Exercise 1.(i) Replacing R by $R/\mathfrak{p}_1$ and R' by $R'/\mathfrak{q}_1$ we can assume that R and R' are integral domains and $\mathfrak{p}_1$ and $\mathfrak{q}_1$ are trivial.
By Noether Normalization there exist algebraically independent $x_1, \ldots, x_r \in R$ such that R is a finite module over $A := k[x_1, \ldots, x_n]$. By Lemma 2.46 we get $(0) \subsetneq \mathfrak{p}_2 \cap A \subsetneq \mathfrak{p}_3 \cap A$. In the following we set $\mathfrak{p}_i' := \mathfrak{p}_i \cap A$. Now as R' integral over R it is also integral over A. Thus $A \subset R'$ is an integral extension of integral domains and A is normal. Hence we can apply the Going-down-Theorem to obtain a prime ideal $\mathfrak{q}_2 \subset \mathfrak{q}_3$ such that $\mathfrak{q}_2 \cap A = \mathfrak{p}_2'$. In particular $\mathfrak{q}_1 = (0) \subsetneq \mathfrak{q}_2 \subsetneq \mathfrak{q}_3$. This finishes the proof of (i).
(ii) Counterexample: $B = K[X, Y], A = \{f \in B | f(0, 0) = f(0, 1)\}$, then as $X, Y(Y-1) \in A$, we have that $A \subset B$ is an integral extension. As $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \in \operatorname{Spec}(A)$ we choose $\mathfrak{p}_1 = (0), \mathfrak{p}_2 = (Y) \cap A$ and $\mathfrak{p}_3 = (X, Y-1) \cap A$. Note that $\mathfrak{p}_2 \subsetneq \mathfrak{p}_3$ since $X \in \mathfrak{p}_3$. It is left to show that there is no prime ideal $\mathfrak{q} \subsetneq (X, Y-1)$ such that $\mathfrak{q} \cap A = \mathfrak{p}_2$. So if such a $\mathfrak{q}$ exists, then since $Y(Y-1), XY \in \mathfrak{q}$ we must have either $X, (Y-1) \in \mathfrak{q}$ or $Y \in \mathfrak{q}$. The former gives $\mathfrak{q} = (X, Y-1)$, the latter $\mathfrak{q} \not\subset (X, Y-1)$. So in both cases we get a contradiction to the assumption $\mathfrak{q} \subsetneq (X, Y-1)$.
Exercise 2. We may assume that p is not constant. So let p be explicitely given by

$$p(X_1, \ldots, X_n) = \sum_{\substack{\nu \in \mathbb{N}^n \\ |\nu| \leq m}} a_\nu X_1^{\nu_1} \ldots X_n^{\nu_n},$$

where $m \geq 1$ and $|\nu| := \nu_1 + \ldots + \nu_n$ and some $a_\nu \neq 0$ with $|\nu| = m$. So the stated substitution yields

$$\begin{aligned} p(X_1, \ldots, X_n) &= p(Y_1 + r_1 X_n, \ldots, Y_{n-1} + r_{n-1} X_n, X_n) \\ &= \sum_{\substack{\nu \in \mathbb{N}^n \\ |\nu| \leq m}} a_\nu (Y_1 + r_1 X_n)^{\nu_1} \ldots (Y_{n-1} + r_{n-1} X_n)^{\nu_{n-1}} X_n^{\nu_n} \end{aligned}$$

Unraveling these terms yields that the leading term in the variable X_n is of the form

$$\sum_{\substack{\nu \in \mathbb{N}^n \\ |\nu| = m}} a_\nu r_1^{\nu_1} \ldots r_{n-1}^{\nu_{n-1}}.$$

As the field K was assumed to be infinite we generally have that the map

$$K[X_1, \ldots, X_{n-1}] \to \operatorname{Maps}(K^{n-1}, K), \quad f \mapsto ((x_1, \ldots, x_{n-1}) \mapsto f(x_1, \ldots, x_{n-1}),$$

is injective. Thus, as some $a_\nu \neq 0$ with $|\nu| = m$, we have that

$$(r_1, \ldots, r_{n-1}) \mapsto \sum_{\substack{\nu \in \mathbb{N}^n \\ |\nu| = m}} a_\nu r_1^{\nu_1} \ldots r_{n-1}^{\nu_{n-1}}$$

is not the zero function. So we can choose $r_1, \ldots, r_{n-1}$ such that the leading term of X_n in $p(Y_1 + r_1 X_n, \ldots, Y_{n-1} + r_{n-1} X_n, X_n)$ does not vanish. This yields the claim.
Exercise 3.(i) We clearly have $(X_1 X_2) = \operatorname{span}_K(X_1^{i_1} X_2^{i_2} X_3^{i_3} | i_1, i_2 \geq 1) \subset K[X_1, X_2, X_3]$. From this follows that X_3 and $X_1 - X_2$ are algebraically independent elements in A. As we have the following relations in A:

$$X_2^2 + (X_1 - X_2) X_2 = 0 \quad \text{and} \quad X_1^2 - (X_1 - X_2) X_1,$$

we obtain that $K[X_1 - X_2, X_3] \subset A$ is an integral extension.
(ii) Note that $A \cong \mathbb{Z}[X]_{2X}$ and A is in particular an integral domain. For the sake of contradiction we assume there exist a Noether Normalization for A. As dim $A = 1$ we have there exist one (over $\mathbb{Z}$) algebraically independent element $t \in A$ such that $\mathbb{Z}[t] \subset A$ is integral. But since $2 \in A^\times$, there exist no prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap A = 2\mathbb{Z}[t]$. In particular $\mathbb{Z}[t] \subset A$ does not satisfy the Going-up property - contradiction.
Exercise 4.(Solution 1). We have to show $f \in \sqrt{I}$, where $I = (f_1, \ldots, f_r)$ and the radical is taken in $k[X_1, \ldots, X_r$. As K/k is algebraic we have that $K[X_1, \ldots, X_n]/k[X_1, \ldots, X_n]$ is integral. Thus we obtain a 1:1 correspondence

$$\{ \text{ max. ideals containing } I\} \leftrightarrow \{\mathfrak{m} \cap k[X_1, \ldots, X_n] | \mathfrak{m} \text{ max. in } K[X_1, \ldots, X_n] \text{ and } I \subset \mathfrak{m}\}.$$

As both $k[X_1, \dots, X_n], K[X_1, \dots, X_n]$ are Jacobson-rings, we obtain that the radical of I in $k[X_1, \dots, X_n]$ is the radical of I in $K[X_1, \dots, X_n]$ intersected with $k[X_1, \dots, X_n]$. So it suffices to show that f lies in the radical of I in $K[X_1, \dots, X_n]$. Given a maximal ideal $\mathfrak{m} \subset K[X_1, \dots, X_n]$ with $I \subset \mathfrak{m}$. By the weak Nullstellensatz we have $\mathfrak{m} = (X_1 - \lambda_1, \dots, X_n - \lambda_n)$ for $\lambda_1, \dots, \lambda_n \in K$ with

$$f_1(\lambda_1, \dots, \lambda_n) = \dots = f_r(\lambda_1, \dots, \lambda_n) = 0.$$

But by assumption this yields $f(\lambda_1, \dots, \lambda_n) = 0$ and hence $f \in \mathfrak{m}$.

Exercise 4.(Solution 2). This is the standard way deducing the assertion from the Weak Nullstellensatz, which can be found in many books on Algebraic Geometry.

Consider the ideal $\mathfrak{a} = (f_1, \dots, f_r, 1 - f \cdot X_{n+1}) \subset k[X_1, \dots, X_{n+1}]$. We are going to show that $\mathfrak{a}$ is not a proper ideal. So for the sake of contradiction suppose that $\mathfrak{a}$ is a proper ideal of $k[X_1, \dots, X_{n+1}]$. Then also $\mathfrak{a}K[X_1, \dots, X_{n+1}]$ is a proper ideal.

To see this we use basic methods of linear algebra. For the sake of contradiction assume that $\mathfrak{a}K[X_1, \dots, X_{n+1}]$ is a proper ideal, in particular there are $p_1, \dots, p_{r+1} \in K[X_1, \dots, X_{n+1}]$ such that

$$1 = \sum_{i=1}^{r} f_i p_i + (1 - X_{n+1} f) p_{r+1}.$$

Let $(v_i)_{i \in I}$ be a k-basis of K and without loss of generality let $i_0 \in I$ such that $v_{i_0} = 1$. We then have that $(v_i X_1^{j_1} \dots X^{j_{n+1}})_{(i,j,\dots,j_{n+1}) \in I \times \mathbb{N}^{n+1}}$ is a basis for $K[X_1, \dots, X_{n+1}] \to k[X_1, \dots, X_{n+1}$. Now let $\pi : K[X_1, \dots, X_{n+1}]$ be the projection with respect to this basis. We then have

$$1 = \pi(1) = \sum_{i=1}^{r} f_i \pi(p_i) + (1 - X_{n+1} f) \pi(p_{r+1}).$$

As $\pi(p_1), \dots, \pi(p_{r+1}) \in k[X_1, \dots, X_{n+1}$ we get that also $1 \in \mathfrak{a}$ - contradiction.

So alltogether $\mathfrak{a}K[X_1, \dots, X_{n+1}]$ is a proper ideal and hence contained in a maximal ideal $\mathfrak{m} \subset K[X_1, \dots, X_{n+1}]$. By the Weak Nullstellensatz there exist $\lambda_1, \dots, \lambda_{n+1} \in K$ such that $\mathfrak{m} = (X_1 - \lambda_1, \dots, X_{n+1} - \lambda_{n+1})$. This means $\mathfrak{a}K[X_1, \dots, X_{n+1}]$ lies inside the kernel of the evaluation

$$\mathrm{ev}_{(\lambda_1, \dots, \lambda_{n+1})} : K[X_1, \dots, X_{n+1}] \to K, \quad g \mapsto g(\lambda_1, \dots, \lambda_{n+1}).$$

In particular we have

$$f_1(\lambda_1, \dots, \lambda_n) = \dots = f_r(\lambda_1, \dots, \lambda_n) = 0 \quad \text{and} \quad f(\lambda_1, \dots, \lambda_n)\lambda_{n+1} = 1.$$

But by our assumptions on f we must have $f(\lambda_1, \dots, \lambda_n) = 0$ - contradiction.

So we have $1 \in \mathfrak{a}$, i.e. there exist $h_1, \dots, h_{r+1} \in k[X_1, \dots, X_{n+1}]$ such that $1 = \sum_{i=1}^{r} h_i f_i + h_{r+1}(1 - X_{n+1} f)$. Now consider the evaluation

$$\mathrm{ev}_{X_1, \dots, X_n, f^{-1}} : k[X_1, \dots, X_{n+1}] \to k[X_1, \dots, X_n]_f, \quad g \mapsto g(X_1, \dots, X_n, f^{-1}).$$

Then we have

$$1 = \mathrm{ev}_{X_1, \dots, X_n, f^{-1}}(1) = \sum_{i=1}^{r} h_i(X_1, \dots, X_n, f^{-1}) f_i(X_1, \dots, X_n).$$

Finally clearing denominators yields $f^N = \sum_{i=1}^{r} g_i f_i$, for some natural number N and polynomials $g_1, \dots, g_r \in k[X_1, \dots, X_n]$.