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Exercises for the lecture Algebra 1
—Exercise sheet 6—

Exercise 1 (10 Punkte). (Going between) Let R ebe a finitely generated K-algebra over a field K. Let $R \subset R'$ be an integral ring extension. Let $p_1 \subsetneq p_3$ be prime ideals in R and $q_1 \subsetneq q_3$ prime ideals in R' with $q_1 \cap R = p_1$ and $q_3 \cap R = p_3$.

(i) Show: If p_2 is a prime ideal in R with $p_1 \subsetneq p_2 \subsetneq p_3$, then there is a prime ideal q_2 in R' with $q_1 \subsetneq q_2 \subsetneq q_3$.

(ii) Can one always find q_2 such that $q_2 \cap R = p_2$ holds?

Exercise 2 (10 Punkte). (Noether Normalization over an infinite field) In the lecture we showed that after the substitution

$$
\begin{aligned}
Y_1 &= X_1 - X_n^{r_1}, \\
Y_2 &= X_2 - X_n^{r_2} \\
&\dots \\
Y_n &= X_n
\end{aligned}
$$

every polynomial $p \in K[X_1, \dots, X_n]$ (K a field) for certain r_i can be written in the form

$$cX_n^m + h_1(Y_1, \dots, Y_{n-1})X_n^{m-1} + \dots + h_m(Y_1, \dots, Y_{n-1}), \quad c \in K^*.$$

Show that if K is an infinite field the linear variable substitution

$$
\begin{aligned}
Y_1 &= X_1 - r_1 X_n, \\
Y_2 &= X_2 - r_2 X_n \\
&\dots \\
Y_n &= X_n
\end{aligned}
$$

already suffices to bring any non-zero $p \in K[X_1, \dots, X_n]$ after appropriate choices of the r_i into the form

$$cX_n^m + h_1(Y_1, \dots, Y_{n-1})X_n^{m-1} + \dots + h_m(Y_1, \dots, Y_{n-1}), \quad c \in K^*.$$

Exercise 3 (10 Punkte). (Explicit Noether normalization)

(i) Let K be a field. According to the Noether Normalization we can find for every finitely generated K-algebra A algebraically independent elements $a_1, \dots, a_n$ such that A is integral over $K[a_1, \dots, a_n]$. Find such elements for the K-algebra $A = K[X_1, X_2, X_3]/(X_1 X_2)$.

(ii) Show that the statement of the Noether normalization cannot hold over $\mathbb{Z}$ by considering the $\mathbb{Z}$-algebra $A = \mathbb{Z}[X_1, X_2]/(2X_1 X_2 - 1)$.

Exercise 4 (10 Punkte). (Hilbert's theorem of zeros) Let k be a field and K an algebraic closure of k. Let $f, f_1, \dots, f_r \in k[X_1, \dots, X_n]$. Wea assume that every simultaneous zero $a \in K^n$ of the f_i

$$f_1(a) = f_2(a) = \dots = f_r(a) = 0$$

is also a zero of f. Show that there are polynomials $g_1, \dots, g_r \in k[X_1, \dots, X_n]$ and an $N \in \mathbb{N}$ such that

$$f^N = g_1 f_1 + g_2 f_2 + \dots g_r f_r.$$

Due date: Thursday, 16.05.2019, around 2pm before the lecture.