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Exercises for the lecture Algebra 1
—Exercise sheet 12—

Remark: In a tensor product of two R-modules M, N, the R-dependence is sometimes indicated via $M \otimes_R N$. This subscript will be omitted if the underlying ring is clear.

Exercise 1 (15 points). Let M be an R-module and R' an R-algebra. Then $R' \otimes M$ becomes an R'-Modul via

$$a'(a \otimes m) = a'a \otimes m$$

for $a' \in R'$ (the scalar extension of M to R').

(i) If $f : M \to N$ is R-linear,

$$id_{R'} \otimes f : R' \otimes M \to R' \otimes N$$

is R'-linear.

(ii) Let M be an R-module and N an R'-module (which can be considered as an R-module). There exists a unique R'-lineare map

$$f_{R'} : R' \otimes M \to N$$

with $f_{R'}(b \otimes x) = bf(x)$.

(iii) The map $\varphi : Hom_R(M, N) \to Hom_{R'}(R' \otimes M, N)$, $f \mapsto f_{R'}$, is R-linear and bijective.

(iv) Let M be an R-module and N an R'-module. Show: $N \otimes M$ is an R'-Modul via

$$b(\sum y_i \otimes x_i) = \sum by_i \otimes x_i$$

and there exists a unique isomorphism

$$N \otimes_R M \to N \otimes_{R'} (R' \otimes_R M)$$

of R-modules with

$$y \otimes x \mapsto y \otimes (1 \otimes x).$$

(v) Let M be an R-module which has one of the following properties:

(a) free
(b) finitely generated
(c) finitely presented
(d) flat

Then $R' \otimes M$ has the same property. Recall that M is called finitely presented if there is a surjection $R^n \to M$ with $n \in \mathbb{N}$ and finitely generated kernel.

Exercise 2 (10 points). Let $S \subset R$ multiplicative and M an R-module. Then there exists a unique isomorphism of $S^{-1}R$-modules

$$\varphi : S^{-1}R \otimes M \to S^{-1}M, \frac{a}{s} \otimes x \mapsto \frac{ax}{s}$$

Conclude that $S^{-1}R$ is a flat R-module.

Exercise 3 (5 points). Let $\mathbb{Z}[\sqrt{5}i] \subset \mathbb{Q}(\sqrt{5}i)$ be the ring of integers of $\mathbb{Q}(\sqrt{5}i)$. Show: The element 2 cannot be written as a product of prime elements. Determine the (unique) decomposition of the principal ideal (2) as a product of prime ideals.

Exercise 4 (10 points). Let R be a noetherian domain with quotient field K. For all fractional ideals I, J and all multiplicative subsets S of R

(i) $S^{-1}(IJ) = S^{-1}I \cdot S^{-1}J$.

(ii) $S^{-1}(I :_K J) = S^{-1}I :_K S^{-1}J$.

Due date: Thursday, 04.07.2019, around 2pm before the lecture.