Thorsten Heidersdorf
Universität Bonn

Summer turn 2019

Exercises for the lecture Algebra 1
—Exercise sheet 4—

For $p \in Spec(R)$ let M_p be the localization of an R-mdoule M at $S = R \setminus p$.

Exercise 1 (5 points). (Local-Global for modules) Let M be an R-module. The following statements are equivalent:

(i) $M = 0$.

(ii) For all $p \in Spec(R)$ is $M_p = 0$.

(iii) For all maximal m is $M_p = 0$.

Exercise 2 (10 points). (Local-Global for morphisms) Let M, N be two R-modules and $f : M \to N$ R-linear. Then the following statements are equivalent:

(i) f is injective respectively surjective.

(ii) For all $p \in Spec(R)$ $f_p : M_p \to N_p$ is injective respektively surjective.

(iii) For all maximal p $f_p : M_p \to N_p$ it injective respektively surjective.

Exercise 3 (20 Punkte). (Annulator and Torsion) For an R-module M let

$$Ann(M) = \{x \in R \mid xm = 0 \forall m \in M\},$$

the *annulator* of M. If $m \in M$, we write $Ann(m)$ für $Ann(Rm)$, where Rm is the submodule of M, which is spanned by m. Then $m \in M$ is called a *torsion element* if $Ann(m) \neq 0$. M is called a *torsion module* if $Ann(m) \neq 0$ for all $m \in M$. M is called *torsion free* if $Ann(m) = 0$ for all $m \in M$. The set of torsion elements is denoted by $T(M)$.

Let R be an integral domain. Show:

(i) $T(M)$ is a submodule of M. Give an example of a ring R' (necessarily not an integral domain) and an R'-module M such that $T(M)$ is not a submodule of M.

(ii) If M is a torsion module, every non-empty subset of M is R-linear dependent.

(iii) $M/T(M)$ is torsion free.

(iv) Let $f : M \to N$ be R-linear. Then $f(T(M)) \subset T(N)$.

(v) If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M''$$

is exact, the sequence

$$0 \longrightarrow T(M') \longrightarrow T(M) \longrightarrow T(M'')$$

is exakt.

(vi) Let $S \subset R$ multiplicative. Then $T(S^{-1}M) = S^{-1}(T(M))$.

(vii) The following are equivalent:

  (a) M is torsion free.
  (b) M_p ist torsion free for all $p \in Spec(R)$.
  (c) M_p is torsion free for all maximal p.

Due date: Thursday, 02.05.2019, 2pm before the lecture.