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Exercises for the lecture Algebra 1
—Exercise sheet 2—

Exercise 1 (5 points). (Idempotents and local rings) Let $e \in R$ be an idempotent element in a local ring R. Then $e \in \{0, 1\}$ (and in particular $\operatorname{Spec} R$ connected).

Exercise 2 (10 points). (Idempotente) Let $\varphi : R \to R'$ be a ring homomorphism and $Idem(R)$ the set of idempotents R. Show:

(i) The ring homomorphism φ induces a map $Idem(\varphi) : Idem(R) \to Idem(R')$.

(ii) Let $I \subseteq R$ an ideal and $q : R \to R/I$ the quotient map. Then $Idem(q)$ is injective if $I \subset JacR$.

(iii) If additionally $I \subset NilR$, then $Idem(q)$ is bijective.

Exercise 3 (10 points). (Local ring at p) Let R be a ring and p a prime ideal. Show that the image of $Spec(R_p)$ in $Spec(R)$ is the intersection of all open neighbourhoods of p in $Spec(R)$.

Exercise 4 (10 points). (Irreducibility) A topological space X is called irreducible if every decomposition $X = X_1 \cup X_2$ in closed subsets X_i implies that either X_1 or X_2 are equal to X.

(i) If $Z \subset X$ is irreducible and $U \subset X$ an open subset with $U \cap Z \neq \emptyset$, then $\overline{U \cap Z} = Z$.

(ii) A closed subset $Z \subset SpecR$ (R Ring) is irreducible if and only if $Z = Z(p)$ for a uniquely determined prime ideal p.

Exercise 5 (10 Punkte). (Density) A subset $S \subset X$ in a topological space X is said to be *dense* if $\overline{S} = X$. Let $\varphi : R \to R'$ be a ring homomorphism and $Spec(\varphi) : SpecR' \to SpecR$ the induced map. Show: The image of $Spec(\varphi)$ is dense in $Spec(R)$ if and only if $ker(\varphi) \subset Nil(R)$.

Due date: Thursday, 18.04.2019, about 2pm before the lecture..