SPHERE PACKINGS AND OPTIMAL CONFIGURATIONS

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1 Kissing numbers and the centered maximal operator (after J. M. Aldaz [1])

A summary written by Ljudevit Palle

Abstract

Aldaz showed in [1] an interesting connection between kissing numbers and uniform bounds (independent of measure) for weak type estimates of centered maximal operators. The concept which plays a key role is the Besicovitch constant, an integer invariant of a metric space. As an application one obtains improved uniform bounds for centered maximal operators in the Euclidean space with the standard norm and even sharp bounds in the case of the ℓ_∞ norm.

1.1 Preliminaries

Let us fix a metric space (X, d). We shall be concerned with Borel measures on X which are locally finite and τ-additive. A Borel measure μ on X is *locally finite* if it assigns finite measure to bounded Borel sets, and it is *τ-additive* if for any collection $\{O_\alpha : \alpha \in \Lambda\}$ of open sets it satisfies

$$\mu(\cup_\alpha O_\alpha) = \sup_{\mathcal{F}} \mu(\cup_{i=1}^n O_{\alpha_i}),$$

where the supremum is taken over all finite subcollections $\mathcal{F} = \{O_{\alpha_1}, \ldots, O_{\alpha_n}\}$ of $\{O_\alpha : \alpha \in \Lambda\}$. In the Euclidean case when the metric is generated by a norm Borel measures which are locally finite and τ-additive correspond precisely to Radon measures.

To each locally finite, τ-additive Borel measure μ on X such that $\mu \neq 0$ we can associate the centered maximal function

$$M_\mu f(x) := \sup_{\{r>0:\mu(B(x,r))>0\}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \mathrm{d}\mu.$$

Since balls in general in a metric space have neither unique centers nor unique radii, whenever we choose or consider a ball, we shall fix one of its possible centers and radii.

Our goal is to relate the centered maximal functions to the following concept. We say that a collection of balls $\mathcal{C}$ in a metric space (X, d) is a *Besicovitch family* if for every pair of distinct balls $B(x, r)$ and $B(y, s)$ from $\mathcal{C}$ it necessarily holds $x \notin B(y, s)$ and $y \notin B(x, r)$. The *Besicovitch constant* $L(X, d)$ of (X, d) is defined by

$$L(X, d) := \sup \left\{ \sum_{B \in \mathcal{C}} 1_B(x) : x \in X,\ \mathcal{C} \text{ a Besicovitch family} \right\}.$$

One says that (X, d) has the *Besicovitch intersection property* if $L(X, d)$ is finite.

Finally, we need to recall what a translative kissing number is. Let $(\mathbb{R}^d, \|\cdot\|)$ be a normed space. The *Hadwiger number* or *translative kissing number* $H(d, \|\cdot\|)$ is the maximum number of translates of the (metrically) closed unit balls $B^{cl}(0, 1)$ that touch $B^{cl}(0, 1)$ with the condition that the translates do not have overlapping interiors with each other. The *strict Hadwiger number* or *strict translative kissing number* $H^*(d, \|\cdot\|)$ is the maximum number of disjoint translates of the closed unit balls $B^{cl}(0, 1)$ that touch $B^{cl}(0, 1)$. In the case $(\mathbb{R}^d, \|\cdot\|_2)$ the number $H(d, \|\cdot\|_2)$ is just called the *kissing number*.

1.2 Results

The main result is contained in the following theorem.

Theorem 1. *For any metric space (X, d) one has*

$$L(X, d) = \sup_{\mu} \|M_\mu\|_{L^1 \to L^{1,\infty}},$$

where the supremum is taken over all locally finite, τ-additive Borel measures μ on (X, d).

The connection with translative kissing numbers is given by the following observation.

Theorem 2. *Consider any normed space $(\mathbb{R}^d, \|\cdot\|)$. Then*

$$L(\mathbb{R}^d, \|\cdot\|) = H^*(d, \|\cdot\|).$$

where $H^*(d, \|\cdot\|)$ is the strict Hadwiger number.

As an application, by using the asymptotics for kissing numbers from [2] and [3], we get the following uniform estimates for centered maximal functions in the Euclidean case with the standard norm.

Corollary 3. *Let us consider the normed space* $(\mathbb{R}^d, \|\cdot\|_2)$*. Then we have*

$$(1+o(1))1.1547^d \leq L(\mathbb{R}^d, \|\cdot\|_2) = \sup_\mu \|M_\mu\|_{L^1 \to L^{1,\infty}} \leq 1.3205^{(1+o(1))d},$$

where the supremum is taken over all Radon measures.

This is a significant improvement over the estimates for centered maximal functions obtained through the use of the Besicovitch covering theorem. Namely, if we denote the number of collections appearing in the Besicovitch covering theorem in dimension d by β_d, then from [4] we know that β_d grows exponentially with d to base at least $8/\sqrt{15}$ and at most 2.641.

A further corollary we get is

Corollary 4. *In the normed space* $(\mathbb{R}^d, \|\cdot\|_\infty)$ *we have*

$$L(\mathbb{R}^d, \|\cdot\|_\infty) = \sup_\mu \|M_\mu\|_{L^1 \to L^{1,\infty}} = 2^d,$$

where the supremum is taken over all Radon measures.

We also note the following result, which in combination with Theorem 1 yields the extrapolation result that a uniform weak type (p,p) estimate for some $1 < p < \infty$ implies a uniform weak type $(1,1)$ estimate.

Theorem 5. *Let* (X,d) *be a metric space. If there exists a* p *satisfying* $1 < p < \infty$ *and an integer* $N \geq 1$ *such that for every locally finite,* τ*-additive Borel measure* μ *we have* $\|M_\mu\|_{L^p \to L^{p,\infty}} \leq N$*, then* (X,d) *has the Besicovitch intersection property with the Besicovitch constant* $\lfloor p^p(p-1)^{1-p}N^p \rfloor$*.*

1.3 The structure of the proof of Theorem 1

The proof has two steps. The first step is to prove the theorem for finite positive linear combinations of Dirac delta measures.

Lemma 6. *For a metric space* (X,d) *the following are equivalent:*

1) $L = L(X,d) < \infty$

2) For any measure μ of the form $\sum_{i=1}^{N} c_i\delta_{x_i}$ with $c_i > 0$ it holds $\|M_\mu\|_{L^1\to L^{1,\infty}} \leq L < \infty$.

The second and more difficult step is to prove

Theorem 7. *Let (X, d) be a metric space and let μ be a locally finite, τ-additive Borel measure μ such that $\|M_\mu\|_{L^1\to L^{1,\infty}} > L$ for some finite positive number L. Then there exists a finite positive linear combination of Dirac delta measures ν such that $\|M_\nu\|_{L^1\to L^{1,\infty}} > L$.*

Combining Lemma 6 and Theorem 7 one easily obtains Theorem 1.

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2 New upper bounds for kissing numbers from semidefinite programming (after C. Bachoc and F. Vallentin [1])

A summary written by Carlos Andrés Chirre

Abstract

We explain the semidefinite programming approach to show new upper bounds for the kissing numbers.

2.1 Introduction

We denote the standard inner product of the Euclidean space R^n by $x \cdot y$ and we denote the unit sphere as $S^{n-1} = \{x \in R^n : x \cdot x = 1\}$. We want to establish a bound for the maximal number

$$A(n, \theta) = \max\{card(C) : C \subset S^{n-1}, c \cdot c' \leq \cos\theta \text{ for } c, c' \in C, c \neq c'\}$$

of points on the unit sphere with minimal angular distance θ. These configurations will be called *spherical codes with minimal angular distance* θ. The kissing number problem is equivalent to the problem of finding $A(n, \pi/3)$.

We consider the action restricted to a subgroup H of the ortogonal group $O(R^n)$, chosen to be the stabilizer group of a fixed point $e \in S^{n-1}$, lead us to some symmetric matrices S_k^n whose coefficients are symmetric polynomials in three variables such that:
For all finite $C \subset S^{n-1}$, $\sum_{(c,c',c'') \in C^3} S_k^n(c \cdot c', c \cdot c'', c' \cdot c'')$ *positive semidefinite.*

2.2 Preliminaries

2.2.1 Semidefinite zonal matrices

Note that the orthogonal group $O(R^n)$ acts homogeneously on the unit sphere. We write the space of real polynomial functions on S^{n-1} of degree at most d by $Pol_{\leq d}(S^{n-1})$. It is endowed with the induced action of $O(R^n)$, and equipped with the standard $O(R^n)$-invariant inner product

$$(f, g) = \frac{1}{w_n} \int_{S^{n-1}} f(x) g(x) dw_n(x),$$

where w_n is the surface area of S^{n-1} for the standard measure dw_n. A classical result shows the descomposition, under the action of $O(R^n)$,

$$Pol_{\leq d}(S^{n-1}) = H_0^n \perp H_1^n \perp \cdots \perp H_d^n,$$

where H_k^n is isomorphic to the $O(R^n)$ irreducible space of homogeneous harmonic polynomials of degree k in n variables, denoted by $Harm_k^n$. We denote the dimension of these spaces by h_k^n.

Let e be a fixed point of S^{n-1}. Under the restricted action of the subgroup $H := Stab(e, O(R^n)) = \{M \in O(R^n) : Me = e\}$ (note that $H \simeq O(R^{n-1})$), we have the following descomposition into isotypic components:

$$Pol_{\leq d}(S^{n-1}) = L_0 \perp L_1 \perp \cdots \perp L_d,$$

where $L_k \simeq (d-k+1)Harm_k^{n-1}$, for $0 \leq k \leq d$. In fact,

$$L_k = H_{k,k}^{n-1} \perp \cdots \perp H_{k,d}^{n-1},$$

where, for $i \geq k$, $H_{k,i}^{n-1}$ is the unique subspace of H_i^n isomorphic to $Harm_k^{n-1}$. Let $\{e_{0,1}^k, e_{0,2}^k, \cdots, e_{0,h_k^{n-1}}^k\}$ be an orthonormal basis of $H_{k,k}^{n-1}$ and let $\phi_s : H_{k,k}^{n-1} \to H_{k,k+s}^{n-1}$ be some H-isomorphism. Then, we will write $\phi_s(e_{0,i}^k) = e_{s,i}^k$, where $1 \leq i \leq h_k^{n-1}$, the ortonormal basis of $H_{k,k+s}^{n-1}$. For $x \in S^{n-1}$ we define the following matrix

$$E_k^n(x) = \frac{1}{\sqrt{h_k^{n-1}}} \begin{pmatrix} e_{0,1}^k(x) & e_{0,2}^k(x) & \cdots & e_{0,h_k^{n-1}}^k(x) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d-k,1}^k(x) & e_{0,2}^k(x) & \cdots & e_{d-k,h_k^{n-1}}^k(x) \end{pmatrix}.$$

Finally, for $x, y \in S^{n-1}$ we define

$$Z_k^n(x,y) = E_k^n(x) E_k^n(y)^t \in R^{(d-k+1)x(d-k+1)}.$$

It is possivel to prove that for $M \in H$, $Z_k^n(Mx, My) = Z_k^n(x,y)$. This implies that the coefficients of Z_k^n can be expressed as polynomials in the variables $u = e \cdot x$, $v = e \cdot y$ and $t = x \cdot y$. We can write $Z_k^n(x,y) = Y_k^n(e \cdot x, e \cdot y, x \cdot y)$, where $Y_k^n(u,v,t)$ is a square matrix $R^{(d-k+1)x(d-k+1)}$.

For $n \geq 3$ we will write $P_k^n(t)$ the Gegenbauer polynomial of degree k with parameter $n/2 - 1$, with $P_k^n(1) = 1$. On another hand, for $n \geq 2$, $P_k^n(t)$ will be the Chebyshev polynomial of the first kind with degree k. The following theorem establishs the relation between this polynomials and the matrices Y_k^n.

Theorem 1. *We have, for all $0 \le i, j \le d-k$,*

$$(Y_k^n)_{i,j}(u,v,t) = \lambda_{i,j} P_i^{n+2k}(u) P_j^{n+2k}(v) Q_k^{n-1}(u,v,t),$$

where $Q_k^{n-1}(u,v,t) := ((1-u^2)(1-v^2))^{k/2} P_k^{n-1}\left(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\right)$ *and*

$$\lambda_{i,j} = \frac{w_n}{w_{n-1}} \frac{w_{n+2k-1}}{w_{n+2k}} (h_i^{n+2k} h_j^{n+2k})^{1/2}.$$

This result allows to have especial semidefinite positive matrices. We write $A \succeq 0$ to express that the matrix A is positive semidefinite.

Corollary 2. *For all $d \ge 0$, and for all $k \ge 0$, let Y_k^n be the matrix above mentioned. We define the matrix*

$$S_k^n = \frac{1}{6} \sum_{\sigma} \sigma Y_k^n,$$

where σ runs through the group of all permutations of the variables u, v, t which acts on matrix coefficients in the obvious way. Then the matrices S_k^n are symmetric and have symmetric polynomials as coefficients. We have that

1. *For all finite $C \subset S^{n-1}$, $\sum_{(c,c') \in C^2} Y_k^n(e \cdot c, e \cdot c', c' \cdot c'') \succeq 0$.*
2. *For all finite $C \subset S^{n-1}$, $\sum_{(c,c',c'') \in C^3} S_k^n(c \cdot c', c \cdot c'', c' \cdot c'') \succeq 0$.*

2.3 The semidefinite programming bound

Now, we set up and semidefinite programming to give an upper bound for $A(n, \theta)$. For a spherical code C we define the three-points distance distribution:

$$x(u,v,t) = \frac{1}{card(C)} \{(c,c',c'') \in C^3 : c \cdot c' = u, c \cdot c'' = v, c' \cdot c'' = t\},$$

where $u, v, t \in [-1, 1]$ such that $1 + 2uvt - u^2 - v^2 - t^2 \ge 0$. Note that $x(u,v,t) \ge 0$, $x(1,1,1) = 1$, there is a finite number of triple (u,v,t) such that $x(u,v,t) \ne 0$. Also, if the minimal angular distance of C is θ, we have $x(u,v,t) = 0$ whenever $u, v, t \notin [-1, \cos\theta] \cup \{1\}$. Finally, note that

$$\sum_u x(u,u,1) = card(C).$$

Therefore we can obtain a semidefinite program in the variables $x'(u,v,t)$, where $x'(u,v,t) = m(u,v,t)x(u,v,t)$ such that the function $m(u,v,t)$ contain the information about the permutations of the variables. In fact, $m(u,v,t) = 6$ if $u \neq v \neq t$, $m(u,v,t) = 3$ if $u = v \neq t$ or similar cases, and $m(u,v,t) = 1$ if $u = v = t$. This semidefinite program allows to obtain an upper bound for $A(n,\theta)$, because we want to maximize

$$1 + \frac{1}{3} \sum_{u \in [-1, \cos\theta]} x'(u,u,1),$$

under certain conditions. Using the principle of duality we obtain the following result:

Theorem 3. *Any feasible solution of the following semidefinite problem gives an upper bound on $A(n,\theta)$:*

$$
\begin{aligned}
1 + \min \Bigg\{ & \sum_{k=1}^{d} a_k + b_{11} + Trace(F_0 \,.\, S_0^n(1,1,1)) : \\
& \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \succeq 0 \\
& a_k \geq 0, \;\; for \;\; k = 1, \ldots d, \\
& F_k \succeq 0, \;\; for \;\; k = 0, \ldots d, \\
& \sum_{k=1}^{d} a_k P_k^n(u) + 2b_{12} + b_{22} + 3 \sum_{k=0}^{d} Trace(F_k \,.\, S_k^n(u,u,1)) \leq -1, \\
& b_{22} + \sum_{k=0}^{d} Trace(F_k \,.\, S_k^n(u,v,t)) \leq 0 \Bigg\},
\end{aligned}
$$

where the last inequality holds for all (u,v,t) such that $-1 \leq u \leq v \leq t \leq \cos\theta$ and $1 + 2uvt - u^2 - v^2 - t^2 \geq 0$ and the second last to last inequality holds for all $u \in [-1, \cos\theta]$.

2.3.1 The kissing numbers

We consider the polynomials $p(u) = -(u+1/4)^2 + 9/16$, $p_1(u,v,t) = p(u)$, $p_2(u,v,t) = p(v)$, $p_3(u,v,t) = p(t)$ and $p_4(u,v,t) = 1 + 2uvt - u^2 - v^2 - t^2$. The last two conditions of the semidefinite program in the above theorem are satisfied if the following two equalities hold:

$$-1-\sum_{k=1}^{d} a_k P_k^n(u)-2b_{12}-b_{22}-3\sum_{k=0}^{d} Trace(F_k \,.\, S_k^n(u,u,1)) = q(u)+p(u)q_1(u),$$

and

$$-b_{22} - \sum_{k=0}^{d} Trace(F_k \,.\, S_k^n(u,v,t)) = r(u,v,t) + \sum_{i=1}^{4} p_i(u,v,t) r_i(u,v,t),$$

where q, q_1 and $r, r_1, ...r_4$ are sums of squares of polynomials.

Finally, we fix d and restrict the polynomials $q, q_1, r, r_1, ..., r_4$ to polynomials having degree at most N, with $N \geq d$. Using the computer we find a feasible soluton of this semidefinite program. This implies an upper bound on the kissing number τ_n. We denote the best previous result known about kissing numbers with τ_n' and the new result with semidefinite programming with τ_n.

1. $\tau_3' = 12$ and $\tau_3 = 12$.
2. $\tau_4' = 24$ and $\tau_4 = 24$.
3. $\tau_5' = 46$ and $\tau_5 = 45$.
4. $\tau_6' = 82$ and $\tau_6 = 78$.
5. $\tau_7' = 140$ and $\tau_7 = 135$.
6. $\tau_8' = 240$ and $\tau_8 = 240$.
7. $\tau_9' = 379$ and $\tau_9 = 366$.
8. $\tau_{10}' = 594$ and $\tau_{10} = 567$.

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3 Geodesic distance Riesz energy on the sphere (after D. Bilyk and F. Dai [BD19])

A summary written by Ganesh Ajjanagadde

Abstract

We study energy integrals and discrete energies on the sphere, in particular, analogues of the Riesz energy with the geodesic distance in place of the Euclidean, and we determine that the range of exponents for which uniform distribution optimizes such energies is different from the classical case. We also obtain a very general form of the Stolarsky principle, which relates discrete energies to certain L^2 discrepancies.

3.1 Introduction

Let $\mathcal{B}$ denote the collection of all Borel probability measures on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. Given such a measure $\mu \in \mathcal{B}$, and a measurable function $F : [-1, 1] \to \mathbb{R}$, we define the *energy integral*:

$$I_F(\mu) \triangleq \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} F(x \cdot y) d\mu(x) d\mu(y).$$

In general, we are interested in maximizing or minimizing $I_F(\mu)$ over all $\mu \in \mathcal{B}$, and also over $\mu \in \mathcal{S} \subset \mathcal{B}$ for interesting subsets $\mathcal{S}$. For example, an important case is that of N-atom uniform measures, or equivalently discrete energy of N-point configurations for some fixed N. Further specializing to $d = 2, F(x \cdot y) = |x - y|^{-1}$ and minimization of $I_F(\mu)$ over such N-atom measures yields the famous Thomson's problem, which physically amounts to finding the equilibrium distribution of N electrons on a sphere subject to Coulomb interactions. A more general version studies the so-called *Riesz potential* $F(x \cdot y) = |x - y|^{-s}$.

The primary subject matter of this paper is a twist of the Riesz energy, namely using geodesic distances as opposed to Euclidean ones. More precisely, the geodesic distance between x, y is:

$$\rho(x, y) = \arccos(x \cdot y).$$

Then we study energies induced by:

$$F_\delta(x \cdot y) \triangleq \rho(x, y)^\delta,$$

for arbitrary $\delta \in \mathbb{R} \setminus \{0\}$; for $\delta = 0$ the natural modification is to use the logarithmic potential $F_0(t) \triangleq -\log(\frac{\arccos(t)}{\pi})$. In summary, we wish to characterize extremizers of *geodesic distance Riesz energy*:

$$I_{d,\delta}(\mu) \triangleq I_{F_\delta}(\mu), \tag{1}$$

over $\mu \in \mathcal{B}$. In order to favor repulsion, it is natural to focus on minimization for $\delta \leq 0$, and maximization for $\delta > 0$.

Our focus in this summary is on the following:

Theorem 1. *Let $I_{d,\delta}(\mu)$ be given by* (1). *Then:*

1. *For $-d < \delta \leq 0$, the unique minimizer of $I_{d,\delta}(\mu)$ is $\mu = \sigma$ (the normalized uniform surface measure).*

2. *For $0 < \delta < 1$, the unique maximizer of $I_{d,\delta}(\mu)$ is $\mu = \sigma$.*

3. *For $\delta = 1$, $I_{d,\delta}(\mu)$ is maximized iff μ is centrally symmetric ($\mu(-E) = \mu(E)$ for measurable E).*

4. *For $\delta > 1$, $I_{d,\delta}(\mu)$ is maximized iff $\mu = \frac{1}{2}(\delta_p + \delta_{-p})$ for some $p \in \mathbb{S}^d$. In other words, the mass is equally concentrated at a pair of antipodes.*

Remark 2.

1. *The restriction to $\delta > -d$ is natural, since for $\delta \leq -d$, $I_{F_\delta}(\mu)$ is infinite for all $\mu \in \mathcal{B}$.*

2. *For classical Riesz energy $F(x \cdot y) = |x - y|^\delta$, the first statement is still true (see e.g. [KS98]), while for $\delta > 0$, the phase transition occurs at $\delta = 2$ as opposed to $\delta = 1$ [Bjö56]. More precisely, for classical Riesz energy, for $\delta \in (0, 2)$, σ is the unique maximizer. For $\delta > 2$, the maximizers collapse to the symmetric antipodal ones. At the critical point $\delta = 2$, the maximizers are precisely the measures μ with $\mathbb{E}[\mu] = 0$.*

3.2 Methods

We first give a brief, impressionistic, sketch of the proof of Theorem 1. As with most results on this subject, the basic approach is via spherical harmonics/positive definiteness. This reduces the proof of optimality of the uniform measure to checking certain sign conditions on the Gegenbauer coefficients of the function F, modulo some technicalities such as the singularity of $F(x \cdot y)$

at $x = y$. The check of these sign conditions utilizes the Taylor/Maclaurin series of the function $\arccos(t)$ in $[-1, 1]$. Uniqueness statements amount to strict positivity of the relevant Gegenbauer coefficients as opposed to mere nonnegativity. We now proceed to some of the key concepts.

First, we explain what we mean by Gegenbauer coefficients. Let $w_\lambda(t) = (1-t^2)^{\lambda-\frac{1}{2}}$ where $\lambda = \frac{d-1}{2}$ in our setting. Define $L^1_{w_\lambda}$ to be the space of integrable functions on $[-1, 1]$ with respect to the weight w_λ:

$$F \in L^1_{w_\lambda} \Leftrightarrow ||F||_{1,\lambda} \triangleq \int_{-1}^{1} |F(t)| w_\lambda(t) dt < \infty.$$

Let

$$C_n^\lambda(z) \triangleq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda) k!(n-2k)!} (2z)^{n-2k}$$

be the Gegenbauer polynomials.

Then every function $F \in L^1_{w_\lambda}$ has a Gegenbauer expansion:

$$F(t) \approx \sum_{n=0}^{\infty} \widehat{F}(n;\lambda) \frac{n+\lambda}{n} C_n^\lambda(t), \tag{2}$$

where

$$\widehat{F}(n;\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} F(t) w_\lambda(t) \frac{C_n^\lambda(t)}{C_n^\lambda(1)} dt.$$

We have used "$\approx$" as a-priori we do not know convergence of the expansion. However, it turns out that positivity of the coefficients ($\widehat{F}(n;\lambda) \geq 0$) and F being continuous is sufficient for uniform and absolute convergence of the expansion (2), see [BD19, Lemma 2.3] for details. The singularities of the functions F_δ may be treated by standard continuous modifications:

$$F_{\delta,\epsilon}(t) = \begin{cases} (\epsilon + \arccos(t))^\delta & \text{if } \delta \neq 0, \\ \log\left(\frac{\pi}{\epsilon + \arccos(t)}\right) & \text{if } \delta = 0. \end{cases} \tag{3}$$

We now explain what we mean by positive definiteness. In this context, it is simply the statement that for any $\mu \in \mathcal{B}$ and $n \geq 0$:

$$\int_{\mathbb{S}^d} \int_{\mathbb{S}^d} C_n^\lambda(x \cdot y) d\mu(x) d\mu(y) \geq 0. \tag{4}$$

The inequality (4) is at the heart of quite a few of the topics here; the proof cited by [BD19] utilizes the addition formula for spherical harmonics.

Positive definiteness (4) together with the Gegenbauer expansion (2) allow us to write:

$$\begin{aligned} I_F(\mu) &= \sum_{n=0}^{\infty} \widehat{F}(n;\lambda) \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \frac{n+\lambda}{\lambda} C_n^{\lambda}(x\cdot y)d\mu(x)d\mu(y) \\ &= \widehat{F}(0;\lambda) + \sum_{n=1}^{\infty} \widehat{F}(n;\lambda) b_{n,\mu}, \end{aligned}$$

where $\forall n,\ b_{n,\mu} \geq 0$ by positive definiteness (4). Again by (4), and taking $\mu = \frac{1}{2}(\delta_{x_0} + \delta_{y_0})$ for arbitrary x_0, y_0, we see that

$$|C_n^{\lambda}(x_0 \cdot y_0)| \leq C_n^{\lambda}(1). \tag{5}$$

Integrating (5) over x, y wrt $\mu \in \mathcal{B}$, we get $b_{n,\mu} \leq a_n^d \triangleq \frac{n+\lambda}{\lambda} C_n^{\lambda}(1)$. At this stage we would like to use the "nonnegativity of Gegenbauer coefficients" alluded to in the impressionistic sketch ($\forall n \geq 1,\ \widehat{F}(n;\lambda) \geq 0$) to obtain extrema of $I_F(\mu)$ over $\mu \in \mathcal{B}$. This is certainly not obvious for the relevant δ ranges for the functions F given in (3), and forms the content of [BD19, Lemma 3.2]. This nonnegativity is established via the Rodrigues formula for Gegenbauer polynomials together with the Maclaurin series for $F_{\delta,\epsilon}$ given in (3).

Here we content ourselves with obtaining extrema assuming the above nonnegativity of coefficients. First, we have:

$$I_F(\mu) \geq \widehat{F}(0;\lambda) = I_F(\sigma),$$

since $b_{n,\mu} \geq 0$.

Similarly, we have:

$$I_F(\mu) \leq \widehat{F}(0;\lambda) + \sum_{n=1}^{\infty} \widehat{F}(n;\lambda) a_n^d = F(1) = I_F(\delta_e),$$

where e is an arbitrary unit vector.

The above proves one direction of [BD19, Propn. 2.1], namely nonnegativity of Gegenbauer coefficients implying that δ_p, σ are extrema of $I_F(\mu)$. The converse direction is based on a perturbative argument, where if $\widehat{F}(n;\lambda) < 0$

for some $n > 0$, one can take μ defined via $d\mu(x) = (1 - \epsilon Y_n(x))d\sigma(x)$ for sufficiently small ϵ. Here, Y_n is a spherical harmonic of degree n on $\mathbb{S}^d$.

Uniqueness statements are contained in [BD19, Propn. 2.2]. Basically, one examines conditions for equality in the above inequalities on $I_F(\mu)$. Strict positivity of the nonzero Gegenbauer coefficients allows one to conclude for the lower bound on $I_F(\mu)$ that $\forall n \geq 1,\ b_{n,\mu} = 0$, and a similar statement for the upper bound. This rather strong information forces $\mu = \sigma, \delta_p$ respectively.

For ease of exposition the above focuses on the first two statements of Theorem 1, though it is possible to obtain the last two statements (covering the symmetric, antipodal measures) by very similar methods. Essentially, one needs alternation of signs of the Gegenbauer coefficeints $\widehat{F}(n;\lambda)$ with n; the relevant precise statement is [BD19, Lemma 2.6]. There are alternative approaches to these latter statements via the Stolarsky principle as obtained in [BDM18] and a particularly simple one due to Tan [Tan17].

3.3 Stolarsky principle

We briefly discuss another theme present in [BD19], namely the classical Stolarsky principle [Sto73]. This relates the difference between discrete and continuous energies with a notion of "L^2 discrepancy" over spherical caps. As mentioned in [BD19], Stolarsky [Sto73] established an identity relating the L^2 discrepancy with the sums of pairwise Euclidean distances between points of the finite point configuration $Z \triangleq \{z_1, \ldots, z_N\}$. In [BD19, Theorem 4.2], a general form of this principle for energies is presented. However, we note that such a form is already implicit in [Sto73]; see in particular the remarks at the top of [Sto73, p. 577].

First, we define L^2 discrepancy of $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ with respect to $f : [-1, 1] \to \mathbb{R}$:

$$D_{L^2,f}(Z) \triangleq \left(\int_{\mathbb{S}^d} \left| \int_{\mathbb{S}^d} f(x \cdot y) d\sigma(y) - \frac{1}{N} \sum_{j=1}^{N} f(x \cdot z_j) \right|^2 d\sigma(x) \right)^{\frac{1}{2}}.$$

We may also define the optimal L^2 discrepancy:

$$D_{L^2,f,N} = \inf_{Z} D_{L^2,f}(Z)$$

over sets Z with $|Z| = N$.

Next, any positive definite function F on $\mathbb{S}^d$ has an associated $f \in L^2_{w_\lambda}[-1,1]$ satisfying (see e.g. [BD19, Lemma 4.1]):

$$\forall x, y \in \mathbb{S}^d,\ F(x \cdot y) = \int_{\mathbb{S}^d} f(x \cdot z) f(z \cdot y) d\sigma(z). \tag{6}$$

Then we have [BD19, Theorem 4.2], [Sto73, Theorem 2]:

$$N^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} F(z_i \cdot z_j) = D^2_{L^2,f}(Z) + I_F(\sigma). \tag{7}$$

Together with some Fourier analysis and methods of discrepancy theory, one may use (7) to obtain [BD19, Theorem 4.2]:

$$C_d \min_{1 \le k \le c_d N^{1/d}} \widehat{F}(k;\lambda) \le D^2_{L^2,f,N} \le N^{-1} \max_{0 \le t \le c_d N^{-1/d}} (F(1) - F(\cos(t))),$$

for some $c_d, C_d > 0$. The notion of a small diameter area regular partition of $\mathbb{S}^d$ plays an important role in the proof. Further results are obtained in [BD19].

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4 Optimal asymptotic bounds for spherical designs (after A. Bondarenko, D. Radchenko and M. Viazovska [3])

A summary written by Stefanos Lappas

Abstract

The goal of this paper is to prove the conjecture of Korevaar and Meyers: for each $N \geq c_d t^d$ there exists a spherical t-design in the sphere S^d consisting of N points, where c_d is a constant depending only on d.

4.1 Introduction

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with the Lebesgue measure μ_d normalized by $\mu_d(S^d) = 1$. Delsarte, Goethals and Seidel [6] introduced the notion of a spherical design:

Definition 1. *A set of points $x_1, \ldots, x_N \in S^d$ is called a spherical t-design if*

$$\int_{S^d} P(x)\, d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d+1$ variables, of total degree at most t.

For each $t, d \in \mathbb{N}$ denote by $N(d,t)$ the minimal number of points in a spherical t-design in S^d. The following lower bound

$$N(d,t) \geq \begin{cases} \binom{d+k}{d} + \binom{d+k-1}{d} & \text{if } t = 2k, \\ \\ 2\binom{d+k}{d} & \text{if } t = 2k+1, \end{cases} \tag{1}$$

is proved in [6].

Definition 2. *A spherical t-design is called* tight *if the bound (1) is attained.*

Example 3. *The vertices of a regular $t+1$-gon form a tight spherical t-design in the circle, that is $N(1,t) = t+1$.*

In higher dimensions tight desings rarely exist. In particular, Bannai and Damerell [1, 2] have shown that tight spherical designs with $d \geq 2$ and $t \geq 4$ may exist only for $t = 4$, 5, 7 or 11.

On the other hand, Seymour and Zaslavsky [10] have proved that spherical t-designs exist for all d, $t \in \mathbb{N}$. However, this proof is nonconstructive and gives no idea of how big $N(d,t)$ is. So, a natural question is to ask how $N(d,t)$ differs from the tight bound (1). Generally, to find the exact value of $N(d,t)$ even for small d and t is a surprisingly hard problem. For example, everybody believes that 24 minimal vectors of the D_4 root lattice form a 5-design with minimal number of points in S^3, although it is only proved that $22 \leq N(3,5) \leq 24$; see [5].

The conjecture of Korevaar and Meyers attracted the interest of many mathematicians. For instance, Kuijlaars and Saff [7] emphasized the importance of this conjecture for $d = 2$, and revealed its relation to minimal energy problems.

In order to prove the conjecture of Korevaar and Meyers we employ the following result from the Brouwer degree theory [9].

Theorem 4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and Ω an open bounded subset, with boundary $\partial\Omega$, such that $0 \in \Omega \subset \mathbb{R}^n$. If $\langle x, f(x)\rangle > 0$ for all $x \in \partial\Omega$, then there exists $x \in \Omega$ satisfying $f(x) = 0$.*

4.2 Preliminaries and auxiliary results

Let $\mathcal{P}_t$ be the Hilbert space of polynomials P on S^d of degree at most t such that

$$\int_{S^d} P(x)d\mu_d(x) = 0,$$

equipped with the usual inner product

$$\langle P, Q\rangle = \int_{S^d} P(x)Q(x)d\mu_d(x).$$

By the Riesz representation theorem, for each point $x \in S^d$ there exists a unique polynomial $G_x \in \mathcal{P}_t$ such that

$$\langle G_x, Q\rangle = Q(x) \text{ for all } Q \in \mathcal{P}_t.$$

Then a set of points $x_1, \ldots, x_N \in S^d$ forms a spherical t-design if and only if

$$G_{x_1} + \cdots + G_{x_N} = 0. \tag{2}$$

The gradient of a differentiable function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is denoted by

$$\frac{\partial f}{\partial x} := \left(\frac{\partial f}{\partial \xi_1}, \ldots, \frac{\partial f}{\partial \xi_{d+1}} \right), \qquad x = (\xi_1, ..., \xi_{d+1}).$$

For a polynomial $Q \in \mathcal{P}_t$ we define the *spherical gradient* as follows:

$$\nabla Q(x) := \frac{\partial}{\partial x} Q\left(\frac{x}{|x|} \right), \tag{3}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{d+1}$.

Remark 5. *Let us define the following open subset Ω of a vector space $\mathcal{P}_t$,*

$$\Omega := \left\{ P \in \mathcal{P}_t \,\middle|\, \int_{S^d} |\nabla P(x)| d\mu_d(x) < 1 \right\}. \tag{4}$$

The proof of our main result is based on observing that the existence of a continuous mapping $F : \mathcal{P}_t \to (S^d)^N$, such that for all $P \in \partial\Omega$

$$\sum_{i=1}^{N} P(x_i(P)) > 0, \text{ where } F(P) = (x_1(P), ..., x_N(P)). \tag{5}$$

readily implies the existence of a spherical t-design in S^d consisting of N points.
Indeed, consider a mapping $L : (S^d)^N \to \mathcal{P}_t$ defined by

$$(x_1, \ldots, x_N) \xrightarrow{L} G_{x_1} + \cdots + G_{x_N},$$

and the following composition mapping $f = L \circ F : \mathcal{P}_t \to \mathcal{P}_t$. Clearly

$$\langle P, f(P) \rangle = \sum_{i=1}^{N} P(x_i(P))$$

for each $P \in \mathcal{P}_t$. Thus, applying Theorem 4 to the mapping f, the vector space $\mathcal{P}_t$, and the subset Ω defined by (4), we obtain that $f(Q) = 0$ for some $Q \in \mathcal{P}_t$. Hence, by (2), the components of $F(Q) = (x_1(Q), ..., x_N(Q))$ form a spherical t-design in S^d consisting of N points.

To construct the corresponding mapping F we extensively use the following notion of an area-regular partition.

Definition 6. *Let $\mathcal{R} = \{R_1, \ldots, R_N\}$ be a finite collection of closed sets $R_i \subset S^d$ such that $\cup_{i=1}^N R_i = S^d$ and $\mu_d(R_i \cap R_j) = 0$ for all $1 \leq i < j \leq N$. The partition $\mathcal{R}$ is called area-regular if $\mu_d(R_i) = 1/N$, $i = 1, \ldots, N$. The partition norm for $\mathcal{R}$ is defined by*

$$\|\mathcal{R}\| := \max_{R \in \mathcal{R}} \operatorname{diam} R,$$

where $\operatorname{diam} R$ *stands for the maximum geodesic distance between two points in R.*

We need the following fact on area-regular partitions (see Bourgain, Lindenstrauss [4] and Kuijlaars, Saff [7])

Theorem 7. *For each $N \in \mathbb{N}$ there exists an area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| \leq B_d N^{-1/d}$ for some constant B_d large enough.*

We will also use the following spherical Marcinkiewicz–Zygmund type inequality:

Theorem 8. *There exists a constant r_d such that for each area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m}$, each collection of points $x_i \in R_i$ ($i = 1, \ldots, N$), and each algebraic polynomial P of total degree m, the inequality*

$$\frac{1}{2} \int_{S^d} |P(x)| d\mu_d(x) \leq \frac{1}{N} \sum_{i=1}^N |P(x_i)| \leq \frac{3}{2} \int_{S^d} |P(x)| d\mu_d(x) \qquad (6)$$

holds.

Theorem 8 follows naturally from the proof of Theorem 3.1 in [8].

Corollary 9. *For each area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m+1}$, each collection of points $x_i \in R_i$ ($i = 1, \ldots, N$), and each algebraic polynomial P of total degree m,*

$$\frac{1}{3\sqrt{d}} \int_{S^d} |\nabla P(x)| d\mu_d(x) \leq \frac{1}{N} \sum_{i=1}^N |\nabla P(x_i)| \leq 3\sqrt{d} \int_{S^d} |\nabla P(x)| d\mu_d(x). \quad (7)$$

4.3 Main result

Theorem 10. *For each $N \geq C_d t^d$ there exists a spherical t-design in S^d consisting of N points, where C_d is sufficiently large positive constant depending only on d.*

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5 The Bourgain–Milman theorem

A summary written by Constantin Bilz and Gianmarco Brocchi after Bourgain–Milman [2] and Nazarov [5]

Abstract

We present the Bourgain–Milman theorem on Mahler's conjecture. We explain both the original proof [2] based on the geometry of normed spaces and Nazarov's proof [5] based on Hörmander's theorem.

5.1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex centrally symmetric bounded open and absorbing set and let $K^\circ = \{x \in \mathbb{R}^n : |\langle x, y\rangle| \leq 1 \text{ for all } y \in K\}$ be the *polar set* of K. Let vol denote n-dimensional volume and let B_n be the n-dimensional euclidean ball.

Consider the affine invariant quantity $\operatorname{vol} K \cdot \operatorname{vol} K^\circ$. It holds that

$$\frac{4^n}{(n!)^2} \leq \operatorname{vol} K \cdot \operatorname{vol} K^\circ \leq (\operatorname{vol} B_n)^2.$$

The upper bound is sharp and was obtained by Santaló [7], improving on the upper bound 4^n established earlier by Mahler [4]. The lower bound was also proved by Mahler and he conjectured that it can be improved to

$$\operatorname{vol} C_n \cdot \operatorname{vol} C_n^\circ = \frac{4^n}{n!} \leq \operatorname{vol} K \cdot \operatorname{vol} K^\circ \tag{1}$$

so that the symmetric hypercube C_n would be minimising. He proved this for $n = 2$. Partial progress towards (1) in higher dimensions has been made by several authors, see e.g. [1] and the citations in [2]. We will present two proofs of the following

Theorem 1 (Bourgain–Milman)**.** *There exists a constant $c > 0$ independent of the dimension d such that*

$$\operatorname{vol} K \cdot \operatorname{vol} K^\circ \geq c^n \operatorname{vol} C_n \cdot \operatorname{vol} C_n^\circ. \tag{2}$$

We remark that the largest known constant for which Theorem 1 holds is $c = \frac{\pi}{4}$ and this is due to Kuperberg [3].

5.2 The proof of Bourgain–Milman

Let μ_{n-1} be the normalized surface measure on the euclidean unit sphere S^{n-1}. We denote the norm on $\mathbb{R}^n$ with unit ball K by $\|\cdot\|_K$. We write $E(K)$ for the normed space $(\mathbb{R}^n, \|\cdot\|_K)$ and we write

$$M_K = \int_{S^{n-1}} \|x\|_K \, d\mu_{n-1}(x), \qquad d_K = \frac{\sup_{x\in S^{n-1}} \|x\|_K}{\inf_{x\in S^{n-1}} \|x\|_K}.$$

The (multiplicative) *Banach–Mazur distance* between $(\mathbb{R}^n, |\cdot|_2)$ and $E(K)$ is

$$d_{E(K)} = \inf\{d_{u(K)} \mid u : \mathbb{R}^n \to \mathbb{R}^n \text{ linear isomorphism}\}.$$

The proof is based on an analysis of the linear structure of the convex body K starting with the following result.

Proposition 2. *Let $\lambda \in (0,1)$. There exists a subspace F of $E(K)$ such that*

$$\dim F \geq \lambda n \quad \textit{and} \quad \|x\|_K \geq c(1-\lambda) M_{K^\circ}^{-1} |x| \quad \textit{for any } x \in F.$$

Proof sketch. We apply the isoperimetric inequality on S^{n-1} to the geodesic $\pi/4$-neighbourhood $A_{\pi/4}$ of the set $A = \{\|x\|_{K^\circ} \leq 2M_{K^\circ}\}$. For any $k < n$ we hence find a k-dimensional subspace F that has a large intersection with $A_{\pi/4}$, namely

$$\mu_{k-1}(A_{\pi/4} \cap F) \geq 1 - \frac{\mathrm{vol}_{n-2} S^{n-2}}{\mathrm{vol}_{n-1} S^{n-1}} \int_0^{\pi/4} \sin^{n-2} t \, dt.$$

If $\tau \sim 1 - k/n$ and $x \in F \cap S^{n-1}$, then this implies

$$\mu_{k-1}(A_{\pi/4} \cap F) > 1 - \mu_{k-1}(B_{\pi/4-\tau}(x))$$

where $B_\epsilon(x) \subset F \cap S^{n-1}$ is the ball with respect to geodesic distance. Then we have $F \cap S^{n-1} \subset A_{\pi/2-\tau}$. This implies the proposition. □

We will combine this with an upper bound on M_{K°. Such a bound is provided by the following result which is well-known in the geometry of Banach spaces.

Proposition 3. *There is a linear isomorphism $u : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$M_{u(K)} \cdot M_{u(K)^\circ} \leq C(1 + \log d_{E(K)})^2.$$

We can now prove the following "subspace of quotient" result.

Lemma 4. *Let $\lambda \in (0,1)$. Then there exists a subspace F of $\mathbb{R}^n$ and a quotient space G of F such that*

$$\dim G \geq \lambda n \quad \textit{and} \quad d_G \leq C(1-\lambda)^{-2}(1+\log d_{E(K)})^2.$$

Proof sketch. We apply Proposition 2 twice. First, we find a subspace F of $E(K)$ with $\dim F \geq \sqrt{\lambda} n$ and by duality

$$\|x\|_{K^\circ} \leq C(1-\sqrt{\lambda})^{-1} M_{K^\circ}|x| \quad \text{for any } x \in F^*.$$

Here F^* denotes the dual space of F. Secondly, we find a subspace G of F^* such that $\dim G \geq \lambda n$ and

$$\|x\|_{K^\circ} \geq c(1-\sqrt{\lambda}) M_K^{-1}|x| \quad \text{for any } x \in G.$$

Now we replace K by the $u(K)$ from Proposition 3 and use the definition of d_G to complete the proof. □

Sketch of proof of Theorem 1. Fix an integer N. For $n \leq N$ let $\mathcal{C}_n(t)$ be the class of convex bodies K in $\mathbb{R}^n$ for which $d_{E(K)} \leq t$. We write

$$\sigma(t) = \inf_{\substack{n \leq N \\ K \in \mathcal{C}_n(t)}} \left(\frac{\mathrm{vol}_n K \cdot \mathrm{vol}_n K^\circ}{(\mathrm{vol}_n B_n)^2} \right)^{1/n}.$$

Using Lemma 4 we will show in the talk that

$$\sigma(t) \geq c^{\frac{1}{\log t}} \sigma(C(\log t)^6)$$

with constants independent of N. This inequality implies a uniform lower bound for $\sigma(t)$ which proves the theorem. □

5.3 An alternative proof via Hörmander's theorem

We can prove (2) constructing an analytic function on $\mathbb{C}^n$ with good decay property. By the Paley–Wiener theorem, given any $g \in L^2(K^\circ)$ its Fourier transform $f(w) = \int_{K^\circ} g(v) e^{-i\langle w,v \rangle} \mathrm{d}v$ extends to an entire function on

$\mathbb{C}^n$. Applying Cauchy–Schwarz $|f(0)|^2 \leq \|g\|^2_{L^2(K^\circ)} \operatorname{vol} K^\circ$, and Plancherel $\|f\|^2_{L^2(\mathbb{R}^n)} = (2\pi)^n \|g\|^2_{L^2(K^\circ)}$ we have the lower bound

$$\operatorname{vol} K^\circ \geq (2\pi)^n \frac{|f(0)|^2}{\|f\|^2_{L^2(\mathbb{R}^n)}}.$$

We want an entire function which $L^2(\mathbb{R}^n)$-norm is not too large compared with its value at the origin. We look for such a function in a Bergman space with Hörmander type weight, i.e. $L^2(\mathbb{C}^n, e^{-\varphi})$ where φ is plurisubharmonic.

Let T_K be the (horizontal) tube domain $\{x + iy : x \in \mathbb{R}^n, y \in K\}$ and consider the Bergman space $A^2(T_K) = \{\text{analytic functions on } T_K\} \cap L^2(T_K)$.

This is a Hilbert space with reproducing kernel

$$\mathcal{K}(z, w) = \int_{\mathbb{R}^n} \frac{e^{i\langle z - \bar{w}, v\rangle}}{\int_K e^{-2\langle x,v\rangle} \mathrm{d}x} \frac{\mathrm{d}v}{(2\pi)^n}.$$

An application of Cauchy–Schwarz gives

$$|f(0)|^2 = \left| \int_{T_K} \mathcal{K}(0, w) f(w) \mathrm{d}w \right|^2 \leq \int |\mathcal{K}(0, w)|^2 \int |f(w)|^2 = \mathcal{K}(0, 0) \|f\|^2_{A^2(T_K)}$$

from which we have the lower bound for $\mathcal{K}(0, 0)$

$$\frac{|f(0)|^2}{\|f\|^2_{A^2(T_K)}} \leq \mathcal{K}(0, 0) = \int_{\mathbb{R}^n} \frac{1}{\int_K e^{-2\langle x,v\rangle} \mathrm{d}x} \frac{\mathrm{d}v}{(2\pi)^n} \leq \frac{n!}{\pi^n} \frac{\operatorname{vol} K^\circ}{\operatorname{vol} K}$$

and the upper one by using the convexity of $x \mapsto e^{-\langle x,v\rangle}$ and optimising in v.

Up to affine linear transformations, we can assume that K contains the ball $B(0, r)$. By the John's ellipsoid theorem, $K \subset B(0, R)$ with $R/r \leq \sqrt{n}$. For any $t \in K^\circ$, the Hermitian product $z \mapsto \langle z, t\rangle$ maps T_K in the strip $S = \{\zeta \in \mathbb{C} : |\Im(\zeta)| < 1\}$, while the conformal map

$$\phi(\zeta) = \frac{4}{\pi} \frac{e^{\frac{\pi}{2}\zeta} - 1}{e^{\frac{\pi}{2}\zeta} + 1}$$

maps the strip S to the disk $D(0, \frac{4}{\pi})$. Consider the set

$$K_{\mathbb{C}} := \{z \in \mathbb{C}^n : |\langle z, t\rangle| \leq 1, \forall t \in K^\circ\} \subset T_K.$$

Note that $K_{\mathbb{C}}$ contains $\frac{1}{\sqrt{2}}(K + iK)$. It is enough to construct an analytic function inside $K_{\mathbb{C}}$. For this purpose we will use the Hörmander's theorem.

Definition 5. *A function* $\varphi\colon \Omega \subset \mathbb{C}^n \to \mathbb{R}$ *is* strictly plurisubharmonic *if there exists* $\tau > 0$ *such that*

$$\langle H(z)w, w\rangle \geq \tau|w|^2, \quad \forall w \in \mathbb{C}^n\,, \forall z \in \Omega$$

where H *is the Hermitian matrix* $H = \left(\dfrac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right)_{i,j=1}^n$.

Theorem 6 (Hörmander). *Let* $\Omega \subset \mathbb{C}^n$ *be an open, pseudoconvex domain, and let* $\varphi\colon \Omega \to \mathbb{R}$ *be strictly plurisubharmonic for a* $\tau > 0$*. For any* $(0,1)$*-form* ω *on* Ω *with* $\bar{\partial}\omega = 0$*, there exists a solution* h *of* $\bar{\partial}h = \omega$ *in* Ω *satisfying*

$$\int_\Omega |h|^2 e^{-\varphi} \mathrm{d}z \leq \tau^{-1} \int_\Omega |\omega|^2 e^{-\varphi} \mathrm{d}z.$$

We take the plurisubharmonic function φ on a shrunk version of $K_{\mathbb{C}}$:

$$\varphi(z) = \frac{|\Im(z)|^2}{R^2} + \log \sup_{t \in K^\circ} |\phi(\langle z, t\rangle)|^{2n}.$$

The first term enforces the strict plurisubharmonicity on any ball of radius $\delta < R$ with $\tau = \delta^2/R^2$. The second term ensures that the function h promised by the theorem will vanish at 0, as soon as $\int |\omega|^2 e^{-\varphi}$ is finite. Indeed, since $\phi(0) = 0$ and $\phi'(0) = 1$, using Taylor we see that $|\phi(\zeta)| \sim |\zeta|$ near the origin, and so $e^{-\varphi} \sim |z|^{-2n}$ which is not locally integrable at 0. Also note that $\varphi(z) \leq 2n \log(4/\pi) + 1$ for $z \in K_{\mathbb{C}}$.

Fix a small δ and let g be a cut-off function on $\delta K_{\mathbb{C}}$. Applying the Hörmander theorem to $-\bar{\partial}g$ produces h such that $\bar{\partial}(h+g) = 0$. Call $f = h+g$ this holomorphic extension of g. Then $f(0) = 1$ and

$$\begin{aligned}\|f\|^2_{A^2(T_K)} &\leq 2(\|h\|^2_{L^2(T_K)} + \|g\|^2_{L^2(T_K)}) \\ &\leq 2(\|e^\varphi\|_{L^\infty} R^2 \delta^{-2} \|\bar{\partial}g\|^2_{L^2(e^{-\varphi})} + \|g\|^2_{L^2}).\end{aligned}$$

One can choose g appropriately so that $\|f\|^2_{A^2(T_K)} \leq \left(\frac{4}{\pi}\right)^{2n} e^{o(n)} (\operatorname{vol} K)^2$ as $\delta \to 0$. This gives the lower bound

$$\left(\frac{\pi}{4}\right)^{2n} \frac{e^{-o(n)}}{(\operatorname{vol} K)^2} \leq \mathcal{K}(0,0) \leq \frac{n!}{\pi^n} \frac{\operatorname{vol} K^\circ}{\operatorname{vol} K}.$$

One can remove the exponential factor with a "tensor power trick" to obtain

$$\left(\frac{\pi}{4}\right)^{2n} \leq \frac{n!}{\pi^n} \operatorname{vol} K^\circ \operatorname{vol} K$$

which gives the value $c = \left(\frac{\pi}{4}\right)^3$ in (2).

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6 Gaussian subordination for the Beurling-Selberg extremal problem (after E. Carneiro, F. Littmann and J. Vaaler [1])

A summary written by Nuria Storch-de-Gracia

Abstract

We solve a variant of the Beurling-Selberg extremal problem by determining real valued entire functions of exponential type that minorize and majorize the real Gaussian function $x \mapsto e^{-\pi\lambda x^2}$, where λ is a positive parameter.

6.1 Introduction

We say that an entire function $F : \mathbb{C} \to \mathbb{C}$ is of exponential type at most $2\pi\delta$ for some $\delta > 0$ if for each $\varepsilon > 0$ there exists $C > 0$ such that $|F(z)| \leq Ce^{(2\pi\delta+\varepsilon)|z|}$, for every $z \in \mathbb{C}$. An interesting remark is that these functions are closely related to the family of entire functions whose restriction to the real line has compactly supported Fourier tranform. For instance, if we assume that F is entire and $F \in L^p(\mathbb{R})$ with $1 \leq p \leq 2$, then F has exponential type at most $2\pi\delta$ if and only if the Fourier tranform of F restricted to the real line is supported on $[-\delta, \delta]$. This is known as the Paley-Wiener theorem, and a proof of this statement may be found in [8].

Now we can present the main problem that will be treated here. Let $f : \mathbb{R} \to \mathbb{R}$, then the Beurling-Selberg extremal problem for f consists of finding an entire function $F : \mathbb{C} \to \mathbb{C}$ of exponential type at most $2\pi\delta$ such that the integral

$$\int_{-\infty}^{\infty} |f(x) - F(x)| dx$$

is minimized. We can also define an important variant of this problem by imposing additional conditions on the function $F(z)$. For instance, we say that F is an extreme minorant of f if it solves the Beurling-Selberg problem for f, it is real valued on $\mathbb{R}$ (we will just say real valued for simplicity) and it satisfies $F(x) \leq f(x)$ for each $x \in \mathbb{R}$. Analogously, if F satisfies $f(x) \leq F(x)$ for all $x \in \mathbb{R}$ instead we say that F is an extreme majorant of f.

This extremal problem was introduced by A. Beurling in the late 1930's for the function $f(x) = \operatorname{sgn}(x)$. The reader may find information about the early development of this theory in [7].

In this summary, we are devoted to determining the extreme minorants and majorants of the Gaussian function $G_\lambda(x) := e^{-\pi\lambda x^2}$, where $\lambda > 0$ is a parameter. Some interesting functions which have applications on analytic number theory arise from these results. The authors presented in [1] a useful technique to determine extremal minorants and majorants for various functions based on these solutions for the Gaussian and tempered distribution arguments. More specifically, they proved that if ν is a finite non-negative Borel measure on $(0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ is a function given by $g(x) = \int_0^\infty G_\lambda(x) d\nu(x)$, then there exists a unique minorant $l(z)$ of exponential type at most 2π for $g(x)$ which interpolates the values of $g(x)$ at $\mathbb{Z} + \frac{1}{2}$. Moreover, this result allows us to determine the value of such function and the corresponding minimal integral. The case for the majorant is analogous. This procedure has recent applications to the theory of the Riemann zeta-function. Assuming that the Riemann hypothesis holds, E. Carneiro and V. Chandee obtained in 2011 upper and lower bounds of $|\zeta(\alpha + it)|$ for large t and α in the critial strip using the extremal minorants and majorants of the function $x \mapsto \log\left(\frac{4+x^2}{\left(\alpha-\frac{1}{2}\right)^2+x^2}\right)$, and improving the bounds known before (see [2]). These were obtained noting that the measure $d\nu(\lambda) := \frac{G_\lambda(\alpha)-G_\lambda(\beta)}{\lambda}$ satifies the assumptions of the theorem for $0 < \alpha < \beta$ and $-\log\left(\frac{\alpha^2+x^2}{\beta^2+x^2}\right) = \int_0^\infty G_\lambda(x) d\nu(\lambda)$ holds.

A similar situation occurs when bounding the argument function $S(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right)$ for large t assuming RH, where the argument is defined by continuous variation along the line segments joining $2, 2 + it$ and $\frac{1}{2} + it$, taking the argument of $\zeta(s)$ at 2 to be zero. In 2013, E. Carneiro, V. Chandee and M.B. Milinovich bounded $S_1(t) := \int_0^t S(u)du$. They used the extremal minorant and majorant of the function $x \mapsto 1 - x \arctan\left(\frac{1}{x}\right)$, which could be obtained from [1] integrating the parameter λ of the Gaussian with respect to a suitable non-negative Borel measure on $(0, \infty)$. Other examples of minorants and majorants obtained from those of the Gaussian can be found in [1].

6.2 Preliminaries

In order to approach the Beurling-Selberg extremal problem for G_λ, we first need to recall the definitions of some basic theta functions and some of its properties. If we denote $q := e^{\pi i\tau}$ and $e(z) := e^{2\pi i z}$ we may define, following the notation used in [1], the functions given by

$$\theta_2(v,\tau) := \sum_{n=-\infty}^{\infty} q^{n^2} e(nv) \quad \text{and} \quad \theta_3(v,\tau) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e(nv),$$

where $v, \tau \in \mathbb{C}$ with $\operatorname{Im}(\tau) > 0$. For fixed τ with $\operatorname{Im}(\tau) > 0$ the functions $v \mapsto \theta_i(v,\tau)$ are even entire functions of v for $i = 2, 3$. Recall that the maps $v \mapsto \theta_i(v,\tau)$ are periodic with period 1 for $i = 2, 3$, and they satisfy the equality $\theta_2\left(v + \frac{1}{2}, \tau\right) = \theta_3(v,\tau)$, for all $v \in \mathbb{C}$. One can alternatively give expressions of these theta functions in terms of the Gaussian $G_\lambda(z)$ (see [4]). In particular, we are interested in the following:

$$\begin{aligned} \lambda^{-\frac{1}{2}}\theta_2(v, i\lambda^{-1}) &= \sum_{n=-\infty}^{\infty} G_\lambda\left(n + \frac{1}{2} - v\right), \\ \lambda^{-\frac{1}{2}}\theta_3(v, i\lambda^{-1}) &= \sum_{n=-\infty}^{\infty} G_\lambda(n - v), \end{aligned} \tag{1}$$

for all $v \in \mathbb{C}$. We also need to introduce three functions which are crucial for solving our main problem. We first define

$$\begin{aligned} L_\lambda(z) &:= \left(\frac{\cos \pi z}{\pi}\right)^2 \left(\sum_{m=-\infty}^{\infty} \frac{G_\lambda\left(m + \frac{1}{2}\right)}{\left(z - m - \frac{1}{2}\right)^2} + \sum_{n=-\infty}^{\infty} \frac{G'_\lambda\left(n + \frac{1}{2}\right)}{z - n - \frac{1}{2}}\right), \\ M_\lambda(z) &:= \left(\frac{\cos \pi z}{\pi}\right)^2 \left(\sum_{m=-\infty}^{\infty} \frac{G_\lambda(m)}{(z - m)^2} + \sum_{n=-\infty}^{\infty} \frac{G'_\lambda(n)}{z - n}\right), \end{aligned}$$

where $\lambda > 0$ is a parameter and $z \in \mathbb{C}$. These two entire functions are of exponential type at most 2π. L_λ interpolates G_λ and G'_λ on the coset $\mathbb{Z} + \frac{1}{2}$, and so does M_λ on $\mathbb{Z}$. The definition of these functions is motivated by the work of S. W. Graham and J.D. Vaaler (see [5]). They described a method to produce minorants and majorants of more general special functions f by constructing entire functions of exponential type which interpolate $f(x)$ and $f'(x)$ at the points of some appropriate set. E. Carneiro et al. showed the following theorem in [1]:

Theorem 1. *For all real values of x we have*

$$0 \leq \left(\frac{\pi}{\cos \pi x}\right)^2 (G_\lambda(x) - L_\lambda(x)) \text{ and } 0 \leq \left(\frac{\pi}{\sin \pi x}\right)^2 (M_\lambda(x) - G_\lambda(x)).$$

In particular $L_\lambda(x) \leq G_\lambda(x) \leq M_\lambda(x)$ holds for every $x \in \mathbb{R}$.

Morever, one can prove that the functions L_λ and M_λ are the unique extreme minorant and majorant of G_λ, respectively, for each $\lambda > 0$ which is the main purpose of this summary, and it is be the aim of next section.

6.3 Extreme minorants and majorants of $G_\lambda(z)$

The following theorems formalize the ideas previously presented. Theorem 2 solves the problem of minorizing $G_\lambda(z)$ on $\mathbb{R}$ by real valued entire functions of exponential type at most 2π.

Theorem 2. *Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function of exponential type at most 2π which is real valued on $\mathbb{R}$ and*

$$F(x) \leq G_\lambda(x) \tag{2}$$

for all $x \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} F(x)dx \leq \lambda^{\frac{1}{2}} \theta_2(0, i\lambda^{-1}), \tag{3}$$

with equality if and only if $F(z) = L_\lambda(z)$ for all $z \in \mathbb{C}$.

Analogously, Theorem 3 solves the problem of majorizing $G_\lambda(z)$ on $\mathbb{R}$ by real valued entire functions of exponential type at most 2π.

Theorem 3. *Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function of exponential type at most 2π which is real valued on $\mathbb{R}$ and*

$$G_\lambda(x) \leq F(x) \tag{4}$$

for all $x \in \mathbb{R}$. Then

$$\lambda^{\frac{1}{2}} \theta_3(0, i\lambda^{-1}) \leq \int_{-\infty}^{\infty} F(x)dx, \tag{5}$$

with equality if and only if $F(z) = M_\lambda(z)$ for all $z \in \mathbb{C}$.

More generally, we might be interested in obtaning solutions of exponential type at most $2\pi\delta$ instead. It is now elementary to deduce from Theorems 2 and 3 that the unique extremal minorant and majorant of $G_\lambda(z)$ are given by the maps $z \mapsto L_{\lambda\delta^{-2}}(\delta z)$ and $z \mapsto M_{\lambda\delta^{-2}}(\delta z)$, respectively.

Proof of Theorem 2. We first prove the lower bound given by (3). Let $F(z)$ be an entire function of exponential type at most 2π that satisfies (2). We may assume that the function $x \mapsto F(x)$ is integrable for each $x \in \mathbb{R}$. Otherwise, the bound (3) is trivial. Then, applying the Poisson summation formula together with the fact that $\widehat{F}$ is continuous and compactly supported on $[-1, 1]$, (1) and condition (2) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} F(x)dx &= \lim_{N\to\infty} \sum_{n=-N}^{N} F(v+n) \\ &\leq \lim_{N\to\infty} \sum_{n=-N}^{N} G_\lambda(v+n) = \lambda^{-\frac{1}{2}}\theta_2\left(\frac{1}{2} - v, i\lambda^{-1}\right)\end{aligned} \tag{6}$$

for each $v \in \mathbb{R}$. Note that using the Poisson summation formula is justified since by the Plancherel-Pólya theorem (see [6]) the sequence $\{F(n+v)\}_{n\in\mathbb{N}}$ is summable. This implies, by a generalization on Bernstein's inequality, that $F'(x)$ is integrable and therefore F has bounded variation. From (1) one deduces that $v \mapsto \theta_2\left(\frac{1}{2} - v, i\lambda^{-1}\right)$ takes its minimum value at $v = \frac{1}{2}$. Therefore, the inequality (3) is satisfied.

Let us now prove that (3) is satisfied with equality if $F(z) = L_\lambda(z)$ for all $z \in \mathbb{C}$. Note that by Theorem 1 L_λ is a minorant of G_λ, i.e. $L_\lambda(x) \leq G_\lambda(x)$ for all real x. As L_λ interpolates G_λ on the coset $\mathbb{Z} + \frac{1}{2}$ then the inequality (6) is an equality in this case. Therefore we have

$$\int_{-\infty}^{\infty} L_\lambda(x)dx = \lambda^{\frac{1}{2}}\theta_2(0, i\lambda^{-1}).$$

Conversely, let us now assume that $F(z)$ is an entire function of exponential type at most 2π that satisfies (2) for each real x and there is equality in the inequality (6). The last assumption implies that $v = \frac{1}{2}$ and this gives $F(n+\frac{1}{2}) = G_\lambda(n+\frac{1}{2})$ for all $n \in \mathbb{Z}$. Moreover, since we have assumed that (2) holds for each real and L_λ interpolates G_λ on the coset $\mathbb{Z}+\frac{1}{2}$ then necessarily $F'(n+\frac{1}{2}) = G'_\lambda(n+\frac{1}{2})$ for all $n \in \mathbb{Z}$. Therefore, the entire function

$$z \mapsto F(z) - L_\lambda(z)$$

has exponential type at most 2π, vanishes at each point of $\mathbb{Z}+\frac{1}{2}$, and so does its derivative.

S. W. Graham and J.D. Vaaler proved in Lemma 4 in [5] that if there exists a real number x_0 such that $G(x_0+n) = G'(x_0+n) = 0$ for every integer n, then $G(z) = 0$ for all $z \in \mathbb{C}$ given that $G(z)$ is an entire function such that G is integrable on the real line and $\text{supp}(\widehat{G}) \subseteq [-1, 1]$. Therefore, an application of the Paley-Wiener theorem gives $F(z) = L_\lambda(z)$ for each $z \in \mathbb{C}$ as desired.

□

One observes easily that the proof of Theorem 3 follows similarly using the properties of the theta functions convinently.

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7 Fourier optimization and prime gaps (after E. Carneiro, M. B. Milinovich, and K. Soundararajan [1]

A summary written by Zirui Zhou

Abstract

This paper studies a new set of extremal problems in Fourier analysis, motivated by a problem in prime number theory.

7.1 Introduction

We establish a connection between the extremal problems in Fourier analysis we study in this paper and the problem of bounding the largest possible gap between consecutive primes (assuming the Riemann hypothesis).

We are interested in the whether the universal constant c in the classical result

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{p_n} log(p_n)} \leq c \tag{1}$$

can get smaller than the current best form of this bound $c = 1$ due to Dudek[3]. This paper gives an affirmative answer to this question and improved other bounds by (i) using Guinand-Weil explicit formula connecting the prime numbers to the root of Riemann-Zeta function, and (ii) known bounds on Brun-Titchmarsh constant and (iii) transforming the above question to an extremization problem in Fourier analysis.

Define the Brun-Titchmarsh constant

$$B = \limsup_{n\to\infty} \frac{\pi(x+\sqrt{x}) - \pi(x)}{\sqrt{x}/log(x)}.$$

Known bounds for B are $1 \leq B \leq \frac{36}{11}$,the first by prime number theorem $\pi(x) \sim x/log(x)$, and the latter bound by works of Iwaniec[5].

Define the following two Fourier optimization problems:

Definition 1. *Given* $1 \leq A < \infty$*, define*

$$C(A) := \sup_{F\in\mathcal{A}, F\neq 0} |F(0)| - A\int_{[-1,1]^c} |\hat{F}(t)|dt \tag{2}$$

where the supremum is taken over the class $\mathcal{A}$ of continuous functions $F : \mathbb{R} \to \mathbb{C}$, with $F \in L^1(\mathbb{R})$. In the case $A = \infty$, determine $C(\infty) = \sup\limits_{F\in\mathcal{E}, F\neq 0} \frac{|F(0)|}{\|F\|_1}$ where the supremum is over the subclass $\mathcal{E} \subset \mathcal{A}$ of continuous functions $F : \mathbb{R} \to \mathbb{C}$, with $F \in L^1(\mathbb{R})$ and supp$(\hat{F}) \subset [-1, 1]$.

Definition 2. *Given $1 \leq A < \infty$, define*

$$C^+(A) := \sup_{F\in\mathcal{A}^+, F\neq 0} \frac{1}{\|F\|} \left(F(0) - A \int_{[-1,1]^c} (\hat{F}(\xi))_+ d\xi \right) \tag{3}$$

where the supremum is taken over the class $\mathcal{A}^+$ of even and continuous functions $F : \mathbb{R} \to \mathbb{R}$, with $F \in L^1(\mathbb{R})$. In the case $A = \infty$, set $C^+(\infty) = \sup_{F\in\mathcal{E},F\neq 0} \frac{F(0)}{\|F\|_1}$ where $\mathcal{E}^+ \subset \mathcal{A}^+$ is the set of even and continuous functions $F : \mathbb{R} \to \mathbb{R}$ with $\hat{F}(t) \leq 0, \forall |t| \geq 1$.

Here comes the main theorem that connects the Brun-Titchmarsh constant and the fourier optimization problem to our question of interest concerning prime gaps.

Theorem 3. *Assume the Riemann hypothesis. Let $C^+(\cdot)$ be defined as above and B be the Brun Titchmarch constant. Then, for any $\alpha \geq 0$, we have*

$$\inf \left\{ c; \liminf_{x\to\infty} \frac{\pi(x + c\log(x)\sqrt{x}) - \pi(x)}{\sqrt{x}} > \alpha \right\} \leq \frac{1 + 2\alpha}{C^+(B)} \leq \frac{21}{25}(1 + 2\alpha) \tag{4}$$

Plugging in $\alpha = 0$ gives an affirmative answer to the question posed at the beginning. The last inequality follows from the characterizations of the optimizer $C^+(\cdot)$ (see claims and theorem 5 in the last section). We use the asymptotic analysis of Guinand-Weil explicit formula to prove theorem 3.

7.2 G-W Explicit formula and asymptotic estimates

Lemma 4. *(Guinand-Weil explicit formula) Let $h(s)$ be analytic in the strip $|Im(s)| \leq \frac{1}{2} + \epsilon$ for some $\epsilon > 0$,and assume that $|h(s)| << (1 + |s|)^{-(1+\delta)}$ for* *some $\delta > 0$ when $|Re(s)| \to \infty$. Then*

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\hat{h}(0)\log(\pi)$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u) Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) du$$
$$- \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\left(\hat{h}\left(\frac{\log(n)}{2\pi}\right) + \hat{h}\left(-\frac{\log(n)}{2\pi}\right)\right)$$

where $\rho = \beta + i\gamma$ are the non-trivial zeros of $\zeta(s)$, Γ'/Γ is the logarithmic derivative of the Gamma function, and $\Lambda(n)$ is the Von-Mangoldt function defined to be $\log(p)$ if $n = p^m$ with p a prime number and $m \geq 1$ an integer, and zero otherwise.

Motivated by the optimization problem, from now on we fix $F : \mathbb{R} \to \mathbb{R}$ to be an even and bandlimited ((i.e. functions with compactly supported Fourier transforms) Schwartz function, with $F(0) > 0$. Let us assume that $supp(\hat{F}) \subset [-N, N]$ for some parameter $N \geq 1$. It then follows that F extends to an entire function, which we continue calling F, and the fact that $x^2F(x) \in L^{\infty}(\mathbb{R})$ implies, via the Phragmen-Lindelof principle, that $|F(s)| << (1 + |s|)^{-2}$ when $|Re(s)| \to \infty$.

The idea of the proof is to translate and dilate $\hat{F}$ (as appearing on the righthand side of the formula) to concentrate the mass of F on the interval that is a large prime gap. We then try to understand the effect of this localization in all the terms of the formula through an asymptotic analysis. Define $f(x) = bF(bx)$, where parameter $0 < b < 1$ will be chosen small enough such that for another parameter $a \to \infty$, $2\pi bN \leq \log a$. This way $supp\hat{f} \subset [-bN, bN]$. let function h be defined by $h(z) = f(z)a^{iz}$. Then assuming Riemann Hypothesis, as we let $b \to 0^+$ and $a \to \infty$, the lemma gives the following inequality:

$$bF(0)(a^{1/2} + a^{-1/2}) + O(b^2\sqrt{a}) + O(1) \tag{5}$$
$$\leq \sum_{\gamma} |f(\gamma)| + \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\left(\hat{f}\left(\frac{\log(n/a)}{2\pi}\right) + \hat{f}\left(-\frac{\log(n/a)}{2\pi}\right)\right) \tag{6}$$

Let $N(x)$ denote the the number of zeroes $\rho = \beta + i\gamma$ of ζ such that $0 \leq \gamma \leq x$.

Using the fact that $N(x) = \frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} + O(\log x)$, we get

$$\sum_{\gamma} |f(\gamma)| = \frac{\log(1/2\pi b)}{2\pi} \|F\|_1 + O(1). \tag{7}$$

The summation over primes is where the Brun-Titchmarsh constant B comes in (To be specified in the talk). Fix $\alpha \geq 0$ and assume that c is a positive constant such that

$$\liminf_{x\to\infty} \frac{\pi(x + c\log(x)\sqrt{x}) - \pi(x)}{\sqrt{x}} \leq \alpha.$$

Then for any $\epsilon > 0$, there exists a sequence of $x_i \to \infty$ such that

$$\frac{\pi(x_i + c\log(x_i)\sqrt{x_i}) - \pi(x_i)}{\sqrt{x_i}} \leq \alpha + \epsilon$$

along this sequence. For each x_i, choose a_i and b_i such that $[x_i, x_i + c\sqrt{x_i}\log x_i] = [a_i e^{-2\pi b_i}, a_i e^{2\pi b_i}]$ (To symplify the notation, we will drop the subscript i in the next two equations, but we should keep in mind that they hold for different pairs of a, b chosen according to specific x). For any $B' > B$, we have

$$\sum_{1 \leq |\frac{\log(p/a_i)}{2\pi b_i}| \leq N} \frac{\log(p)}{\sqrt{p}} \hat{F}\left(\frac{\log(p)}{2\pi b}\right) \leq B' \int_{1 \leq |\frac{\log(p/a)}{2\pi b}| \leq N} \hat{F}\left(\frac{\log(t/a)}{2\pi b}\right)_+ \frac{dt}{\sqrt{t}} \tag{8}$$

$$= B' 2\pi b_i \sqrt{a_i} \int_{[-1,1]^c} \hat{F}(t)_+ dt + O(1) \tag{9}$$

Putting the assymptotic estimates together, we get that for any a, b chosen according to an x in the sequence,

$$b\sqrt{a}\left(F(0) - B' \int_{[-1,1]^c} (\hat{F}(t))_+ dt\right) \leq \frac{log(1/2\pi b)}{2\pi} \|F\|_1 + \frac{1}{2\pi} \|F\|_1 (\alpha + \epsilon) \log x + O(1).$$

Sending $x \to \infty$ and then $\epsilon \to 0^+$, $B' \to B^+$, and plugging in a and b in terms of c and $x(a = x + O(\sqrt{x}\log x), 4\pi b = c\frac{\log x}{\sqrt{x}} + O(\log^2 x))$, we get

$$c \leq (1 + 2\alpha) \frac{\|F\|_1}{\left(F(0) - B\int_{[-1,1]^c} (\hat{F}(t))_+ dt\right)}.$$

This suffices to prove the main theorem on prime gaps.

7.3 Fourier optimization

Recall the definition of $C(\cdot)$ and $C^+(\cdot)$ from introduction.
Claim: There exist extremizers for $C(A), 1 < A \leq \infty$ and $C^+(A)$ for $1 < A \leq \infty$, $C(1) = C^+(1) = 2$. The extremizers for $C(\infty)$ is unique up to a constant scalar.

Theorem 5. *(i) Let* $c_0 = \frac{4}{\pi}(\int_{-1}^{1} \frac{sin(\pi t)}{\pi t} dt)^{-1} = 1.07995...$, $d_0 = 1.09769$ *be the upper bound of* $C(\infty)$ *obtained in [4], and* $\lambda = \lambda(A)$ *be the unique solution of* $1 - \frac{1}{A} = \sin(\frac{\pi\lambda}{2}) - \frac{\pi\lambda}{2}\cos(\frac{\pi\lambda}{2})$ *with* $0 < \lambda < 1$, *then*

$$\max\left\{2A - 2\sqrt{A(A-1)}, \frac{\pi A c_0}{2}\cos\left(\frac{\pi\lambda A}{2}\right)\right\} \leq C(A) \leq \min\left\{\left(\frac{d_0}{1 - \frac{0.3}{A-2}}\right), 2\right\}, \tag{10}$$

where the first upper bound on the right-hand side of 10 is only available in the range $2.6 \leq A < \infty$. *(ii)The sharp constant* $C^+(A)$ *verifies the inequality*

$$C(A) \leq C^+(A) \leq \min\left\{\left(\frac{1.2}{1 - \frac{0.222}{A-1}}\right), 2\right\} \tag{11}$$

where the first upper bound on the right-hand side of (11) *is only available in the range* $1.222 < A < \infty$. *In particular, if* $A = \frac{36}{11}$ *a numerical example yields the lower bound*

$$\frac{25}{21} < C^+\left(\frac{36}{11}\right)$$

Remark 1: Without loss of generality, we can assume that (1) $|F(0)| = \|F\|_\infty$, by translating F; (2) $\|F\|_1 = 1$ by dilating F; (3) $F(0) > 0$ by multiplying F by a unimodular complex number; (4) F is real-valued by replacing F by $(F + \overline{F(x)})/2$; and (5) F is even by replacing F by $(F(x) + \overline{F(-x)})/2$. **Remark 2:** As a consequence $C^+(A) \geq C(A)$. Fejer kernel $F(x) = (sin(\pi x)/(\pi x))^2$ establishes $C(\infty) \geq 1$. We also have the trivial bound $C(A) \leq 2$. In summary

$$1 \leq C(\infty) \leq C(A) \leq C^+(A) \leq C^+(1) \leq 2.$$

(Proof for the existence of extremizers: omitted.)

Sketch of proof for theorem 5: Without loss of generality, we can assume $\hat{F} \in C_c^\infty(\mathbb{R})$ (i.e., F is bandlimited). Construct $\eta \in C^\infty$ such that $\hat{\eta}$ is an even, nonnegative, and radially non-increasing function and $\hat{\eta}(0) = 1, supp(\eta) = [-1,1], \int \hat{\eta} = 1$ (Fejer kernel $K(x) = (\sin(\pi x)/(\pi x))^2$ is an example). Let $\eta_\lambda(x) = \lambda\eta(x/\lambda)$, we see that $\limsup_{\lambda\to 0} \frac{\eta(0) - A\int_{[-1,1]^c}(\hat{\eta}_\lambda(t))_+ dt}{\|\eta_\lambda\|_1} \geq C^+(A)$. The case of $A = \infty$ needs slight modification and the same reasoning holds for $C(A)$. This shows that we may replace F by some $F * \eta_\lambda$ which is bandlimited.

Lower bounds: Let $H(x) = (cos 2\pi x)/(1 - 16x^2)$, Examining $F(x) = H(x/\lambda)$ for $\lambda \in (0,1]$ gives $C(A) \geq \frac{\pi A c_0}{2}\cos(\frac{\pi\lambda A}{2})$. Note that as $A \to 1^+$, this lower bound goes to $\pi c_0/2$ and is not very effective. Alternatively, we can then use a dilation of the Fejer kernel $K(x) = (sin(\pi x)/(\pi x))^2$, and obtain the other bound $C^+(A) \geq C(A) \geq 2A - 2\sqrt{A(A-1)}$.

Upper bounds: We already know that $C(A) \leq C^+(A) \leq C^+(1) = 2$. The other upper bounds all come from duality considerations, so we just show the case $C^+(A)$ in detail. Suppose that $\Phi \in L^\infty(\mathbb{R})$ is a real-valued function such that its distributional Fourier transform is identically equal to 1 on the interval $(-1,1)$ and $-A \leq \Phi(t) - 1 \leq 0$ for all $t \in \mathbb{R}$, then for $F \in \mathcal{A}^+$ such that $\hat{F} \in C_c^\infty$ we have

$$\|\Phi\|_\infty \int_{-\infty}^{\infty} |F(x)|dx \geq \int F(x)\Phi(x)dx = \int \hat{F}(t)\hat{\Phi}(t)dt \geq F(0) - A\int_{[-1,1]^c} F(t)dt. \tag{12}$$

Now it remains to minimize $\|\Phi\|_\infty$ with its restrictions.

7.4 related optimization problems

For the multi-dimensional generalizations, if we let the unit cube $Q = [-1,1]^d$ be in the place of $[-1,1]$ in our problem, $C_{d,Q}(\infty) = C(\infty)^d$. However, we do not have a sharp estimate for general compact set K.

Another extremal problem in Fourier analysis was proposed by Cohn and Elkies [2] in connection to the sphere packing problem. Find $C = \sup_{F\in\mathcal{E}_d^+, F\neq 0} \frac{F(0)}{\hat{F}(0)}$ where the supremum is taken over the class $\mathcal{E}_d^+$ of real-valued, continuous, and integrable functions $F : \mathbb{R}^d \to \mathbb{R}$ with $F \geq 0$ and $F(y) \leq 0$ for $|y| \geq 1$.

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8 New upper bounds on sphere packings I (after H. Cohn and N. Elkies [1])

A summary written by Oleksandr Vlasiuk

Abstract

An analogue of the linear programming bounds for error-correcting codes is developed for sphere packing and used to prove upper bounds for the density of sphere packings in a range of dimensions.

8.1 Introduction

The problem of sphere packing consists in finding the densest arrangement of equal non-overlapping spheres in the Euclidean space $\mathbb{R}^n$. For a collection of non-overlapping balls $\mathcal{P}$, its upper density is defined as the quantity

$$\Delta = \limsup_{r\to\infty} \sup_{p\in\mathbb{R}^n} \frac{\operatorname{vol}(B(p,r)\cap\mathcal{P})}{\operatorname{vol} B(p,r)},$$

where $B(p,r)$ denotes the ball of radius r, centered at p, and $B(p,r)\cap\mathcal{P}$ consists of those parts of the balls in $\mathcal{P}$ that lie within $B(p,r)$. To obtain an upper bound on the density of sphere packings, it suffices to produce such a bound for periodic packings, as any general packing can be approximated by a periodic one with a sufficiently large fundamental domain. Furthermore, for periodic packings, the lim sup above can be replaced by a lim, so that

$$\Delta = \lim_{r\to\infty} \frac{\operatorname{vol}(B(p,r)\cap\mathcal{P})}{\operatorname{vol} B(p,r)},$$

where the right-hand side does not depend on $p \in \mathbb{R}^n$. In addition to the density Δ, defined above as the portion of volume contained inside the packed spheres, the so-called *center density* is also of interest. Denoted by δ, it is defined as the number of spheres per unit volume, if spheres of unit radius are used. Thus

$$\Delta = \frac{\pi^{n/2}}{\Gamma(n/2+1)}\delta = \operatorname{vol} B(0,1)\cdot\delta.$$

We shall further need several standard definitions. Recall that a *lattice* Λ is an additive subgroup of $\mathbb{R}^n$, $\Lambda = \{Ax \mid x \in \mathbb{Z}^n\}$, for a fixed matrix $A \in \mathbb{R}^{n^2}$. Given a lattice $\Lambda \subset \mathbb{R}^n$, the *dual lattice* Λ^* is defined by

$$\Lambda^* = \{y \mid \langle x, y\rangle \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

The *covolume* $|\Lambda| = \mathrm{vol}(\mathbb{R}^n/\Lambda)$ of a lattice Λ is the volume of any fundamental parallelotope. Given a lattice Λ with shortest nonzero vectors of length l, the density of the corresponding lattice packing is

$$\Delta = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \left(\frac{l}{2}\right)^n \frac{1}{|\Lambda|},$$

and its center density is therefore $\delta = (l/2)^n/|\Lambda|$.

The Fourier transform of an L^1 function $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\widehat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle x,t\rangle}\, dx.$$

An essential step in the proof of Theorem 2 below is Poisson summation formula, that is, the identity

$$\sum_{x\in\Lambda} f(x+v) = \frac{1}{|\Lambda|} \sum_{t\in\Lambda^*} e^{-2\pi i\langle v,t\rangle} \widehat{f}(t), \tag{1}$$

which holds for every lattice $\Lambda \subset \mathbb{R}^n$ and vector $v \in \mathbb{R}^n$ under suitable assumptions on f. It is therefore reasonable to consider a class of functions for which convergence in the above formula is absolute. A convenient example of such a class is given in the following

Definition 1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called* admissible *if there exist a pair of constants $C, \delta > 0$, such that $f(x)$ and $\widehat{f}(x)$ satisfy*

$$|f(x)| \le C(1+|x|)^{-n-\delta}, \qquad |\widehat{f}(x)| \le C(1+|x|)^{-n-\delta}.$$

8.2 Main results

The following result has also been established independently in [4].

Theorem 2. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an admissible function, not identically equal to zero, and satisfies:*

(1) *$f(x) \le 0$ for $|x| \ge 1$, and*

(2) *$\widehat{f}(t) \ge 0$ for all t.*

Then the center density of n-dimensional sphere packings is bounded above by

$$\frac{f(0)}{2^n \widehat{f}(0)}.$$

Since $f \not\equiv 0$ and $\widehat{f}$ is nonnegative, $f(0) = \|\widehat{f}\|_{L^1} > 0$. If $\widehat{f}(0) = 0$, it is understood that $f(0)/\widehat{f}(0) = +\infty$, and the theorem is vacuously true.

Proof. It suffices to consider periodic packings, as their densities are arbitrarily close to the largest value of Δ over all packings. Consider a packing $\mathcal{P}$ given by the translates of a lattice Λ by vectors $v_1, \dots, v_N$, with $v_i - v_j \notin \Lambda$ when $i \neq j$. Assume spheres in $\mathcal{P}$ have radius $1/2$, so no two centers are at distance less than 1. Then a fundamental parallelotope of Λ contains N balls of radius $1/2$ from $\mathcal{P}$, and so the central density of the latter satisfies $\delta = N/(2^n|\Lambda|)$.

By the Poisson summation formula (1),

$$\sum_{x \in \Lambda} f(x+v) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} e^{-2\pi i \langle v,t \rangle} \widehat{f}(t)$$

for all $v \in \mathbb{R}^n$. It follows that

$$\sum_{1 \le j,k \le N} \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \widehat{f}(t) \left| \sum_{1 \le j \le N} e^{2\pi i \langle v_j, t \rangle} \right|^2. \qquad (2)$$

Every term on the right is nonnegative, so the right-hand side is bounded from below by the term with $t = 0$, equal to $N^2 \widehat{f}(0)/|\Lambda|$.

On the left, the vector $x + v_j - v_k$ is the difference between two centers in the packing $\mathcal{P}$, so $|x + v_j - v_k| < 1$ if and only if $x = 0$ and $j = k$. Whenever $|x + v_j - v_k| \ge 1$, the corresponding term is nonpositive by the first assumption of the theorem, so the left-hand side is at most $N f(0)$. Thus,

$$N f(0) \ge \frac{N^2 \widehat{f}(0)}{|\Lambda|},$$

and using the equality $\delta = N/(2^n|\Lambda|)$ gives

$$\delta \le \frac{f(0)}{2^n \widehat{f}(0)},$$

as desired. ∎

An alternative form of the above theorem can be given as follows.

Theorem 3. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is an admissible function satisfying the following three conditions*:

(1) $f(0) = \widehat{f}(0) > 0$,

(2) $f(x) \le 0$ *for* $|x| \ge r$, *and*

(3) $\widehat{f}(t) \ge 0$ *for all* t.

Then the center density of sphere packings in $\mathbb{R}^n$ *is bounded above by* $(r/2)^n$.

8.3 Sharpness of the bound and uniqueness of its solution

Let Λ_8 and Λ_{24} denote the E_8 and the Leech lattice, respectively. An inspection of the proof of Theorem 2 reveals that for a lattice packing, the bound becomes sharp if and only if f vanishes on $\Lambda\backslash\{0\}$ and $\widehat{f}$ vanishes on $\Lambda^*\backslash\{0\}$. Since both Λ_8 and Λ_{24} coincide with their duals, to prove their optimality, both f and $\widehat{f}$ must vanish on $\Lambda\backslash\{0\}$. The nonzero vectors in Λ_8 and Λ_{24} have lengths $\sqrt{2n}$ with $n \ge 1$ and $n \ge 2$, respectively. The functions constructed in [5, 2] have a single root at the length of the shortest vector of the lattice, and double roots afterwards.

Let Λ_2 denote the isodual scaling of the hexagonal lattice in $\mathbb{R}^2$ (so that the rescaled version is isometric with its dual). Assuming existence of a function f satisfying the hypotheses of Theorem 3, proving that Λ_n is the densest packing in $\mathbb{R}^n$, and such that both f and $\widehat{f}$ have roots only at the vector lengths in Λ_n, $n \in \{2, 8, 24\}$, Cohn and Elkies [1] demonstrate that Λ_n is unique among periodic packings of maximal density.

Namely, let $\mathcal{P}$ be a periodic packing achieving equality in the bound, scaled to have one sphere center per unit volume. From (2) with the function f as above, and our discussion of sharpness, each vector $x + v_j - v_k$ occurs at a root of f. Recall the following lemma [1, Lemma 8.2]:

Lemma 4. *Suppose S is a subset of $\mathbb{R}^n$ such that $0 \in S$, there are n linearly independent vectors in S, and for all $x, y \in S$, $|x - y| = \sqrt{2k}$, $k \in \mathbb{Z}$. Then the subgroup of $\mathbb{R}^n$ generated by S is an even integral lattice.*

An *integral lattice* is one in which the inner product of each pair of vectors is an integer; it is *even* if every vector has even norm. The proof of uniqueness then proceeds as follows. Let L be the subgroup of $\mathbb{R}^n$ generated by centers in $\mathcal{P}$. By the lemma, it is an even integral lattice, so its covolume is the square root of an integer. Thus, L has at most one center per unit volume, with equality if and only if L is unimodular. Notice that as L is the subgroup generated by $\mathcal{P}$, it contains more spheres. By the assumption, $\mathcal{P}$ has exactly one center per unit volume, so it must coincide with L, and therefore be an even unimodular lattice. The lattice must have minimal vector norm 2 in $\mathbb{R}^8$ and 4 in $\mathbb{R}^{24}$ to have this density; such lattices are unique, which completes the proof.

8.4 2-point homogeneous spaces

The Euclidean space $\mathbb{R}^n$ is an example of non-compact 2-point homogeneous space. That is, for any two pairs of points equal distance apart $|x_1 - x_2| = |y_1 - y_2|$, there exists an isometry of $\mathbb{R}^n$ that maps x_i to y_i, $i = 1, 2$. This allows to put Theorem 2 in the perspective of similar results on compact 2-point homogeneous spaces. For example, on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, let $t(x, y) = (1 + \langle x, y\rangle)/2$, and P_j be the Jacobi polynomial $P_j^{(\alpha,\beta)}(t)$, where $\alpha = \beta = (n-3)/2$, then there holds the following theorem due to Delsarte.

Theorem 5 ([3])**.** *Let $C \subset \mathbb{S}^{n-1}$ be a finite subset. Suppose $f(t) = \sum_{j=0}^{m} a_j P_j(t)$ with $a_j \geq 0$ for all j and $f(t) \leq 0$ for $0 \leq t \leq \tau$. If $t(x, y) \leq \tau$ whenever x and y are distinct points of C, then $|C| \leq f(1)/a_0$.*

8.5 Subsequent progress

Based on extensive numerical experiments, authors of [1] formulate the following conjecture.

Conjecture 6. *There exist functions that satisfy the hypotheses of Theorem* 2 *and solve the sphere packing problem in dimensions* 2, 8, *and* 24.

This statement has been proved true in the works by Viazovska and Cohn, Kumar, Miller, Radchenko, and Viazovska [5, 2] in dimensions 8 and 24. The f constructed in these two cases are Schwartz functions, and thus admissible. In addition, they only have zeros at vector lengths from Λ_8 and Λ_{24}, respectively, hence the uniqueness argument given above is applicable. The auxiliary functions are constructed as integral transforms of certain modular forms. For both Λ_8 and Λ_{24}, it is essential that the vector lengths in the lattice are of the form $\sqrt{2k}$, a property that does not hold for Λ_2, optimality of which for spherical packing remains an open question.

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9 An optimal uncertainty principle in twelve dimensions via modular forms (after H. Cohn and F. Gonçalves [3])

A summary written by Josiah Park

Abstract

In [1] a new uncertainty principle was derived and relations made with the theory of zeta functions. We summarize recent progress in understanding the behavior of certain extremizers for this and related uncertainty principles.

9.1 Introduction

Heisenberg's uncertainity principle states that

$$\int |x|^2|f(x)|^2dx \int |\xi|^2|\hat{f}(\xi)|^2d\xi \geq 1/16\pi^2$$

when $\|f\|_2 = 1$ and $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle x,\xi\rangle}dx$, equality holding precisely for $f(x) = 2^{d/4}e^{-\pi|x|^2}$. The importance of this inequality in modern science is impossible to downplay, being absolutely fundamental for quantum mechanics.

In [1] another question about trade-offs between localization of a function and its Fourier transform was formulated through study of the value

$$r(f) = \inf\{R \geq 0 : f(x) \text{ has the same sign for } |x| \geq R\}.$$

Setting $\mathcal{A}_+(d)$ to be the set of all functions $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying

1. $f \in L^1(\mathbb{R}^d)$, $\hat{f} \in L^1(\mathbb{R}^d)$, and $\hat{f}$ is real-valued (i.e., f is even),
2. f is eventually nonnegative while $\hat{f}(0) \leq 0$, and
3. $\hat{f}$ is eventually nonnegative while $f(0) \leq 0$,

this uncertainty principle takes the form

$$\mathrm{A}_+(d) := \inf_{f\in\mathcal{A}_+(d)\setminus\{0\}} \sqrt{r(f)r(\hat{f})} > 0.$$

The exact value of $\mathrm{A}_+(d)$ remains unknown generally, however one of the main results from [3] proves optimality of a "magic" function for this uncertainty principle when $d = 12$. They show $\mathrm{A}_+(12) = \sqrt{2}$ and that this value is attained for a radial eigenfunction of the Fourier transform having a double root at $|x| = 0$, a single root at $|x| = \sqrt{2}$, and double roots at $|x| = \sqrt{2j}$ for integers $j \geq 2$.

A clean exact solution is possible in part due to the connections the authors make with well-established linear programming bounds for sphere packing [2]. Recall that for suitable f, these upper bounds for the sphere packing density Δ_d are given as

$$\Delta_d \leq \mathrm{vol}(B^d_{r(f)/2}).$$

The conditions satisfied by feasible f in the above setting naturally give rise to another uncertainty principle closely related to the one previously mentioned. When the set $\mathcal{A}_+(d)$ is replaced with $\mathcal{A}_-(d)$, a set of functions satisfying conditions (1) and (2) in the definition of $\mathcal{A}_+(d)$, but also containing only f with $\hat{f}$ eventually nonpositive and $f(0) \geq 0$, one obtains another uncertainty principle

$$\mathrm{A}_-(d) := \inf_{f \in \mathcal{A}_-(d) \setminus \{0\}} \sqrt{r(f) r(\hat{f})} > 0.$$

This inequality follows from an extension of the main result in [6].

Theorem 1. *Let $s \in \{\pm 1\}$. Then there exist positive constants c and C such that*

$$c \leq \frac{\mathrm{A}_s(d)}{\sqrt{d}} \leq C$$

for all d. Moreover, for each d there exists a radial function $f \in \mathcal{A}_s(d) \setminus \{0\}$ with $\hat{f} = sf$, $f(0) = 0$, and

$$r(f) = \mathrm{A}_s(d).$$

Furthermore, any such function must vanish at infinitely many radii greater than $\mathrm{A}_s(d)$.

9.2 Optimality in twelve dimensions

9.2.1 $\mathrm{A}_+(12) \geq \sqrt{2}$

The inequalities $\mathrm{A}_+(12) \geq \sqrt{2}$ and $\mathrm{A}_+(12) \leq \sqrt{2}$ follow from a summation formula from the Eisenstein series E_6 and from construction of an optimal function via modular forms using techniques from [7]. $E_6(z) = 1 - 504\sum_{j=1}^{\infty}\sigma_5(j)q^j$, where $q = e^{2\pi i z}$ and $\sigma_5(j)$ is the sum of fifth powers of the divisors of j. So, $E_6(z) = z^{-6}E_6(-\frac{1}{z})$, meaning E_6 is a modular form of weight 6 for $SL_2(\mathbb{Z})$. This fact translates to a summation formula for radial Schwartz functions $f : \mathbb{R}^{12} \to \mathbb{C}$.

Lemma 2. *For all radial Schwartz functions* $f : \mathbb{R}^{12} \to \mathbb{C}$*,*

$$f(0) - \sum_{j\geq 1} c_j f(\sqrt{2j}) = -\hat{f}(0) + \sum_{j\geq 1} c_j \hat{f}(\sqrt{2j}).$$

This equation can be checked first for the Gaussian $f(x) = e^{\pi i z|x|^2}$ and then verfied in full by checking its validity on compactly supported radial C^∞ functions by density of such functions in $\mathcal{S}_{\text{rad}}(\mathbb{R}^{12})$. The proof for the latter functions just involves an application of Fourier inversion. Significantly, this lemma may be used to show the following result.

Lemma 3. *Let* $f \in \mathcal{A}_+(12)$*. If both* $r(f)$ *and* $r(\hat{f})$ *are at most* $\sqrt{2}$*, then* $f(x) = \hat{f}(x) = 0$ *whenever* $|x| = \sqrt{2j}$ *with* j *a nonnegative integer.*

This lemma is established by natural use of the conditions that a function $f \in \mathcal{A}_+(12)$ satisfies, namely that $f(0) \leq 0$, $\hat{f}(0) \leq 0$ and that f and $\hat{f}$ are eventually nonnegative. The above summation formula along with a mollification argument establish the result.

Together, these lemmas may be used to establish the final inequality $\mathrm{A}_+(12) \geq \sqrt{2}$ which is equivalent to the next statement.

Lemma 4. *Suppose* $f \in \mathcal{A}_+(12)$*. If* $r(f)r(\hat{f}) < 2$*, then* f *vanishes identically.*

The idea here is to rescale f, $g_\lambda(x) = f(\lambda x)$ ($g \in \mathcal{A}_+(12)$) applying the previous lemma to g_λ. Fitting what is known about each of these functions together, one sees that f and $\hat{f}$ must both be compactly supported, thus vanishing entirely.

A quick remark is suitable now. The observations above carry over generally to other dimensions and both signed quantities $\mathrm{A}_s(d)$. By calculation, this implies in particular that $\mathrm{A}_-(8)$ and $\mathrm{A}_-(24)$ are greater than $\sqrt{2}$ and 2 respectively. Since $\mathrm{A}_-(d)$ is the optimal value for a broader optimization problem than that associated with the LP upper bounds for sphere packing, denoted $r(g)$, $r(g) \geq \mathrm{A}_-(d)$. The known existence of optimal functions in dimensions 8 and 24 [7, 4] imply finally that $\mathrm{A}_-(8) = \sqrt{2}$ and $\mathrm{A}_-(24) = 2$.

9.2.2 $A_+(12) \leq \sqrt{2}$

For the upper bound, the "magic" optimal function $f : \mathbb{R}^{12} \to \mathbb{R}$ establishing $A_+(12) \leq \sqrt{2}$ is constructed through a generalization of the procedure developed in [7] to prove optimality of the E_8 lattice for sphere packing. Setting

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}$$

and using the same notation from [7] for theta functions $\Theta_{00}(z), \Theta_{01}(z)$, and $\Theta_{10}(z)$, by taking an integral transform of the weakly holomorphic modular form

$$\psi = \frac{\big(\Theta_{00}^4 + \Theta_{10}^4\big)\Theta_{01}^{12}}{\Delta},$$

one arrives at a radial function $f = \hat{f}$ satisfying that

$$f(x) = \sin\big(\pi|x|^2/2\big)^2 \int_0^\infty \psi(it) e^{-\pi|x|^2 t}\, dt$$

for $|x| > \sqrt{2}$. Further analysis of the series for ψ establishes that f behaves as desired in the range $0 < |x| < 2$ (f belongs to $\mathcal{A}_+(12)$) and the location of the zeros of f shows finally that $A_+(12) \leq \sqrt{2}$.

The choice of ψ here was guided by ensuring that $\psi\Delta$ be a holomorphic modular form of weight 8 for $\Gamma(2)$. This implies the form should be a linear combination of five different products of theta functions, and the condition used in Viazovska's method removes three degrees of freedom. When it is observed that the coefficient of $e^{-\pi iz}$ in the Fourier expansion for $\psi(z)$ should vanish, this pins down the choice of ψ up to scaling.

9.3 Negative uncertainty principle

Since Theorem 1 was demonstrated to hold for the "positive" uncertainty principle in [6], it remains to consider only the negative case. As mentioned earlier, the problem of computing good LP upper bounds for sphere packing density is closely related to the problem of minimizing $A_-(d)$. These bounds come from $g \in L^1(\mathbb{R}^d) \setminus \{0\},\ g = \hat{g},\ g(0) = 0$, and for g eventualy positive. We noted earlier that each such g satisfies $r(g) \geq A_-(d)$, however it also holds that computing $A_-(d)$ reduces to minimizing $r(g)$ also.

Lemma 5. *For each* $f \in \mathcal{A}_-(d) \setminus \{0\}$*, there exists a radial function* $g \in \mathcal{A}_-(d) \setminus \{0\}$ *such that* $\hat{g} = -g$*,* $g(0) = 0$*, and* $r(g) \leq \sqrt{r(f)r(\hat{f})}$*.*

To show this, one can make first a few simple reductions to considering only f radial and satisfying $r(f) = r(\hat{f})$. Then the function $g = f - \hat{f}$ satisfies $g \in \mathcal{A}_-(d)$ and $r(g) \leq r(f)$. To show that g may be taken so that $g(0) = 0$, proceed by contradiction, peturbing a g with $g(0) > 0$ by an auxiliary function, $h = g + g(0)(\varphi_t - \hat{\varphi}_t)$ where

$$\varphi_t(x) = \frac{e^{-t\pi|x|^2} - e^{-2t\pi|x|^2}}{t^{-d/2} - (2t)^{-d/2}}.$$

One will see that $r(h) \leq r(g)$ while $h(x) > g(x)$ for large x, a contradiction.

The arguments now for the lower bound for $\mathrm{A}_-(d)$ from Theorem 1 come in the same way they do in [6, Theorem 7] (which in turn is heavily influenced by the proof in [1] for lower bounds). Similarly, the upper bound too follows generally the same as those employed in the above works. Kabatiansky-Levenshtein's bounds (through an observation from [5]) alternatively show that $r(g) \leq (0.3194... + o(1))\sqrt{d}$, and it is expected that this bound should hold too for $\mathrm{A}_+(d)$ but has yet to be demonstrated.

The remaining ingredients of Theorem 1 are existence of extremizers for $\mathrm{A}_-(d)$ and a proof that such extremizers have infinitely many roots. Both arguments employed here closely resemble those employed in [6]. The first follows from Mazur's lemma, Fatou's lemma, and a version of Nazarov's uncertainty principle, while the latter uses again the function φ_t to construct perturbations of a optimal function.

If the optimal function vanishes only finitely many times beyond $r(f)$ one obtains a contradiction by considering the constructed perturbation. Poisson summation formula excludes the case $d = 1$ automatically, so the proof in this line (using Laguerre polynomials similar to [6]) is given for $d \geq 2$.

9.4 Numerics

Results from extensive numerical experiments using exact rational arithmetic were collected to derive the (rigorous) upper bounds presented in the paper (see the ancillary files at [3] for a larger database of bounds). This data supports the conjecture that:

Conjecture 6. *The limits* $\lim_{d\to\infty} \frac{\mathrm{A}_+(d)}{\sqrt{d}}$ *and* $\lim_{d\to\infty} \frac{\mathrm{A}_-(d)}{\sqrt{d}}$ *exist and are equal.*

The value of $A_+(28)$ might be given extra attention since $A_+(12) = A_-(8)$, although numerically it is clear this value cannot be equal to $A_-(24) = 2$. The big open conjecture from [2] must be emphasized too of finding a feasible function $g : \mathbb{R}^2 \to \mathbb{R}$ for the LP bounds for which $r(g) = (4/3)^{1/4}$.

A few conjectures which do not appear in print are that $A'_-(8) = \frac{\sqrt{2}}{30}$, $A'_-(24) = \frac{368}{12285}$ and $A'_+(12) = \frac{\sqrt{8}}{63}$, where the derivative here is with respect to the dimension d. H. Cohn proposed these at an AIM meeting in 2018.

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10 Universally optimal distribution of points on spheres (after H. Cohn and A. Kumar [1])

A summary written by Juan Criado del Rey and Alan Groot

Abstract

We prove that certain configurations on the sphere attain the minimal potential energy among all other configurations of sphere with the same number of points. Furthermore, we give a lower bound for the potential energy of periodic point configurations in Euclidean space.

10.1 Introduction

The paper investigates the question which finite configurations $\mathcal{C}$ on the unit sphere S^{n-1} in $\mathbb{R}^n$ minimize the *(f-)potential energy* given by

$$\sum_{x,y\in\mathcal{C},x\neq y} f(|x-y|^2).$$

Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a C^∞ function. The function f is *completely monotonic* if $(-1)^k f^{(k)}(x) \geq 0$ for all $x \in I$ and all $k \geq 0$, where derivatives at the endpoints are one-sided derivatives.

A finite subset of the sphere is called a *spherical M-design* if every polynomial on $\mathbb{R}^n$ of total degree at most M has the same average over the design as over the sphere. Moreover, such a finite subset $\mathcal{C}$ of the sphere S^{n-1} is said to be a *sharp configuration* if there are m inner products between distinct points in it and it is a spherical $(2m-1)$-design.

There is a list of all known sharp configurations on the unit sphere. (Even though the 600-cell is not sharp, the arguments in the paper apply to this case as well.) The main theorem of the paper is the following theorem, which states that essentially these sharp configurations and the 600-cell are the only configurations that minimize the potential energy on the sphere.

Theorem 1. *Let $f : (0,4] \to \mathbb{R}$ be completely monotonic and let $\mathcal{C} \subset S^{n-1}$ be a sharp arrangement or the vertices of a regular 600-cell. If $\mathcal{C}' \subset S^{n-1}$ is any subset satisfying $|\mathcal{C}| = |\mathcal{C}'|$, then*

$$\sum_{x,y\in\mathcal{C}',x\neq y} f(|x-y|^2) \geq \sum_{x,y\in\mathcal{C},x\neq y} f(|x-y|^2). \tag{1}$$

If f is strictly completely monotonic, then equality in (1) *implies that $\mathcal{C}'$ is also a sharp configuration or the vertices of a 600-cell (whichever $\mathcal{C}$ is) and the same distances occur in $\mathcal{C}$ and $\mathcal{C}'$. In that case, $\mathcal{C}'$ can generally (see remark below) be obtained by an orthogonal transformation of $\mathcal{C}$.*

Remark 2. *There is a particular family of sharp configurations coming from totally isotropic subspaces of a Hermitian vector space over a finite field for which uniqueness does not hold. We will comment on this during the lecture.*

The proof of the above theorem is split into two parts: the case in which $\mathcal{C}$ is a sharp arrangement and the case of the 600-cell. The proof of the first part uses Hermite interpolation, the ultraspherical (or Gegenbauer) polynomials and further properties of orthogonal polynomials and linear programming bounds.

The ultraspherical polynomials are denoted by C_i^λ, where i is the degree of the polynomial and $\lambda \in \mathbb{R}$ is a parameter. They satisfy orthogonality with respect to the measure $(1-t^2)^{\lambda-1/2}dt$ on the interval $[-1,1]$. This measure is (up to scaling) the projection of the surface measure of the sphere to a line passing through two antipodal points. The ultraspherical coefficients of a function on $[-1,1]$ are the coefficients in terms of ultraspherical polynomials. A function is positive definite if all its ultraspherical coefficients are nonnegative, while a polynomial is strictly positive definite if all its ultraspherical coefficients (up to its degree) are strictly positive.

One useful application of ultraspherical polynomials is that it gives a criterion for a configuration to be a spherical design, namely a configuration $\mathcal{C} \subset S^{n-1}$ is a spherical M-design if and only if

$$\sum_{x,y\in\mathcal{C}} C_i^{n/2-1}(\langle x,y\rangle) = 0$$

for all $1 \le i \le M$.

The part on linear programming bounds uses an auxiliary positive definite polynomial h to give a bound on the f-potential energy of a configuration $\mathcal{C}$.

Proposition 3. *Let $f:(0,4]\to\mathbb{R}$ be any function. Suppose $h:[-1,1]\to\mathbb{R}$ is a polynomial such that*

$$h(t) \le f(2-2t)$$

for all $t \in [-1,1)$*, and suppose there are nonnegative coefficients* $\alpha_0, \dots, \alpha_d$ *such that* h *has the expansion*

$$h(t) = \sum_{i=0}^{d} \alpha_i C_i^{n/2-1}(t)$$

in terms of ultraspherical polynomials (i.e., h *is positive definite). Then every set of* N *points on* S^{n-1} *has potential energy at least*

$$N^2\alpha_0 - Nh(1)$$

with respect to the potential function f*.*

The bound given by h is sharp for $\mathcal{C}$ if and only if two conditions hold: $h(t)$ must equal $f(2-2t)$ at every inner product t that occurs between distinct points in $\mathcal{C}$ and whenever the ultraspherical coefficient α_i of h is positive with $i > 0$, we must have

$$\sum_{x,y\in\mathcal{C}} C_i^{n/2-1}(\langle x,y\rangle) = 0.$$

In particular, if h is strictly positive definite, then $\mathcal{C}$ must be a spherical $\deg(h)$-design.

To complete the proof of the first part of the the theorem, an auxiliary function h is introduced. Let $\mathcal{C}$ be a sharp arrangement with N points and let $-1 \le t_1 < \cdots < t_m < 1$ be the ordered inner products that occur between distinct points in $\mathcal{C}$. The auxiliary function is a Hermite interpolation polynomial that agrees with $f(2-2t)$ to order 2 at each t_i. The authors prove that h is (strictly) positive definite if f is (strictly) completely monotonic and then apply the criterion above to prove the first part of the theorem.

The proof of the second part of the theorem does not follow directly from the proof of the first part of the theorem, because the 600-cell is only a spherical 11-design, but the auxiliary function h constructed in the proof of the first part would have degree at least 14. Nevertheless, the same techniques apply with some modifications.

Finally, the paper discusses a generalization of Proposition 3 to periodic point configurations $\mathcal{C}$ in Euclidean space that are a union of finitely many translates of a lattice. If the configuration consists of the N translates $\Lambda + v_1, \dots, \Lambda + v_N$ of a lattice Λ in $\mathbb{R}^n$, with $v_j - v_k \notin \Lambda$ for $j \neq k$, then the density is $N/\mathrm{vol}(\mathbb{R}^n/\Lambda)$. For a potential function $f : (0, \infty) \to [0, \infty)$, the ($f$-)potential energy of the configuration is defined as

$$\frac{1}{N} \sum_{1 \leq j,k \leq N} \sum_{x \in \Lambda, x+v_j-v_k \neq 0} f(|x + v_j - v_k|^2).$$

The potential energy may be infinite.

Proposition 4. *Let $f : (0, \infty) \to [0, \infty)$ be any function. Suppose $h : \mathbb{R}^n \to \mathbb{R}$ satisfies $h(x) \leq f(|x|^2)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and is the Fourier transform of a function $g \in L^1(\mathbb{R}^n)$ such that $g(t) \geq 0$ for all $t \in \mathbb{R}^n$. Then every periodic configuration in $\mathbb{R}^n$ with density δ has f-potential energy at least*

$$\delta \left(\liminf_{t \to 0} g(t) \right) - h(0).$$

The proof uses elements of Fourier analysis. In contrast to the proposition in the case of configurations on the sphere, it is not known if there exists a function h that satisfies the hypotheses of the above proposition. For three types of lattices, the hexogonal lattice Λ_2 in $\mathbb{R}^2$, the E_8 root lattice Λ_8 in $\mathbb{R}^8$ and the Leech lattice Λ_{24} in $\mathbb{R}^{24}$, the authors conjecture that such functions h exist.

Conjecture 5. *Let $n \in \{2, 8, 24\}$, and let $f : (0, \infty) \to \mathbb{R}$ be completely monotonic and satisfy $f(x) = O(|x|^{-n/2-\epsilon})$ as $|x| \to \infty$ for some $\epsilon > 0$. Then there exists a function h that satisfies the hypothesis of Proposition 2 and proves that Λ_n has the least f-potential energy of any periodic configuration in $\mathbb{R}^n$ with its density.*

As an argument supporting this conjecture, the authors are able to prove that in one dimension, Proposition 4 proves a sharp bound for the minimal potential energy.

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11 Universal optimality of the E_8 and Leech lattices and interpolation formulas (after H. Cohn, A. Kumar, S. Miller, D. Radchenko and M. Viazovska[1])

A summary written by Matthew de Courcy-Ireland and Giuseppe Negro

Abstract

We apply the method of linear programming to show that the E_8 and Leech lattices have minimal energy, among all point configurations of unit density, and with respect to any completely monotonic radially symmetric potential.

11.1 Introduction

Given a *potential function* $p\colon (0,\infty) \to \mathbb{R}$, we associate to any discrete point configuration $\mathcal{C} \subset \mathbb{R}^d$ its *lower p-energy*, defined by

$$E_p(\mathcal{C}) := \liminf_{r\to\infty} \frac{1}{|\mathcal{C} \cap B_r(0)|} \sum_{x \neq y \in \mathcal{C} \cap B_r(0)} p(|x-y|) \tag{1}$$

where the sum is over pairs of distinct points $x \neq y$ lying in the intersection of $\mathcal{C}$ with a larger and larger ball $B_r(0)$ centered at the origin. The problem of optimal configuration consists in finding those $\mathcal{C}$ that minimize (1) with a prescribed *density*, that is a given value of

$$\lim_{r\to\infty} \frac{|\mathcal{C} \cap B_r(0)|}{\mathrm{vol}(B_r(0))}. \tag{2}$$

We denote by Λ_d, with $d = 8$ or 24, the E_8 and Leech lattices, respectively. These are additive subgroups of $\mathbb{R}^d$ with the further properties of being *even* and *self-dual*. This means that a vector y satisfies $x \cdot y \in \mathbb{Z}$ for all $x \in \Lambda_d$ if and only if $y \in \Lambda_d$, and moreover $|x|^2 \in 2\mathbb{N}$ for all $x \in \Lambda_d$. These arithmetic constraints are quite restrictive: even, self-dual lattices exist only in dimensions d divisible by 8. We remark that all self-dual lattices have density equal to 1. The E_8 and Leech lattices are exceptional structures with exotic symmetries and many remarkable properties. The main result summarized in these notes is the following theorem proved by Cohn, Kumar, Miller, Radchenko, and Viazovska.

Theorem 1 (Universal optimality)**.** *Let $p(r) = g(r^2)$, where g is a smooth function satisfying $(-1)^k g^{(k)} \geq 0$ for all $k \geq 0$; we say that g is* completely monotonic*. For every discrete subset $\mathcal{C} \subset \mathbb{R}^d$ of unit density, where the dimension d is 8 or 24,*

$$E_p(\mathcal{C}) \geq E_p(\Lambda_d). \tag{3}$$

By a theorem of Bernstein, every completely monotonic function is the Laplace transform of a positive measure. It therefore suffices to prove Theorem 1 with $p(r) = e^{-\pi\alpha r^2}$, where $\alpha > 0$, the general case being a superposition of these with different values of α.

11.2 Fourier interpolation and linear programming

Denote by $\mathcal{S}_{\text{rad}}(\mathbb{R}^d)$ the space of radially symmetric Schwartz functions; for any such function g, we let $\widehat{g}$ denote its Fourier transform and g' its derivative in the radial direction. The proof of Theorem 1 is based on the following lemma, introduced by Cohn and Kumar [2].

Lemma 2 (Linear programming bounds)**.** *Let $p\colon (0,\infty) \to [0,\infty)$. If there is $g \in \mathcal{S}_{\text{rad}}(\mathbb{R}^d)$ satisfying*

$$g(x) \leq p(|x|), \quad \widehat{g}(x) \geq 0, \quad \forall x \in \mathbb{R}^d, \tag{4}$$

then, for every discrete $\mathcal{C} \subset \mathbb{R}^d$ with unit density,

$$E_p(\mathcal{C}) \geq \widehat{g}(0) - g(0).$$

If, moreover, $\Lambda \subset \mathbb{R}^d$ is a self-dual lattice, and

$$g(x) = p(|x|), \quad \widehat{g}(x) = 0, \quad \forall x \in \Lambda, \tag{5}$$

then $E_p(\Lambda) = \widehat{g}(0) - g(0)$.

We remark that (4) and (5) imply $g'(x) = p'(|x|)$ at all $x \in \Lambda_d$. The proof of Theorem 1 is therefore complete if we can construct a function $g \in \mathcal{S}_{\text{rad}}(\mathbb{R}^d)$ satisfying (4) and (5), with $\Lambda = \Lambda_8$ and $\Lambda = \Lambda_{24}$, and with a Gaussian potential $p(r) = e^{-\pi\alpha r^2}$. This construction is done via the following interpolation theorem, which is the core result of Cohn-Kumar-Miller-Radchenko-Viazovska [1]

Theorem 3. *Let (d, n_0) be $(8,1)$ or $(24,2)$. There are $a_n, b_n, \tilde{a}_n, \tilde{b}_n \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$, with $n \geq n_0$, such that every $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ satisfies*

$$f(x) = \sum_{n=n_0}^{\infty} f(\sqrt{2n})a_n(x) + f'(\sqrt{2n})b_n(x) + \widehat{f}(\sqrt{2n})\tilde{a}_n(x) + \widehat{f}\,'(\sqrt{2n})\tilde{b}_n(x)$$

for all $x \in \mathbb{R}^d$, where the series converges absolutely.

Here $\sqrt{2n_0}$ is the length of the shortest nonzero vector in Λ_8 or Λ_{24}. By this theorem, the function

$$g(x) = \sum_{n=n_0}^{\infty} p(\sqrt{2n})a_n(x) + p'(\sqrt{2n})b_n(x) \tag{6}$$

satisfies the condition (5) of Lemma 2.

11.3 Functional equations

The strategy of [1] is to solve for the unknowns $a_n, b_n, \widetilde{a}_n, \widetilde{b}_n$ using generating functions. Let

$$F(\tau, x) = \sum_{n \geq n_0} a_n(x)e^{2\pi i n\tau} + 2\pi i\tau \sum_{n \geq n_0} \sqrt{2n}b_n(x)e^{2\pi i n\tau} \tag{7}$$

and

$$\widetilde{F}(\tau, x) = \sum_{n \geq n_0} \widetilde{a}_n(x)e^{2\pi i n\tau} + 2\pi i\tau \sum_{n \geq n_0} \sqrt{2n}\widetilde{b}_n(x)e^{2\pi i n\tau} \tag{8}$$

where τ is a complex variable in the upper half-plane $\Im(\tau) > 0$ and $x \in \mathbb{R}^d$ (although only $|x|$ will be relevant). Ultimately, $g(x) = F(i\alpha, x)$ will be the test function (6), while $\widetilde{F}(i\alpha, x) = \widehat{g}(x)$. The interpolation formula is equivalent to certain functional equations that must be obeyed by F and $\widetilde{F}$. To determine them, consider interpolating the function $f(x) = e^{\pi i\tau|x|^2}$, which decays rapidly as long as $\Im(\tau) > 0$. The Fourier transform of a Gaussian is

$$\widehat{e^{\pi i\tau|x|^2}}(\xi) = \left(\frac{i}{\tau}\right)^{d/2} e^{-\pi i|\xi|^2/\tau} \tag{9}$$

that is, another Gaussian but with a change of parameter $\tau \mapsto -1/\tau$. The radial derivatives are given by

$$f'(r) = 2\pi i\tau r e^{\pi i\tau r^2}, \quad \widehat{f}'(r) = \left(\frac{i}{\tau}\right)^{d/2} \left(-\frac{2\pi i r}{\tau}\right) e^{-\pi i r^2/\tau} \tag{10}$$

In this case, the interpolation formula claims that

$$
\begin{aligned}
e^{\pi i\tau|x|^2} = \sum_{n=n_0}^{\infty} e^{2\pi in\tau} a_n(x) + \sum_{n=n_0}^{\infty} 2\pi i\tau\sqrt{2n}e^{2\pi in\tau} b_n(x) \\
+ \left(\frac{i}{\tau}\right)^{d/2} \sum_{n=n_0}^{\infty} e^{-2\pi in/\tau}\widetilde{a}_n(x) + \left(\frac{i}{\tau}\right)^{d/2} \sum_{n=n_0}^{\infty} -2\pi i\sqrt{2n}/\tau e^{-2\pi in/\tau}\widetilde{b}_n(x).
\end{aligned} \tag{11}
$$

In terms of the generating functions, we can restate (11) as

$$
F(\tau, x) + \left(\frac{i}{\tau}\right)^{d/2} \widetilde{F}(-1/\tau, x) = e^{\pi i|x|^2} \tag{12}
$$

and Theorem 3 will be proved by solving this functional equation. Note that proving the interpolation formula for all Gaussians is enough to conclude it for all radial Schwartz functions.

In addition to (12), we have implicitly imposed two further conditions on the generating functions. In order for F and $\widetilde{F}$ to have an expansion in terms of $e^{2\pi in\tau}$ and $\tau e^{2\pi in\tau}$ as above, they must also satisfy

$$
F(\tau+2, x) - 2F(\tau+1, x) + F(\tau, x) = 0 \tag{13}
$$

$$
\widetilde{F}(\tau+2, x) - 2\widetilde{F}(\tau+1, x) + \widetilde{F}(\tau, x) = 0 \tag{14}
$$

The required functional equations describe how F and $\widetilde{F}$ must behave under the transformations $S : \tau \mapsto -1/\tau$ and $T : \tau \mapsto \tau + 1$. They can be written more easily with the help of the "slash notation" for the action of S, T, and other linear fractional transformations $\tau \mapsto (a\tau + b)/(c\tau + d)$ on functions. Given a "weight" k, which will either be $d/2$ or $2 - d/2$ in the proof of the interpolation formula for $\mathbb{R}^d$, we write

$$
f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)
$$

for any real matrix obeying $ad - bc = 1$. Note that $\Im((a\tau + b)/(c\tau + d))$ is positive. The slash notation extends linearly to expressions such as $(T - 1)^2$ or $S - 1$ in the group ring of SL_2. Thus the functional equations can be written concisely as

$$
F + i^{d/2}\widetilde{F}|_{d/2} = e^{\pi i|x|^2} \tag{15}
$$

$$
F|(T-1)^2 = 0 \tag{16}
$$

$$
\widetilde{F}|(T-1)^2 = 0 \tag{17}
$$

where the slash acts on the variable τ, and the identities must hold pointwise for all x. Note that $i^{d/2} = 1$ for both $d = 8$ and $d = 24$, so this factor can be omitted from (15).

11.4 Construction with modular forms

The solution F produced by Cohn-Kumar-Miller-Radchenko-Viazovska is an integral transform

$$F(\tau, r) = e^{\pi i \tau r^2} + 4\left(\sin \frac{\pi r^2}{2}\right)^2 \int_0^\infty K(\tau, it) e^{-\pi r^2 t} dt \tag{18}$$

of a certain kernel $K = K^{(d)}$ that can be written explicitly in terms of modular forms. For reasons of space, these notes will not describe the construction in detail or review the theory of modular forms. Suffice it to say that modular forms are classical special functions including theta functions and Eisenstein series, and obeying special transformation laws under S and T. The integral in (18) converges for real values $r \neq 0$ in dimension $d = 8$, or for $|r| > 2$ in case $d = 24$, and for τ in a fundamental domain for the group generated by T^2 and ST^2S. An analytic continuation is needed to define $F(\tau, r)$ for complex values of r, and then for τ throughout $\mathbb{H}$. This ansatz and continuation argument are adapted from Viazovska's solution of the 8-dimensional sphere-packing problem [3], with a new ingredient in the modular forms used as building blocks. The factor $\sin(\pi r^2/2)$ vanishes when $r = \sqrt{2n}$, which guarantees that $F(\tau, r)$ interpolates $e^{\pi i \tau r^2}$ at the required radii. The complementary function $\widetilde{F}$ is also given by an integral transform involving another kernel $\widehat{K}$, designed to interpolate the function 0 instead of $e^{\pi i \tau r^2}$.

11.5 Positivity of kernels and end of proof

The proof is achieved by showing that $f(x) = F(i\alpha, x)$ is a valid test function for the Cohn-Kumar linear program. In other words, it is a radial Schwartz function on $\mathbb{R}^d$ satisfying $\widehat{f} \geq 0$ and $f(x) \leq e^{-\pi\alpha|x|^2}$. In both of these inequalities, equality holds if and only if $|x|$ is the length of a nonzero vector in the lattice E_8 in the case $d = 8$ or the Leech lattice in the case $d = 24$. Thus f is an optimal test function, as the bound it gives matches the energy achieved by a known configuration.

By an appeal to duality, one can deduce the inequailty $f(x) \leq e^{-\pi\alpha|x|^2}$ from $\widehat{f} \geq 0$. Thus it is enough to show that

$$0 \leq \widetilde{F}(\tau, r) = 4 \left(\sin \frac{\pi r^2}{2}\right)^2 \int_0^\infty \widehat{K}(\tau, it) e^{-\pi t r^2} dt$$

using the integral formula for $\widetilde{F}$. For $d = 8$, the integral converges absolutely for all $r > 0$. The positivity will follow by showing that

$$\widehat{K}^{(8)}(\tau, it) > 0.$$

The condition for equality then follows in light of the factor $(\sin \pi r^2/2)^2$.

For $d = 24$, the integral converges absolutely for $|r| > \sqrt{2}$. For small values of r, the integral does not converge and it is necessary to truncate $K(\tau, it)$ by subtracting certain leading terms. To confirm positivity, one must check two-variable inequalities on the truncated kernel as a function of both τ and t, as well as (comparatively easy) single-variable inequalities involving the error term as a function of τ. Cohn-Kumar-Miller-Radchenko-Viazovska do all of this by rigorous numerical verification.

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12 Upper bounds for packings of spheres of several radii (after D. de Laat, F. De Oliveira Filho and F. Vallentin)

A summary written by Maria Dostert and Martin Stoller

Abstract

We give a summary of the paper [2] by de Laat, Oliveira, and Vallentin in which they present theorems to obtain upper bounds on the density of translative packings of different convex bodies in $\mathbb{R}^n$ as well as for packings of different spherical caps on the unit sphere.

12.1 Translative packings of convex bodies and multiple-size packings

In this section, we study the density of translative packings of different convex bodies in Euclidean space. Let $\mathcal{K}_1, \ldots, \mathcal{K}_N \subset \mathbb{R}^n$ be convex bodies. A *translative packing* $\mathcal{P}$ is a union of translated copies of the bodies $\mathcal{K}_1, \ldots, \mathcal{K}_N$ such that any two copies have disjoint interiors. In other words, a translative packing $\mathcal{P}$ is given by

$$\mathcal{P} = \bigcup_{i=1}^{m} \left(x_i + \mathcal{K}_{r(i)}\right) \text{ for some } x_1, \ldots, x_m \in \mathbb{R}^m,$$

and some function $r : \{1, \ldots, m\} \to \{1, \ldots, N\}$ such that $\left(x_i + \mathcal{K}^{\circ}_{r(i)}\right) \cap \left(x_j + \mathcal{K}^{\circ}_{r(j)}\right) = \emptyset$ whenever $i \neq j$. The *density* of $\mathcal{P}$ is Δ, if for all $p \in \mathbb{R}^n$

$$\Delta = \lim_{r\to\infty} \frac{\mathrm{vol}(B(p,r) \cap \mathcal{P})}{\mathrm{vol}(B(p,r))},$$

where $B(p,r)$ is the ball of radius r centered at p. Since not every packing has a density, we also define the *upper density* of $\mathcal{P}$ by

$$\limsup_{r\to\infty} \sup_{p\in\mathbb{R}^n} \frac{\mathrm{vol}\left(B(p,r) \cap \mathcal{P}\right)}{\mathrm{vol}(B(p,r))}.$$

If there exists a lattice $L \subseteq \mathbb{R}^n$ such that $\mathcal{P} = x + \mathcal{P}$ for all $x \in L$, then $\mathcal{P}$ is called a *periodic packing*. A periodic packing has a density. Since the supremum of the upper density of any packing can be approximated arbitrary well by periodic packings, we can restrict ourselves to periodic packings.

In 2003, Cohn and Elkies [1] gave a linear programming method to obtain upper bounds for the density of sphere packings. In 2014, de Laat, Oliveira and Vallentin [2] extended the Cohn-Elkies method to obtain new upper bounds of packings of spheres with different radii.

Before stating the corresponding theorem, we have to define the Fourier transform of an L^1 function. Let $f : \mathbb{R}^n \to \mathbb{C}$ be an L^1 function. The Fourier transform of f at $u \in \mathbb{R}^n$ is

$$\widehat{f}(u) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i u \cdot x} \mathrm{d}x.$$

The function f is called a *Schwartz function* if it is infinitely differentiable and if any derivative of f multiplied by any power of the variables $x_1, \ldots, x_n$ is a bounded function. The Fourier transform of a Schwartz function is a Schwartz function as well.

Theorem 1. *Let $\mathcal{K}_1, \ldots, \mathcal{K}_N$ be convex bodies in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}^{N \times N}$ be a matrix-valued function such that every component f_{ij} is a Schwartz function. Suppose that f satisfies the following conditions.*

i) The matrix $\left(\widehat{f}_{ij}(0) - (\mathrm{vol}\ \mathcal{K}_i)^{1/2} (\mathrm{vol}\ \mathcal{K}_j)^{1/2}\right)_{i,j=1}^N$ is positive semidefinite.

ii) The matrix of Fourier transforms $\left(\widehat{f}_{ij}(u)\right)_{i,j=1}^N$ is positive semidefinite for every $u \in \mathbb{R}^n \setminus \{0\}$.

iii) $f_{ij}(x) \leq 0$ whenever $\mathcal{K}_i^\circ \cap \left(x + \mathcal{K}_j^\circ\right) = \emptyset$.

Then the density of any packing of translates of $\mathcal{K}_1, \ldots, \mathcal{K}_N$ in the Euclidean space $\mathbb{R}^n$ is at most $\max\{f_{ii}(0) : i = 1, \ldots, N\}$.

In the proof of Theorem 1 we apply the Poisson summation formula. By considering L^1 functions instead of Schwartz functions this part of the proof would not work anymore. However, the Poisson summation formula also holds for continuous functions of bounded support and positive type.

Using this property, de Laat, Oliveira and Vallentin develop a generalization of Theorem 1 for continuous L^1 functions of positive type.

Note that if we consider packings of spheres of different radii, which means all $\mathcal{K}_i$ are spheres of a radius r_i, then the last condition in Theorem 1 just depends on the norm of x. The intersection $\mathcal{K}_i^\circ \cap \left(x + \mathcal{K}_j^\circ\right)$ is empty if and only if $\|x\| \geq r_i + r_j$. Therefore, one can restrict f_{ij} to be radial Schwartz functions. Furthermore, the Fourier transform of a radial function is radial as well. Using the above observations, de Laat, Oliveira and Vallentin give a version of Theorem 1 to obtain upper bounds for the packing density of spheres of different radii. Moreover, they computed new upper bounds for the density of sphere packings with two different radii called *binary sphere packings* by using semidefinite optimization.

12.2 Multiple-size spherical cap packings

The *spherical cap* with *center* $x \in S^{n-1}$ and *angle* $\alpha \in (0, \pi]$ on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ is defined as $C(x, \alpha) := \{y \in S^{n-1} : x \cdot y \geq \cos(\alpha)\}$. The *normalized volume* of such a cap $C(x, \alpha)$ is independent of its center and given by

$$w(\alpha) := \frac{\omega_{n-1}(S^{n-2})}{\omega_n(S^{n-1})} \int_{\cos(\alpha)}^{1} (1 - u^2)^{(n-3)/2} du, \qquad (1)$$

where $\omega_n(S^{n-1}) = (2\pi^{n/2})/\Gamma(n/2)$ is the surface area of the unit sphere. We have

$$C(x_1, \alpha_1)^\circ \cap C(x_2, \alpha_2)^\circ = \emptyset \quad \Leftrightarrow \quad x \cdot y \leq \cos(\alpha_1 + \alpha_2).$$

For the remainder of the section, we fix a dimension $n \geq 3$ and a set of angles $\mathcal{A} = \{\alpha_1, \ldots, \alpha_N\} \subseteq (0, \pi]$. We are interested in packing spherical caps with angles from $\mathcal{A}$ on S^{n-1} and in obtaining upper bounds on the density of such packings. This can be done by translating the problem into graph theory. Define a conflict graph $G = (V, E)$ with set of vertices $V = S^{n-1} \times \{1, \ldots, N\}$ and set of edges E defined by

$$\{(x, i), (y, j)\} \in E \quad :\Leftrightarrow \quad x \cdot y > \cos(\alpha_1 + \alpha_2).$$

A vertex $(x, i) \in V$ corresponds to the spherical cap $C(x, \alpha_i) \subseteq S^{n-1}$ and a packing of spherical caps corresponds to a finite independent set of vertices I. The density of such a packing is $\sum_{(x,i) \in I} w(\alpha_i)$. We equip V with the product topology and we let the orthogonal group $H := \mathrm{O}(n)$ act on V by

$h \cdot (x, i) := (h \cdot x, i)$. This action is continuous, the weight function w is H-invariant and H acts by continuous graph automorphisms. Consider, for a moment, a general triple $(G = (V, E), H, w)$ with these properties. We would like to bound the *weighted independence number*

$$\alpha_w(G) := \sup \left\{ \sum_{x \in I} w(x) \; : \; I \subseteq V \text{ is finite and independent} \right\}.$$

To that end we will use certain *kernels* on $V \times V$, i.e. continuous functions $K : V \times V \to \mathbb{R}$. Such a kernel K is *symmetric* if $K(x, y) = K(y, x)$ for all $x, y \in V$. A symmetric kernel is *positive*, if for all $m \geq 1$ and all $x_1, \ldots, x_m \in V$, the real symmetric matrix $(K(x_i, x_j))_{1 \leq i,j \leq m}$ is positive semidefinite. Let $\mathcal{K}_w$ denote the set of all symmetric kernels K on $V \times V$ having the property that

- $K - w^{1/2} \otimes w^{1/2}$ is a positive kernel,
- $K(x, y) \leq 0$ for all $\{x, y\} \notin E$.

Let $\mathcal{K}_w^H \subseteq \mathcal{K}_w$ denote the subset consisting of H-invariant kernels. We define the *weighted theta-prime numbers*

$$\vartheta'_w(G) := \inf \{ \max_{x \in V} K(x, x) \; : \; K \in \mathcal{K}_w \},$$
$$\vartheta'_w(G)^H := \inf \{ \max_{x \in V} K(x, x) \; : \; K \in \mathcal{K}_w^H \}.$$

By averaging over the group H and using the assumptions on the action $H \times V \to V$ one can show that $\vartheta'_w(G) = \vartheta'_w(G)^H$. It is also not hard to see that $\alpha_w(G) \leq \vartheta'_w(G)$ (adapt the proof of [2, Theorem 1.1]).

We now return to the example of the spherical-cap-conflict graph G from above and the group $H = \mathrm{O}(n)$. For $k \geq 0$ we denote by $P_k = P_k^\lambda$ the kth Gegenbauer polynomial with parameter $\lambda = (n-2)/2$. It is shown in [2, Theorem 2.1] that any H-invariant positive kernel $Q : V \times V \to \mathbb{R}$ is of the form

$$Q((x, i), (y, j)) = \sum_{k=0}^{\infty} g_{ij,k} P_k(x \cdot y),$$

where $((g_{ijk})_{1 \leq i,j \leq N})_{k \geq 0}$ is an absolutely summable sequence of positive semidefinite matrices in $M_N(\mathbb{R})$. One can combine these results to give a short proof of the following theorem [2, Theorem 1.2].

Theorem 2. *Let $\mathcal{F}_w$ be the set of all functions $F : [-1,1] \to M_N(\mathbb{R})$ of the form $F(u) = \sum_{k=0}^{\infty} P_k(u) F_k$, where $(F_k)_{k\geq 0}$ is an absolutely summable sequence of real symmetric matrices such that:*

i) $F_0 - (w(\alpha_i)^{1/2} w(\alpha_j)^{1/2})_{1\leq i,j\leq N}$ *is positive semidefinite,*

ii) F_k *is positive semidefinite for all* $k \geq 1$,

iii) For all $i, j \in \{1, \ldots, N\}$ *and all* $u \in [-1,1]$ *we have*

$$u \leq \cos(\alpha_i + \alpha_j) \quad \Rightarrow \quad F(u)_{ij} \leq 0.$$

Then the density of every packing of spherical caps with angles from $\mathcal{A}$ on S^{n-1} is bounded by $\max_{1\leq i\leq N} (F(1)_{ii})$.

Proof. Let $F \in \mathcal{F}_w$. The kernel $Q((x,i),(y,j)) := F(x\cdot y)_{ij} - w(\alpha_i)^{1/2} w(\alpha_j)^{1/2}$ is in $\mathcal{K}_w^H$ and $\vartheta'_w(G)^H \leq \max_{(x,i)} Q((x,i),(x,i)) \leq \max_i F(1)_{ii}$. □

One can also show that

$$\inf\left\{ \max_{1\leq i\leq N} (F(1)_{ii}) \;:\; F \in \mathcal{F}_w \right\} = \vartheta'_w(G)^H.$$

This says that Theorem 2 gives the sharpest possible upper bound in the framework of the weighted-theta prime number $\vartheta'_w(G)^H$.

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13 Bounds for unrestricted codes, by linear programming (after P. Delsarte [1])

A summary written by Louis Brown

Abstract

We give upper bounds on the number of codewords in unrestricted additive codes with specified minimum distance δ between codewords.

13.1 Preliminaries

An (n, M) q-ary *additive code* C is a sub**group** of $V = (F, +)^n$ of order M, where F is an arbitrary abelian group of order q. We say C has *length* n.

Remark 1.
Note that q need not be a prime power, and F need not be the additive group of a field, hence "unrestricted". Many of the results in this paper generalize previously known results on linear codes (subspaces of vector spaces $\mathbb{F}_q^n$) to additive codes, which need not be linear even if q is a prime power.

The weight $w(a)$ of $a \in V$ is the number of non-zero entries in a.
$A_i(C)$ is the number of elements in C with weight i.
C has *designed minimum distance* δ if $A_i(C) = 0$ for all $i < \delta$.
The *Hamming distance* $d(a, b)$ on F^n is the number of coordinates at which a and b differ (or, equivalently, $w(a - b)$).

Remark 2. *For any $a \in C$, the number of codewords b in C with $d(a, b) = i$ is $A_i(C)$, since C is additive and d is translation-invariant.*

Y_i is the subset of F^n with weight i.
$|Y_i| = v_i = \binom{n}{i}\lambda^i$, where $\lambda = |F \backslash \{0\}| = q - 1$.
The *dual code* of C is denoted C'.
The *Krawtchouk polynomial* $P_k(x)$ is given by

$$P_k(x) = \sum_{j=0}^{k} (-1)^j \lambda^{k-j} \binom{x}{j}\binom{n-x}{k-j}.$$

The *MacWilliams transform* of a homogeneous polynomial

$$A(y,z) = \sum_{i=0}^{n} A_i y^i z^{n-i}$$

is

$$A'(y,z) = A(z-y, z+\lambda y) = \sum_{i=0}^{n} A'_i y^i z^{n-i}.$$

We apply this where the A_i are $A_i(C)$ for some code C with length n.

Remark 3. *Using generating functions, we see that the MacWilliams transform is described by Krawtchouk polynomials as follows:*

$$A'_k = \sum_{i=0}^{n} A_i P_k(i).$$

The kth *characteristic matrix* of C, H_k, is the $M \times v_i$ matrix of inner products between C and Y_k, arbitrarily ordered.

13.2 Results

Theorem 4. *For any $a \in V$,*

$$\sum_{h \in Y_k} \langle a, h \rangle = P_k(w(a)).$$

Theorem 5. *For all k,*

$$(H_k H_k^*)_{a,b} = P_k(d(a,b)).$$

Proof.

$$(H_k H_k^*)_{a,b} = \sum_{h \in Y_k} \langle a, h \rangle \overline{\langle b, h \rangle} = \sum_{h \in Y_k} \langle a-b, h \rangle = P_k(w(a-b)) = P_k(d(a,b)),$$

by Theorem 4. Note that $\overline{\langle b, h \rangle} = \langle b, h \rangle^{-1} = \langle -b, h \rangle$ since the inner product is defined by mapping $(V, +)$ to multiplicative characters in $\mathbb{C}$. □

Theorem 6. *For any code C, the MacWilliams transform of the weight distribution is non-negative. That is, $A'_i(C) \geq 0$ for all i.*

Proof. Let $j = (1, \ldots, 1)$. Then

$$\|H_k^* j\|^2 = j^* H_k H_k^* j = \sum_{a,b \in C} P_k(d(a,b)) = M \sum_{i=0}^{n} A_i(C) P_k(i) = M A'_k(C),$$

by Remarks 2 and 3, and Theorem 5. Then, since the left side is non-negative, so must be the right side. □

Theorem 7. *Up to a constant factor, the MacWilliams transform yields the weight distribution on the dual code. Specifically, for $0 \leq k \leq n$,*

$$M A_k(C') = A'_k(C).$$

Proof.

$$\|H_k^* j\|^2 = \sum_{h \in Y_k} \sum_{a,b \in C} \langle a - b, h \rangle = M \sum_{h \in Y_k} \left(\sum_{u \in C} \langle u, h \rangle \right).$$

It is well-known that the parenthetical term is 1 if $h \in C'$ and 0 if not, so we may simplify the expression to

$$M \sum_{h \in Y_k \cap C'} \sum_{u \in C} 1 = M^2 |Y_k \cap C'| = M^2 A_k(C').$$

Combining with the argument in Theorem 6, we have

$$M A'_k(C) = \|H_k^* j\|^2 = M^2 A_k(C'),$$

yielding the desired result. □

13.3 Linear Programming

Given a set $D \subseteq N = \{1, \ldots, n\}$ of possible codeword weights, we associate the following linear program P-I:

$$\forall k \in N, \ \sum_{i \in D} A_i P_k(i) \geq -v_k \tag{1}$$

$$\forall i \in D, \ A_i \geq 0 \tag{2}$$

$$\text{maximize } z = \sum_{i \in D} A_i \tag{3}$$

Note that any code satisfies (1), by Remark 3 and the fact that $P_k(0) = v_k$, and satisfies (2) by Theorem 6.
Moreover, letting $\bar{z}(D)$ be the maximum value of z satisfying these 3 equations, $\bar{M}(D) = \bar{z}(D) + 1$ is an upper bound on the number of codes with weights in D.

However, a solution to this linear program does not necessarily yield a code, and indeed for certain q, n, D values, $\bar{M}(D)$ is far from tight. For instance, when $q = 2$, $n = 13$, and $D = \{6, 8, 10, 12\}$, we have $\bar{M}(D) = 40$ but the classical Johnson bound is 35 and there is a known code with $M = 32$.

We can then define the dual program P-II by

$$\forall i \in D, \ \sum_{k \in N} \alpha_k P_k(i) \leq -1 \tag{4}$$

$$\forall k \in N, \ \alpha_k \geq 0 \tag{5}$$

$$\text{minimize } \zeta = \sum_{k \in N} \alpha_k v_k \tag{6}$$

We similarly denote the minimum value of ζ as $\bar{\zeta}(D)$.

Theorem 8.

1. $\bar{z}(D) = \bar{\zeta}(D)$
2. *For any pair of* $(A_i), (\alpha_k)$ *of solutions to P-I and P-II, respectively, we have*

$$\forall k \in N,\ \ \alpha_k A'_k = 0 \tag{7}$$

$$\forall i \in D,\ \ A_i \alpha(i) = 0, \tag{8}$$

where

$$\alpha(i) = \sum_{k \in N} \alpha_k P_k(i)$$

(i.e., $\alpha(x)$ *is a polynomial in Krawtchouk polynomials with coefficients* α_i*).*

Conversely, if $(A_i), (\alpha_i)$ *satisfy equations (1),(2),(4), (5), (7), and (8), then they are necessarily optimal and satisfy (3), (6).*

Theorem 9. *For any* (n, M) *code with weights in* D*, and any polynomial*

$$\alpha(x) = \sum_{k \in N} \alpha_i P_k(x)$$

with $\alpha_0 = 1$*,* $\alpha_k \geq 0$ *for all* $k \in N$*, and* $\alpha(i) \leq 0$ *for all* $i \in D$*, we have*

$$M \leq \alpha(0).$$

13.4 The Bounds

Theorem 10 (Plotkin Bound)**.**
For any C *of designed minimum distance* $\delta > n\lambda/q$*, we have*

$$M \leq \frac{q\delta}{q\delta - n\lambda}.$$

If equality is achieved, then C *is* equidistant *with distance* δ*. That is, every pair of codewords has distance exactly* δ*, and* $A_i(C) = 0$ *for* $i \neq \delta$*.*

Theorem 11 (Singleton Bound)**.**
For any C of designed minimum distance δ, we have

$$M \leq q^{n-\delta+1}.$$

When equality is obtained, C is a maximum distance separable code*, and*

$$A_{n-j}(C) = \sum_{i=j}^{n-\delta} (-1)^{i-j} \binom{i}{j} \binom{n}{i} (q^{n-\delta+1-i} - 1).$$

Theorem 12 (Sphere-Packing Bound / Lloyd Theorem)**.** *For any C of designed minimum distance δ, we have*

$$M \leq q^n \left(\sum_{i=0}^{t} v_i \right)^{-1},$$

where $t = \lfloor \delta - 1)/2 \rfloor$. If some (n, M) q-ary code obtains equality above, it is a perfect t-error-correcting code*, and the* Lloyd Polynomial

$$Q_t(x) = \sum_{i=0}^{t} P_i(x)$$

has t distinct integral zeros in $N = \{1, \ldots, n\}$.

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14 Spherical codes and designs (after Delsarte, Goethals and Seidel [6])

A summary written by Hans Parshall

Abstract

The linear programming bound of Delsarte, Goethals and Seidel [6] provides an upper bound on the cardinality of a spherical code in terms of its angles. We provide some motivation for their approach, discuss their main results, and consider some applications.

14.1 Spherical codes

A **spherical code** is a finite set X of unit vectors in Euclidean space $\mathbf{R}^d$. Fundamental problems in discrete geometry and communication theory are concerned with the interplay between the size $|X|$ and the set of inner products $A(X) := \{\langle x, y\rangle : x, y \in X, x \neq y\}$. For instance, the **kissing number** $\tau(d)$ is the largest number of unit spheres in $\mathbf{R}^d$ that touch S^{d-1} without overlapping. Equivalently, $\tau(d)$ is the size of the largest spherical code $X \subseteq S^{d-1}$ with $A(X) \subseteq [-1, 1/2]$.

More generally, we call a spherical code $X \subseteq S^{d-1}$ an A-**code** when $A(X) \subseteq A \subseteq [-1, 1)$. Delsarte, Goethals and Seidel [6] obtain upper bounds for the size $|X|$ of an arbitrary A-code $X \subseteq S^{d-1}$ in terms of polynomials that interact nicely with the set A. To see the utility of such a strategy, consider their absolute bound:

Theorem 1 ([6, Theorem 4.8]). *If $X \subseteq S^{d-1}$ is an A-code with $s = |A| < \infty$,*

$$|X| \leq \binom{d+s-1}{s} + \binom{d+s-2}{s-1}. \tag{1}$$

Proof. Observe that the polynomial $F(t) = \prod_{\alpha \in A(X)} (t-\alpha)/(1-\alpha)$ vanishes on A and has degree s. For each $y \in X$, define the polynomial function $F_y : S^{d-1} \to \mathbf{R}$ by $F_y(x) = F(\langle x, y\rangle)$. For each $x, y \in X$, notice $F_y(x) = \delta_{x=y}$, and so each F_y is linearly independent. Moreover, each F_y resides in the vector space of real-valued functions on S^{d-1} that can be represented by a polynomial of degree at most s. It follows that $|X|$ is at most the dimension of this vector space, which is given by $\binom{d+s-1}{s} + \binom{d+s-2}{s-1}$; see [7, §2.2]. □

Theorem 1 is clearly sharp for $s = 1$ by considering the $d+1$ vertices $X \subseteq S^{d-1}$ of a regular simplex. However, already for $s = 2$, equality in (1) is only known to occur for $d \in \{2, 6, 22\}$, where the corresponding spherical code $X \subseteq S^{d-1}$ is constructed from a set of $\binom{d+2}{2}$ equiangular lines in $\mathbf{R}^{d+1}$. In what follows, we derive improved upper bounds on $|X|$ for arbitrary A-codes $X \subseteq S^{d-1}$ that depend on more detailed information about A.

14.2 Gegenbauer polynomials

The linear programming bound for spherical codes in $\mathbf{R}^d$ is stated in terms of **Gegenbauer polynomials** $\{Q_k\}_{k=0}^{\infty} \subseteq \mathbf{R}[t]$. These can be described recursively by $Q_0(t) = 1$, $Q_1(t) = dt$, and, for $k \geq 2$,

$$\frac{k}{d+2k-2} Q_k(t) = tQ_{k-1}(t) - \frac{d+k-4}{d+2k-6} Q_{k-2}(t). \tag{2}$$

Note the dependence on the ambient dimension d. This recursion has the benefit of being concrete and the drawback of being completely unmotivated. Before we see how these polynomials are useful, we give some motivation as to why they naturally appear in the context of spherical codes. The rest of this section is based loosely on the excellent lecture notes by Vallentin [14].

We want an upper bound on $|X|$, where $X \subseteq S^{d-1}$ is an arbitrary A-code. Equivalently, we want an upper bound on the clique number of the infinite graph with vertices S^{d-1} and an edge between $x, y \in S^{d-1}$ exactly when $\langle x, y\rangle \in A$. One such influential bound for finite graphs is given by the Lovász theta number [9], which is defined as a semidefinite program. This was strengthened by Schrijver [13] and subsequently extended to infinite graphs by Bachoc, Nebe, Oliveira and Vallentin [1]. To state a specialized version of their bound, let $\mathcal{C}(S^{d-1} \times S^{d-1})$ denote the set of continuous functions $K : S^{d-1} \times S^{d-1} \to \mathbf{C}$, which we call **kernels**. A kernel K is called **positive**, and we write $K \succeq 0$, if for every set of m points $x_1, \ldots, x_m \in S^{d-1}$, the matrix $(K(x_i, x_j))_{i,j\in[m]}$ is positive semidefinite. We have the following bound.

Proposition 2. *If $X \subseteq S^{d-1}$ is an A-code, then*

$$\begin{aligned} |X| \leq \inf\{\lambda : \exists K \in \mathcal{C}(S^{d-1} \times S^{d-1}) \text{ such that } K \succeq 0, \\ K(x,x) = \lambda - 1\ \forall x \in S^{d-1}, \text{ and } K(x,y) \leq -1\ \forall \langle x, y\rangle \in A\}. \end{aligned} \tag{3}$$

Proof. Let $K \in \mathcal{C}(S^{d-1} \times S^{d-1})$ be feasible for (3). Since $K \succeq 0$, we have

$$\sum_{x,y\in X} K(x,y) = \mathbf{1}^T (K(x,y))_{x,y\in X} \mathbf{1} \geq 0,$$

and so

$$0 \le \sum_{x,y\in X} K(x,y) = \sum_{x\in X} K(x,x) + \sum_{\substack{x,y\in X \\ x\neq y}} K(x,y) \le |X|(\lambda-1) - |X|(|X|-1).$$

Rearranging yields $|X| \le \lambda$ as desired. □

In Proposition 2, we may without loss of generality restrict our attention to $O(d)$**-invariant** kernels K, where $K(Rx, Ry) = K(x,y)$ for all $R \in O(d)$. In particular, if K is $O(d)$-invariant, then $K(x,y) = F(\langle x,y\rangle)$ for some continuous function $F \in \mathcal{C}(S^{d-1})$. The Peter–Weyl theorem provides the decomposition $\mathcal{C}(S^{d-1}) = \bigoplus_{k=0}^{\infty} H_k$, where H_k is the vector space of restrictions of homogenous degree-k harmonic polynomials to S^{d-1}. Hence, we may express $O(d)$-invariant kernels in terms of the orthogonal projections $\pi_k : \mathcal{C}(S^{d-1}) \to H_k$. Set $h_k := \dim(H_k)$ and let $\{v_{ik}\}_{i=1}^{h_k}$ be a real orthonormal basis for H_k. A kernel representation for π_k is given by

$$(\pi_k f)(x) = \sum_{i=1}^{h_k} \langle f, v_{ik}\rangle v_{ik} = \int_{S^{d-1}} f(y) \sum_{i=1}^{h_k} v_{ik}(x) v_{ik}(y)\, d\sigma(y) \quad (f \in \mathcal{C}(S^{d-1})).$$

The upshot is that every positive $O(d)$-invariant kernel K can be expressed as $\sum_{k=0}^{\infty} f_k K_k$ with $f_k \ge 0$, with each kernel defined by the addition formula $K_k(x,y) := \sum_{i=1}^{h_k} v_{ik}(x) v_{ik}(y)$. Moreover, each kernel K_k can be expressed as $K_k(x,y) = F_k(\langle x,y\rangle)$ for a polynomial $F_k \in \mathbf{R}[t]$ of degree k, and the orthogonality of the spaces $\{H_k\}_{k=0}^{\infty}$ leads to the orthogonality relation

$$\int_{-1}^{1} F_j(t) F_k(t) (1-t^2)^{(d-3)/2}\, dt = 0 \text{ for } j \neq k.$$

This determines the polynomials $\{F_k\}_{k=0}^{\infty}$ recursively, and rescaling each F_k appropriately yields the Gegenbauer polynomials $\{Q_k\}_{k=0}^{\infty}$ described by (2). The choice of scaling is irrelevant for our applications.

14.3 The linear programming bound for spherical codes

While we could derive the linear programming bound from Proposition 2, we instead give a concrete proof in the spirit of Delsarte, Goethals, and Seidel [6].

Theorem 3 ([6, Theorem 4.3])**.** *If $X \subseteq S^{d-1}$ is an A-code, then*

$$|X| \le \inf\{F(1) : F = \sum_{k=0}^{\infty} f_k Q_k,\ f_0 = 1,\ f_k \ge 0\ \forall k,\ F(\alpha) \le 0\ \forall \alpha \in A\}. \quad (4)$$

Equality in (4) *occurs if and only if $F(\alpha) = 0$ for all $\alpha \in A(X)$ and*

$$f_k \sum_{x,y \in X} Q_k(\langle x, y\rangle) = 0 \text{ for all } k \ge 1.$$

Proof. Let $F = \sum_k f_k Q_k$ be feasible for (4). The key idea is to consider bounding $\sum_{x,y\in X} F(\langle x, y\rangle)$ from above and below. To begin, expand

$$\begin{aligned}\sum_{x,y\in X} F(\langle x, y\rangle) &= \sum_{x,y\in X} Q_0(\langle x, y\rangle) + \sum_{k\ge 1} f_k \sum_{x,y\in X} Q_k(\langle x, y\rangle)\\ &= |X|^2 + \sum_{k\ge 1} f_k \sum_{x,y\in X} Q_k(\langle x, y\rangle).\end{aligned}$$

For $k \ge 1$, $Q_k(\langle x, y\rangle)$ is a positive kernel, and so $\sum_{x,y\in X} F(\langle x, y\rangle) \ge |X|^2$. For an upper bound, the constraint $F(\alpha) \le 0$ for all $\alpha \in A$ provides

$$\sum_{x,y\in X} F(\langle x, y\rangle) = \sum_{x\in X} F(\langle x, x\rangle) + \sum_{\substack{x,y\in X\\ x\ne y}} F(\langle x, y\rangle) \le |X|F(1).$$

All together, we have $|X| \le F(1)$ with equality exactly when claimed. $\square$

Observe that the only property of the Gegenbauer polynomials that we used for Theorem 3 was that, for each k, $\sum_{x,y\in X} Q_k(\langle x, y\rangle) \ge 0$ for all finite $X \subseteq S^{d-1}$. This is weaker than each $Q_k(\langle x, y\rangle)$ being a positive kernel, and Pfender [12] obtained slight improvements based on this observation.

The case of equality in (4) motivates the following definition. A **t-design** is a spherical code $X \subseteq S^{d-1}$ with $\sum_{x,y\in X} Q_k(\langle x, y\rangle) = 0$ for all $1 \le k \le t$. Equivalently, for every polynomial $f \in \mathbf{R}[x_1, \dots, x_d]$ of degree at most t,

$$\int_{S^{d-1}} f(x)d\sigma(x) = \frac{1}{|X|} \sum_{x\in X} f(x);$$

see [7, §9.6]. These highly uniform sets are good candidates for A-codes of maximal cardinality. Indeed, if $X \subseteq S^{d-1}$ is a t-design with $A(X) \subseteq A$

and $F = \sum_{k=0}^{t} f_k Q_k$ is a polynomial that is feasible for (4) that vanishes on $A(X)$, then every A-code has size at most $|X|$.

This strategy gives the exact values for the kissing numbers $\tau(8) = 240$ and $\tau(24) = 196560$. For $d = 8$, the 240 shortest vectors $X_8 \subseteq S^7$ of the E_8 lattice have inner products $A(X_8) = \{-1, -1/2, 0, 1/2\}$. Hence, X_8 is a $[-1, 1/2)$-code and $\tau(8) \geq 240$. Delsarte, Goethals and Seidel [6] showed that X_8 is a 7-design, and later Levenshtein [8] and Odlyzko and Sloane [11] independently showed that

$$F(t) = (320/3)(t+1)(t+1/2)^2 t^2 (t-1/2) \tag{5}$$

satisfies $F = \sum_{k=0}^{6} f_k Q_k$ with $f_0 = 1$ and $f_k \geq 0$. Applying Theorem 3 proves $\tau(8) \leq F(1) = 240$. A similar strategy with the Leech lattice yields $\tau(24) = 196560$.

Delsarte, Goethals and Seidel again use Gegenbauer polynomials to give a linear programming lower bound [6, Theorem 5.10] on the size $|X|$ of an arbitrary t-design $X \subseteq S^{d-1}$. In some sense, this lower bound is dual to Theorem 3 and the proof is similar. They give an upper bound on $|X|$ for spherical t-designs $X \subseteq S^{d-1}$ with fixed $s = |A(X)|$ and show that, in all cases, $t \leq 2s$ [6, Theorem 6.6].

The linear programming method has been generalized beyond Theorem 3. Musin [10] developed a nonconvex extension to prove $\tau(4) = 24$. Cohn and Elkies [3] extended the linear programming method to noncompact settings, leading to the resolution of the sphere packing problems in $\mathbf{R}^8$ by Viazovska [15] and $\mathbf{R}^{24}$ by Cohn, Kumar, Miller, Radchenko, and Viazovska [4]. De Laat and Vallentin [5] identified a general semidefinite programming hierarchy for problems in discrete geometry, the lowest level of which is the linear programming method. For more on the development of these methods, the reader is encouraged to consult the notes of Vallentin [14] and Cohn [2].

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15 Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds (after D. P. Hardin, E. B. Saff [1])

A summary written by Ryan W Matzke

Abstract

We give the asymptotic behavior of the minimal Riesz s-energy over any compact $A \subset \mathbb{R}^d$, for $s \geq d$, as $N \to \infty$, and show that if A has nonzero d-dimensional Hausdorff measure, then N-point configurations on A that minimize this energy are asymptotically uniformly distributed, with respect to the Hausdorff measure.

15.1 Introduction

Let $A \subseteq \mathbb{R}^d$ and $\omega_N = \{x_1, ..., x_N\} \subset A$. For $s > 0$, we define the s-energy (also called the Riesz s-energy) of ω_N as

$$E_s(\omega_N) := \sum_{x \neq y \in \omega_N} \frac{1}{|x-y|^s} = \sum_{y \in \omega_N} \sum_{\substack{x \in \omega_N \\ x \neq y}} \frac{1}{|x-y|^s} \tag{1}$$

and the minimal N-point s-energy over A by

$$\mathcal{E}_s(A, N) := \inf_{\omega_N \subset A} E_s(\omega_N). \tag{2}$$

We are interested in the asymptotic behavior of $\mathcal{E}_s(A, N)$ as $N \to \infty$ and $s \geq d$. In this setting, standard potential theoretic arguments do not apply, so alternative methods are necessary. On the sphere, $\mathbb{S}^d$, $\mathcal{E}_s(\mathbb{S}^d, N)$ behaves like N^2 if $0 < s < d$, $N^2 \log(N)$ if $s = d$, and $N^{1+s/d}$ if $s > d$ [2, 3]. Roughly speaking, at $s = d$, there is a transition from the domination of global effects (when $s < d$) to the domination of more local interactions. Such behave generalizes to a much larger collection of sets. In this summary, we denote the d-dimensional Hausdorff measure by $H_d(\cdot)$, the unit cube $[0,1]^d$ by U^d, and define

$$\tau_{s,d}(N) := \begin{cases} N^2 \log(N) & \text{if } s = d \\ N^{1+s/d} & \text{if } s > d \end{cases}, \qquad G_{s,d}(A, N) := \frac{\mathcal{E}_s(A, N)}{\tau_{s,d}(N)},$$

and $g_{s,d}(A) := \lim_{N\to\infty} G_{s,d}(A,N)$, if it exists ($\overline{g}_{s,d}$ and $\underline{g}_{s,d}$ are the lim sup and lim inf, respectively).

The following three theorems are the main results of [1]:

Theorem 1. *Suppose $A \subset \mathbb{R}^d$ is compact. Then*

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{\tau_{s,d}(N)} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}}, \tag{3}$$

where $C_{s,d}$ is a positive, finite constant, independent of A (and is in fact $g_{s,d}(U^d)$).

Theorem 2. *Let $A \subset \mathbb{R}^d$ be compact, with $\mathcal{H}_d(A) > 0$, and $\omega_N = \{x_{k,N}\}_{k=1}^N$ be a sequence of asymptotically optimal N-point configurations in A in the sense that for some $s \geq d$,*

$$\lim_{N\to\infty} \frac{E_s(\omega_N)}{\tau_{s,d}(N)} = \frac{g_{s,d}(U^d)}{\mathcal{H}_d(A)^{s/d}}, \tag{4}$$

Let δ_x denote the unit point mass in the point x. Then in the weak-star topology of measures, we have

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_{i,N}} \to \frac{\mathcal{H}_d(\cdot)|_A}{\mathcal{H}_d(A)} \quad \text{as } N \to \infty. \tag{5}$$

We call $A \subseteq \mathbb{R}^{d'}$ a d-rectifiable manifold if

$$A = \cup_{k=1}^{n} \phi_k(K_k)$$

where, for $k = 1, ..., n$, $K_k \subset \mathbb{R}^d$ is compact and ϕ_k is bi-Lipschitz on $G_k \supset K_k$ open.

Theorem 3. *Suppose $A \subset \mathbb{R}^{d'}$ is a d-rectifiable manifold and $s \geq d$. If $s = d$, suppose further that A is a subset of a d-dimensional C^1-manifold. Then* (3) *holds. If $\mathcal{H}_d(A) > 0$, then* (5) *holds for any asymptotically minimal sequence of N-point configurations ω_N for A satisfying* (4).

In this summary, we will give a sketch of the first theorem, and then prove the second. The first part of Theorem 3 is proven by carefully determining bounds on the energy using the bi-Lipschitz constants and Theorem 1, and the second part is proven the same way at Theorem 2.

15.2 Energy Asymptotics

To sketch the proof of Theorem 1, we first need (3) to hold if A is the unit cube or a bounded almost clopen set.

15.2.1 The Cube and Almost Clopen Sets

Theorem 4. *The limit* $\lim_{N\to\infty} \frac{\mathcal{E}_s(U^d,N)}{\tau_{s,d}(N)}$ *exists and is finite and positive.*

While we will not go into the details of the proof of Theorem 4, we will give a rough outline. For $s > d$, the self-similarity of U^d, as well as the scaling and translation properties of E_s and the monotonicity of $\mathcal{E}_s(U^d, N)$ and $\tau_{s,d}(N)$, allow one to obtain estimates relating $G_{s,d}(U^d, N)$ at different values N. A careful handling of the asymptotics of these relations shows that $\overline{g}_{s,d}(U^d) \leq \underline{g}_{s,d}(U^d)$, giving us existence. Positivity and finiteness follow from U^d being bounded and having nonempty interior (see Lemma 3.1 in [1]). When $s = d$, one can make use of stereographic projection and the scaling and translation properties of $\mathcal{E}_d$ to relate the minimal d-energy of U_d to the minimal d-energy of a collection of subsets, $\{A_\gamma\}_{\gamma\in(0,1/d)}$, of the sphere, $\mathbb{S}^d$. A careful handling of the asymptotics of $\mathcal{G}_{d,d}(A_\gamma)$ (see [1], [2]) shows that $g_{d,d}(U^d) = \mathcal{H}_d(B^d)$, where B^d is the unit ball in $\mathbb{R}^d$.

Due to the scaling and translation properties of the energy function, we see that if $A = \gamma U^d + x$, with $\gamma > 0$ and $x \in \mathbb{R}^d$, $g_{s,d}(A) = \frac{g_{s,d}(U^d)}{\gamma^s} = \frac{g_{s,d}(U^d)}{\mathcal{H}_d(A)^{s/d}}$. Applying the next lemma inductively, it follows that if A is the finite union of cubes with disjoint interiors, Theorem 1 holds. We call a set A scalable if it is closed and for each $\varepsilon > 0$, there is some bi-Lipschitz mapping $h : A \to A^\circ$ with constant $(1 + \varepsilon)$.

Lemma 5. *Suppose $s \geq d$ and A and B are compact subsets of $\mathbb{R}^d$ such that $A^\circ \cap B^\circ = \emptyset$, $g_{s,d}(A)$ and $g_{s,d}(B)$ exist, and A is scalable. Then*

$$g_{s,d}(A \cup B) = \left(g_{s,d}(A)^{-d/s} + g_{s,d}(B)^{-d/s}\right)^{-s/d}.$$

We call a measurable set $A \subset \mathbb{R}^d$ almost clopen if $\mathcal{H}_d(\partial A) = 0$.

Theorem 6. *Let A be a bounded almost clopen set in $\mathbb{R}^d$. Then $g_{s,d}(A)$ exists for $s \geq d$ and*

$$g_{s,d}(A) = \frac{g_{s,d}(U^d)}{\mathcal{H}_d(A)^{s/d}}.$$

Proof. Let Q_n be the collection of cubes of the form $\left[\frac{k_1}{n}, \frac{k_1+1}{n}\right] \times \cdots \times \left[\frac{k_n}{n}, \frac{k_n+1}{n}\right]$, with $k_1, ..., k_n \in \mathbb{Z}$. We define $\underline{A}_n$ as the union of cubes in Q_n that are contained in A and $\overline{A}_n$ as the union of cubes in Q_n that intersect the closure of A. Since $\mathcal{H}_d(\partial A) = 0$, it follows that $\lim_{n\to\infty} \mathcal{H}_d(\overline{A}_n \cap \underline{A}_n^c) = 0$, so

$$\lim_{n\to\infty} \mathcal{H}_d(\underline{A}_n) = \lim_{n\to\infty} \mathcal{H}_d(\overline{A}_n) = \mathcal{H}_d(A).$$

Our claim then follows from

$$\frac{\mathcal{H}_d(U^d)}{\mathcal{H}_d(\underline{A}_n)^{s/d}} = g_{s,d}(\underline{A}_n) \leq \underline{g}_{s,d}(A) \leq \overline{g}_{s,d}(A) \leq g_{s,d}(\overline{A}_n) = \frac{\mathcal{H}_d(U^d)}{\mathcal{H}_d(\overline{A}_n)^{s/d}}.$$

□

15.2.2 Compact Sets

To prove Theorem 1, we first require the following lemma, relating the energies of two disjoint sets to the energy of their union:

Lemma 7. *If A and B are bounded sets in $\mathbb{R}^d$ such that dist$(A, B) > 0$, then*

$$\overline{g}_{s,d}(A \cup B) \leq \left(\overline{g}_{s,d}(A)^{-d/s} + \overline{g}_{s,d}(B)^{-d/s}\right)^{-s/d}. \tag{6}$$

Sketch of proof of Theorem 1. Let $\varepsilon > 0$ and G be the union of a finite collection of open balls such that $A \subset G$ and $\mathcal{H}_d(G \setminus A) = \varepsilon$. The set G is almost clopen, so Theorem 6 gives us

$$\underline{g}_{s,d}(A) \geq g_{s,d}(G) = \frac{\mathcal{H}_d(U^d)}{\mathcal{H}_d(G)^{s/d}} \geq \frac{\mathcal{H}_d(U^d)}{(\mathcal{H}_d(A) + \varepsilon)^{s/d}}. \tag{7}$$

If $\mathcal{H}_d(A) = 0$, then $\underline{g}_{s,d}(A) = \overline{g}_{s,d}(A) = \infty$. For the rest of the proof, we may assume that $\mathcal{H}_d(A) > 0$. Since (7) holds for $\varepsilon > 0$, we have

$$\underline{g}_{s,d}(A) \geq \frac{\mathcal{H}_d(U^d)}{\mathcal{H}_d(A)^{s/d}}. \tag{8}$$

Now we need only show that $\overline{g}_{s,d}(A) \leq \frac{\mathcal{H}_d(U^d)}{\mathcal{H}_d(A)^{s/d}}$. Let

$$A^* := \left\{ x \in A : \limsup_{r\to 0^+} \frac{\mathcal{H}_d(\overline{B}(x,r) \cap A)}{\mathcal{H}_d(\overline{B}(x,r))} \right\}.$$

The Lebesgue Density Theorem states that $\mathcal{H}_d(A \setminus A^*) = 0$. For $0 < \varepsilon < 1$, set

$$C_\varepsilon := \left\{ \overline{B}(x,r) : x \in A^*, r \in (0,1), \frac{\mathcal{H}_d(\overline{B}(x,r) \cap A)}{\mathcal{H}_d(\overline{B}(x,r))} > 1 - \varepsilon \right\}. \tag{9}$$

The Besicovitch covering theorem guarantees a countable collection of pairwise disjoint closed balls $\{B_i = \overline{B}(x_i, r_i)\} \subset C_\varepsilon$ that covers almost all of A^*, and therefore almost all of A. We may choose n sufficiently large so that

$$\mathcal{H}_d\left(\bigcup_{j=1}^{n} A \cap B_i\right) = \sum_{j=1}^{n} \mathcal{H}_d(A \cap B_i) \geq (1-\varepsilon)\mathcal{H}_d(A). \tag{10}$$

The energy over $A \cap B_i$ can be bounded in terms of the energy over B_i, and one finds that

$$\overline{g}_{s,d}(A \cap B_i) \leq \frac{1}{(1-2v)^s}\left(\frac{1}{1-4C\varepsilon v^{-d}}\right)^{1+s/d} g_{s,d}(B_i). \tag{11}$$

Lemma 7, Theorem 6, (11), and (10) combine to give us

$$\begin{aligned}
\overline{g}_{s,d}(A) \leq \overline{g}_{s,d}\left(\bigcup_{i=1}^{n} A \cap B_i\right) &\leq \left(\sum_{i=1}^{n} \overline{g}_{s,d}(A \cap B_i)^{-d/s}\right)^{-s/d} \\
&\leq \frac{1}{(1-2v)^s}\left(\frac{1}{1-4C\varepsilon v^{-d}}\right)^{1+s/d} g_{s,d}(U^d)\left(\sum_{i=1}^{n} \mathcal{H}_d(B_i)\right)^{-s/d} \\
&\leq \frac{1}{(1-2v)^s}\left(\frac{1}{1-4C\varepsilon v^{-d}}\right)^{1+s/d} g_{s,d}(U^d)(1-\varepsilon)^{-s/d}\mathcal{H}_d(A)^{-s/d}
\end{aligned}$$

for $\varepsilon > 0$ and $(4C\varepsilon)^{1/d} < v < \frac{1}{2}$. Taking $\varepsilon \to 0$, then $v \to 0$, we see that $\overline{g}_{s,d}(A) \leq \frac{g_{s,d}(U^d)}{\mathcal{H}_d(A)^{s/d}}$, giving us our claim. □

15.3 Proof of Theorem 2

To prove Theorem 2, we need Theorem 1 and the following lemma:

Lemma 8. *Suppose $s \geq d$ and that A and B are bounded subsets of $\mathbb{R}^d$. If $\underline{g}_{s,d}(A)$ or $\underline{g}_{s,d}(B)$ are finite and $(\omega_{N_j})_{j\in\mathbb{N}}$ is a sequence of sets $\omega_{N_j} \subset A \cup B$ such that*

$$\lim_{j\to\infty} \frac{E_s(\omega_{N_j})}{\tau_{s,d}(N)} = \left(\underline{g}_{s,d}(A)^{-d/s} + \underline{g}_{s,d}(B)^{-d/s}\right)^{-s/d},$$

then

$$\lim_{j\to\infty} \frac{|\omega_{N_j} \cap A|}{N} = \frac{\underline{g}_{s,d}(B)^{d/s}}{\underline{g}_{s,d}(B)^{d/s} + \underline{g}_{s,d}(A)^{d/s}}.$$

Proof of Theorem 2. Let $B \subseteq A$ be a measurable set, with $\mathcal{H}_d(\partial_r B) = 0$, where $\partial_r B := \partial B \cap \overline{A \setminus B}$ is the relative boundary of B. Then $A = \partial_r B \cup \overline{A \setminus \partial_r B} = A_1 \cup A_2$, where $A_1 := B \cup \partial_r B$ and $A_2 := (A \setminus B) \cup \partial_r (A \setminus B)$. Since $\mathcal{H}_d(\partial_r(A \setminus B)) = 0$, we have $\mathcal{H}_d(A) = \mathcal{H}_d(A_1) + \mathcal{H}_d(A_2)$. Theorem 1 and Lemma 8 then give us that

$$\lim_{N\to\infty} \frac{|\omega_N \cup A_1|}{N} = \frac{\mathcal{H}_d(B)}{\mathcal{H}_d(A)} \quad \text{and} \quad \lim_{N\to\infty} \frac{|\omega_N \cup \partial_r B|}{N} = 0.$$

Thus, for all measurable $B \subseteq A$ with $\mathcal{H}_d(\partial_r B) = 0$, $\lim_{N\to\infty} \frac{|\omega_N \cup B|}{N} = \frac{\mathcal{H}_d(B)}{\mathcal{H}_d(A)}$, which is equivalent to (5), giving us our claim. □

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16 On kissing numbers and spherical codes in high dimensions (after M. Jenssen, F. Joos, and W. Perkins [1])

A summary written by Nina Zubrilina

Abstract

We prove a lower bound of $\Omega\left(d^{3/2} \cdot (2/\sqrt{3})^d\right)$ on the kissing number in dimension d, improving the classical lower bound of Chabauty, Shannon, and Wyner by a linear factor. We obtain a similar linear factor improvement to the best known lower bound on the maximal size of a spherical code of acute angle θ in high dimensions.

16.1 Introduction

Let S_{d-1} denote the unit sphere in $\mathbb{R}^d$. A spherical code of angle θ in dimension d is a set of vectors $x_1, \ldots, x_k \in S_{d-1}$ such that $\langle x_i, x_j \rangle \leq \cos\theta$ for all $i \neq j$, i.e., a collection of unit vectors in $\mathbb{R}^d$ such that every pair forms an angle at least θ. We call k the *size* of the code, and we let $A(d, \theta)$ be the maximal size of a spherical code of angle θ in dimension d.

Spherical codes are closely related to kissing configurations. The kissing number in dimension d, $K(d)$, is the maximum number of non-overlapping unit spheres that can touch a single unit sphere in $\mathbb{R}^d$. By radially projecting the centers of the spheres onto the unit sphere, it is easy to see that $K(d) = A(d, \pi/3)$.

We will give a lower bound on $A(d, \theta)$ by analyzing the expected size of a random spherical code. Bounding it from below will allow us to prove the following result:

Theorem 1.

$$K(d) \geq (1 + o(1))\sqrt{\frac{3\pi}{8}} \log \frac{3}{2\sqrt{2}}\; d^{3/2} \left(\frac{2}{\sqrt{3}}\right)^d \approx 0.0639\; d^{3/2} \left(\frac{2}{\sqrt{3}}\right)^d$$

as $d \to \infty$.

16.2 The hard cap model

For $x \in S_{d-1}$, let $C_\theta(x) := \{y \in S_{d-1} : \langle x, y\rangle \geq \cos\theta\}$ be the spherical cap of angular radius θ centered at x. Observe that a spherical code $x_1, \ldots, x_k$ corresponds to a covering of S_{d-1} by non-overlapping caps $C_{\theta/2}(x_i)$.

Let $\mathcal{P}_k(d,\theta) := \{\{x_1, \ldots, x_k\} \in S_{d-1}^k : \langle x_i, x_j\rangle \leq \cos\theta \text{ for all } i \neq j\}$ be the set of all spherical codes in dimension d of size k and angle θ. The *canonical hard cap model with k caps* is a spherical code drawn from $\mathcal{P}_k(d,\theta)$ uniformly at random. For a measurable subset $A \subseteq S_{d-1}$, let

$$\hat{Z}_A^\theta(k) := \frac{1}{k!} \int_{A^k} \mathbf{1}_{\mathcal{D}(x_1,\ldots,x_k)} ds(x_1) \cdots ds(x_k),$$

where $\mathcal{D}(x_1, \ldots, x_k)$ is the event that $\langle x_i, x_j\rangle \leq \cos\theta$ for all $i \neq j$, and $s(\cdot)$ denotes the normalized surface area, i.e., $s(A) := \frac{\hat{s}(A)}{\hat{s}(S_{d-1})}$, where $\hat{s}(\cdot)$ is the surface area. Then $\hat{Z}_d^\theta(k) := \hat{Z}_{S_{d-1}}^\theta(k)$ is the partition function for the hard cap model. Note that the probability that k random points on S_{d-1} form a spherical code is $k!\ \hat{Z}_d^\theta(k)$.

For a measurable subset $A \subseteq S_{d-1}$, we let $\mathbf{X}_A$ be a Poisson point process of intensity λ on A conditioned on $\langle x, y\rangle \leq \cos\theta$ for all distinct $x, y \in \mathbf{X}_A$. The partition function for this process is

$$Z_A^\theta(\lambda) := \sum_{k\geq 0} \lambda^k \hat{Z}_A^\theta(k).$$

The *grand canonical hard cap model* at fugacity λ is the Poisson process $\mathbf{X} := \mathbf{X}_{S_{d-1}}$ with a partition function $Z_d^\theta(\lambda) := Z_{S_{d-1}}^\theta(\lambda)$. This corresponds to a random process where we choose a random integer k with probability proportional to $\lambda^k \hat{Z}_d^\theta(k)$ and then choose a spherical code $\mathbf{X}$ from the canonical hard cap model with k caps. We will write $\mathbb{E}_\lambda, \mathbb{P}_\lambda, \mathrm{var}_\lambda$ to indicate the dependence of the model on the fugacity λ.

Our goal will be to give a lower bound on the expected size $\alpha_d^\theta(\lambda)$ of a random spherical code in this model (stated in the next section). Specifically, we let

$$\alpha_A^\theta(\lambda) := \sum_{k\geq 0} k\cdot\mathbb{P}_\lambda[|\mathbf{X}_A| = k] = \sum_{k\geq 0} k\, \frac{\lambda^k \hat{Z}_A^\theta(\lambda)}{Z_A^\theta(\lambda)} = \lambda\, \frac{\partial Z_A^\theta(\lambda)/\partial\lambda}{Z_A^\theta(\lambda)} = \lambda\, \frac{\partial\left(\log Z_A^\theta(\lambda)\right)}{\partial\lambda},$$

and let $\alpha_d^\theta(\lambda) := \alpha_{S_{d-1}}^\theta(\lambda)$.

We define $s_d(\theta) := s(C_\theta(x))$ to be the normalized surface area of a cap of angular radius θ in S_{d-1}. Let $q(\theta)$ denote the minimal angular radius of a cap that contains the intersection of two spherical caps of angular radius θ whose centers are at angle θ, that is, $q(\theta) := \arcsin\left(\frac{\sqrt{(\cos\theta-1)^2(1+2\cos\theta)}}{\sin\theta}\right)$ (note $q(\theta) < \theta$ for $\theta \in (0, \pi/2)$). We let $c_\theta := \log\left(\frac{\sin\theta}{\sin q(\theta)}\right)$ (note this is positive for $\theta \in (0, \pi/2)$). We prove the following theorem:

Theorem 2. *Let $\theta \in (0, \pi/2)$ and let $\lambda \geq \dfrac{1}{d \cdot s_d(q(\theta))}$. Then:*

$$\alpha_d^\theta(\lambda) \geq (1 + o(1))\frac{c_\theta \cdot d}{s_d(\theta)}.$$

Since $K(d) = A(d, \pi/3) \geq \alpha_d^{\pi/3}(\lambda)$, we can obtain Theorem 1 by plugging in $\theta = \pi/3$ and using that $s_d(\theta)^{-1} = (1 + o(1))\sqrt{2\pi d} \cdot \cos(\theta)/\sin^{d-1}(\theta)$ (from the work of Chabauty, Shannon, and Wyner).

We let the *free area*, $F_A^\theta(\lambda)$, be the expected normalized surface area of points $y \in A$ that form a valid spherical code when added to $\mathbf{X}_A$, that is,

$$F_A^\theta(\lambda) := \mathbb{E}_\lambda\left[s\left(\{y \in A : \langle y, x\rangle \leq \cos\theta \text{ for all } x \in \mathbf{X}_A\}\right)\right].$$

For a spherical code $\mathbf{X}_A$ and a random vector $\mathbf{v} \in A$, we let

$$\mathbf{T}_A := \{x \in C_\theta(\mathbf{v}) \cap A : \langle x, y\rangle \leq \cos\theta \text{ for all } y \in \mathbf{X}_A \cap C_\theta(\mathbf{v}^c\},$$

i.e. $\mathbf{T}_A$ is the set of points in $A \cap C_\theta(\mathbf{v})$ which are not blocked from being in the spherical code by a vector outside the cap $C_\theta(\mathbf{v})$. We let $\mathbf{T} := \mathbf{T}_{S_{d-1}}$.

16.3 Outline of the proof of Theorem 2

The first step of the proof is to outline some basic properties enjoyed by some of the objects defined above.

Lemma 3.

(i) $\alpha_A^\theta(\lambda)$ *is strictly increasing in* λ.

(ii) $\alpha_A^\theta(\lambda) = \lambda \cdot s(A) \cdot \mathbb{E}\left[\frac{1}{Z_{\mathbf{T}_A}^\theta}(\lambda)\right]$.

(iii) $\alpha_A^\theta(\lambda) = \frac{1}{s_d(\theta)}\mathbb{E}\left[\alpha_{\mathbf{T}}^\theta(\lambda)\right]$.

(iv) $\log Z_A^\theta(\lambda) \le \lambda s(A)$. .

Sketch of proof. To see (i), note

$$\lambda\cdot\alpha_A^\theta(\lambda)' = \lambda\left(\frac{\lambda(Z_A^\theta(\lambda))'}{Z_A^\theta(\lambda)}\right)' = \mathbb{E}_\lambda[|\mathbf{X}_A|]+\mathbb{E}_\lambda[|\mathbf{X}_A|(\mathbf{X}_A-1)]-(\mathbb{E}_\lambda[|\mathbf{X}_A|])^2 = \mathrm{var}_\lambda(|\mathbf{X}_A|) > 0.$$

To see (ii), first, note that

$$\begin{aligned}\alpha_A^\theta &= \sum_{k=0}^{\infty}\mathbb{P}_\lambda[|\mathbf{X}_A| = k+1] = \frac{1}{Z_A^\theta(\lambda)}\sum_{k=0}^{\infty}\int_{A^{k+1}}\frac{\lambda^{k+1}}{k!}\mathbf{1}_{\mathcal{D}_\theta(x_0,\ldots,x_k)}ds(x_0)\cdots ds(x_k)\\ &= \frac{1}{Z_A^\theta(\lambda)}\int_A\left(1+\sum_{k=1}^{\infty}\int_{A^k}\frac{\lambda^k}{k!}\mathbf{1}_{\mathcal{D}_\theta(x_0,\ldots,x_k)}ds(x_1)\cdots ds(x_k)\right)ds(x_0) = \lambda\cdot F_A^\theta(\lambda),\end{aligned}$$

and then note

$$F_A^\theta(\lambda) = \int_A \mathbb{P}[\max_{y\in\mathbf{X}}\langle v,y\rangle \le \cos\theta]ds(v) = s(A)\cdot\mathbb{E}\left[\mathbf{1}_{\mathbf{T}_A\cap\mathbf{X}_A=\emptyset}\right] = s(A)\cdot\mathbb{E}\left[\frac{1}{Z_{\mathbf{T}_A}^\theta(\lambda)}\right].$$

The last equality uses the spatial Markov property of the hard cap model: conditioned on $X\cap C_\theta(\mathbf{v})^c$, the distribution of $X\cap C_\theta(\mathbf{v})$ is exactly that of the hard cap model on the set $\mathbf{T}_A$.

To see (iii), let $\mathbf{v}$ be a random chosen point on S_{d-1}. Let $\mathbf{X}$ be a spherical code chosen independently from the hard cap model on S_{d-1}. Let $\mathbf{T}\subseteq C_\theta(\mathbf{v})$ be the random set described above. Using the Markov property again,

$$\alpha_d^\theta(\lambda) = \frac{1}{s_d(\theta)}\mathbb{E}\left[\mathbf{X}\cap C_\theta(\mathbf{v})\right] = \frac{1}{s_d(\theta)}\mathbb{E}\left[\alpha_{\mathbf{T}}^\theta(\lambda)\right].$$

Lastly, by definitions of Z and $\hat{Z}$ we have, $Z_A^\theta(\lambda) \le \sum_{k\ge 0}\frac{1}{k!}s(A)^k\lambda^k = e^{\lambda s(A)}$, proving (iv).

□

We will also use the following lemma.

Lemma 4. *Let $x \in S_{d-1}$, let $A \in C_\theta(x)$ be measurable with positive measure. Let $\mathbf{u}$ be a uniformly random point in A. Then:*

$$\mathbb{E}\left[s\left(C_\theta(\mathbf{u} \cap A)\right] \leq 2s_d(q(\theta)).$$

The proof of this lemma is a geometric calculation which we are going to leave out. Now, from (ii), (iv) and Jensen's inequality, we can see that $\alpha_A^\theta(\lambda) \geq \lambda s(A) e^{-\lambda \mathbb{E}[s(\mathbf{T}_A)]}$, so combined with the lemma above, we get that

$$\alpha_A^\theta(\lambda) \geq \lambda s(A) e^{-2\lambda s_d(q(\theta))}. \tag{1}$$

We are now ready to prove the main result.

Proof of Theorem 2. Let $\theta \in (0, \pi/2)$, and let $\alpha := \alpha_d^\theta(\lambda)$. From Jensen's inequality and (ii),

$$\alpha = \lambda \mathbb{E}\left[\frac{1}{Z_{\mathbf{T}}^\theta(\lambda)}\right] \geq \lambda e^{-\mathbb{E}\left[\log Z_{\mathbf{T}}^\theta(\lambda)\right]}.$$

On the other hand, by inequality (1) combined with (iii) and (iv),

$$\alpha = \frac{\mathbb{E}\left[\alpha_{\mathbf{T}}^\theta(\lambda)\right]}{s_d(\theta)} \geq \frac{1}{s_d(\theta)} \mathbb{E}\left[\lambda s(\mathbf{T}) e^{-2\lambda s_d(q(\theta))}\right] \geq \frac{e^{-2\lambda s_d(q(\theta))}}{s_d(\theta)} \mathbb{E}\left[\log Z_{\mathbf{T}}^\theta(\lambda)\right].$$

The idea now is to play these two lower bounds on α against each other. Let $z := \mathbb{E}\left[\log Z_{\mathbf{T}}^\theta(\lambda)\right]$. From the two inequalities above,

$$\alpha \geq \inf_z \max\left\{\lambda e^{-z}, \frac{z}{s_d(\theta)} e^{-2\lambda s_d(q(\theta))}\right\}.$$

As the first expression is decreasing in z and the second increasing, the infimum occurs when they are equal, so $\alpha \geq \lambda e^{-z^*}$, where z^* is the solution to $\lambda e^{-z} = \frac{z}{s_d(\theta)} e^{-2\lambda s_d(q(\theta))}$. It remains to compute z^* asymptotically and set $\lambda := \dfrac{1}{d s_d(q(\theta))}$ to get the desired bound (this is because although α increases with λ, the obtained bound does not necessarily become sharper as λ grows; the chosen value of λ gives the best lower bound). □

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17 Bounding sphere packings and spherical codes (after G. A. Kabatiansky and V. I. Levenshtein)

A summary written by Itamar Oliveira and Mateus Sousa

Abstract

We prove upper bounds for the spherical code problem, and we obtain upper bounds for the sphere packing problem as a byproduct via a simple geometric argument that associates these two quantities.

17.1 Introduction

We call a collection $\mathcal{P}$ of disjoint balls $B \subset \mathbb{R}^d$ such that every ball $B \subset \mathcal{P}$ has the same radius a *sphere packing*. The sphere packing problem consists of maximizing the proportion of the spaces that such a collection $\mathcal{P}$ can occupy. More precisely, we define the density of a packing $\mathcal{P}$ as

$$\Delta_{\mathcal{P}} = \limsup_{R\to\infty} \frac{|[-R,R]^d \cap \left[\bigcup_{B\in\mathcal{P}} B\right]|}{(2R)^d}.$$

The sphere packing problem consists of determining the value

$$\Delta_{\mathbb{R}^d} = \Delta = \sup_{\mathcal{P} \text{ packing}} \Delta_{\mathcal{P}},$$

and if this value is attained by some packing $\mathcal{P}$.

A *spherical code* with separation $\varphi \in [0,\pi]$ is a set $\mathcal{C} = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$ such that $|x_i| = 1$ for any $i = 1, \dots, N$ and $\langle x, y\rangle \leq \cos\varphi$. The spherical code problem is to determine the value of

$$m[d,\varphi] = \sup_{\mathcal{C}} \#\mathcal{C},$$

where the supremum is taken over all spherical codes with separation φ. In other words, this problem consists of finding the maximum number of points one can place inside the unit sphere $\mathbb{S}^{d-1}$, assuming the points are separated by an angle bigger than φ.

Intuitively, one might expect that the spherical code problem and the sphere packing are connected. This is indeed the case, and one way to quantify it is the following result

Proposition 1 (Cohn and Zhao [1]). *Let $d \geq 1$ and $\pi/3 \leq \varphi \leq \pi$. Then*

$$\Delta_{\mathbb{R}^d} \leq \sin^{d/2}(\varphi/2) m[d, \varphi]. \tag{1}$$

The main result we present is the following bound for spherical codes.

Theorem 2 (Kabatiansky, Levenshtein [2]). *Fix $0 \leq \varphi \leq \pi/2$. For big enough d, one has*

$$\frac{1}{d} m[d, \varphi] \lesssim \frac{1+\sin\varphi}{2\sin\varphi} \log\left[\frac{1+\sin\varphi}{2\sin\varphi}\right] - \frac{1-\sin\varphi}{2\sin\varphi} \log\left[\frac{1-\sin\varphi}{2\sin\varphi}\right] \tag{2}$$

Theorem 2 is a direct consequence of a result of Delsarte which we will present in the next section. Due to the following inequality for $\varphi' \leq \varphi$

$$(1-\cos\varphi)^{(d-1)/2} m[d, \varphi] \leq \sqrt{2\pi d}(1-\cos\varphi')^{(d-1)/2}, m[d+1, \varphi'],$$

we get the following result as a corollary of Theorem 2.

Corollary 3. *Fix $0 \leq \varphi \leq \varphi^*$, and big enough d, one has*

$$\frac{1}{d} m[n, \varphi] \lesssim -\frac{1}{2} \log(1-\cos\varphi) - 0.099,$$

where φ^* solves the equation

$$\cos\varphi \log\left[\frac{1+\sin\varphi}{1-\sin\varphi}\right] + (1+\cos\varphi)\sin\varphi = 0$$

Corollary 3 together with (1) will imply the following sphere packing bound.

Corollary 4. *For big enough d, one has*

$$\Delta_{\mathbb{R}^d} \lesssim 2^{-n(0.599\ldots+o(1))}$$

17.2 Preliminaries

17.2.1 Spherical harmonics and Gegenbauer Polynomials

We let $\mathbb{H}_k^d$ denote the space of spherical harmonics of degree k over the sphere $\mathbb{S}^{d-1}$, which is known to be vector space of dimension $h_{d,k} = \binom{d+k-1}{k} - \binom{d+k-3}{k-2}$.

If one chooses an orthonormal basis $\{Y_{k,\ell}\}_{\ell=1}^{h_{d,k}}$ of $\mathbb{H}_k^d$, it follows from [4, Chapter IV, Lemma 2.8 and Corollary 2.9] we know that the function

$$Z_k(\xi, \eta) := \sum_{\ell=1}^{h_{d,k}} Y_{k,\ell}(\xi)\, \overline{Y_{k,\ell}(\eta)}$$

is real-valued and does not depend on the choice of the orthonormal basis $\{Y_{k,\ell}\}_{\ell=1}^{h_{d,k}}$. It is indeed the reproducing kernel of the finite dimensional space $\mathbb{H}_k^d$, i.e, for every $f \in \mathbb{H}_k^d$

$$f(\xi) = \int_{\mathbb{S}^{d-1}} f(\omega)\overline{g(\omega)}\mathrm{d}\sigma(\omega).$$

When $d > 2$, the function $Z_k(\xi, \eta)$ has a particularly simple expression in terms of the Gegenbauer (or ultraspherical) polynomials C_k^λ. For $\lambda > 0$, these are orthogonal polynomials in the interval $[-1, 1]$ with respect to the measure $(1 - t^2)^{\lambda - \frac{1}{2}}\mathrm{d}t$ (in particular, C_k^λ has degree k), and are defined by the generating function

$$(1 - 2rt + r^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k^\lambda(t)\, r^k.$$

From [4, Chapter IV, Theorem 2.14], if $d > 2$ we have

$$Z_k(\xi, \eta) = c_{d,k}\, C_k^{(d-2)/2}(\xi \cdot \eta),$$

for some constant c_k

17.2.2 Delsarte's bound

The key to get Theorem 2 is the following bound:

Theorem 5 (Delsarte's Lemma)**.** *Consider a function* $f : [-1, 1] \to \mathbb{R}$ *such that* *(i)* $f(t) = \sum_{k=0}^{\ell} f_k C_k^{(d-2)/2}(t)$.

(ii) $f_k \geq 0$, *when* $k > 0$, *and* $f_0 > 0$.

(iii) $f(t) \leq 0$, *for* $-1 \leq t \leq \cos\varphi$.

Then

$$m[d, \varphi] \leq \frac{f(1)}{f_0}.$$

Delsarte's results is a simple consequence of the aforementioned connection between spherical harmonics and with Gegenbauer polynomials. Despite its simplicity, it is a powerful tool, and a key ingredient of the proof of Theorem 2.

17.3 A few words about the proof of Theorem 2

The proof of Theorem 2 consists of a optimization over the possible choices of f in Delsarte's lemma. The functions we are gonna consider are of the form

$$f(t) = \frac{\left[C_{l+1}^{(d-2)/2}(t)C_l^{(d-2)/2}(s) - C_l^{(d-2)/2}(t)C_{l+1}^{(d-2)/2}(s)\right]^2}{t-s}, \tag{3}$$

where $s = \cos\varphi$. It is clear that for any l, a function of the form (3) satisfies automatically condition (iii) of Delsarte's lemma, so now we are left with analysing the other conditions. There are essentially three main steps:

Step 1: An application of the Christofell-Darboux formula [3] allows one to express f in (3) as sum of Gegenbauer polynomials in the following fashion

$$f(t) = \sum f_k(s) C_k^{(d-2)/2}(t), \tag{4}$$

and in fact one gets

$$f_0 = f_0(s) = \frac{d-2}{l+1}[-C_{l+1}^{(d-2)/2}(s)C_l^{(d-2)/2}(s)]\binom{l+d-1}{l}. \tag{5}$$

Step 2: Once one decomposes f as in (4), we have to prove the values of $f_k(s)$ are non negative numbers in order to apply Delsarte's lemma. That task is performed by choosing l big enough so the s lies between the last zero of $C_l^{(d-2)/2}(t)$ and the last zero of $C_{l+1}^{(d-2)/2}(t)$ to use the sign changes of the polynomials in order for that f fullfills condition (ii) of Delsarte's lemma.

Step 3: Now that f fullfills all the needed hypothesis, it becomes a problem of optimization of the value of $\frac{f(1)}{f_0}$, and it can be deduced to be of the form

$$\frac{f(1)}{f_0} = \binom{l+d-2}{l} \frac{(1+\tau)^2}{(1-s)\tau},$$

where

$$\tau = -\frac{l+1}{l+d-2} \frac{C_{l+1}^{(d-2)/2}(s)}{C_l^{(d-2)/2}(s)}.$$

The bound in Theorem 2 is now a consequence of the optimization process over the degree.

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18 Asymptotics for Minimal Discrete Energy on the Sphere (after A. B. J. Kuijlaars and E. B. Saff [2])

A summary written by Changkeun Oh

Abstract

We investigate the energy of arrangements of N points on the surface of the unit sphere S^d in $\mathbb{R}^{d+1}$ that interact through a potential $V = 1/r^s$. In the cases when $0 < s < d$ or $2 \le d \le s$, we obtain bounds for the minimal energy for such N-point arrangements. For $s = d$, we determine the precise asymptotic behavior of the minimal energy as $N \to \infty$.

18.1 Introduction

We are interested in the asymptotics for minimal discrete energy on the sphere. Let $S^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ be the unit sphere in $\mathbb{R}^{d+1}$. We denote by σ the normalized Lebesgue measure on S^d.

For a given $s > 0$, the discrete s-energy associated with a finite subset $w_N = \{x_1, \ldots, x_N\}$ of points of S^d is

$$E_d(s, w_N) := \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|^s}.$$

The minimal s-energy for N points on the sphere is

$$\mathcal{E}_d(s, N) := \inf_{w_N} E_d(s, W_N),$$

where the infimum is taken over all N-points subsets of S^d. Any configuration w_N for which the minimum is attained is called an s-extremal configuration.

We try to obtain the asymptotic behavior of the minimal s-energy $\mathcal{E}_d(s, N)$. We first consider the energy integral. For $0 < s < d$,

$$I_{d,s}(\mu) := \int_{S^d} \int_{S^d} \frac{1}{|x - y|^s} d\mu(x) d\mu(y),$$

the measure is taken for probability measures μ on S^d. We define

$$V_d(s) := I_{d,s}(\sigma).$$

The main theorems are as follows.

Theorem 1. *Let $d \geq 2$ and $0 < s < d$. There is a constant $C > 0$ such that*

$$\mathcal{E}_d(s, N) \leq \frac{1}{2} V_d(s) N^2 - C N^{1+\frac{s}{d}}.$$

Theorem 2. *Let $d \geq 2$ and $s > d$. There are constants $C_1, C_2 > 0$ such that*

$$C_1 N^{1+\frac{s}{d}} \leq \mathcal{E}_d(s, N) \leq C_2 N^{1+\frac{s}{d}}.$$

Theorem 3. *Let $d \geq 2$ and $s = d$. Then*

$$\lim_{N \to \infty} \frac{\mathcal{E}_d(d, N)}{N^2 \log N} = \frac{1}{2d} \gamma_d,$$

where

$$\gamma_d = \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2})}.$$

Remark 4. *Theorem 2 are improved and generalized by Hardin and Saff [1]. They also generalize Theorem 3 in [1].*

18.2 Some conjectures for $d = 2$

The Voronoi cell associated with x_i of a configuration $w_N = \{x_1, \ldots, x_N\}$ is

$$\{x \in S^2 : |x - x_i| \leq |x - x_j| \text{ for all } j\}.$$

We define the hexagonal lattice in $\mathbb{R}^2$

$$L = \{m(1, 0) + n(\frac{1}{2}, \frac{\sqrt{3}}{2}) : m, n \in \mathbb{Z}\}.$$

We define the zeta function for L

$$\zeta_L(s) := \sum_{0 \neq X \in L} |X|^{-s} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} (m^2 + mn + n^2)^{-s/2}.$$

Conjecture 5. *For $s > 2$ the limit*

$$\lim_{N \to \infty} N^{-1-\frac{s}{2}} \mathcal{E}_2(s, N) = C_s$$

exists, and

$$C_s := \frac{1}{2} (\frac{\sqrt{3}}{8\pi})^{\frac{s}{2}} \zeta_L(s).$$

Recall that the Riemann zeta function ζ is defined by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

and L_{-3} is a Dirichlet L-function

$$L_{-3}(s) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} + \dots.$$

Conjecture 6. *Let* $0 < s < 2$. *Then*

$$\mathcal{E}_2(s,N) = \frac{1}{2}V_2(s)N^2 + 3\Big((\frac{\sqrt{3}}{8\pi})^{\frac{s}{2}}\zeta(\frac{s}{2})L_{-3}(\frac{s}{2})\Big)N^{1+\frac{s}{2}} + o(N^{1+\frac{s}{2}}).$$

18.3 Preliminaries

18.3.1 Surface measure

A spherical cap with center $x_0 \in S^d$ and radius r is

$$C(x_0,r) := \{x \in S^d : |x - x_0| \leq r\}.$$

It is not difficult to obtain that as $r \to 0$,

$$\int_{S^d\setminus C(x,r)} |x-y|^{-s} d\sigma(y) = \begin{cases} (s-d)^{-1}\gamma_d r^{d-s} + O(r^{d-s}) & \text{for } s > d \\ -\gamma_d\big(\log r\big) + O(1) & \text{for } s = d. \end{cases} \tag{1}$$

18.3.2 Ultraspherical polynomials

We denote by $P_n^\lambda(t)$ the ultrashperical polynomials. They are orthogonal polynomials on the interval $[-1,1]$ with respect to the weight $(1-t^2)^{\lambda-1/2}$. By Rodrigues formula, it is explicitly given by

$$P_n^\lambda(t) = \frac{(-2)^n}{n!}\frac{\Gamma(n+\lambda)\Gamma(n+2\lambda)}{\Gamma(n)\Gamma(2n+2\lambda)}(1-t^2)^{\frac{1}{2}-\lambda}(\frac{d}{dt})^n(1-t^2)^{n+\lambda-\frac{1}{2}}. \tag{2}$$

Suppose that K is a continuous function on $[-1,1]$ with

$$K(t) = \sum_{n=0}^{\infty} a_n P_n^{(d-1)/2}(t).$$

We need the following property of ultraspherical polynomials: if the coefficients $a_1, a_2, \dots,$ are all non-negative, the following holds true

$$\sum_{i\neq j} K(\langle x_i, x_j\rangle) \geq a_0 N^2 - K(1)N. \tag{3}$$

18.4 Sketch of the proof of Theorem 2 and 3

The upper bound for $s \geq d$.

Consider the function

$$U_i(x) := \sum_{j \neq i} |x - x_j|^{-s}.$$

Note that $\sum_i U_i(x_i) = 2\mathcal{E}_d(s, N)$. It suffices to show that for each i

$$U_i(x_i) \leq \begin{cases} CN^{s/d} & \text{for } s > d \\ \frac{1}{2d}\gamma_d N \log N + O(N) & \text{for } s = d. \end{cases}$$

Let $w_N = \{x_1, \ldots, x_N\}$ be a configuration of N points on the unit sphere that minimizes the s-energy. For each i we put

$$D_i(r) = S^d \setminus C(x_i, rN^{-1/d}), \quad D(r) := \bigcap_{i=1}^{N} D_i(r).$$

Since w_N minimizes the s-energy, we obtain

$$U_i(x_i) \leq \frac{1}{\sigma(D)} \int_D U_i(x)\, d\sigma(x) \leq \sum_{j \neq i} \frac{1}{\sigma(D(r))} \int_{D_j} |x - x_j|^{-s}\, d\sigma(x).$$

We apply (1) to estimate the integration. Take $r = 1$ (resp. $r \to 0$) for the case $s > d$ (resp. $s = d$) gives the desired bounds.

The lower bound for $s > d$.

Let $w_N = \{x_1, \ldots, x_N\}$ be any configuration of N points on S^d. We define

$$r_i = \min_{j \neq i} |r_j - r_i|.$$

Then the caps $C(x_i, r_i/2)$ are disjoint. Since $\sigma(C(x_i, r_i/2)) \geq Ar_i^d$, we obtain

$$A \sum_{i=1}^{N} r_i^d \leq \sum_{i=1}^{N} \sigma(C(x_i, r_i/2)) \leq 1.$$

By Lagrange multipliers, it implies

$$\sum_{i=1}^{N} r_i^{-s} \geq A^{s/d} N^{1+s/d}.$$

Since A is independent of a configuration, it gives $\mathcal{E}_d(s, N) \geq C_1 N^{1+s/d}$.

The lower bound for $s = d$.

We define two functions

$$K(t) = (2-2t)^{-d/2}, \quad K_\epsilon(t) = (2-2t+\epsilon)^{-d/2},$$

and expand K_ϵ in a series with respect to $P_n^{(d-1)/2}$:

$$K_\epsilon(t) = \sum_{n=0}^{\infty} a_n P_n^{(d-1)/2}(t).$$

Since the ultraspherical polynomials $P_n^{(d-1)/2}$ are orthogonal with respect to $(1-t^2)^{d/2-1}$, the coefficients are given by

$$a_n(\epsilon) = A_{n,d} \int_{-1}^{1} (2-2t+\epsilon)^{-\frac{d}{2}} P_n^{\frac{(d-1)}{2}}(t)(1-t^2)^{\frac{d}{2}-1}\, dt.$$

By using Rodrigues formula (2), we can check that a_i are all non-negative. Thus, by using (3) and $K(t) \geq K_\epsilon(t)$, we obtain

$$\mathcal{E}_d(d,N) \geq \frac{1}{2}(-\frac{1}{2}\gamma_d(\log\epsilon)N^2 - \epsilon^{-d/2}N).$$

Taking $\epsilon = N^{-2/d}$, we find that

$$\mathcal{E}_d(d,N) \geq \frac{1}{2d}\gamma_d N^2 \log N + O(N^2).$$

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19 Quasicrystals and Poisson's summation formula (after N. Lev and A. Olevskii [3])

A summary written by Lenka Slavíková

Abstract

Following Lev and Olevskii [3], we show that each measure μ on $\mathbb{R}$ with uniformly discrete support and spectrum has a periodic structure. In the higher-dimensional setting, an analogous result is obtained under the additional assumption that μ is a positive measure.

19.1 Introduction

We say that a set $\Lambda \subseteq \mathbb{R}^n$ is uniformly discrete (u.d.) if

$$d(\Lambda) = \inf_{\lambda,\lambda' \in \Lambda,\ \lambda \neq \lambda'} |\lambda - \lambda'| > 0.$$

Let μ be a (complex) measure on $\mathbb{R}^n$ supported on a u.d. set Λ. Then μ has the form

$$\mu = \sum_{\lambda \in \Lambda} \mu(\lambda)\delta_\lambda, \quad \mu(\lambda) \neq 0, \quad d(\Lambda) > 0. \tag{1}$$

We shall assume that μ is a temperate distribution and that its Fourier transform

$$\widehat{\mu}(x) = \sum_{\lambda \in \Lambda} \mu(\lambda) e^{-2\pi i \langle \lambda, x \rangle}$$

(in the sense of distributions) is a measure supported on a u.d. set S, that is,

$$\widehat{\mu} = \sum_{s \in S} \widehat{\mu}(s)\delta_s, \quad \widehat{\mu}(s) \neq 0, \quad d(S) > 0. \tag{2}$$

The set S is called the spectrum of the measure μ and we shall use the notation $S = \operatorname{spec}(\mu)$.

An example of a measure μ satisfying the above-mentioned assumptions is the measure

$$\mu = \sum_{m \in \mathbb{Z}^n} \delta_m. \tag{3}$$

Then $\widehat{\mu} = \mu$, a fact which can be expressed in terms of the classical Poisson summation formula

$$\sum_{m \in \mathbb{Z}^n} f(m) = \sum_{m \in \mathbb{Z}^n} \widehat{f}(m),$$

where f is a function on $\mathbb{R}^n$ satisfying some mild smoothness and decay conditions.

There is a conjecture (see, e.g., [1, page 79]) that the measure μ given by (3) is essentially the only example of a measure satisfying (1) and (2). More precisely, it is conjectured that the support of every such measure is contained in a finite union of translates of a certain lattice (let us recall that by a lattice we mean the image of $\mathbb{Z}^n$ under some invertible linear transformation). The main goal of this summary is to outline a proof of this conjecture in the case when either $n = 1$, or $n > 1$ and μ is a positive measure. Our exposition follows that of Lev and Olevskii [3] (see also [2]).

Theorem 1. *Let μ be a measure on $\mathbb{R}^n$ satisfying* (1) *and* (2)*. Assume that either $n = 1$, or $n > 1$ and μ is a positive measure. Then the support Λ of μ is contained in a finite union of translates of a certain lattice. The same is true for the spectrum S (with the dual lattice).*

The following proposition, whose proof will be omitted, describes the explicit form of the measure from Theorem 1.

Proposition 2. *Let μ be a measure on $\mathbb{R}^n$ satisfying* (1) *and* (2) *whose support Λ is contained in a finite union of translates of a certain lattice L. Then μ takes the form*

$$\mu = \sum_{j=1}^{N} P_j \sum_{\lambda \in L + \theta_j} \delta_\lambda, \tag{4}$$

where $\theta_j \in \mathbb{R}^n$ and $P_j(x)$ is a trigonometric polynomial (that is, a finite linear combination of exponentials $\exp 2\pi i \langle \omega, x \rangle$*).*

In addition, let us point out that every measure of the form (4) satisfies (1) and (2). Thus, a measure μ on $\mathbb{R}$ has uniformly discrete support and spectrum if and only if it is of the form (4), and a similar equivalence holds for positive measures on $\mathbb{R}^n$.

19.2 Proofs

In this subsection we describe several ingredients which, when combined, lead to the proof of Theorem 1. We do not prove each particular claim; the reader is referred to [3] for more details.

19.2.1 Spectral gaps

Given $R > 0$, we denote by B_R the open ball in $\mathbb{R}^n$ with radius R centered at the origin. For a set $\Lambda \subseteq \mathbb{R}^n$ we introduce the following two variants of density:

$$D_{\#}(\Lambda) = \liminf_{R\to\infty} \frac{\#(\Lambda \cap B_R)}{|B_R|}$$

and

$$D^{+}(\Lambda) = \limsup_{R\to\infty} \sup_{x\in\mathbb{R}^n} \frac{\#(\Lambda \cap (x + B_R))}{|B_R|}.$$

Clearly, $D_{\#}(\Lambda) \leq D^{+}(\Lambda)$.

We say that a measure μ has a spectral gap of size $a > 0$ if its Fourier transform $\widehat{\mu}$ vanishes on a ball of radius a. It turns out that, in the one-dimensional case, a u.d. set Λ which supports a measure with a spectral gap needs to have a positive density.

Lemma 3. *Let $\Lambda \subseteq \mathbb{R}$ be a u.d. set which supports a non-zero measure μ whose Fourier transform vanishes on the open interval $(0, a)$ for some $a > 0$. Then*

$$D_{\#}(\Lambda) \geq c(a, d(\Lambda)) > 0.$$

Lemma 3 fails in the higher-dimensional case. Indeed, the set $\Lambda = \mathbb{Z} \times \{0\} \subseteq \mathbb{R}^2$ has density zero and, at the same time, it supports the measure

$$\mu = \sum_{m\in\mathbb{Z}} (-1)^m \delta_{(m,0)}$$

which has a spectral gap around the origin. Nevertheless, if the measure μ has not only a spectral gap, but an isolated atom in its spectrum, then the conclusion of Lemma 3 is valid even in the higher-dimensional setting.

Lemma 4. *Let $\Lambda \subseteq \mathbb{R}^n$ be a u.d. set which supports a measure μ satisfying*

$$\sup_{\lambda\in\Lambda} |\mu(\lambda)| < \infty$$

and having the property that $\mathrm{spec}(\mu) \cap B_a = \{0\}$ *for some* $a > 0$. *Then*

$$D_{\#}(\Lambda) \geq c(a, n) > 0.$$

19.2.2 Delone and Meyer sets

A set $\Lambda \subseteq \mathbb{R}^n$ is called relatively dense if there is $R > 0$ such that every ball of radius R intersects Λ. We say that Λ is a Delone set if Λ is both u.d. and relatively dense. Further, Λ is a Meyer set if it is a Delone set and there is a finite set F such that $\Lambda - \Lambda \subseteq \Lambda + F$.

It can be shown that the support Λ of any non-zero measure μ satisfying (1) and (2) is a relatively dense set, and, therefore, a Delone set.

Below we present a sufficient condition for a Delone set to be a Meyer set.

Lemma 5. *Assume that* $\Lambda \subseteq \mathbb{R}^n$ *is a Delone set satisfying* $D^+(\Lambda - \Lambda) < \infty$. *Then* Λ *is a Meyer set.*

19.2.3 The set Λ_h

Let μ be a measure on $\mathbb{R}^n$ satisfying (1) and (2). For $h \in \Lambda - \Lambda$ we denote

$$\Lambda_h = \{\lambda \in \Lambda : \ \lambda + h \in \Lambda\}$$

and introduce the new measure

$$\mu_h = \sum_{\lambda \in \Lambda_h} \mu(\lambda)\overline{\mu(\lambda + h)}\delta_\lambda.$$

Then μ_h is a non-zero measure supported in the set Λ_h and having bounded atoms, so it is a tempered distribution. Further, we claim that each measure μ_h has a spectral gap.

Lemma 6. *For any* $h \in \Lambda - \Lambda$, *we have* $\mathrm{spec}(\mu_h) \cap B_{d(S)} \subseteq \{0\}$.

In addition, if μ is a positive measure then so is μ_h. This implies that $0 \in \mathrm{spec}(\mu_h)$ which, in combination with Lemma 6, yields that the spectrum of μ_h has an isolated atom at the origin.

Let us now present the last two ingredients needed for the proof of Theorem 1.

Lemma 7. *Let Λ be a u.d. set in $\mathbb{R}^n$ such that*

$$\inf_{h \in \Lambda - \Lambda} D_{\#}(\Lambda_h) > 0. \tag{5}$$

Then $D^+(\Lambda - \Lambda) < \infty$.

Lemma 8. *Let Λ be a Meyer set in $\mathbb{R}^n$ satisfying* (5)*. Then Λ is contained in a finite union of translates of some lattice.*

19.2.4 Proof of Theorem 1

We shall now combine the results of the previous three subsections to prove Theorem 1. We assume that μ is a measure on $\mathbb{R}^n$ which satisfies (1) and (2) and which is positive if $n > 1$. Let $h \in \Lambda - \Lambda$. In the one-dimensional case, it follows from Lemma 6 that the Fourier transform of the measure μ_h vanishes on the interval $(0, d(S))$. In the higher-dimensional case, we use the remark after Lemma 6 to conclude that $\operatorname{spec}(\mu_h) \cap B_{d(S)} = \{0\}$. According to Lemma 3 (if $n = 1$) or Lemma 4 (if $n > 1$), condition (5) is satisfied. Thus, by Lemma 7, we obtain that $D^+(\Lambda - \Lambda) < \infty$, and Lemma 5 in turn implies that Λ is a Meyer set. An application of Lemma 8 then concludes the proof.

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20 Measures with locally finite support and spectrum. (after Y. F. Meyer)

A summary written by Marco Fraccaroli and Milan Kroemer

Abstract

We answer an important question in harmonic analysis: is the Poisson summation formula unique or does it belong to a wider class? Are there measures combining the conflicting properties of locally finite support and locally finite spectrum which are not lattices?

20.1 Introduction

The celebrated Poisson summation formula

$$\sum_{n\in\mathbb{Z}^n} f(n) = \sum_{m\in\mathbb{Z}^n} \widehat{f}(m), \tag{1}$$

establishes that the Fourier transform of the *Dirac comb* associated with the lattice $\mathbb{Z}^n$ is itself, yielding the basic example of a *crystalline measure* on $\mathbb{R}^n$.

Definition 1. *A tempered distribution on $\mathbb{R}^n$ given by a linear combination of Dirac masses*

$$\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda, \tag{2}$$

is called a crystalline measure *if both its support, Λ, and the support of its Fourier transform, the* spectrum *S, are locally finite sets.*

Applying affine transformations both in space and in frequency to the Dirac comb associated with the lattice $\mathbb{Z}^n$, these properties are preserved. Finite sums of such measures produce the *generalized Dirac combs.*

The cancellation properties between the waves associated to the Dirac masses in the point of the lattice come from the structure of Λ itself. A first step in questioning its necessity was established by Lev and Olevskii.

Theorem 2. *There exists a crystalline measure μ on $\mathbb{R}^n$ such that it is not a generalized Dirac comb.*

The construction described by Lev and Olevskii produces crystalline measures such that the $\mathbb{Q}$-linear spans of Λ and S, respectively $\Lambda_{\mathbb{Q}}$ and $S_{\mathbb{Q}}$, have finite dimension over $\mathbb{Q}$. An improvement was obtained by Kolountzakis.

Theorem 3. *There exists a crystalline measure μ on $\mathbb{R}$ such that both $\Lambda_{\mathbb{Q}}$ and $S_{\mathbb{Q}}$ have infinite dimension.*

In both cases, the proof is not given by a closed-form expression. Then, the first contribution of Meyer is to recall a summation formula discovered by Guinand in [1] and produce from it an explicit example for both of the results. However, both the construction described by Kolountzakis and the Guinand's distribution provide crystalline measures such that both Λ and S are not $\mathbb{Q}$-linearly independent.

The work on the Guinand's distribution inspired the construction improving the previous results.

Theorem 4. *There exists an odd crystalline measure μ on $\mathbb{R}$ such that, for $\Lambda_+ = \Lambda \cap (0, \infty), S_+ = S \cap (0, \infty)$, each finite subset of both Λ_+ and S_+ is $\mathbb{Q}$-linearly independent.*

This theorem extends to higher dimension.

Theorem 5. *There exists a crystalline measure μ on $\mathbb{R}^n$, odd in the last variable x_n, such that*

1. *the support Λ of μ is the union of $\Lambda_+ = \Lambda \cap \{x_n > 0\}$ and $\Lambda_- = \Lambda \cap \{x_n < 0\}$. Moreover, each finite subset of Λ_+ is $\mathbb{Q}$-linearly independent and similarly for Λ_-;*

2. *the analogous property holds true for the spectrum S.*

20.2 Guinand's distribution

Let $r_3(n)$ be the number of decompositions of the integer $n \geq 1$ into a sum of three squares, 0^2 being admitted, with $r_3(n) = 0$ if no such a decomposition exists. In [1], Guinand introduced the odd distribution

$$\sigma = -2\partial\delta_0 + \sum_{n=1}^{\infty} \frac{r_3(n)}{\sqrt{n}} (\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}). \tag{3}$$

The behaviour of the mean of $r_3(n)$ is regular enough to guarantee that σ is a tempered distribution. In particular, Guinand proved that $\widehat{\sigma} = -i\sigma$.

To get rid of the derivative of δ_0, one can consider, for $\alpha \in (0, 1)$,

$$\tau_\alpha(t) = (\alpha^2 + \alpha^{-1})\sigma(t) - \alpha\sigma(\alpha t) - \sigma(\alpha^{-1}t), \tag{4}$$

so that $\widehat{\tau}_\alpha = -i\tau_\alpha$. Fixing $\alpha = 1/2$, we obtain a crystalline measure τ with support Λ given by the square roots of the numbers in

$$\mathbb{N} \setminus (\{4^j(8k + 7)\colon j = 0, 1, k \in \mathbb{N}\} \cup 16\mathbb{N}). \tag{5}$$

The set in the display contains infinitely many primes p, whose square roots are $\mathbb{Q}$-linearly independent. The tensor product between n copies of τ gives a crystalline measure on $\mathbb{R}^n$ satisfying the analogous property.

It is worth noting the identity

$$\sum_{n=1}^{\infty} \frac{r_3(n)}{\sqrt{n}}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{1}{|k|}(\delta_{|k|} - \delta_{-|k|}), \tag{6}$$

which, relating a measure in $\mathbb{R}$ with a Dirac comb in $\mathbb{R}^3$, inspired the construction used to prove Theorem 4.

20.3 Proof of Theorem 4 and Theorem 5

The main ingredient in the proof of Theorem 4 is the following theorem.

Theorem 6. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3) \notin \mathbb{Z}^3$. Then, for*

$$\sigma_{(\alpha,\beta)} = \sum_{k\in\mathbb{Z}^3} \frac{\exp(2\pi i k \cdot \beta)}{|k + \alpha|} \left(\delta_{|k+\alpha|} - \delta_{-|k+\alpha|}\right), \tag{7}$$

its Fourier transform is $-i \exp(-2\pi i\alpha \cdot \beta)\sigma_{(\beta,-\alpha)}$.

When α, β tend to zero, then the limit $\sigma_{(0,0)}$ is the Guinand's distribution. The key part in the construction of the measures in Theorem 6 is the use of the generalized Dirac comb with support $\alpha + \mathbb{Z}^3$ and spectrum $\beta + \mathbb{Z}^3$. The idea, described by Theorem 7 below, is to produce a measure on $\mathbb{R}$ by associating δ_λ to $(\delta_{|\lambda|} - \delta_{-|\lambda|})/|\lambda|$. In the same way, an odd test function ϕ on $\mathbb{R}$ can be lifted to a radial function $\Phi(x) = \phi(|x|)/|x|$ on $\mathbb{R}^3$, for which the Fourier transform behaves nicely. Since the origin doesn't belong to either the support or the spectrum of the generalized Dirac comb, we can allow for non smoothness of Φ in it.

Sketch of proof of Theorem 4. Observe that $\sigma_{(\alpha,\beta)}$ is an odd measure. Choosing α such that $1, \alpha_1, \alpha_2, \alpha_3$ are $\mathbb{Q}$-linearly independent guarantees that the support of $\sigma_{(\alpha,\beta)}$ is $\mathbb{Q}$-linearly independent. This condition on α implies

$$\frac{1}{x}\int_x^{x+1} d|\sigma_{(\alpha,\beta)}| \xrightarrow{x\to\infty} 1. \tag{8}$$

so that $\sigma_{(\alpha,\beta)}$ is a tempered measure. □

Theorem 6 is a corollary of a more general statement.

Theorem 7. *Let μ be a crystalline measure on $\mathbb{R}^3$. Then we have $\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda$ and $\widehat{\mu} = \sum_{y\in S} b(y)\delta_y$. Suppose $0 \notin \Lambda$ and $0 \notin S$ and consider the measures on $\mathbb{R}$ given by*

$$\sigma_\Lambda = \sum_{\lambda\in\Lambda} \frac{a(\lambda)}{|\lambda|}\left(\delta_{|\lambda|} - \delta_{-|\lambda|}\right), \quad \sigma_S = \sum_{y\in S} \frac{b(y)}{|y|}\left(\delta_{|y|} - \delta_{-|y|}\right) \tag{9}$$

Then σ_Λ is a crystalline measure and its Fourier transform is $-i\sigma_S$.

Sketch of proof. The measures σ_Λ and σ_S are odd. Therefore it suffices to check the identity

$$\langle\sigma_\Lambda, \widehat{\phi}\rangle = -i\langle\sigma_S, \phi\rangle \tag{10}$$

for every odd test function ϕ. This is verified considering the radial function $\Phi(x) = \phi(|x|)/|x|$ on $\mathbb{R}^3$, its Fourier transform and the definition of μ. □

Analogously to what we have seen above, the main ingredient in the proof of Theorem 5 is an extension of Theorem 6. Consider a lattice $\Gamma \subset \mathbb{R}^{n-1}\times\mathbb{R}^3$ such that the projections $p_1 : \Gamma \to \mathbb{R}^{n-1}$ and $p_2 : \Gamma \to \mathbb{R}^3$ are injective with dense range. Let Γ^* be its dual lattice and assume $\mathrm{vol}(\Gamma) = 1$.

Theorem 8. *Let $\alpha \notin \Gamma$ and $\beta \notin \Gamma^*$. Then the measure defined on $\mathbb{R}^n$ by*

$$\sigma_\Gamma^{[\alpha,\beta]} = \sum_{\gamma\in\Gamma+\alpha} \frac{\exp(2\pi i\beta\cdot\gamma)}{|p_2(\gamma)|}\left(\delta_{(p_1(\gamma),|p_2(\gamma)|)} - \delta_{(p_1(\gamma),-|p_2(\gamma)|)}\right) \tag{11}$$

is a crystalline measure with Fourier transform $-i\exp(2\pi i\alpha\cdot\beta)\sigma_{\Gamma^*}^{[\beta,-\alpha]}$.

Sketch of the proof. It suffices to show that

$$\langle\sigma_{(\alpha,\beta)}, \widehat{\phi}\rangle = -i\exp(2i\pi\alpha\cdot\beta)\langle\sigma_{(\beta,-\alpha)}, \phi\rangle \tag{12}$$

holds for every test function $\phi(u,v) = \phi_1(u)\otimes\phi_2(v), u \in \mathbb{R}^{n-1}, v \in \mathbb{R}$, with ϕ_2 odd. The proof is analogous to the one of Theorem 6. □

Sketch of the proof of Theorem 5. If Λ is the support of $\sigma_\Gamma^{[\alpha,\beta]}$, then for almost every α the set

$$\Lambda \cap \{x_n > 0\} = \{(p_1(\gamma), |p_2(\gamma+\alpha)|) : \gamma \in \Gamma\}$$

is $\mathbb{Q}$-linearly independent. The same holds for the spectrum of $\sigma_\Gamma^{[\alpha,\beta]}$. $\square$

20.4 The crystalline measures of Kolountzakis

Kolountzakis ([2]) and Meyer ([6]) independently came up with two similar constructions that provide a different proof of Theorem 3. Let $\{N_j\} \subset \mathbb{N}$ be a strictly increasing sequence, $\{a_j\}$ a bounded $\mathbb{Q}$-linearly independent sequence. Then there exist a sequence of atomic measures $\{\sigma_j\}$ such that both $\sigma_j, \widehat{\sigma}_j$ are $10N_j$-periodic measures with support contained in $(10N_j)^{-1}\mathbb{Z}$ and vanishing in the interval $(-N_j, N_j)$. Moreover, there exists a sequence $\{\varepsilon_j\} \subset [0,\infty)$ that ensures the following is a well-defined Borel measure,

$$\sum_{j=1}^{\infty} \varepsilon_j M_{a_j} T_{a_j} \sigma_j, \tag{13}$$

where T_a, M_a are respectively the translation and the modulation operator.

The construction of σ_j is based on the following observation.

Lemma 9. *For every $N \in \mathbb{N}$, there is a function $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ not identically zero such that both the function and its Fourier transform $\widehat{f}\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ vanish in the interval $\{x \in \mathbb{Z}/N\mathbb{Z}\colon |x| \le N/10\}$.*

20.5 The crystalline measures of Lev and Olevskii

In [5], Lev and Olevskii gave a different proof of Theorem 2. Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ an *oblique lattice*, namely such that the projections $p_1\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $p_2\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ restricted to Γ are injective, hence they have a dense range. For $I = [-a, a]$, let the *model set* Λ_I be defined by the standard *cut and projection scheme*

$$\Lambda_I = \{\lambda = p_1(\gamma)\colon \gamma \in \Gamma \cap p_2^{-1}(I)\}. \tag{14}$$

Let $\{h_j\}, \{a_j\} \subset (0,\infty)$ be two strictly increasing sequences tending to infinity, and set $a_0 = 0$. Let $\{\Lambda_j\}$ be the sequence of model sets Λ_j associated to $[-h_j, h_j]$. The associated *enriched model set* is defined by

$$\widetilde{\Lambda} = \bigcup_{j=1}^{\infty} \widetilde{\Lambda}_j, \ \widetilde{\Lambda}_j = \{\lambda \in \Lambda_j \colon |\lambda| \geq a_{j-1}\}. \tag{15}$$

An example is given by setting $h_j = a_j = j$ and considering the lattice

$$\Gamma = \{(k + m\sqrt{2}, k - m\sqrt{2}) \colon (k, m) \in \mathbb{Z}^2\} \tag{16}$$

The associated enriched model set is then $\{k + m\sqrt{2} \colon (k, m) \in \mathbb{N}^2 \cup (-\mathbb{N})^2\}$.

Lev and Olveskii proved the following existence result.

Theorem 10. *Every enriched model set $\widetilde{\Lambda}$ contains the support of a measure μ that is not a generalized Dirac comb and such that the Fourier transform of μ is also supported by an enriched model set S.*

20.6 Geometry of crystalline measures

Envisioning all these constructions, one may ask what does the set $\mathcal{L}$ of possible supports of a crystalline measure look like. In the examples, the support Λ is neither a lattice nor a *uniformly discrete*, namely

$$\inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| = 0. \tag{17}$$

However, it is worth noting that in all the constructions it was involved a lattice structure in equal or higher dimension.

Some basic questions can be the following:

- is every locally finite set in $\mathcal{L}$? No, in fact $\{1/2\} \cup \mathbb{Z} \setminus \{0\} \notin \mathcal{L}$;
- do there exist model sets in $\mathcal{L}$ a part from lattices?
- do there exist uniformly discrete sets in $\mathcal{L}$ except lattices? This was partially answered by Lev and Olevskii in [3], [4].

  Theorem 11. *On $\mathbb{R}$, if the support and the spectrum of a crystalline measure μ are uniformly discrete, then μ is a generalized Dirac comb. On $\mathbb{R}^n, n \geq 2$, the additional hypothesis μ non negative is required.*

- do there exist non negative crystalline measures except Dirac combs?

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21 Fourier interpolation on the real line (after D. Radchenko and M. Viazovska)

A summary written by Gevorg Mnatsakanyan and João P. G. Ramos

Abstract

Using weakly holomorphic modular forms for the Hecke theta group, we prove an interpolation result for even Schwartz functions on the real line, recovering the function from its data at $\{\pm\sqrt{n}\colon n \in \mathbb{N}\}$.

21.1 Main results

We first start with a little digression. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable with Fourier transform $\widehat{f}$ supported in $[-1/2, 1/2]$. It then holds that

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\text{sinc}(x-n),$$

where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. By dilating our original function, we get an interpolation formula recovering a function g whose Fourier support is contained in $[-w/2, w/2]$ by its values at n/w, $n \in \mathbb{Z}$. This unique property unfortunately does not hold for general Schwartz functions f, such as the gaussian $e^{-\pi x^2}$. The main results here exposed are resolving this issue by interpolating at a somewhat denser subset of $\mathbb{R}$ than multiples of $\mathbb{Z}$.

Theorem 1. *There exists a collection of Schwartz functions $\{a_n\}$ such that, for each even Schwartz function f, it holds that*

$$f(x) = \sum_{n=0}^{\infty} f(\sqrt{n})a_n(x) + \sum_{n=0}^{\infty} \widehat{f}(\sqrt{n})\widehat{a_n}(x),$$

where the right hand side coverges absolutely.

We then have the following immediate corollary to this theorem:

Corollary 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an even Schwartz function, so that f and $\widehat{f}$ vanish at $\sqrt{n}$, $\forall n \in \mathbb{N}$. Then $f \equiv 0$.*

Our second main theorem deals with a converse to the Poisson summation formula. In fact, let $\mathfrak{s}$ be the space of sequences $\{x_n\}_{n\in\mathbb{Z}}$ such that $n^k x_n \to 0$ as $n \to \infty$, for any $k > 0$. Let $\mathcal{S}_{even}$ denote the space of even Schwartz functions. The map

$$\begin{aligned}\Psi :&\mathcal{S}_{even} \to \mathfrak{s} \oplus \mathfrak{s}\\ &f \mapsto (f(\pm\sqrt{n}))_{n\geq 0} \oplus (\widehat{f}(\pm\sqrt{n}))_{n\geq 0}\end{aligned}$$

is well-defined. The following result is a detailed description of the map Ψ.

Theorem 3. *Ψ is an isomorphism from $\mathcal{S}_{even}$ to the ker L, where*

$$L(\{x_n\},\{y_n\}) = \sum_{n\geq 0} x_{n^2} - \sum_{n\geq 0} y_{n^2}.$$

21.2 The Hecke theta group and modular forms

Define the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These are obviously elements of the special linear group $SL_2(\mathbb{Z})$ of matrices with integer coefficients and determinant 1. We define then the *Hecke theta group* Γ_θ to be the subgroup of $SL_2(\mathbb{Z})$ generated by S and T^2. Alternatively, the following characterization holds:

$$\Gamma_\theta = \left\{ A \in SL_2(\mathbb{Z})\colon A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2 \right\}.$$

With this subgroup in mind, we define the following classical Jacobi theta series: if $q = e^{2\pi i z}$, and

$$\eta(z) = q^{1/24} \prod_{n\geq 1} (1-q^n)$$

denotes the Dedekind $\eta-$function, we let

$$\Theta_2(z) = 2\frac{\eta(2z)^2}{\eta(z)},\ \Theta_3(z) = \frac{\eta(z)^5}{\eta(z/2)^2\eta(2z)^2},\ \Theta_4(z) = \frac{\eta(z/2)^2}{\eta(z)}.$$

Among many properties, these functions are used to define the following modular $\lambda-$invariant:

$$\lambda(z) = \frac{\Theta_2^4(z)}{\Theta_3^4(z)}.$$

This function satisfies that

$$\lambda(\gamma z) = \lambda(z),\, \forall \gamma \in SL_2(\mathbb{Z}),\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2.$$

We then define the Hauptmodul associated to Γ_θ as

$$J(z) = \frac{1}{16}\lambda(z)(1-\lambda(z)).$$

This function, in particular, is invariant under action of elements of Γ_θ.

21.3 Weakly modular forms of weight $3/2$

The definition of J above enables us to define, implicitly, a sequence of holomorphic functions on the upper half plane that will be crucial to our construction of the sequence a_n in Theorem 1. Explicitly, let $\epsilon \in \{-,+\}$ be a sign. We find functions $\{g_n^\epsilon\}_{n\geq 0}$ satisfying

$$g_n^\epsilon(z+2) = g_n^\epsilon(z),\ (-iz)^{-3/2} g_n^\epsilon(-1/z) = \epsilon g_n^\epsilon(z),$$

together with appropriate behaviour near the cusps of Γ_θ. That is,

$$g_n^+(z) = q^{-n/2} + O(q^{1/2}),\, g_n^-(z) = q^{-n/2} + O(1),\, z \to i\infty;$$

$$g_n^\epsilon(1+i/t) \to 0 \text{ as } t \to \infty.$$

The construction of these functions is evident by considering

$$g_n^+(z) = \Theta_3^3(z) P_n^+(J^{-1}(z)),\, g_n^-(z) = \Theta_3^3(z)(1-2\lambda(z)) P_n^-(J^{-1}(z)),$$

where we let $P_n^\pm$ be monic, rational polynomials of degree n, with $P_n^-(0) = 0$. These are uniquely determined by the fact that the Fourier expansion of J^{-1} near $i\infty$ is $q^{-1/2} + 24 + O(q^{1/2})$. We remark that both the modularity and the vanishing of Fourier coefficients of these functions are of crucial importance, as they will provide us, respectively, with eigenfunction properties for the Fourier transform and delta-like behaviour at the interpolation nodes.

21.4 Construction of the interpolation basis

Next, we use the functions g_n^ϵ above to build our interpolation basis. As we would like to first obtain eigenfunctions of the Fourier transform with good properties at the interpolation nodes $\pm\sqrt{n}$, we define the sequence of functions

$$b_n^\epsilon(x) = \frac{1}{2}\int_{-1}^{1} g_n^\epsilon(z) e^{i\pi x^2 z}\, dz,$$

where integration is defined on the semicircle contained in the upper half plane $\mathfrak{H}$ connecting -1 and 1. By the modularity of g_n^ϵ, the good decay properties of J^{-1} and the Fourier expansion conditions we have imposed on g_n^ϵ, we have the following:

Proposition 4. *The function $b_n^\epsilon : \mathbb{R} \to \mathbb{R}$ is an even Schwartz function satisfying*

$$\widehat{b_n^\epsilon} = \epsilon b_n^\epsilon,$$

as well as

$$b_n^\epsilon(\sqrt{m}) = \delta_{m,n},\ \textit{for}\ m \geq 1, n \geq 0.$$

Additionally, we have $b_0^+(0) = 1$.

21.5 Growth estimate

Next, we need to estimate the growth of b_n^ϵ, in order to conclude the convergence of the right hand side of Theorem 1.

Theorem 5. *For $\epsilon \in \{+, -\}$ we have $|b_n^\epsilon(x)| = O(n^2)$ uniformly in x.*

In order to prove this Theorem, we will work instead with the generating function $F_\epsilon(\tau, x)$ of b_n^ϵ. This object is initially defined through certain contour integrals of the generating function of g_n^ϵ, but we will omit this definition here for shortness. The bottomline is that the representation

$$F_\epsilon(\tau, x) = \sum_{n=0}^{\infty} b_n^\epsilon(x) e^{i\pi n\tau}$$

holds in the upper half space, and the following equations hold

$$F_\epsilon(\tau, x) - F_\epsilon(\tau + 2, x) = 0, \tag{1}$$

$$F_\epsilon(\tau, x) + \epsilon(-i\tau)^{-1/2} F_\epsilon(-\frac{1}{\tau}, x) = e^{i\pi\tau x^2} + \epsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}. \tag{2}$$

The last identity induces the definition of a $\Gamma_\theta-$cocycle associated to the generating function by

$$\phi_{T^2}(\tau) = 0, \ \phi_S(\tau) = e^{i\pi x^2\tau} + \epsilon(-i\tau)^{-1/2}e^{i\pi x^2(-1/\tau)},$$

In an overly simplified way, the main strategy to prove Theorem 5 is to use the following result by Hecke:

Theorem 6 (Hecke)**.** *If a 2-periodic function $f : \mathfrak{H} \to \mathbb{C}$ has Fourier expansion $\sum_{n\geq 0} a_n e^{i\pi n\tau}$ and satisfies*

$$|f(\tau)| \leq C Im(\tau)^{-\alpha},$$

for $Im(\tau) < c$, then the coefficients satisfy $|a_n| \leq C(e\pi/\alpha)^\alpha n^\alpha$.

We want to use this result for $f(\tau) = F_\epsilon(\tau, x)$. We need bounds uniform on x, so the idea is to exploit the generating function relation (1) to split our task into two parts: the first deals with bounding a cocyle term – which corresponds to complex numbers τ with small imaginary part – and the second deals with a straight analytical property of the generating function, which can be proved directly for τ in the fundamental domain of Γ_θ. The inequality we arrive is $|F_\epsilon(\tau; x)| \leq C_0 \mathrm{Im}(\tau)^{-2}$, with C_0 independent of x. By Theorem 6, we conclude Theorem 5.

21.6 Proof of the main results

Let us define

$$a_n(x) = \frac{b_n^+(x) + b_n^-(x)}{2}.$$

The properties of b_n^ϵ imply

$$\widehat{a_n}(x) = \frac{b_n^+(x) - b_n^-(x)}{2}.$$

Modularity properties of g_n^ϵ imply the representation

$$e_\tau(x) = \sum_{n=0}^{\infty} a_n(x) e_\tau(\sqrt{n}) + \sum_{n=0}^{\infty} \widehat{a_n}(x) \widehat{e_\tau}(\sqrt{n})$$

for $e_\tau = e^{i\pi\tau x^2}$. Thus, Theorem 1 holds for the subspace spanned by $\{e_\tau\}_{Im(\tau)>0}$. Noting that the functional

$$\phi_x(f) = f(x) - \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) - \sum_{n=0}^{\infty} \widehat{a_n}(x)\hat{f}(\sqrt{n})$$

is a tempered distribution by Theorem 5, the rest of the proof is just a standard approximation argument.

For Theorem 3, first note that, by the Poisson summation formula, the image of Ψ is in ker L. We then define $\Phi : \ker L \to \mathcal{S}_{\text{even}}$, the inverse map of Ψ, by

$$(\{x_n\}, \{y_n\}) \mapsto \sum_n x_n a_n(x) + y_n \widehat{a_n}(x).$$

The right hand side above is a Schwartz functions, as long as we prove a polynomial growth estimate for norms $(\|b_n^\epsilon\|_{\alpha,\beta})_n$. This, however, is just a modification of the proof of Theorem 5 using derivatives of the generating function.

21.7 A possible extension?

Finally, we remark that it is expected for Theorem 1 to be extendable to a larger class than Schwartz functions. More specifically, the most general question in this setting remains open:

Question 7. *Does Theorem 1 still hold whenever the right hand side converges absolutely? Say, for instance, whenever $f, \widehat{f}$ are both continuous so that pointwise values are well-defined.*

In this regard, we have the following generalization to our main theorems:

Theorem 8. *Let f be even and integrable. If both $f, \widehat{f}$ are bounded pointwise by (an absolute constant times) $(1+|x|)^{-13}$, then the conclusion of Theorem 1 still holds.*

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22 Lower bounds for the optimal density of sphere packings (after A. Venkatesh and P. Moustrou [2, 4])

A summary written by Cristian Gonzalez-Riquelme and Oscar Quesada

Abstract

Venkatesh improves on the lower bounds for the optimal density of sphere packings. In all sufficiently large dimensions, the improvement is by a factor of at least 10,000; along a sparse sequence of dimensions n_i, the improvement is by roughly $\log\log n_i$. Moustrou proves an explicit version of this last result, in the sense of exhibiting, for the same set of dimensions, finite families of lattices containing a lattice reaching this bound. Moustrou's construction uses codes over cyclotomic fields, lifted to lattices via Construction A.

22.1 Introduction

We denote the optimal density of spheres packings in dimension d by ρ_d.

22.1.1 A brief historical perspective

The first result that implied the existence of a lower bound for ρ_d in every dimension was due to Minkowski, who proved that it is bounded below by 2^{-d}. Minkowski's proof relies in the existence of an origin-centered ellipsoid of volume 1, containing no nonzero vector in $\mathbb{Z}^d$. Here, by origin-centered ellipsoid we mean the image under a linear transformation of the standard unit ball. This result can be reformulated as the existence of a lattice Λ of covolume 1 in $\mathbb{R}^d$ such that $B(0,1) \cap \Lambda = \{0\}$, thus providing a periodic sphere packing with density 2^{-d}. We define here

$$c_d = \sup\{Volume(E); E \text{ is an origin centered ellipsoid with } E \cap \mathbb{Z}^d = \{0\}\}.$$

Minkowski's result implies $c_d \geq 1$. It is possible then to get better lower bounds for ρ_d by getting better lower bounds for c_d. The first substantial improvement on Minkowski's work was given by Rogers in 1947; he showed that $c_d > 0.73d$ for d large enough. Later K. Ball proved that

$$c_d \geq 2(d-1),$$

providing therefore an sphere packing with density at least $2(d-1)2^{-d}$; and Vance proved that $c_d > (2.2)d$ when d is divisible by 4, providing an sphere packing with density at least $(2.2d)2^{-d}$. These results far surpass the density of any "explicitly known" sphere packing. It is in this context where A. Venkatesh made the improvements that are the central results of our talk.

22.1.2 Codes, cyclotomic lattices, and geometry of numbers

The concept of error-correcting codes comes from the design of signals for data transmission systems (see chapter 3 of [1]). One model for these systems encodes information via sequences of 0's and 1's, and due to noise, there is a probability that when a symbol is transmitted, the other one is received. A *codeword* is a sequence of d symbols, and a *binary code* of length d is a set of codewords, with d coordinates each.

To be able to correct errors, we choose codewords that are sufficiently distinct from each other under some given metric. One would like codes with a large number of codewords, given some mininum distance between them. This is analogous to the sphere packing problem, and one can exploit this similarity to construct sphere packings and lattices from binary codes. *Construction A* is one such construction ([1], Chapter 7). Similarly, instead of binary digits, one can instead consider as symbols the elements of $\mathbb{F}_s = \{0, 1, ..., s-1\}$, the finite field of order s, where s is either a prime or a prime power. This motivates the following

Definition 1. *A s-ary code is a subset of $\mathbb{F}_s^d$. A* linear *s-ary code is a subset which is also a vector space over $\mathbb{F}_s$.*

We can obtain finite fields as the quotient of the integers modulo a prime, and this can be done in general algebraic number fields. Minkowski's "geometry of numbers" insight is that algebraic number fields can be seen as points in an euclidean space (see [3], Section 5 of Chapter 1), and, for cyclotomic fields, this lies at the heart of the constructions in both papers.

22.2 Summary of article: A note on sphere packings in high dimension. Akhsay Venkatesh.

22.2.1 Main results

A. Venkatesh proved the following theorem:

Theorem 2. *There exist infinitely many dimensions d for which*

$$c_d > \frac{1}{2} d \log(\log(d)).$$

Also, in every sufficiently large dimension, $c_d > 65967d$.

The results are proved with a similar idea: One consider random lattices but constrained in some algebraic way. Now we elaborate a little more in the proofs of the first assertion of this theorem:

22.2.2 Proof of the first assertion:

Let us define the basic objects of the proof. Let $K = \mathbb{Q}(\mu_d)$ be the ciclotomic field, o be its ring of integers, and V a 2-dimensional vector space over K. Let $V_{\mathbb{R}} = V \bigotimes_Q \mathbb{R}$, $\Lambda_0 = o^2 \subset V_{\mathbb{R}}$, $G = Sl_2(K \bigotimes_Q \mathbb{R})$ and $\Gamma = Sl_2(o) \subset G$. Endow G/Γ with the G invariant probability measure, denoted by μ, and endow $V_{\mathbb{R}}$ with the Lebesgue measure for which Λ_0 has covolume 1. For $f \in C_c(V_{\mathbb{R}})$ and $\Lambda \subset V_{\mathbb{R}}$ any lattice, we put

$$E_f(\Lambda) = \sum_{v \in \Lambda - \backslash 0} f(v)$$

The crucial tool of the proof is the following lemma

Lemma 3. *We have:* $\int_{g \in G/\Gamma} E_f(g\Lambda_0) d\mu(g) = \int_{V_{\mathbb{R}}} f(x) dx$

After proving that we can conclude the proof of the assertion. Let us consider an quadratic form q_0 invariant under the action of μ_d. If the ellipsoid $E = \{x \in VR : q_0(x) \leq T\}$ has volume less than n, it follows from 3 that there exists a lattice $\Lambda = g\Lambda_0$ so that the size of $\Lambda \cap E \setminus \{0\}$ is less than n; since E is μ_d invariant and μ_d acts without fixed points, $\Lambda \cap E = \{0\}$. Thus we have proved that there is a lattice Λ of covolume 1 in dimension $2\phi(d)$, and an ellipsoid of volume $d - \epsilon$ so that $\Lambda \cap E = 0$. This means, in the notation of the introduction, that $c_{2\phi(d)} \geq d$. Using estimates on $\phi(d)$ for an appropiate sequence of dimensions, we conclude.

22.2.3 Discussion of the proof of the second assertion:

The proof of the second assertion is significantly more involved; we will briefly discuss here the ideas behind the proof. Here one considers random lattices with fixed discriminant but subject to the constraint that all the lenghts are integers.

We discuss the basic definitions here. We define $k = \frac{d}{2}$, $\Gamma(k+1) =: (k)!$. The jth minima γ_j of a quadratic form q on a free abelian group Λ is the smallest number for which $\{x \in \Lambda : \sqrt{q(x)} \leq \gamma_j\}$ spans a real space of dimension $\geq j$. We also define $V_j = Vol(\{\sqrt{q} \leq \gamma_j\})$.

The proof of the second assertion is based in the following result due to Minkowski: If Λ is a lattice equipped with a quadratic form q in $\Lambda \otimes \mathbb{R}$ with sucessive minima $\gamma_1, ...\gamma_n$, then there is a Lattice $\Lambda_2 \subset \Lambda \otimes \mathbb{R}$ with $\Lambda_2 \setminus \{0\}$ disjoint from an ellipsoid of volume $\sqrt[n]{V_1...V_d}$. The idea is to find a positive definite quadratic form q in $\mathbb{Z}^d$ with

$$\sqrt[n]{V_1...V_d} \geq \delta^2 \frac{sinh^2(\pi e)}{\pi^2 e^3} 2d \tag{1}$$

where δ is a constant, $\delta < 1$.

In order to do that, the main intermediate results are the following: First we consider a class of positive definite quadratic forms $\mathcal{Q}$ (taking representatives under isomorphisms) with the following properties: For every $q \in \mathcal{Q}$ we have $q(\mathbb{Z}^d) \subset 2\mathbb{Z}$, and q has discriminant D (where D has a certain arithmetic restriction). If $N_q(m)$ is the number of representation of m by q, we consider the following weighted average:

$$N_{\mathcal{Q}}(m) := \frac{\sum_{\mathcal{Q}} N_q(m) |Aut(q)|^{-1}}{\sum_{\mathcal{Q}} |Aut(q)|^{-1}} = \frac{2km^{k-1}\pi^k}{k!\sqrt{D}}(1 + O(2^{\frac{k}{2}})).$$

Then we have to find the "Transitional point" m_1 such that m_1 is sufficiently large, and if $V(T) := Vol(\{Q(x) \leq T\}$, we have that for $\delta < 1$ there exists a constant $D(\delta)$ with the following property: in any sufficiently large dimension, there exists an admissible $D \leq D(\delta)$ such that

$$\sum_{m \text{ even } < m_1} N_{\mathcal{Q}}(m) log\left(\frac{\delta V(m_1)}{V(m)}\right) < 2d.$$

This implies that there exists $q \in \mathbb{Q}$ with

$$\sum_{m \text{ even } < m_1} N_q(m) log\left(\frac{\delta V(m_1)}{V(m)}\right) < 2d$$

and to conclude we use the previous assumption together with some estimates to conclude that q satisfies the desired property (1).

22.3 Summary of article: On the density of cyclotomic lattices constructed from codes

Like before, let $K = \mathbb{Q}(\mu_d)$ and $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R}$. The trace form $\mathrm{tr}(x\bar{y})$ induces a scalar product on $K_{\mathbb{R}}$, denoted by $\langle\,,\rangle$, giving $K_{\mathbb{R}}$ the structure of an Euclidean space of dimension $\phi(d)$. We have a natural embedding $K \to K_{\mathbb{R}}$, and the image of the ring of integers $o := \mathbb{Z}[\mu_n]$ is a lattice in $K_{\mathbb{R}}$ (see Proposition 5.2 in [3]).

22.3.1 Construction A for cyclotomic fields

Given a linear s-ary code, one can obtain an associated lattice via *Construction A*, which takes the following form in our context, from [2]:

"Let $\mathcal{B}$ be a prime ideal of o lying over a prime number p which does not divide d. Then $F = o/\mathcal{B}$ is a finite field of cardinality $s = p^f$. Consider $E = K_{\mathbb{R}}^2$ and denote by $\langle\,,\rangle$ the scalar product $\langle x,\, y\rangle = \langle x_1,\, y_1\rangle + \langle x_2,\, y_2\rangle$ induced on the $2\phi(d)$-dimensional $\mathbb{R}$-vector space E by that of $K_{\mathbb{R}}$, and denote the associated norm by $||\cdot||$. Consider the lattice $\Lambda_0 = o^2$ in E, it is also a o-submodule of E. Consider the projection

$$\pi : \Lambda_0 \to \Lambda_0/\mathcal{B}\Lambda_0.$$

Define the *weight* of an element $c \in \Lambda_0/\mathcal{B}\Lambda_0$:

$$wt(c) = \min\{||z||, \pi(z) = c\}$$

Denote the discrete ball of radius r by $\bar{B}(r) = \{c \in \Lambda_0/\mathcal{B}\Lambda_0, \mathrm{wt}(c) \leq r\}$. Note that the quotient $\Lambda_0/\mathcal{B}\Lambda_0$ is a 2-dimensional vector space over F, so that a linear subspace $C \subset \Lambda_0/\mathcal{B}\Lambda_0$ is a linear s-ary code. The lattice obtained from C is defined by

$$\Lambda_C = \pi^{-1}(C)\text{''}.$$

Its properties are given in [2], Lemma 2.

22.3.2 Density of a family of lattices

Each line in $\Lambda_0/\mathcal{B}\Lambda_0$ is a 1-dimensional subspace, and therefore a code. Moustrou uses this family of codes:

Definition 4. *Denote by $\mathcal{C}$ the set of $(s+1)$ lines of $\Lambda_0/\mathcal{B}\Lambda_0$, and $\mathcal{L}_{\mathcal{C}} = \{\Lambda_C, C \in \mathcal{C}\}$ the associated lattices from the previous construction.*

The first ingredient is a *mean value theorem* for this family, analogous to Lemma 3. Since the family $\mathcal{C}$ is finite, it is a straigthforward counting argument.

Lemma 5.

$$\mathbb{E}(|\bar{B}(r) \cap (C \backslash 0)|) < \frac{|\bar{B}(r)}{s}$$

The second ingredient is the symmetry argument with the d roots of unity: If $|\bar{B}(r) \cap (C\backslash 0)| < d$, then necessarily $\bar{B}(r) \cap (C\backslash 0) = \emptyset$. This would prove the existence of a code C such that its minimal weight is at least r. It turns out that the minimal weight of a code can be related to the minimal norm of the associated lattice, and therefore to its density (see [2], Lemma 3). The problem then reduces to finding, for a given d, the largest possible radius r_d such that $\frac{|\bar{B}(r_d)|}{s} < d$, in order to apply the symmetry argument. $|\bar{B}(r)|$ can be estimated in terms of r and d from properties of Λ_0. Allowing s to also depend on d and imposing adequate growth conditions on s, after some asymptotic analysis one finds the optimal radius r_d and arrives at the main result of [2]:

Theorem 6. *For every $1 > \epsilon > 0$, if $\phi(d)^2 d = o(s^{\frac{1}{\phi(d)}})$, then for d big enough, the family of lattices $\mathcal{L}_{\mathcal{C}}$ contains a lattice $\Lambda \subset \mathbb{R}^{2\phi(d)}$ satisfying*

$$\Delta(\Lambda) > \frac{(1-\epsilon)d}{2^{2\phi(d)}}$$

Venkatesh's bound is recovered from his same bounds on $\phi(d)$, as before.

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23 The Sphere Packing Problem in Dimensions 8 and 24 (after M. Viazovska)

A summary written by Tainara Borges and Cynthia Bortolotto

Abstract

Using Cohn and Elkies linear programming bounds we solve the sphere packing problem in dimension 8 and 24 by constructing suitable functions.

23.1 Main results

In dimension 8 and 24 we know two special lattice sphere packings, associated to the E_8 lattice and the Leech lattice Λ_{24}, respectively. Computing their densities, we conclude that the sphere packing constant in dimension 8 and 24 are at least $vol(B_8(\sqrt{2}/2))$ and $vol(B_8(1))$, respectively.

Cohn and Elkies showed that we can give upper bounds for the sphere packing constant by finding functions satisfying given properties. Precisely,

Theorem 1 (Cohn and Elkies)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Schwartz function and r a positive real number such that $f(0) = \hat{f}(0) = 1$, $f(x) \leq 0$ for $||x|| \geq r$ and $\hat{f}(y) \geq 0$ for $y \in \mathbb{R}^n$. Then, the sphere packing density in $\mathbb{R}^n$ is at most $vol(B_n(r/2))$.*

Viazovska constructed such a function in dimension 8, with $r = \sqrt{2}$, proving the optimality of the E_8 lattice. Using similar ideas, Cohn, Kumar, Miller, Radchenko and Viazovska found the function in dimension 24 that proves that the Leech lattice is the unique periodic lattice that achieves the optimal sphere packing density, up to scaling and isometries.

23.2 Constructing the functions

We first deduce some properties that the functions that we want to construct must satisfy. Assuming that f is radial and using the Poisson summation formula, in dimension 8, we conclude f and $\hat{f}$ must have a simple zero at $\sqrt{2}$ and double zeros at $\sqrt{2m}$, for $m \geq 2$. Analogously, in dimension 24, the function and its Fourier transform must have a simple zero at 2 and double zeros at $\sqrt{2m}$, for $m \geq 3$. Also, we can write $f = a + b$, with $\hat{a} = a$ and $\hat{b} = -b$, so a and b must satisfy the conditions stated above.

Then, the main task is to construct good eigenfunctions a and b, in a way that a convenient combination of them give us the function with the desired properties. We state two theorems that lead to their construction. Both theorems are not presented in Viazovska's paper in the way we do, but one could infer them given her results. We give an idea of how the formulation of the functions can be deduced given the conditions.

23.2.1 The eigenfunctions

Theorem 2. *Let $n \geq 4$ be a positive integer such that $4|n$ and $k = 4 - \frac{n}{2}$. Consider φ a weakly holomorphic quasimodular form of weight k and depth 2 over $\Gamma(1)$, satisfying the conditions:*

1. *There exists $\epsilon > 0$ such that $|\varphi(it)| = O(e^{-\epsilon t})$, for $t \to \infty$;*
2. *There exists $P \geq 0$ such that $|t^{n/2-2}\varphi(i/t)| = O(e^{P\pi t})$, for $t \to \infty$.*

   Then, the function

$$a(x) = \sin^2\left(\frac{\pi|x|^2}{2}\right)\int_0^\infty t^{\frac{n}{2}-2}\varphi(i/t)e^{-\pi t|x|^2}dt$$

   is well defined for $|x|^2 > P$ and extends analytically for $|x| \geq 0$. Furthermore, a is a radial Schwartz function and $\hat{a} = (-1)^{n/4}a$.

Theorem 3. *Let $n \geq 4$ be a positive integer such that $4|n$ and $k = 2 - \frac{n}{2}$. Consider ψ a weakly holomorphic modular form of weight k over $\Gamma(2)$ and, satisfying the conditions:*

1. *There exists $\epsilon > 0$ such that $|\psi(i/t)t^{n/2-2}| = O(e^{-\epsilon t})|$, for $t \to \infty$;*
2. *The exists $P \geq 0$ such that $|\psi(x + iy)| = O(e^{P\pi t})|$, for $t \to \infty$.*
3. *$z^{n/2-2}\psi(-1/z) + \psi(z + 1) = \psi(z)$, for all $z \in \mathbb{H}$.*

   Then, the function

$$b(x) = \sin^2\left(\frac{\pi|x|^2}{2}\right)\int_0^\infty \psi(it)e^{-\pi t|x|^2}dt$$

is well defined for $|x|^2 > P$ and extends analytically for $|x| \geq 0$. Furthermore, b is a radial Schwartz function and $\hat{b} = (-1)^{n/4+1} b$.

The starting point for the construction of a and b given by Viazovska is the use of the Laplace transform to create eigenfunctions of the Fourier transform. If we define

$$f(x) = \int_0^\infty g(it) e^{-\pi t |x|^2} dt,$$

and assume that g is sufficiently well-behaved, we get that

$$\hat{f}(y) = \int_0^\infty g(it) t^{-n/2} e^{-\pi |x|^2/t} dt = \int_0^\infty g(i/t) t^{n/2-2} e^{-\pi t |x|^2} dt.$$

One can notice that if g is a modular form of weight $2-n/2$, then $g(it) = (i)^{\frac{n}{2}-2} g(i/t) t^{n/2-2}$ and, consequently, the function f is a -1 eigenfunction in dimension n, if $\frac{n}{2} - 2 \equiv 2 \mod 4$, which is the case when $n = 8$ or $n = 24$.

However, defining the functions a or b this way will not solve our problem, since we have no control of the zeros of the function f. Viazovska's great idea was to multiply the Laplace transform by $\sin^2(\pi |x|^2/2)$, that is, we let

$$f(x) = \sin^2\left(\frac{\pi |x|^2}{2}\right) \int_0^\infty g(it) e^{-\pi t |x|^2} dt.$$

The factor $\sin^2(\pi r^2/2)$ gives us some of the desired roots and also information about the poles of the Laplace transform and, consequently, about the q-expansion of the function g.

To get the remaining information about g that we need, we consider f in the form

$$f(x) = \frac{i}{4}\left(\int_1^{1+i\infty} g(t-1) e^{i\pi r^2 t} dt + \int_{-1}^{-1+i\infty} g(t+1) e^{i\pi r^2 t} dt - 2\int_0^{i\infty} g(t) e^{i\pi r^2 t} dt\right),$$

and take its Fourier transform. We change variables and the contours of integration, so we can compare f with $\hat{f}$ and derive some identities that g must satisfy so that f is a ± 1-eigenfunction.

23.2.2 The magic functions

We now give the explicit form for the magic function that solves the sphere packing problem in dimension 8 and 24. The spaces of quasi-modular forms that are considered in both theorems stated above have finite dimension, so our job is to find the right linear combination of the generators that satisfies all the desired conditions.

For $n = 8$,

$$a(x) = 4i \sin^2\left(\frac{\pi|x|^2}{2}\right) \int_0^\infty t^2 \varphi(i/t) e^{-\pi t|x|^2} dt, \text{ for } |x| > \sqrt{2}.$$

where φ is the quasimodular form of weight 0 and depth 2 given by

$$\varphi = \frac{1728E_4^2}{E_4^3 - E_6^2} E_2^2 + 2\frac{-1728E_4E_6}{E_4^3 - E_6^2} E_2 + \frac{1728E_4^3}{E_4^3 - E_6^2} - 1728,$$

and

$$b(x) = -4i \sin^2\left(\frac{\pi|x|^2}{2}\right) \int_0^\infty \psi(it) e^{-\pi t|x|^2} dt, \text{ for } |x| > \sqrt{2},$$

where the weakly holomorphic modular form ψ over $\Gamma(2)$ is

$$\psi(z) := 128\frac{\theta_{00}^4(z) + \theta_{01}^4(z)}{\theta_{10}^8(z)} + 128\frac{\theta_{01}^4(z) - \theta_{10}^4(z)}{\theta_{00}^8(z)}.$$

Then, linear combination of a and b

$$g(x) := \frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x),$$

solves the problem in dimension 8.

In dimension $n = 24$, we construct a and b as in dimension 8, but now in terms of the quasi-modular form

$$\varphi = \frac{25E_4^4 - 49E_6^2E_4^2 + 48E_6E_4^2E_2 + (-49E_4^3 + 25E_6^2)E_2^2}{\Delta^2},$$

and the weakly modular form

$$\psi = \frac{7\theta_{01}^{20}\theta_{10}^8 + 7\theta_{10}^{24}\theta_{01}^4 + 2\theta_{01}^{28}}{\Delta^2}.$$

The solution of the problem is given by

$$g(x) = -\frac{\pi i}{113218560}a(x) - \frac{i}{262080\pi}b(x).$$

In order to check that both functions, in dimensions 8 and 24, satisfy all the conditions of Cohn and Elkies theorem, one needs to show that associated functions are eventually non-negative/non-positive and computer assistance was needed to this purpose.

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24 Lattices with exponentially large kissing numbers (after Serge Vlăduţ [4])

A summary written by Philippe Moustrou

Abstract

In [4] it is shown that there exist, in any dimension, lattices with exponentially large kissing numbers. These lattices are obtained from algebraic geometric codes. Here we give the main results of the paper and describe the strategy of the proofs, introducing the main ingredients needed.

24.1 Introduction

What is the largest number τ_n of non-overlapping unit spheres in a real space of dimension n that can simultaneously touch a central unit sphere? This is the celebrated *kissing number* problem. Even though it has been extensively studied, the number τ_n is only known for a very small set of dimensions: $\tau_1 = 2$, $\tau_2 = 6$, $\tau_3 = 12$ (Schütte and van der Waerden), $\tau_4 = 24$ (Musin), $\tau_8 = 240$ and $\tau_{24} = 196560$ (Levenshtein and independently Odlyzko and Sloane).

Here we are interested in the asymptotic behaviour of τ_n. Kabatianski and Levenstein proved that $\frac{\log_2(\tau_n)}{n} \leq 0.4041...$. The classical lower bound is $\frac{\log_2(\tau_n)}{n} \geq 0.2075...$, due independently to Chabauty, Shannon, and Wyner. Note that it has been improved lately by Jenssen, Joos and Perkins. However, these arguments only relate to non-lattice configurations. If we restrict our attention to lattices and denote by τ_n^l the *lattice kissing number* in dimension n, it was not known whether τ_n^l grows exponentially in n. The best lower bound so far, $\log_2(\tau_n^l) \geq c \log_2^2(n)$, was given by the Barnes-Wall lattices in dimensions $n = 2^m$. The main result that we discuss here follows.

Theorem 1. *There exists a constant $c_0 > 0$ such that for every dimension $n \geq 1$,*

$$\frac{\log_2(\tau_n^l)}{n} \geq c_0.$$

We will discuss later the value of c_0.

24.2 Ingredients and sketch of the proof

24.2.1 Lattice packings and kissing numbers

A lattice L in $\mathbb{R}^n$ naturally defines a packing of spheres of diameter d, where d is the minimal norm realized by a non-zero vector of L, also called the minimum of L. An element of L with norm d is called a minimal vector of L. The kissing number of the packing provided by L is then the number of its minimal vectors. In other words, we aim at proving the existence of lattices with an exponentially many minimal vectors.

24.2.2 Linear codes

Linear codes are closely related to lattices. Consider a finite field $\mathbb{F}_q$. A q-ary linear code is a linear subspace $C \subset \mathbb{F}_q^n$. We call n the length of C and $R = k/n$ its rate, where k is the dimension of C as an $\mathbb{F}_q$-vector space. The number of non-zero coordinates of an element c of $\mathbb{F}_q^n$ gives its Hamming weight $wt(c)$. The minimum distance d of C is the minimal Hamming weight realized by the non-zero elements of C, and we denote by $\delta = d/n$ its relative minimum distance. We call C an $[n, k, d]_q$-code.

24.2.3 From codes to lattices: Construction D

There are several ways to construct lattices from codes. One of them is Construction D: Let $n \geq 1$ and $r \geq 1$. Suppose we have a decreasing sequence of linear binary codes $C_0 = \mathbb{F}_2^n \supset C_1 \supset \ldots \supset C_r$, where for $i = 1, \ldots, r$, the code C_i is an $[n, k_i, d_i]_2$-code with $d_i = 4^i$. We see the elements of $\mathbb{F}_2^n$ as elements of $\mathbb{R}^n$ by taking their representatives with coordinates 0 and 1. Take a basis $\{c_1, \ldots, c_n\}$ of $\mathbb{F}_2^n$ such that for every $i = 1, \ldots, r$, the code C_i is generated by $c_1, \ldots, c_{k_i}$. Also define $k_{r+1} = 0$. The lattice L constructed from this sequence of codes is the lattice generated by $2\mathbb{Z}^n$ and all the vectors in

$$\bigcup_{i=1}^{r} \left(\bigcup_{j=k_{i+1}+1}^{k_i} c_j 2^{1-i} \right).$$

By construction, the minimum of L is 2, and every element of weight 4^i in C_i produces a minimal vector of L. Therefore codes with exponentially many elements of minimal weight lead to lattices with exponential kissing numbers through Construction D.

24.2.4 Codes with many light vectors

The problem now boils down to a coding theory problem: Are there codes with exponentially many vectors of minimal weight? Unsurprisingly, this question is not easy. The following was even conjectured by Kalai and Linial: Assume $(C_n)_n$ is a sequence of codes of length n and minimal distance d_n. Denote by A_{d_n} the number of elements of C_n having weight d_n. Then $\log_2(A_{d_n}) = o(n)$. In 2001, fortunately for our purpose, this conjecture was disproved by Ashikhmin, Barg, and Vlăduţ in [1]. Let us describe their result: Let $q = 2^{2s}$ with $s \geq 3$ and consider the function

$$E_s(\delta) = H(\delta) - \frac{2s}{2^s - 1} - \log_2 \frac{2^{2s}}{2^{2s} - 1},$$

where $H(\delta) = -\delta \log_2(\delta) - (1-\delta)\log_2(1-\delta)$ is the entropy function. The function E_s has two zeroes $0 < \delta_1 < \delta_2 < 1 - 2^{-2s}$ and is positive for $\delta_1 < \delta < \delta_2$.

Theorem 2. *Let $q = 2^{2s}$, for $s \geq 3$ a fixed integer. Then for any $\delta_1 < \delta < \delta_2$, there exists a sequence $(C_n)_n$ of binary linear codes of length $n = qN$ with N going to infinity, minimal distance $d_n = n\delta/2$, such that*

$$\frac{\log_2(A_{d_n})}{n} \geq \frac{E_s(\delta)}{2^{2s}} - o(1).$$

Theorem 1 follows then from Theorem 2 by taking families of codes adapted to Construction D.

24.2.5 Algebraic geometric codes

We now say a few more about the codes involved in Theorem 2, the so-called algebraic geometric codes. More details, including all the definitions that we do not give here can be found in [3].

Let X be a smooth projective curve absolutely irreducible over a finite field $\mathbb{F}_q$. Let g be its genus, and let D be an $\mathbb{F}_q$-rational divisor of degree $a \geq g - 1$. Denote by $L(D)$ the associated function space

$$L(D) = \{f \in \mathbb{F}_q(X) : (f) + D \geq 0\}.$$

For $N > a$, take a set $\mathcal{P} = \{P_1, \ldots, P_N\}$ of N distinct $\mathbb{F}_q$-rational points of X, outside the support of D, and consider

$$\begin{array}{rcccc} ev_{\mathcal{P}} & : & L(D) & \to & \mathbb{F}_q^N \\ & & f & \mapsto & (f(P_1), \ldots, f(P_N)) \end{array}$$

the evaluation map. It is well defined, injective, and its image is a linear $[N, k, d]_q$-code, where $k \geq a - g + 1$ and $d > N - a$. The parameters of this code are good if the curve has many rational points compared to its genus.

From these codes one can build appropriate binary codes for Theorem 2. The challenge is then to find good families of curves that produce codes that are suitable for Construction D.

An example of a good family of curves is given by the Garcia-Stichtenoth tower [2], where for $k \geq 1$, the curve X_k is recursively defined by the equations, for $1 \leq i \leq k-1$,

$$x_{i+1}^q + x_{i+1} = \frac{x_i^q}{x_i^{q-1} + 1}.$$

By considering these curves over $\mathbb{F}_{q^2}$, one can construct a sequence of codes leading to Theorem 1.

24.3 Improvements

By using the previous strategy, the constant c_0 obtained in Theorem 1 is very small. We conclude by describing in a few words how it can be optimized.

First, the Garcia-Stichtenoth tower can be replaced by towers of Drinfeld modular curves, leading to a slight improvement in the constant. More importantly, one can use a generalization of Construction D to construct good lattices from good codes: Construction E. It allows to deal with more general sequences of codes, and produces better lattices. With these refinements, one gets the following theorem:

Theorem 3. *We have*

$$\liminf_{n \to \infty} \frac{\log_2(\tau_n^l)}{n} \geq 0.0219.$$

By restriction to particular sequences of dimensions, the constant can be further improved. Among all the results of this flavour given in [4], we only give the one with the best constant:

Theorem 4. *For $N = 5 \cdot 2^{10k+2}$, we have*

$$\frac{\log_2(\tau_N^l)}{N} \geq 0.0338 - o(1).$$

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