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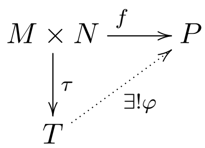
Exercises for the lecture Algebra 1
—Exercise sheet 11—

Exercise sheet 12 (next week) will be the last regular exercise sheet. Exercise sheet 13 will only serve for repitition.

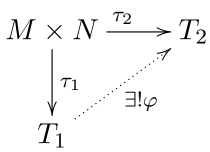
Definition: Let R be a ring and M, N, P three R-modules.

(i) A map $f : M \times N \to P$ is R-bilinear if $f(-, n) : M \to P$ and $f(m, -) : N \to P$ are R-linear for all $m \in M, n \in N$.

(ii) A tensor product of M and N over R is an R-module T together with a bilinear map $\tau : M \times N \to T$, such that for every bilinear map $f : M \times N \to P$ into an R-module P there exists a unique linear map $\varphi : T \to P$ such that $f = \varphi \circ \tau$:



Exercise 1 (15 points). (Tensor products) a) Let $(T_1, \tau_1), (T_2, \tau_2)$ be two tensor products of M and N. Show that there exists a unique R-module isomorphism $\varphi : T_1 \to T_2$ such that $\tau_2 = \varphi \circ \tau_1$:



Therefore one simply talks about *the* tensor product $M \otimes N$ of M and N. The element $\tau(m, n)$ is denoted by $m \otimes n$.

b) Consider the free R-module $R^{M\times N}$ with basis $M \times N$. Its elements are linear combinbations $\sum a_i(x_i, y_i)$ with $a_i \in R$, $x_i \in M$, $y_i \in N$. Let $U \subset R^{M\times N}$ be the submodule which is generated by expressions of the form

$$(ax + a'x', y) - a(x, y) - a'(x', y)$$
$$(x, ay + a'y') - a(x, y) - a'(x', y)$$

for all $a, a' \in R$, $x, x' \in M$, $y, y' \in N$. Define T and τ via $T = R^{M\times N}/U$ and

$$\tau : M \times N \to R^{M\times N}/U, (x, y) \mapsto (x, y) + U.$$

Show that the pair (T, τ) satisfied the universal property of the tensor product.

c) Let $\varphi : M \to N$, $\varphi' : M' \to N'$ be two R-lineare mgaps. Show that there is an induced R-linear map

$$\varphi \otimes \varphi' : M \otimes M' \to N \otimes N'$$

such that $(\varphi \otimes \varphi')(m \otimes m') = \varphi(m) \otimes \varphi'(m')$.

Exercise 2 (10 points). (Tensor products 2) For all R-modules M, N, P exist isomorphisms (independent of the choice of a basis)

(i) $M \otimes N \cong N \otimes M$.

(ii) $M \otimes R \cong M$.

(iii) $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$.

(iv) $M \otimes N \otimes P \cong (M \otimes N) \otimes P \cong M \otimes (N \otimes P)$.

Conclude that for free R-modules M and N with bases $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ the tensor product $M \otimes N$ is free with basis $(x_i \otimes y_j)_{i \in I, j \in J)}$.

Exercise 3 (15 Points). (Exaktness and adjunction) a) For all R-modules M, N, P

$$Hom(M, Hom(N, P)) \cong Hom(M \otimes N, P).$$

b) Let

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0$$

be short exact. For every R-module N the sequence

$$M_1 \otimes N \xrightarrow{\varphi_1 \otimes id} M_2 \otimes N \xrightarrow{\varphi_2 \otimes id} M_3 \otimes N \longrightarrow 0$$

is short exact (i.e. $- \otimes N$ is a right exact functor).

c) Show via an example that $- \otimes N$ is in general not left exact.

d) An R-module N is called flat if $- \otimes N$ is exact. Show: If N is free, then N is flat.

Due date: Monday, 27.06.2019, around 2pm before the lecture.