

Exercises for the lecture Algebra 1

—Exercise sheet 10—

Exercise 1 (10 points). (Minimal prime ideals) Let I be an ideal in a noetherian ring. The isolated prime ideals of I are exactly the minimal prime ideals over I . In particular there are only finitely many minimal prime ideals over a fixed ideal I in a noetherian ring.

Exercise 2 (10 points). (Only one prime ideal) The following are equivalent for a ring R :

- (i) R has exactly one prime ideal.
- (ii) Each element of R is a unit or nilpotent.
- (iii) $\sqrt{(0)}$ is maximal.

Exercise 3 (10 points). (Primary ideals) Let $\varphi : R \rightarrow R'$ be a ring homomorphism between noetherian rings and let I be a p -primary ideal in R' . Show that $\varphi^{-1}(I)$ is a $\varphi^{-1}(p)$ -primary ideal. The converse is true if φ is surjective.

Exercise 4 (10 points). (Primary decomposition)

- (i) Let k be a field. Let $I = \langle XY, X - YZ \rangle$, $I_1 = \langle X, Z \rangle$, $I_2 = \langle Y^2, X - YZ \rangle$ in the polynomial ring $k[X, Y, Z]$. Show that $I = I_1 \cap I_2$ is a minimal primary decomposition.
- (ii) Let R, R' integral domains. Find a minimal primary decomposition of (0) in $R \times R'$.

Due date: Monday, 17.06.2019, around 4pm before the lecture.