SPHERE PACKINGS AND OPTIMAL CONFIGURATIONS

SUMMER SCHOOL: SEPTEMBER 29 - OCTOBER 4, 2019

ORGANIZERS:
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1. Sphere packings

(1) **Oleksander Vlasiuk:**
H. Cohn and N. D. Elkies. *New upper bounds on sphere packings I.* Annals of Mathematics Second Series, vol. 157, No. 2 (2003), pp. 689-714 (26 pages). Remarks: The main results are the results of Section 3 (Theorems 3.1 and 3.2), Section 5 (conditions for when the LP bound is sharp), and Section 8 (uniqueness of optimal sphere packings). Section 4 is optional, as well as the numerics described in Section 7. One should note that Conjecture 7.3 is now a theorem in all cases except in dimension 2.

(2*) **Maria Dostert/Martin Stoller:**
D. de Laat, F. De Oliveira Filho and F. Vallentin. *Upper bounds for packings of spheres of several radii.* Forum of Mathematics, Sigma 2 (2014) E23 (42 pages).
Remarks: Sections 1,2,3 and 6 and all theorems therein should be presented. For instance, Theorems 1.1, 1.2, 1.3 (which is generalized by Thm 3.1 and has a radial version in Thm 5.1) and 1.4 should be covered. Sections 4 and 5 discuss the computational aspects and numerical models, lecturers should present only a brief overview of sections 5.1 and 5.2. Section 4 can be omitted.

(3*) **Cristian González Riquelme/Oscar Emilio Quesada Herrera:**
A. Venkatesh. *A note on sphere packing in high dimension.* International Mathematics Research Notices, vol. 2013, Issue 7, 1 January 2013, pp. 1628-1642 (15 pages).
Remarks: The first speaker should present the part of Theorem 1 of Venkatesh's paper that shows $c_n > \frac{1}{2}n \log\log n$ for infinitely many dimensions n and briefly comment about the ingredients in the proof that $c_n > 65963n$ for all sufficiently

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large dimensions. The second speaker should present Theorem 1 of [P. Moustrou. *On the density of cyclotomic lattices constructed from codes.* International Journal of Number Theory. vol. 13, No. 05, pp. 1261-1274 (2017)], focusing on Sections 2 and 3. The part of the paper that deals with the complexity of binary operations in the construction should be omitted, that is, everything after and including Proposition 1 of Section 3.

(4*) **Itamar Oliveira/Mateus Sousa:**
G. A. Kabatiansky, V. I. Levenshtein, *On Bounds for Packings on a Sphere and in Space*, Probl. Peredachi Inf., 14:1 (1978), 3-25; Problems Inform. Transmission, 14:1 (1978), 1-17.
Remarks: Speakers should focus only in the part of the paper applied to spheres, thus presenting the proofs of Theorem 4 and Corollaries 1 and 2. Only sections 5 and 6 (up to Corollary 2) need to be covered. Note that the authors use Delsarte's linear programming bound for spherical codes (Corollary 2 on section 2), and thus speakers should only briefly state and remind Delsarte's bound, since it will be covered in talk (15). Also note that to go from spherical codes to sphere packings the authors use inequality (2), which has a simple proof without the need to add an extra dimension in Proposition 2.1 of [H. Cohn, Y.Zhao. *Sphere packing bounds via spherical codes.* Duke Math. J., vol. 163, Number 10 (2014), 1965-2002]. The speakers should also present from this last paper Proposition 2.1 and Theorems 3.4. Speakers may only briefly recall the Cohn and Elkies Linear Programming bounds.

(5*) **Tainara G. Borges/Cynthia Bortolotto:**
M. Viazovska. *The sphere packing problem in dimension 8.* Annals of Mathematics. p. 991-1015, vol. 185 (2017), Issue 3 (25 pages).
***AND* .**
Remarks: One should only briefly recall the Cohn-Elkies bound for sphere packing and the conditions for when this bound is attained for E_8 and Λ_{24}, since it will be discussed in another talk. The main part is the construction of Fourier eigenfunctions in Section 4 (Sections 2 and 3 in the Leech lattice paper [H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, M. Viazovska. *The sphere packing problem in dimension 24.* Annals of Mathematics, p. 1017-1033, vol. 185 (2017), Issue 3]. The actual verification of the inequalities should be only sketched. First speaker presents the E_8 case and second speaker presents the Leech lattice case.

2. Kissing numbers

(6) **Carlos Andrés Chirre:**
C. Bachoc and F. Vallentin. *New upper bounds for kissing numbers from semidefinite programming.* J. Amer. Math. Soc. 21 (2008), 909-924 (16 pages).
Remarks: The emphasis should be on Sections 3 and 4 and on proving Theorems 3.1, 3.2 and 4.2. Section 5 should be briefly covered together with the computational results therein.

(7) **Philippe Moustrou:**
S. Vlăduţ. *Lattices with exponentially large kissing numbers.* (2018) arXiv:1802.00886 [math.NT] (16 pages).
Remarks: The main emphasis should be on proving Corollary 1.2 for some positive value of c_0, as described in the first paragraph of Section 6. The choice of which of the other results to present (Theorems 1.1, 1.3, 1.4, and 1.5) is up to the speaker. There is also no need to present results with optimal choices of constants, which also means that the speaker can choose to describe only one of the families of curves (i.e., either Garcia-Stichtenoth or a tower of Drinfeld curves).

(8) **Nina Zubrillina:**
M. Jenssen, F. Joos, W. Perkins. *On kissing numbers and spherical codes in high dimensions.* Advances in Mathematics, vol. 335, 7 September 2018, p. 307-321 (15 pages).
Remarks: The speaker should focus on the proof of Theorem 4 as all the other main results (Theorems 1, 2 and 3) follow from it.

3. Potential energy minimization

(9) **Ganesh Ajjanagadde:**
D. Bilyk and F. Dai. *Geodesic Distance Riez Energy on the Sphere.* Trans. Amer. Math. Soc. https://doi.org/10.1090/tran/7711 (24 pages).
Remarks: The speaker should focus on Theorem 1.1, which is proven in Sections 2 and 3. Section 4 should be only briefly covered, mentioning a general form of the Stolarsky principle (Theorem 4.2 and Corollary 4.3). Section 5 and 6 can be omitted.

(10) **Changkeun Oh:**
A. B. J. Kuijlaars and E. B. Saff. *Asymptotics for Minimal Discrete Energy on the Sphere.* Trans. Amer. Math. Soc., vol. 350, No. 2 (1998), pp. 523-538 (16 pages).

Remarks: The main results to cover are Theorems 1, 2, and 3, as well as the conjectures regarding the two-dimensional case (Section 2). Section 8 is optional. Note that a strengthening of Theorem 2 will be presented in the following talk (11), so it makes sense to coordinate the two.

(11) **Ryan Matzke:**
D. P. Hardin, E. B. Saff. Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds. Advances in Mathematics vol. 193, Issue 1 (2005), pp. 174-204 (31 pages).
Remarks: The setup of the problem is analogous to the previous talk and thus can be very brief; note also that Theorems 1.2 and 1.3 are proved in the above talk (10). The main results to concentrate on are the proofs of Theorem 2.1 and Theorem 2.2.

(12*) **Juan Criado del Rey/Alan Groot:**
H. Cohn and A. Kumar. *Universally optimal distribution of points on spheres.* J. Amer. Math. Soc. 20, Number 1 (2007), pp. 99-148 (50 pages).
Remarks: The main result is Theorem 1.2; the case of 600-cell should only be treated if there is sufficient time, thus Section 7 is optional. Section 8 can be skipped; the second half of Section 9 (everything after Conjecture 9.4) is also optional.

(13&14) **13 Matthew De Courcey-Ireland:**
14 Guiseppe Negro:
H. Cohn, A. Kumar, S. Miller, D. Radchenko, M. Viazovska. *Universal optimality of the E8 and Leech lattices and interpolation formulas.* (2019) arXiv:1902.05438 [math.MG] (88 pages).
Remarks: The introductory section has a considerable overlap with other talks and should be presented very briefly. The same goes for Section 2, with the exception of 2.1.3. Section 2.2 can be skipped, and so can be the proof of Lemma 2.2. Section 3.3 can be omitted (up until 3.3.1) and Proposition 3.8 and 3.9 can be stated without proof. Proof of Proposition 4.2 should be omitted, and so can be the proof of uniqueness in Theorem 4.3. The verification of positivity (sections 6.2-6.7) and Section 7 should be skipped. Preferably, the first talk should go from beginning until Proposition 4.2 (without proof), and the second talk can begin with Theorem 4.3 and go until the end of section 6.1 (excluding Proposition 6.3).

4. Codes and Designs

(15) **Hans Parshall:**
P. Delsarte, J. M. Goethals, J. J. *Seidel. Spherical codes and designs.* Geom Dedicata (1977) 6, Issue 3, pp. 363-388 (26 pages).

<u>*Remarks*</u>: The main results are in Section 4 (Theorems 4.3 and 4.8), Section 5 (Theorems 5.10, 5.11, and 5.12), and Section 6 (Theorems 6.6 and 6.8). The paper contains many examples of codes and spherical designs from which the speaker may choose any selection according to their taste; this mainly applies to Sections 8 and 9. One may also omit the discussion of association schemes and Bose-Mesner algebras, except for the simplest case of strongly regular graphs.

(16) **Louis Brown:**
P. Delsarte. Bounds for unrestricted codes, by linear programming. Philips Res. Repts 27 (1972), pp. 272-289 (18 pages).
<u>*Remarks*</u>: The main results are Theorem 6 and Theorems 14, 15, and 16. Among the three examples of codes given in the paper the speaker may choose to present only one. The results from Appendix are standard and we assume that they are known (the same goes for Theorem 1).

(17) **Stefanos Lappas:**
A. Bondarenko, D. Radchenko, M. Viazovska. Optimal asymptotic bounds for spherical designs. Annals of Mathematics, Vol. 178, issue 2 (2013), pp. 443-452 (10 pages).
<u>*Remarks*</u>: The paper should be presented in full.

5. Quasicrystals and interpolation

(18*) **Gevorg Mnatsakanyan/João Pedro Gonçalves Ramos:**
D. Radchenko, M. Viazovska. *Fourier interpolation on the real line.* Publ. math. IHES (2018), https://doi.org/10.1007/s10240-018-0101-z (31 pages).
<u>*Remarks*</u>: The main results to present are Theorems 1 and 2; one could also mention that analogous results hold for odd functions (Theorem 7), but otherwise Sections 7 and 8 are optional. In Section 5 the proof of Lemma 4 should be either omitted or only sketched.

(19*) **Marco Fraccaroli/Milan Kroemer:**
Y. F. Meyer. *Measures with locally finite support and spectrum.* PNAS March 22, 2016 113 (12) 3152-3158 (7 pages).
<u>*Remarks*</u>: The paper should be presented almost in full. One speaker could focus on the proofs of Theorems 1, 2 and 5 (all are more or less implied by Theorem 7). Theorem 3 will be presented in talk (20). The second speaker could focus examples and main results of sections Guinand's Distribution (Lemmas 3, 4 and Theorem 4), Kolountzakis Theorem and The Crystalline Measures of Lev and Olevskii. The paper [Y. Meyer. *Quasicrystals, Almost Periodic Patterns, Mean-periodic Functions and Irregular Sampling.* Afr. Diaspora J. Math. (N.S.), vol. 13,

Number 1 (2012), 1-45] should be used as a good source of definitions and results, so as to fill any background gaps and give a self-contained presentation.

(20) **Lenka Slavikova:**
N. Lev, A. Olevskii. *Quasicrystals and Poisson's summation formula.* Inventiones mathematicae, vol. 200, Issue 2 (2015), pp. 585-606 (22 pages).
Remarks: The speaker should focus on the proofs of Theorems 1 and 2. Theorem 3 should only be stated, while the proof could only be sketched or omitted.

6. Other related topics

(21) **Josiah M. Park:**
H. Cohn, F. Gonçalves. *An optimal uncertainty principle in twelve dimensions via modular forms.* (2017). To appear in Inventiones Mathematicae. arXiv:1712.04438 [math.CA] (24 pages).
Remarks: The speaker should focus on Theorems 1.2 and 1.4. Section 5 could be only briefly covered or omitted.

(22) **Zirui Zhou:**
E. Carneiro, M. Milinovich, K. Soundararajan. *Fourier optimization and prime gaps.* To appear in Commentarii Mathematici Helvetici. (2017) arXiv:1708.04122 [math.NT] (30 pages).
Remarks: The focus should be on presenting Theorem 3 and Corollary 4, and thus proving also the necessary parts from Theorems 1 and 2. The effective proof of Theorem 5 should only be sketched or omitted.

(23) **Ljudevit Palle:**
J. M. Aldaz. *Kissing numbers and the centered maximal operator.* (2018) arXiv: 1811.10478 [math.CA] (14 pages).
Remarks: The paper should be presented in full.

(24*) **Constantin Bilz/Gianmarco Brocchi:**
J. Bourgain, V. D. Milman. *New volume ratio properties for convex symmetric bodies.* Inventiones mathematicae, Vol. 88, Issue 2 (1987), pp. 319-340 (22 pages).
Remarks: The first speaker should focus solely in the relevant parts for the proof of Theorem 1. Other parts of the paper should be omitted. The second speaker should present a sketch of the alternative proof [Nazarov, F. *The Hörmander proof of the Bourgain-Milman theorem.* In: Klartag, B., Mendelson, S., Milman, V.D. (eds.) Geometric Aspects of Functional Analysis, Israel Seminar 2006–2010. Lecture Notes in Mathematics, vol. 2050, pp. 335-343. Springer (2012)].

(25) **Nuria Storch de Gracia:**
Emanuel Carneiro, Friedrich Littmann and Jeffrey D. Vaaler. ***Gaussian subordination for the Beurling-Selberg extremal problem*** **Journal: Trans. Amer. Math. Soc. 365 (2013), 3493-3534.**
Remarks: The Speaker should present Theorems 2 and 3. Theorem 1 could be sketched, time permitting.

Observations.

- * = Paper is shared by two speakers in consecutive talks of 35min each with no break.
- 13&14 = Two separated but consecutive talks of 45min each and with a break.
- All other talks are of 45min each and given by one speaker only.
- We strongly advise that speakers sharing papers should get in touch before the summer school to organize their joint talk. We leave to the judgment of the speakers to adequately split papers/talks where no specific splitting instruction was given.
- Suggestions on what to cover should not be interpreted as the only things to talk about. Highly relevant remarks, comments or applications should be commented.