras has finite fibres).

Hint: Localize in order to reduce this problem to the previous exercise.

Exercise 5 (15 Punkte). (Not a noetherian ring) Let $R = C[0,1]$ be the ring of continous functions on the unit interval $[0,1] \subset \mathbb{R}$.

(i) Is this a noetherian ring?

(ii) For each $c \in [0,1]$ define $M_c = \{f \in R \mid f(c) = 0\}$. Show that M_c is a maximal ideal in R.

(iii) An ideal $m \subset R$ is maximal if and only if $m = M_c$ for some $c \in [0,1]$.

Due date: Monday, 03.06.2019, around 2pm before the lecture.