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Exercises for the lecture Algebra 1
—Exercise sheet 5—

Exercise 1 (10 points). (Integral closure and localization) Let $R \subset R'$ be a ring extension and $\overline{R}$ the integral closure of R in R'. Let S be a multiplicative closed subset of R. Then $S^{-1}\overline{R}$ is the integral closure of $S^{-1}R$ in $S^{-1}R'$.

Exercise 2 (15 points). (Geometric interpretation of Going Down und Going Down) Let $\varphi : R \subset R'$ be a ring extension. a) The following are equivalent:

(i) φ satisfies Going Up.

(ii) For all prime ideals $q \subset R'$ and $p = q \cap R$ is the map $Spec(R'/q) \to Spec(R/p)$ surjective.

(iii) $Spec(\varphi)$ is closed, i.e. images of closed sets are closed.

b) The following are equivalent:

(i) φ satisfies Going Down.

(ii) For all prime ideals $q \subset R'$ and $p = q \cap R$ is the map $Spec(R'_q) \to Spec(R_p)$ surjective.

Exercise 3 (10 points). (Technical lemma) Let $\varphi : R \to R'$ be a ring homomorphism, $p \in Spec(R)$. The following are equivalent:

(i) There exists a prime ideal $q \subset R'$ with $p = \varphi^{-1}(q)$.

(ii) $\varphi^{-1}((\varphi(p)R')) = p$.

Exercise 4 (5 points). (Number fields) Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be square free. Show that the integral closure $\overline{R}$ of $\mathbb{Z}$ in the quadratic field extension

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$$

of $\mathbb{Q}$ is

$$\overline{R} = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}, -2a \in \mathbb{Z}, a^2 - db^2 \in \mathbb{Z}\}.$$

Is this always equal to $\mathbb{Z} + \mathbb{Z}\sqrt{d}$? You don't have to compute $\overline{R}$ explicitely.

The student council of mathematics will organize the math party on 9/05 in N8schicht. The presale will be held on Mon 6/05, Tue 7/05 and Wed 8/05 in the mensa Poppelsdorf. Further information can be found at fsmath.uni-bonn.de http://fsmath.uni-bonn.de/

Due date: Thursday, 09.05.2019, 2pm before the lecture.