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Sommersemester 2019

Übungen zur Vorlesung Algebra 1
—Blatt 7—

Aufgabe 1 (10 points). („Prime avoidance") Let R be a ring, $p, p_1, \ldots, p_m \in \operatorname{Spec} R$ prime ideals and $I, I_1, \ldots, I_n$ ideals of R. Show:

(i) If $I \subseteq \bigcup_{i=1,\ldots,m} p_i$, then there exists an $i \in \{1, \ldots, m\}$ such that $I \subseteq p_i$.

(ii) If $\bigcap_{i=1,\ldots,m} I_i \subseteq p$, then there exists an $i \in \{1, \ldots, m\}$ such that $I_i \subseteq p$.

(iii) If $\bigcap_{i=1,\ldots,m} I_i = p$, then there exists an $i \in \{1, \ldots, m\}$ such that $I_i = p$.

(iv) For rings R and subsets $J \subseteq \operatorname{Spec} R$ the condition $I \subseteq \bigcup_{p_i \in J} p_i$ implies not necessarily that $I \subseteq p_i$ für ein $p_i \in J$.

Aufgabe 2 (10 points). (Charakterization of Jacobson rings) Let R be a ring. The following are equivalent:

(i) R is a Jacobson ring.

(ii) If B is a field which is finitely generated as an R-algebra, then B is finitely generated as an R-module.

(iii) For $p \in \operatorname{Spec} R$ with $p \notin Specm\ R$

$$p = \bigcap_{q \in \operatorname{Spec} R, q \supsetneq p} q.$$

(iv) For every ring homomorphism $f : R \to R'$ the nilradical of $f(R)$ is equal to the Jacobson radical of $f(R)$.

Conclude that every field, which is a finitely generated ring (i.e. as a $\mathbb{Z}$-algebra), is a finite field.

Aufgabe 3 (5 Punkte). (Polynomial rings) Describe the prime ideals of $\mathbb{C}[X, Y]$. If $p \in Spec\,\mathbb{C}[X, Y]$ is not maximal, how does its closure look geometrically?

Aufgabe 4 (5 Punkte). (Irreducibility) Which of the following varieties are irreducible?

(i) $V(xy^3)$

(ii) $V(x^2 + y^3 + xy)$

Aufgabe 5 (10 Punkte). (Varieties) Let K be algebraically closed. Which of the following subsets of an affine space are varieties?

(i) $\{(x, sin(x)) \mid x \in \mathbb{C}\} \subset \mathbb{A}^2_{\mathbb{C}}$.

(ii) The set of all $n \times n$ matrices of rank k (for $0 \leq k \leq n$) in $\mathbb{A}^{n^2}_K$.

(iii) The set of all nilpotent $n \times n$ matrices in $\mathbb{A}^{n^2}_K$.

Due date: Thursday, 23.05.2019, around 2pm before the lecture.