## Exercises for the lecture Algebra 1 —Exercise sheet 1—

Recall: All rings in this course are commutative with 1.

**Exercise 1 (10 points).** (Examples for the Zariski topology)

- (i) Determine the points of Spec  $\mathbb{Z}$ . Which ones are closed and which ones are not closed?
- (ii) Let k be an algebraically closed field. Determine the points of  $\operatorname{Spec} k[x]$ . Which ones are closed and which ones are not closed?
- (iii) Let k be an algebraically closed field. Give an example of a non-closed point in Spec k[x, y], whose closure is not the entire space Spec k[x, y].

**Exercise 2 (10 points).** (Quasicompactness) Show: SpecR is quasi-compact, i.e. every open cover admits a finite subcover: If

$$SpecR = \bigcup_{\lambda_i \in \Lambda} U_{\lambda_i}$$

is an open cover, then there are finitely many  $\lambda_i$  with

$$SpecR = \bigcup_{i=1}^{n} U_{\lambda_i}.$$

**Exercise 3 (10 points).** (Idempotents) Let R be a ring. An element  $e \in R$  is called an *idempotent* if  $e^2 = e$ . Show:

- (i) A ring R is *decomposable*, i.e.  $R \cong R_1 \oplus R_2$  for two commutative rings  $R_1, R_2$  if and only if R contains an idempotent element  $e \neq 0, 1$ .
- (ii) Spec R, equipped with the Zariski topology, is disconnected, i.e. Spec R = U a disjoint union of two nontrivial closed subsets, if and only if R contains an idempotent element e ≠ 0, 1.
  (Hint: Use the identities Z(I) ∩ Z(J) = Z(I + J) and Z(I) ∪ Z(J) = Z(IJ).)

**Exercise 4 (10 points).** (Polynomial ring) Let R be a ring and R[x] the polynomial ring in one variable with coefficients in R. Let  $f = \sum_{i=0}^{n} a_i x^i \in R[x]$ . Show:

- (i) f is nilpotent if and only if all  $a_i$  are nilpotent.
- (ii) f is a unit if and only if  $a_0$  is a unit and all  $a_i$  with i > 0 are nilpotent.
- (iii) f is a zero divisior if and only if there exists  $a \in R \setminus \{0\}$  with af = 0.

(Hint: Show (i) and (ii) first in the case that R is a domain and deduce the general case from that.)

Due date: Thursday, 11.04.2019, about 2pm before the lecture.

General information about the exercises:

- The **course webpage** is http://www.math.uni-bonn.de/people/thorsten/teaching/comm-alg-ss-19/index.htmpl.
- The first exercise groups take place on Monday, 08.04.2019.
- The new exercise sheet will always be available on Thursday. It can be found on the course webpage
- Exercise sheets should always be handed in between 2pm and 2.15pm before the lecture on Thursday. You can work in groups of two people and submit together.