1

SLE, conformal welding, and random planar maps

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Contents

1 On the Riemann surface type of random planar maps

after J. Gill and S. Rhode [3]
A summary written by Joe Adams

Abstract

Given a sequence of random disk triangulations satisfying some conditions converging in distribution to a random triangulation T, we form a Riemann surface by gluing together planar equilateral triangles with side length 1 according to T. This Riemann surface is almost surely conformally equivalent to $\mathbb{C}$. It is then possible to prove that the UIPT is parabolic almost surely.

1.1 Preliminaries and notation

We will think of triangulations as combinatorial objects, and we will assume that any vertex in a given triangulation is connected to at most finitely many other vertices. Let T be a triangulation. Let Δ be a closed equilateral triangle in $\mathbb{C}$ with side length 1. The **interstice** I of Δ is the equilateral subtriangle whose vertices are the midpoints of the sides of Δ. If the object obtained by gluing copies of Δ according to T is a Riemann surface, we denote it by $R(T)$. We denote by $F(T)$ the set of faces of triangles of $R(T)$. We say that T is a **disk triangulation** if $R(T)$ is simply connected.

A **rooted** triangulation is a pair (T, o), where o is an oriented triangle of T. Let $B_0(T, o)$ denote the unoriented triangle corresponding to o. For any integer $k \geq 1$, let $B_k(T, o)$ denote $B_{k-1}(T, o)$ together with the triangles incident to any vertex of $B_{k-1}(T, o)$. Let $\mathcal{X}$ denote the set of rooted triangulations. We define a metric on $\mathcal{X}$ by defining the distance between (T, o) and (T', o') as $1/(k+1)$, where k is the largest integer such that $B_k(T, o)$ and $B_k(T', o')$ are isomorphic. This metric topology is separable.

A **center embedding** is a triple (T, o, g), where (T, o) is a rooted triangulation and $g : F(T) \to \mathbb{C}$ is an injective map such that $g(o) = 0$ and

$$\inf\{|g(o')| : o' \neq o\} = 1.$$

Let $\mathcal{E}$ denote the set of center embeddings. Abusing notation, we will often think of g as a map defined on the set of centers (which we will denote by

c) of faces of triangles of $R(T)$. We define a topology on $\mathcal{E}$ by saying that $(T_n, o_n, g_n) \to (T, o, g)$ if and only if $(T_n, o_n) \to (T, o)$ and $g_n \to g$ uniformly on finite subtriangulations. This topology is metrizable and separable.

Let $\mathcal{X}_{\text{finite}} \subset \mathcal{X}$ and $\mathcal{E}_{\text{finite}} \subset \mathcal{E}$ denote the subsets arising from finite triangulations.

A Borel probability measure μ on $\mathcal{X}$ is **unbiased** if whenever a random sample (T, o) arises from a finite triangulation, the distribution of o is uniform.

Theorem 1. *Suppose that a sequence of unbiased random rooted finite disk triangulations (T_n, o_n) converge in distribution to a random triangulation (T, o). Suppose that $\operatorname{dist}(o_n, \partial T_n) \to \infty$ and T is one-ended almost surely. Then $R(T)$ is conformally equivalent to $\mathbb{C}$ almost surely.*

Lemma 2. *Let T be a triangulation. There is a constant C such that for any injective holomorphic map $f : R(T) \to \mathbb{C}$ and any adjacent centers c and c',*

$$\frac{1}{C} \leq \frac{|f(c) - f(c')|}{\operatorname{diam} f(I_c)} \leq C. \tag{1}$$

Lemma 3. *Suppose that (T, o) be a rooted disk triangulation satisfying $\operatorname{dist}_T(o, \partial T) = \infty$. Let $\phi : R(T) \to \mathbb{C}$ be an injective holomorphic map. If the image of ϕ restricted to the centers of the triangles of T has at most finitely many accumulation points in $\mathbb{C}$, then ϕ is surjective.*

1.2 A magical lemma

The following magical lemma is due to Benjamini and Schramm [2].

Lemma 4. *Let $\delta \in (0, 1)$. There is a constant $c = c(\delta) > 0$ such that for any number $s \geq 2$ and any finite set $V \subset \mathbb{C}$,*

$$\frac{\#\{v \in V : v \text{ is } (\delta, s)\text{-supported}\}}{\#V} \leq \frac{c}{s}.$$

Given a finite set $V \subset \mathbb{C}$, the probability that a point of V chosen uniformly at random is (δ, s)-supported is the fraction to the left of the inequality in the lemma above.

Lemma 5. *Let $\mathcal{E}_0$ be a subset of $\mathcal{E}_{\text{finite}}$ such that if (T, o, g) and (T', o', g') belong to $\mathcal{E}_0$ and satisfy $T = T'$, then there is an affine map α such that $g = \alpha \circ g'$. Suppose that a sequence of unbiased Borel probability measures μ_n on $\mathcal{E}_0$ converge in distribution on $\mathcal{E}$ to a Borel probability measure μ. Let A denote the set of (T, o, g) in $\mathcal{E}$ such that the image of g has at least two accumulation points in $\mathbb{C}$. Then $\mu(A) = 0$.*

Proof. Let A_{δ,p_1,p_2} denote the set of (T, o, g) in $\mathcal{E}$ such that the image of g has at least one accumulation point in each of $B_{\mathbb{C}}(p_1, \delta)$ and $B_{\mathbb{C}}(p_2, \delta)$. One can check that

$$\bigcup_{\delta} \bigcup_{p_1,p_2} A_{\delta,p_1,p_2} = A, \tag{2}$$

where the first union is taken over all $\delta = 1/n$, $n \in \mathbb{Z}_{\geq 1}$, and the second union is taken over all p_1 and p_2 in $\mathbb{C}$ that have rational real and imaginary parts and satisfy $|p_1 - p_2| \geq 4\delta$ and $p_1, p_2 \in B_{\mathbb{C}}(0, \delta^{-1})$.

Suppose that $\mu(A) > 0$. It follows from (2) that one of the sets A_{δ,p_1,p_2} satisfies $\mu(A_{\delta,p_1,p_2}) =: \varepsilon > 0$. Let $O_{k,s}$ denote the set of all (T, o, g) in $\mathcal{E}$ such that $B_{\mathbb{C}}(p_1, \delta)$ and $B_{\mathbb{C}}(p_2, \delta)$ each contain at least s points of the image of the centers of triangles of $B_k(T, o)$ under g. It is easy to see that the sets $O_{k,s}$ are open, and one can check that

$$\bigcap_{s} \bigcup_{k} O_{k,s} \supset A_{\delta,p_1,p_2}.$$

Consequently, for any integer $s \geq 2$, there is an integer $k \geq 0$ such that $\mu(O_{k,s}) \geq \varepsilon/2$. Since $O_{k,s}$ is open, distributional convergence implies

$$\liminf_{n} \mu_n(O_{k,s}) \geq \mu(O_{k,s}). \tag{3}$$

We will now speak in terms of probabilities. Let (T_n, o_n, g_n) denote a random sample of μ_n. Assume that the inequality

$$\operatorname{Prob}((T_n, o_n, g_n) \in O_{k,s} | T_n = T) \leq c/s \tag{4}$$

has already been established. Then

$$\begin{aligned}
\mu_n(O_{k,s}) &= \mu_n(\{(T, o, g) \in O_{k,s} : T \text{ is a finite triangulation}\} \\
&= \operatorname{Prob}((T_n, o_n, g_n) \in O_{k,s}) \\
&= \sum_{T} \operatorname{Prob}((T_n, o_n, g_n) \in O_{k,s} | T_n = T) \operatorname{Prob}(T_n = T) \\
&\leq \frac{c}{s} \sum_{T} \operatorname{Prob}(T_n = T) = \frac{c}{s}.
\end{aligned}$$

Together with (3), this implies that $c/s \geq \varepsilon$, which is impossible when s is sufficiently large.

Now we will show that (4) is valid. Whenever (T, o, g) and (T, o', g') belong to $\mathcal{E}_0$, there is an affine map α such that $g = \alpha \circ g'$. Choose a preferred map g_T from the set of all injective maps $g : F(T) \to \mathbb{C}$ appearing as the third component of a triple (T, o, g) in $\mathcal{E}_0$ having T as its first component. Let V denote the image of g_T. We see that

$$\begin{aligned}
&\text{Prob}((T_n, o_n, g_n) \in O_{k,s} | T_n = T) \\
&\quad \leq \text{Prob}(0 \text{ is } (\delta, s)\text{-supported in the image of } g_n | T_n = T) \\
&\quad = \text{Prob}(g_T(o_n) \text{ is } (\delta, s)\text{-supported in } V | T_n = T)
\end{aligned}$$

The inequality above follows from the observation that for any (T, o, g) in $O_{k,s}$, $g(o) = 0$ is (δ, s)-supported. The equality above follows from the fact that affine maps preserve the property of a point being (δ, s)-supported. Since μ_n is unbiased, $g_T(o_n)$ is distributed uniformly among the points of V. Then lemma 4 implies that

$$\text{Prob}(g_T(o_n) \text{ is } (\delta, s)\text{-supported in } V | T_n = T) \leq \frac{c}{s}.$$

□

1.3 Proof of theorem 1

Consider a sequence of unbiased Borel probability measures μ_n on $\mathcal{X}_{\text{finite}}$ converging in distribution on $\mathcal{X}$ to a Borel probability measure μ, where

$$\mu(\{(T, o) \in \mathcal{X} : T \text{ has at most one end}\}) = 1,$$

and for every number $D > 0$,

$$\mu_n(\{(T, o) \in \mathcal{X}_{\text{finite}} : \text{dist}(o, \partial T) > D) \to 1.$$

For each finite triangulation T, choose an injective, holomorphic map $\phi_T : R(T) \to \mathbb{C}$. For each $(T, o) \in \mathcal{X}_{\text{finite}}$, choose numbers $a > 0$ and $b \in \mathbb{C}$ such that the map $f_{(T,o)} : R(T) \to \mathbb{C}$ satisfies $f_{(T,o)}(c_o) = 0$ and

$$\inf\{|f_{(T,o)}(c)| : c \neq c_o\} = 1.$$

Having made these choices, we obtain an injective function $h : \mathcal{X}_{\text{finite}} \to \mathcal{E}$ defined by

$$(T, o) \mapsto (T, o, f_{(T,o)}),$$

where we consider the restriction of $f_{(T,o)}$ to the centers of the triangles of T.

It is easy to see that h is measurable, so we can consider the sequence of Borel probability measures $\mathbb{P}_n = h_* \mu_n$ on $\mathcal{E}$.

Remark 6. *We have not arranged for h to be continuous, and it is certainly possible to make choices so that h is not continuous.*

Now, we will show that the collection $\{\mathbb{P}_n\}$ is tight. To this end, given integers $k \geq 0$ and $L \geq 2$, let $U_{k,L}$ denote the set of all $(T, o) \in \mathcal{X}$ such that the degree of any vertex v of $B_k(T, o)$ is at most L. It is easy to see that the sets $U_{k,L}$ are open.

Let $\varepsilon > 0$. For each k, $U_{k,L} \subset U_{k,L+1}$ and $\bigcup_L U_{k,L} = \mathcal{X}$. It follows that we can choose a number $L(k)$ such that

$$\mu(U_{k,L(k)}) \geq 1 - \frac{\varepsilon}{2^k}. \tag{5}$$

Since $U_{k,L(k)}$ is open, the Portmanteau theorem implies that

$$\liminf_n \mu_n(U_{k,L(k)}) \geq \mu(U_{k,L(k)}).$$

Consequently, there is an integer N such that for each $n > N$,

$$\mu_n(U_{k,L(k)}) \geq \mu(U_{k,L(k)}) - \frac{\varepsilon}{2^k}. \tag{6}$$

Choosing a larger value for $L(k)$, we can guarantee that (6) holds for each of the finitely many indices $n \leq N$ and that (5) remains valid. Set $U = \bigcap_k U_{k,L(k)}$. Induction on (5) leads to

$$\mu(U) \geq 1 - \sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} = 1 - 2\varepsilon.$$

Induction on (6) leads to

$$\mu_n(U) \geq \mu(U) - \sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} = \mu(U) - 2\varepsilon.$$

Combining these inqualities, we conclude that for each n,

$$\mu_n(U) \geq 1 - 4\varepsilon. \tag{7}$$

Let $A_{k,L}$ denote the set of all $(T, o, g) \in \mathcal{E}$ such that $(T, o) \in U_{k,L}$, $g(c_o) = 0$, $\inf\{|g(c)| : c \neq c_o\} = 1$, and g extends to an injective, holomorphic map $R(T) \to \mathbb{C}$. Set $A = \bigcap_k A_{k,L(k)}$. It is easy to see that for each n, $\mathbb{P}_n(A_{k,L}) = \mu_n(U_{k,L})$, so (7) implies that

$$\mathbb{P}_n(A) \geq 1 - 4\varepsilon.$$

To establish tightness, it remains to show that A is compact. To this end, let $\{(T_n, o_n, g_n)\}$ be a sequence of points in A. We can assume that $(T_n, o_n) \to (T, o)$. (For each k, using finiteness of $L(k)$, we see that there are only finitely many isomorphism classes of $B_k(T, o)$.)

Fix k. For large enough n, $B_k(T_n, o_n)$ is isomorphic to $B_k(T, o)$, so we can assume that the g_n are all defined on $R(B_k(T, o))$. The sequence of maps f_n restricted to $R(B_k(T, o)) \setminus \{o, o'\}$ miss 0 and $g_n(o')$, so they form a normal family.

By a diagonalization process, we can pick a subsequence converging on each $R(B_k(T, o))$, $k \in \mathbb{Z}_{\geq 0}$. From the left inequality in (1), we see that $\operatorname{diam} g_n(I_o) \leq C|g_n(c_o) - g_n(c)|$ for all centers c adjacent to c_o. From our normalization, we then have $\operatorname{diam} g_n(I_o) \leq C$. From the right inequality in (1), we see that $|g_n(c_o) - g_n(c)| \leq C \operatorname{diam} g_n(I_o)$. Hence,

$$1 \leq |g_n(c)| \leq C^2,$$

where the left inequality follows from our normalization, and the right inequality follows from the inequalities in the two preceding sentences. This implies that any limiting map is nonconstant. This completes the proof that A is compact.

We have established that $\{\mathbb{P}_n\}$ is tight. By Prokhorov's theorem, a subsequence converges in distribution on $\mathcal{E}$ to a Borel probability measure $\mathbb{P}$ supported on the set of all (T, o, g) in $\mathcal{E}$ such that g extends to an injective holomorphic map $R(T) \to \mathbb{C}$.

Let (T, o, g) be a random sample of $\mathbb{P}$. By lemma 5, the image of g has at most one accumulation point almost surely. By lemma 3, $R(T)$ is conformally equivalent to $\mathbb{C}$ almost surely.

1.4 Implications for UIPT

Let τ_n denote the uniform measure on the set of rooted triangulations of the sphere having exactly n vertices. Angel and Schramm [1] showed that τ_n converges in distribution to a measure τ, and random samples of τ are almost surely one-ended. Let $M_{n,r}$ denote the maximal degree of any vertex in $B_r(T)$, where T is a random sample of τ_n. Angel and Schramm [1] also showed that for a fixed r, $\text{Prob}(M_{n,r} > t) \to 0$ as $t \to \infty$ uniformly with respect to n.

Theorem 7. *Let T be a sample of UIPT. Then $R(T)$ is conformally equivalent to $\mathbb{C}$ almost surely.*

Proof. Let S_n be a random sphere triangulation chosen according to τ_n. Removing a triangle chosen uniformly at random from S_n, we obtain a disk triangulation T_n. Choose an oriented triangle o_n uniformly at random from T_n. Then (T_n, o_n) is a sequence of random triangulations, converging to UIPT, to which we can apply theorem 1. □

References

[1] Angel, O. and Schramm, O., *Uniform infinite planar triangulations.* arXiv:math/0207153v3

[2] Benjamini, I. and Schramm, O., *Recurrence of distributional limits of finite planar graphs.* arXiv:math/0011019v4

[3] Gill, J. and Rhode, S., *On the Riemann surface type of planar random maps.* arXiv:1101.1320v1

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2 Recurrence of distributional limits of finite planar graphs

after I. Benjamini and O. Schramm [1]
A summary written by Gerandy Brito

Abstract

In their paper [1] the authors proved that the distributional limit of rooted random unbiased finite planar graphs with degrees uniformly bounded is almost surely recurrent. In this note we focus on the case where the graphs are triangulations of the sphere.

2.1 Preliminaries

By a rooted graph, we mean a pair (G, o) where G is a connected graph and o is a vertex of G. Two rooted graphs (G, o) and (G', o') are isomorphic if there is an isomorphism of graphs from G to G' which sends o to o'. Denote by $\mathcal{X}$ the space of isomorphism classes of rooted connected, locally finite graphs. From now on, (G, o) refers to the point of $\mathcal{X}$ corresponding to the isomorphic class of (G, o).

Let $B_G(o, r)$ denote the closed ball of radius r and center o in (G, o). Put:

$$k(G, G') = sup\{k : B_G((o, k), o) \text{ and } B_{G'}((o', k), o') \text{ are isomorphic}\}$$

Then $d((G, o), (G', o')) = 2^{-k(G,G')}$ is a metric in $\mathcal{X}$.

A *random* rooted finite graph is an element of $\mathcal{X}$ chosen according to a Borel probability μ supported on a set of finite graphs. A random rooted finite graph (G, o) with **law** μ ((G, o) is chosen according to μ) is unbiased if, given G, the root o is uniformly distributed among the vertices of G. A precise definition can be found in [1].

The main object of study in this note is a **distributional limit** of a sequence of random elements of $\mathcal{X}$, this can be defined as follows: let $(G, o), (G_j, o_j)_{j\geq 1}$ be random connected rooted graphs. Then (G, o) is the distributional limit of (G_j, o_j) as $j \to \infty$ if the law of (G, o) is the weak limit of the law of (G_j, o_j). An equivalent way to look at it (perhaps more intuitive) is saying that, for every $k > 0$ and every finite rooted graph (H, o') the probability that $(B_{G_j}(o_j, k), o_j)$ is isomorphic to (H, o') converges to the probability that $(B_G(o, k), o)$ is isomorphic to (H, o').

We will restrict our attention to graphs with degrees uniformly bounded by a constant M. One important consequence of this assumption is that the subspace $\mathcal{X}_M$ of such graphs is compact in $(\mathcal{X}, d)$. The reader can easily check this by noticing that the combinatorics of $B_G(o, R)$ is finite. Hence, if a sequence of rooted graphs $(G_j, o_j) \in \mathcal{X}_M$ a Cantor diagonal argument will produce a subsequence with a distributional limit.

A probabilistic audience will immediately ask how the random walk in such graphs will behave. Here we answer the question of whether a distributional limit is recurrent. We recall that, a connected rooted graph is recurrent if, with probability one, the random walk starting at the root will revisit it.

A random rooted graph is a *triangulation* if all its faces are topological triangles. The main goal of this note is to prove the following:

Theorem 1. *Let $M < \infty$, and let (T, o) be a distributional limit of rooted random unbiased finite triangulations of the sphere T_j with degrees bounded by M. Then with probability one T is recurrent.*

This is proposition 2.1 in [1].

A natural choice for the law of (G_j, o_j) is the uniform measure on the set of all rooted triangulations (with uniformly bounded degrees!) with j vertices. The more general case, where the uniform measure is on the set of all triangulations of size j is not included here. Furthermore the set of all isomorphism classes of triangulations is not compact, as one can see by examining the sequence (T_n) of triangulations where T_n is obtained by connecting all the vertices in an n-cycle to two new vertices (forming a double pyramid). Angel and Schramm [2] proved the existence of the limit in this scenario and Gurel and Nachmias [3] showed that the limit is, in fact, recurrent almost surely, as was conjectured in [1].

The proof of the main theorem relies on the theory of circle packing.

2.2 Circle packing theorem and the ring lemma

A packing P is simply a collection of closed disks in the plane with disjoint interiors. The tangency graph of a packing is a graph which vertices are indexed by the disks of P and an edge connects two vertices iff their corresponding disks are tangent. The connection between the circle packing theory and planar graphs goes back to the work of Koebe ([4]) who proved that for every triangulation of the sphere G, there is a packing whose tangency graph is G and the packing is unique up to Möbius transformations. Many years later the theorem was rediscovered by Thurston ([8]) who conjectured ([9]) that finite circle packings can approximate the Riemann map from a simply connected domain to the unit disk. This result was proved by Rodin and Sullivan ([5]). Much research have been done in the theory of circle packing after the work of Rudin and Sullivan.

2.2.1 The ring lemma

If the tangency graph of a packing P is a triangulation, for any circle C that does not intersect the unbounded component of $\mathbb{R}^2 - P$ the neighbors will surround it completely. The ring lemma ([5]) states that, if the neighbors of a circle C with radius r are C_i with radii r_i then the quotient $\frac{r}{r_i}$ is bounded from above by a constant that only depends on the degree of C. We will use this powerful result and the combinatorics of the graphs to produce a packing with tangency graph T as follows: denote by P_j the packing with tangency graph T_j such that C_{o_j} is the unit disk. Let t_j be *the* triangle that intersect the unbounded component of $\mathbb{R}^2 - P_j$. Note that:

- The graph distance from o_j to t_j converges to infinity (in probability) as $j \to \infty$. By the assumption on the bound of the degrees, for any k, the number of vertices at distance at most k from t_j is bounded by a constant, that depends on k and M, uniformly in j.

- For any fixed $k > 0$ there is a constant $c = c(M, k)$ such that all disks in P_j at distant at most k from o_j have radii in $[1/c, c]$ asymptotically almost surely. (one should exclude those disks which intersect the unbounded component of $\mathbb{R}^2 - P$). This is a consequence of the ring lemma.

The last observation implies that, the family of random variables denoting the radii of the disks of P_j at distance at most k from o_j is tight for all k. Hence, one can take a subsequence of these random variables that converges in distribution. Another diagonal argument will produces a sequence of packings that converges to a (random) packing P which tangency graph will be T. The packing P obtained above will be infinite and disks may accumulate on different points of the plane. We say a point z in the plane is an **accumulation point** of a packing if any open set containing z intersects infinitely many disks. The next is proposition 2.2 in [1] which proves that, in our case, there is at most one accumulation point in the plane.

Proposition 2. *With probability 1, there is at most one accumulation point in $\mathbb{R}^2$ of the packing P.*

With this at hand one needs to look at two cases to finish the proof of the theorem: if the packing P has no accumulation point the result follows from [6] and [7]. In these two papers it was shown, independently, that the tangency graph of a packing with no accumulation points in the plane and bounded degrees is recurrent.
If P has one accumulation point, p, consider the graph G_1 spanned by P in the disk $B(p, 1)$. Theorems $2.6, 3.1$ and 8.1 in [6] imply that G_1 is recurrent. Similarly, the graph outside $B(p, 1)$ is recurrent. Since these two graphs are connected by a finite number of edges it follows that T is recurrent.

References

[1] Benjamini I. and Schramm O., *Recurrence of distributional limits of finite planar graphs.* Electronic Journal of Mathematics. **6** (2001), 23, 1–13.

[2] Angel O. and Schramm O., *Uniform infinite planar triangulations.* Communications in Mathematical Physics. **241**(2003), 2-3, 191–213.

[3] Gurel-Gurevich O. and Nachmias A., *Recurrence of planar graphs limits.* Annals of Mathematics. **177** (2013) 2, 761–781.

[4] Koebe P., *Kontaktprobleme der Konformen Abbildung.* Ber. S??achs. Akad. Wiss. Leipzig, Math.-Phys. Kl., **88** (1936) 141–164.

[5] Rodin B. and Sullivan D., *The convergence of circle packings to the Riemann mapping.* Journal of Differential Geometry. **26** (1987) 2,349–360.

[6] He Z. and Schramm O., *Hyperbolic and Parabolic Packings.* Discrete Computational Geometry. **14** (1995) 123–149.

[7] McCaughan G., *A recurrence/transient result for circle packings.* Proceedings of the American Mathematical Society. **26** (1998) 12,3647–3656.

[8] Thurston W.P., *The geometry and topology of 3-manifolds.* Lecture notes, Princeton University. (1980), Chapter 13.

[9] Thurston W.P., *The finite Riemann mapping theorem.* invited address, International Symposium in Celebration of the Proof of the Bieberbach Conjecture, Purdue University, (1985).

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3 An Estimate for the Conformal Radius

after S. Rohde and C. Wong [2]
A summary written by Angel Chavez

Abstract

The half-plane capacity of a subset of the upper-half plane is comparable to the euclidean area of the hyperbolic neighborhood of radius one of this set. This is proved by showing a similar estimate for the conformal radius of a subdomain of the unit disk. We summarize Rohde and Wong's proof (from [2]) of the estimate for the conformal radius.

3.1 Introduction

Throughout this paper we will let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the upper-half plane and open unit-disk, respectively. A *hull* in $\mathbb{H}$ is a subset $A \subset \mathbb{H}$ with the property that $\mathbb{H}\backslash A$ is simply-connected. If $A \subset \mathbb{H}$ is a hull, then we let $g_A : \mathbb{H}\backslash A \longrightarrow \mathbb{H}$ denote the unique conformal map satisfying the normalization

$$g_A(z) = z + O\big(\frac{1}{z}\big) \tag{1}$$

as $z \to \infty$. The *half-plane capacity of* A is defined as the limit

$$\operatorname{hcap}(A) = \lim_{z\to\infty} z\big(g_A(z) - z\big). \tag{2}$$

The half-plane capacity is intrinsically linked to SLE: The evolution of a curve determines a collection of hulls $\{A_t : t \geq 0\}$ in $\mathbb{H}$, and the conformal maps g_{A_t} satisfy the chordal Loewner equation. Naturally, it is of great interest for the study of SLE to estimate $\operatorname{hcap}(A)$ in terms of simpler geometric quantities associated to A. Rohde and Wong proved the following in [2].

Theorem 1 (Theorem 1.1 in [2])**.** *The half-plane capacity and the (euclidean) area of the hyperbolic neighborhood of radius one are comparable,*

$$hcap(A) \asymp |N(A)|.$$

More precisely, there are absolute constants $C_1, C_2 > 0$ so that $C_1|N(A)| \leq hcap(A) \leq C_2|N(A)|$, where $|N(A)|$ denotes the (euclidean) area of the hyperbolic neighborhood of radius one of A and

$$N(A) = \{z \in \mathbb{H} : dist_{hyp}(z, A) \leq 1\}.$$

Theorem 1 is proved by showing that a similar estimate holds for the conformal radius of a subdomain of the unit disk. Recall the *conformal radius* of a simply-connected region $B \subset \mathbb{C}$ relative to a point $z_0 \in B$ is defined as

$$\text{crad}(B, z_0) = |f'(0)|, \tag{3}$$

where $f : \mathbb{D} \longrightarrow B$ is a uniformization of B satisfying $f(0) = z_0$.

As with chordal SLE and the half-plane capacity, we also have an intrinsic relationship between radial SLE and we refer to as disk capacity. In particular, suppose that $B \subset \mathbb{D}$ for which $0 \in \mathbb{D}\backslash B$ and $\mathbb{D}\backslash B$ is simply-connected. The *disk capacity* of B is defined as the positive number

$$\text{dcap}(B) = -\log \text{crad}\big(\mathbb{D}\backslash B, 0\big).$$

Estimating the disk capacity is the context of the next theorem.

Theorem 2 (Theorem 1.2 in [2])**.** *If* $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$ *such that* $\mathbb{D}\backslash B$ *is simply-connected, then*

$$dcap(B) \asymp |N(B)|,$$

where $dcap(B)$ *denotes the disk capacity of* B *and* $|N(B)|$ *denotes the (euclidean) area of the hyperbolic (with respect to* $\mathbb{D}$*) neighborhood of radius one of* B*.*

Theorem 2 actually implies Theorem 1 (see page 8 in [2]). In this paper we will summarize the proof of Theorem 2 given in [2].

3.2 Dyadic Decomposition of $\mathbb{D}$

Here, we state two results related to the dyadic decomposition of the disk. We will provide the ideas for the proofs, closely following [2]. We begin by fixing a subset $B \subset \{z \in \mathbb{D} : \frac{1}{2} < |z| < 1\}$ with the property that $\mathbb{D}\backslash B$ is simply-connected. As before, we let $N(B)$ denote the hyperbolic neighborhood of B of radius 1. Given a *dyadic interval*,

$$J = \Big[\frac{k-1}{2^n}, \frac{k}{2^n}\Big] \;\; (n = 1, 2, \ldots \text{ and } k = 1, 2 \ldots, 2^n),$$

we define the *dyadic square*,

$$Q_j = \{ z \in \mathbb{D} \, : \, \frac{z}{|z|} \in \exp(2\pi i J) \text{ and } 1 - |z| \leq 2^{-n} \}.$$

The *top half* of Q_J is defined as

$$T(Q_J) = \{ z \in Q_J \, : \, 1 - |z| > 2^{-(n+1)} \}.$$

Denote by $Q(B)$ the union of all dyadic squares whose top half intersected with B is non-empty. Lastly, we write $x \lesssim y$ if there exists a constant $C > 0$ for which $x \leq Cy$.

Proposition 3 (Proposition 2.1 in [2])**.** *If $\mathbb{B} \subset \{ z \in \mathbb{D} \, : \, \frac{1}{2} < |z| < 1 \}$ and $\mathbb{D} \backslash B$ is a simply-connected region, then*

$$C_1 |B| \leq dcap(B) \leq dcap\big(Q(B)\big) \leq C_2 |Q(B)|,$$

with absolute constants $C_1, C_2 > 0$.

Idea of Proof. The first inequality follows from Parseval's identity and the second inequality follows from the Schwarz lemma.

To prove the third inequality one writes $Q(B)$ as union of dyadic squares $Q_1, Q_2, \ldots$ satisfying (1) $\{Q_j\}$ is a disjoint family (modulo boundary) and (2) $|Q_1| \geq |Q_2| \geq \cdots$. Define f_m to be the conformal map from $\mathbb{D}$ onto $\mathbb{D} \backslash \bigcup_{j=m+1}^{\infty}$ with the normalization $f_m(0) = 0$, and let $K_m = f_m^{-1}(Q_m)$. Showing

$$\text{dcap}(K_m) \asymp |Q_m| \tag{4}$$

for each $m \geq 1$ implies the third inequality. Inequality (4) is proven in [3] (see Proposition 2) by showing that one can construct two concentric circular hulls in $\mathbb{D}$ with sizes that are comparable and such that one contains K_m and the other is contained in K_m.

□

We will now take $\widehat{N}(B)$ to be the union of $N(B)$ with its complementary components (w.r.t. $\mathbb{D}$) which do not contain the origin.

Proposition 4 (Proposition 2.2 in [2])**.** *If $\mathbb{B} \subset \{ z \in \mathbb{D} \, : \, \frac{1}{2} < |z| < 1 \}$ and $\mathbb{D} \backslash B$ is a simply-connected region, then*

$$dcap\big(\widehat{N}(B)\big) \lesssim dcap(B).$$

Idea of Proof. Let $\varepsilon > 0$ and define $N_\varepsilon(B)$ to be the hyperbolic neighborhood of B of radius ε. Define $\widehat{N}_\varepsilon(B)$ just as $\widehat{N}(B)$ with $N_\varepsilon(B)$ in place of $N(B)$. Define the dyadic layer

$$D_n = \left\{z \in \mathbb{D} \,:\, 2^{-(n+1)} \le 1 - |z| < 2^{-n}\right\}$$

and let $B_n = B \cap D_n$. One first establishes the relationships

$$\operatorname{dcap}(B) \asymp \sum_{n=1}^{\infty} 2^{-n}\omega_n(0) \quad \text{and} \quad \operatorname{dcap}\big(\widehat{N}_\varepsilon(B)\big) \asymp \sum_{n=1}^{\infty} 2^{-n}\widehat{\omega}_n(0), \tag{5}$$

where $\omega_n(z)$ denotes harmonic measure of B_n with respect to the region $\mathbb{D}\backslash B$ and the point z (similarly for $\widehat{\omega}$). The inequality

$$\widehat{\omega}_n(0) \lesssim \omega_{n-1}(0) + \omega_n(0) + \omega_{n+1}(0) \tag{6}$$

can be shown to hold provided $\varepsilon > 0$ is chosen so that for each n one has the property that every hyperbolic ball centered in D_n of radius 2ε is contained in $D_{n-1} \cup D_n \cup D_{n+1}$. Relations (5) and (6) together imply

$$\operatorname{dcap}\big(\widehat{N}_\varepsilon(B)\big) \lesssim \operatorname{dcap}(B).$$

Iteration of this inequality $\frac{1}{\varepsilon}$ times gives $\operatorname{dcap}\big(\widehat{N}(B)\big) \lesssim \operatorname{dcap}(B)$. The Schwarz lemma implies the reverse inequality.

□

3.3 Proof of Theorem 2

Theorem 2 follows in a straightforward manner from Propositions 3 and 4. In particular, $\operatorname{dcap}(B) \gtrsim \operatorname{dcap}\big(\widehat{N}(B)\big)$ by Proposition 4. Applying Proposition 3 to the subset $\widehat{N}(B)$ implies $\operatorname{dcap}\big(\widehat{N}(B)\big) \gtrsim |\operatorname{dcap}\big(\widehat{N}(B)\big)|$. Altogether,

$$\operatorname{dcap}(B) \gtrsim |N(B)|$$

by monotonicity of $|\cdot|$. By Proposition 3 we have $\operatorname{dcap}(B) \le \operatorname{dcap}\big(Q(B)\big)$, wich implies $\operatorname{dcap}(B) \le \operatorname{dcap}\big(Q(\widehat{N}(B))\big)$ since $N \subset \widehat{N}(B)$. Applying Propositon 3 to the subset $\widehat{N}(B)$ now implies $\operatorname{dcap}(B) \lesssim |Q\big(\widehat{N}(B)\big)| \lesssim |N(B)|$. Therefore,

$$\operatorname{dcap}(B) \asymp |N(B)|.$$

References

[1] Lawler, G., *Conformally Invariant Processes in the Plane.* American Mathematical Society, 2005.

[2] Rohde, S. and Wong, C., *Half-Plane Capacity and Conformal Radius.* Proc. Amer. Math. Soc. 142 (2014), 931-938.

[3] Rohde, S. and Zinsmeister, M., *Variation of the Conformal Radius.* J. Anal. Math. **92** (2004), 105-115.

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4 Forward and backward SLE($\kappa; \vec{\rho}$) processes

after O. Schramm, D. B. Wilson [2], S. Rohde and D. Zhan [3]
A summary written by Laurie Field

Abstract

We define the SLE($\kappa; \vec{\rho}$) processes and give the relationship between the disk and half-plane versions of them. We also discuss backward SLE($\kappa; \vec{\rho}$).

4.1 SLE($\kappa; \vec{\rho}$) processes

SLE($\kappa; \vec{\rho}$) processes are variants of SLE(κ) that appear when one or more *force points* are tracked and the measure is weighted in the sense of the Girsanov theorem by certain martingales.

Though some of the motivation for studying them is omitted by doing so, SLE($\kappa; \vec{\rho}$) processes can be defined without using the Girsanov theorem in the following simple manner. This approach was expounded by Schramm and Wilson [2], though the concept of SLE($\kappa; \vec{\rho}$) processes was introduced by Lawler, Schramm and Werner [1]. For simplicity we restrict our discussion to the upper half-plane $\mathbb{H}$ and the unit disk $\mathbb{D}$. Recall the chordal and radial Loewner equations in $\mathbb{H}$ and $\mathbb{D}$ respectively,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z, \qquad z \in \mathbb{H}, \tag{1}$$

$$\partial_t g_t(z) = -g_t(z)\, \frac{g_t(z) + W_t}{g_t(z) - W_t}, \qquad g_0(z) = z, \qquad z \in \mathbb{D}, \tag{2}$$

where W_t is the driving function lying on the real line or unit circle, respectively.

Definition 1. *Let $\rho_1, \ldots, \rho_m \in \mathbb{R}$.*

Chordal SLE($\kappa; \rho_1, \ldots, \rho_m$), often just called SLE($\kappa; \vec{\rho}$), in $\mathbb{H}$ starting from $W_0, V_0^1, \ldots, V_0^m$ is the solution to the chordal Loewner equation (1) *with driving function given by the solution to the SDE system*

$$dW_t = \sqrt{\kappa}\, dB_t + \sum_{i=1}^{m} \rho_i\, \Re \frac{1}{W_t - V_t^i}, \qquad dV_t^i = \frac{2}{V_t^i - W_t}, \quad 1 \le i \le m,$$

which is well-defined until the first time τ that V_t^i approaches W_t for some i.

Radial SLE($\kappa; \rho_1, \ldots, \rho_m$) in $\mathbb{D}$ starting from $W_0, V_0^1, \ldots, V_0^m$ is the solution to the chordal Loewner equation (1) *with driving function given by the solution to the SDE system*

$$dW_t = -(\kappa/2)\, W_t\, dt + i\sqrt{\kappa}\, W_t\, dB_t + \sum_{i=1}^{m} \frac{\rho_i}{4}\, W_t \left(\frac{V_t^i + W_t}{V_t^i - W_t} + \frac{\overline{V_t^i}^{-1} + W_t}{\overline{V_t^i}^{-1} - W_t} \right),$$

$$dV_t^i = -V_t^i\, \frac{V_t^i + W_t}{V_t^i - W_t}, \quad 1 \le i \le m,$$

which is once again well-defined until the first time V_t^i approaches W_t for some i.

We remark that $\overline{V_t^i}^{-1}$ is simply the inversion of V_t^i in the unit circle.

The following theorem shows that the image of chordal or radial SLE($\kappa; \vec{\rho}$) under a Möbius transformation is again SLE($\kappa; \vec{\rho}$) with the same weights, *as long as* the sum of those weights is $\kappa - 6$. In particular, it implies the well-known result that the image of radial SLE in the disk under a Möbius transformation to the upper half-plane is a time change of chordal SLE($\kappa; \kappa - 6$), with the force point at the image of 0. This uses the fact that a force point of any weight can be introduced at 0 (for radial SLE($\kappa; \vec{\rho}$) in $\mathbb{D}$) or ∞ (for chordal SLE($\kappa; \vec{\rho}$) in $\mathbb{H}$) without changing the law of the process.

In the theorem, we adopt the convention that SLE($\kappa; \vec{\rho}$) in $\mathbb{D}$ is always radial SLE($\kappa; \vec{\rho}$) and SLE($\kappa; \vec{\rho}$) in $\mathbb{H}$ is always chordal SLE($\kappa; \vec{\rho}$), both as specified in Definition 1.

Theorem 2. *Let $\psi : X \to Y$ be a Möbius transformation between two domains X and Y, each of which is either $\mathbb{D}$ or $\mathbb{H}$. Suppose that $\rho_1, \ldots, \rho_m \in \mathbb{R}$ with $\rho_1 + \cdots + \rho_m = \kappa - 6$. Then the image under ψ of SLE($\kappa; \vec{\rho}$) in X starting from $(W, V_0^1, \ldots, V_0^m)$ is a time change of SLE($\kappa; \vec{\rho}$) in Y starting from $(\psi(W), \psi(V_0^1), \ldots, \psi(V_0^m))$, up to a stopping time.*

Proof sketch. It suffices to cover the case of $\psi : \mathbb{H} \to \mathbb{D}$. After changing the time parametrization so that the half-plane capacity parametrization in $\mathbb{H}$ becomes the capacity parametrization in $\mathbb{D}$, one can calculate the evolution of $(\psi(W), \psi(V_0^1), \ldots, \psi(V_0^m))$ by applying Itō's formula to the SDE system that appears in the definition. □

4.2 Backward SLE($\kappa; \vec{\rho}$) processes

The concept of backward SLE processes is simple: instead of growing the curve from the tip, grow it from the base, always attaching new segments to the boundary of the reference domain, and mapping the existing curve conformally inwards. Instead of a single evolving curve, this yields a family of curves over time.

As a matter of taste, it is possible to define the radial version of SLE processes in the disk by lifting the disk to the half-plane under the map $z \mapsto -i \log z$. This makes certain formulas simpler, especially when tracking points on the boundary. We follow Rohde and Zhan [3] in this section.

Definition 3. *The backward Loewner equations are the forward Loewner equations flowed in the reverse direction, that is,*

$$\begin{aligned}
\partial_t f_t(z) &= -\frac{2}{f_t(z) - W_t}, & f_0(z) &= z, & z &\in \mathbb{H}, \\
\partial_t f_t(z) &= f_t(z)\,\frac{f_t(z) + W_t}{f_t(z) - W_t}, & g_0(z) &= z, & z &\in \mathbb{D}, \\
\partial_t \tilde{f}_t(z) &= -\cot\Big(\frac{\tilde{f}_t(z) - \tilde{W}_t}{2}\Big) & \tilde{f}_0(z) &= z, & z &\in \mathbb{H},
\end{aligned}$$

respectively being chordal, radial and radial lifted under the map $z \mapsto -i \log z$.

The backwards Loewner equations generate hulls L_t that are the complement of the image of f_t in the original domain; note that these hulls are *not* nested as they were for the hulls of the forward Loewner equation.

Definition 4. *The backward chordal SLE($\kappa; \vec{\rho}$) processes are those generated by the backward chordal Loewner equation driven by the solution to the SDE*

$$dW_t = \sqrt{\kappa}\, dB_t - \sum_{i=1}^{m} \rho_i\, \Re \frac{1}{W_t - V_t^i}, \qquad dV_t^i = -\frac{2}{V_t^i - W_t}\, dt, \quad 1 \le i \le m.$$

The backward radial SLE($\kappa; \vec{\rho}$) processes (lifted by $z \mapsto -i \log z$) are generated by the backward lifted radial Loewner equation driven by the solution to the SDE

$$\begin{aligned}
d\tilde{W}_t &= \sqrt{\kappa}\, dB_t - \sum_{i=1}^{m} \frac{\rho_i}{2} \cot\Big(\frac{\tilde{W}_t - \tilde{V}_t^i}{2}\Big)\, dt, \\
d\tilde{V}_t^i &= -\cot\Big(\frac{\tilde{V}_t^i - \tilde{W}_t}{2}\Big)\, dt, \quad 1 \le i \le m.
\end{aligned}$$

Note the minus sign in front of the ρ_i weighted terms in addition to that in front of the usual Loewner vector field.

The existence of these processes can be deduced from the existence of backward chordal SLE(κ) via the Girsanov theorem.

The analogue to Schramm and Wilson's theorem for backward SLE is as follows.

Theorem 5. *Let L_t by the backward chordal SLE($\kappa;\vec{\rho}$) hulls started from $(W_0, V_0^1, \dots, V_0^m)$. Suppose that $\rho_i \in \mathbb{R}$ with $\sum_i \rho_i = -\kappa - 6$. Let ϕ be a Möbius transformation of the upper half-plane and suppose that $\infty, \phi^{-1}(\infty) \in \{V_0^1, \dots, V_0^m\}$. Then, after a time-change, $\phi^*(L_t)$ are the backward chordal SLE($\kappa;\vec{\rho}$) hulls started from $(\phi(W_0), \phi(V_0^1), \dots, \phi(V_0^m))$.*

In this theorem, the map ϕ^* sends a hull K to $\phi^K(K)$, where ϕ^K is the unique $\mathbb{R}$-symmetric conformal map defined in domain$(\phi)^K$ such that $V_K = \phi$. See [3], Section 2 for definitions of these objects. Once they are understood, the proof proceeds via Itō's lemma as in the forward case.

References

[1] Lawler, G., Schramm, O. and Werner, W., Conformal restriction: the chordal case, J. Amer. Math. Soc. **16** (2003), no. 4, 917–955

[2] Schramm, O. and Wilson, D. B., *SLE coordinate changes*, New York J. Math. **11** (2005), 659–669.

[3] Rohde, S. and Zhan, D., *Backward SLE and the symmetry of the welding*, `arXiv:1307.2532`.

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5 Random curves, scaling limits and Loewner evolutions

after A. Kemppainen and S. Smirnov [5]
A summary written by Alexander Glazman

Abstract

It is shown that a weak estimate on the probability of an annulus crossing implies that a random curve arising from a statistical mechanics model will have scaling limits and those will be well-described by Loewner evolutions with random driving forces. Interestingly, the proofs indicate that existence of a nondegenerate observable with a conformally-invariant scaling limit seems sufficient to deduce the required condition.

The paper serves as an important step in establishing the convergence of Ising and FK Ising interfaces to SLE curves, moreover, the setup is adapted to branching interface trees.

5.1 Introduction

The paper is concerned with sequences of random planar curves and different conditions sufficient to establish their precompactness.

Typically, the random curves we want to consider connect two boundary points $a, b \in \partial U$ in a simply connected domain U. While we work with different domains U, we still prefer to restate our conclusions for a fixed domain. Thus we encode the domain U and the curve end points $a, b \in \partial U$ by a conformal transformation ϕ from U onto the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The domain $U = U(\phi)$ is then the domain of definition of ϕ and the points a and b are preimages $\phi^{-1}(-1)$ and $\phi^{-1}(1)$, respectively.

Because of the above reasons the first fundamental object in our study is a *pair* $(\phi, \mathbb{P})$ where ϕ is a *conformal map* and $\mathbb{P}$ is a *probability measure on curves* with the following restrictions: Given ϕ we define the domain $U = U(\phi)$ to be the domain of definition of ϕ and we require that ϕ is a conformal map from U onto the unit disc $\mathbb{D}$. Therefore U is a simply connected domain other than $\mathbb{C}$. We require also that $\mathbb{P}$ is supported on (a closed subset of)

$$\left\{ \gamma \in X_{\text{simple}}(U) \; : \; \begin{matrix} \text{the beginning and end point of} \\ \phi(\gamma) \text{ are } -1 \text{ and } +1, \text{ respectively} \end{matrix} \right\}, \tag{1}$$

where $X_{\text{simple}}(U)$ denotes the set of Jordan curves $\gamma : [0,1] \to \bar{U}$ such that $\gamma(0,1) \subset U$. The second fundamental object in our study is *some collection* Σ of pairs $(\phi, \mathbb{P})$ satisfying the above restrictions.

We uniformize by a disk $\mathbb{D}$ to work with a bounded domain. As it is shown in the paper, our conditions are conformally invariant, so the choice of a particular uniformization domain is not important.

For any $0 < r < R$ and any point $z_0 \in \mathbb{C}$, denote the annulus of radii r and R centered at z_0 by $A(z_0, r, R)$:

$$A(z_0, r, R) = \{z \in \mathbb{C} \;:\; r < |z - z_0| < R\}. \tag{2}$$

The following definition makes speaking about crossing of annuli precise.

Definition 1. *A curve γ is said to* make a crossing *of the annulus $A = A(z_0, r, R)$ if for some T_0 and T_1 both $\gamma(T_0)$ and $\gamma(T_1)$ lie outside A and they are in the different components of $\mathbb{C} \setminus A$.*

For given domain U and for given simple (random) curve γ on U, we always define $U_\tau = U \setminus \gamma[0, \tau]$ for each (random) time τ. We call U_τ as the domain at time τ.

Definition 2. *For a fixed domain (U, a, b) and for fixed simple (random) curve in U starting from a, define for any annulus $A = A(z_0, r, R)$ and for any (random) time $\tau \in [0,1]$, $A^u_\tau = \emptyset$ if $\partial B(z_0, r) \cap \partial U_\tau = \emptyset$ and*

$$A^u_\tau = \left\{ z \in U_\tau \cap A \;:\; \begin{array}{c} \textit{the connected component of } z \textit{ in } U_\tau \cap A \\ \textit{doesn't disconnect } \gamma(\tau) \textit{ from } b \textit{ in } U_\tau \end{array} \right\} \tag{3}$$

otherwise.

5.2 Main theorem

The main theorem is proven under a specific condition on the crossing probability. Four different ways to state this condition are mentioned in the paper — 2 geometric and 2 conformal. It is proven in the paper that they are equivalent. We will use one of the geometric conditions to state the main theorem.

Condition G2. The family Σ is said to satisfy a *geometric bound on an unforced crossing* if there exists $C > 1$ such that

$$\mathbb{P}\big(\gamma[\tau, 1] \text{ makes a crossing of } A \text{ which is contained in } A^u_\tau \,\big|\, \gamma[0,\tau]\big) < \frac{1}{2}. \tag{4}$$

for any $(\phi, \mathbb{P}) \in \Sigma$, for any stopping time $0 \leq \tau \leq 1$ and for any annulus $A = A(z_0, r, R)$ where $0 < C\, r \leq R$.

Denote by $\phi\mathbb{P}$ the pushforward of $\mathbb{P}$ by ϕ defined by

$$(\phi\mathbb{P})(A) = \mathbb{P}(\phi^{-1}(A)) \tag{5}$$

for any measurable $A \subset X_{\text{simple}}(\mathbb{D})$. In other words $\phi\mathbb{P}$ is the law of the random curve $\phi(\gamma)$. Given a family Σ as above, define the family of pushforward measures

$$\Sigma_{\mathbb{D}} = \{\phi\mathbb{P} \ : \ (\phi, \mathbb{P}) \in \Sigma\}\,. \tag{6}$$

The family $\Sigma_{\mathbb{D}}$ consist of measures on the curves $X_{\text{simple}}(\mathbb{D})$ connecting -1 to 1.

Fix a conformal map

$$\Phi(z) = i\frac{z+1}{1-z} \tag{7}$$

which takes $\mathbb{D}$ onto the *upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} \,:\, \mathrm{Im} z > 0\}$. Note that if γ is distributed according to $\mathbb{P} \in \Sigma_{\mathbb{D}}$, then $\tilde{\gamma} = \Phi(\gamma)$ is a simple curve in the upper half-plane slightly extending the definition of $X_{\text{simple}}(\mathbb{H})$, namely, $\tilde{\gamma}$ is simple with $\tilde{\gamma}(0) = 0 \in \mathbb{R}$, $\tilde{\gamma}((0,1)) \subset \mathbb{H}$ and $|\gamma(t)| \to \infty$ as $t \to 1$. Therefore, if $\tilde{\gamma}$ is parametrized with the half-plane capacity, then it has a continuous driving term $W = W_\gamma : \mathbb{R}_+ \to \mathbb{R}$. As a convention the driving term or process of a curve or a random curve in $\mathbb{D}$ means the driving term or process in $\mathbb{H}$ after the transformation Φ and using the half-plane capacity parametrization.

The following theorem is the main result of this paper.

Theorem 1.3. *If the family Σ of probability measures satisfies Condition G2, then the family $\Sigma_{\mathbb{D}}$ is tight and therefore relatively compact in the topology of the weak convergence of probability measures on $(X, \mathcal{B}_X)$. Furthermore if $\mathbb{P}_n \in \Sigma_{\mathbb{D}}$ is converging weakly and the limit is denoted by $\mathbb{P}^*$ then the following statements hold $\mathbb{P}^*$ almost surely*

1. *the point 1 is not a double point, i.e., $\gamma(t) = 1$ only if $t = 1$,*

2. *the tip $\gamma(t)$ of the curve lies on the boundary of the connected component of $\mathbb{D} \setminus \gamma[0,t]$ containing 1 (having the point 1 on its boundary), for all t,*

3. *if $\hat{K}_t$ is the hull of $\Phi(\gamma[0,t])$, then the capacity $\mathrm{cap}_{\mathbb{H}}(\hat{K}_t) \to \infty$ as $t \to 1$*

4. for any parametrization of γ the capacity $t \mapsto \mathrm{cap}_{\mathbb{H}}(\hat{K}_t)$ is strictly increasing and if $(K_t)_{t\in\mathbb{R}_+}$ is $(\hat{K}_t)_{t\in[0,1)}$ reparametrized with capacity, then the corresponding g_t satisfies the Loewner equation with a driving process $(W_t)_{t\in\mathbb{R}_+}$ which is Hölder continuous for any exponent $\alpha < 1/2$.

Furthermore, there exists $\varepsilon > 0$ such that for any t, $\mathbb{E}^*[\exp(\varepsilon|W_t|/\sqrt{t})] < \infty$.

5.3 Corollaries

The first corollary clarifies the relation between the convergence of random curves and the convergence of their driving processes. For instance, it shows that if the driving processes of Loewner chains satisfying Condition G2 converge, also the limiting Loewner chain is generated by a curve.

Corollary 1. *Suppose that $(W^{(n)})_{n\in\mathbb{N}}$ is a sequence of driving processes of random Loewner chains that are generated by simple random curves $(\gamma^{(n)})_{n\in\mathbb{N}}$ in $\mathbb{H}$ satisfying Condition G2. If $(\gamma^{(n)})_{n\in\mathbb{N}}$ are parametrized by capacity, then the sequence of pairs $(\gamma^{(n)}, W^{(n)})_{n\in\mathbb{N}}$ is tight in the topology of uniform convergence on the compact intervals of $\mathbb{R}_+$ in the capacity parametrization. Furthermore, if either $(\gamma^{(n)})_{n\in\mathbb{N}}$ or $(W^{(n)})_{n\in\mathbb{N}}$ converges (weakly), also the other one converges and the limits agree in the sense that $\gamma = \lim_n \gamma_n$ is driven by $W = \lim_n W_n$.*

For the next corollary let's define the space of *open curves* by identifying in the set of continuous maps $\gamma : (0,1) \to \mathbb{C}$ different parametrizations. The topology will be given by the convergence on the compact subsets of $(0,1)$.

We say that (U_n, a_n, b_n), $n \in \mathbb{N}$, converges to (U, a, b) in the *Carathéodory sense* if there exists conformal and onto mappings $\psi_n : \mathbb{D} \to U_n$ and $\psi : \mathbb{D} \to U$ such that they satisfy $\psi_n(-1) = a_n$, $\psi_n(+1) = b_n$, $\psi(-1) = a$ and $\psi(+1) = b$ (possibly defined as prime ends) and such that ψ_n converges to ψ uniformly in the compact subsets of $\mathbb{D}$ as $n \to \infty$.

The next corollary shows that if we have a converging sequence of random curves in the sense of Theorem 1.3 and if they are supported on domains which converge in the Carathéodory sense, then the limiting random curve is supported on the limiting domain. Note that the Carathéodory kernel convergence allows that there are deep fjords in U_n which are "cut off" as $n \to \infty$. One can interpret the following corollary to state that with high probability the random curves don't enter any of these fjords. This is a desired property of the convergence.

Corollary 2. *Suppose that (U_n, a_n, b_n) converges to (U^*, a^*, b^*) in the Carathéodory sense (here a^*, b^* are possibly defined as prime ends) and suppose that $(\phi_n)_{n\geq 0}$ are conformal maps such that $U_n = U(\phi_n), a_n = a(\phi_n), b_n = b(\phi_n)$ and $\lim \phi_n = \phi^*$ for which $U^* = U(\phi^*), a^* = a(\phi^*), b = b(\phi^*)$. Let $\hat{U} = U^* \setminus (V_a \cup V_b)$ where V_a and V_b are some neighborhoods of a and b, respectively, and set $\hat{U}_n = \phi_n^{-1} \circ \phi(\hat{U})$. If $(\phi_n, \mathbb{P}_n)_{n\geq 0}$ satisfy Condition G2 and γ^n has the law $\mathbb{P}_n$, then γ^n restricted to $\hat{U}_n$ has a weakly converging subsequence in the topology of X, the laws for different $\hat{U}$ are consistent so that it is possible to define a random curve γ on the open interval $(0,1)$ such that the limit for γ^n restricted to $\hat{U}_n$ is γ restricted to the closure of $\hat{U}$. Especially, almost surely the limit of γ_n is supported on open curves of U^* and don't enter $(\limsup \overline{U_n}) \setminus \overline{U}^*$.*

5.4 Proof of the main theorem

There are essentially two parts in the main theorem — tightness of a family of measures satisfying Condition G2, and properties of a weak limit of a converging sequence of measures chosen in this family.

The first part follows from [1]. In [1] the tightness of a family of measures is obtained under a similar assumption — we call it Condition G4. Thus, one just needs to prove that Condition G4 follows from Condition G2. This is done in Propostion 3.5.

As a general strategy in the proof of the properties of the limitting measure, we find an increasing sequence of events $E_n \subset X_{\text{simple}}(\mathbb{D})$ such that

$$\lim_{n\to\infty} \inf_{\mathbb{P}\in\Sigma_{\mathbb{D}}} \mathbb{P}(E_n) = 1$$

and the curves in E_n have some good properties which among other things guarantee that the closure of E_n is contained in the class of Loewner chains. The main tool is Lemma A.5 from the appendix.

Lemma A.5. *Let $T > 0$ and for each $n \in \mathbb{N}$, let $\gamma_n : [0,T] \to \mathbb{C}$ be injective continuous function such that $\gamma_n(0) \in \mathbb{R}$ and $\gamma_n(0,T] \subset \mathbb{H}$. Suppose that*

1. *$\gamma_n \to \gamma$ uniformly on $[0,T]$ and γ is not constant on any subinterval of $[0,T]$*
2. *$W_n \to W$ uniformly on $[0, \upsilon(T)]$.*

3. $F_n \to F$ *uniformly on* $[0,T] \times [0,1]$*, where*

$$F_n(t,y) = g^{-1}_{\gamma_n[0,t]}\big(W_n(\upsilon_n(t)) + iy \big). \tag{8}$$

Then $t \mapsto \nu$ *is strictly increasing and* $g_t := g_{\gamma \circ \upsilon^{-1}[0,t]}$ *satisfies the Loewner equation with the driving term* W*. Furthermore, the sequence of mappings* $(t,z) \mapsto g_{\gamma_n \circ \upsilon_n^{-1}[0,t]}$ *converges to* g_t *uniformly on*

$$S_K(T,\delta) = \{(t,z) \in [0,T] \times \overline{\mathbb{H}} \,:\, \mathrm{dist(z, K_t)} \geq \delta\} \tag{9}$$

for any $\delta > 0$*. Here* K_t *is the hull of* $\gamma[0,t]$*.*

Thus, it is enough to find an increasing sequence of events $E_n \subset X_{\mathrm{simple}}(\mathbb{D})$ mentioned above that satisfies all three conditions in this lemma — then a weak limit of $\{\mathbb{P}_k\}$ will be"almost supported" on $\bigcup \bar{E}_n$, and Lemma A.5 shows that $\bigcup \bar{E}_n$ satisfies all the desired properties.

An E_n will be found as the intersection of several events, each of them having a measure almost 1 according to any $\mathbb{P}_k$ and ensuring one of the conditions of Lemma A.5 being satisfied. The question why these events can be chosen is addressed in Proposition 3.8, Proposition 3.5, Theorem 3.9 and Theorem 3.10.

A tool which makes many of the proofs easier is the fact that we can use always the most suitable form of the equivalent conditions. In particular, if Condition G2 can be verified in the original domain then Condition G2 (or any equivalent condition) holds in any reference domain where we choose to map the random curve as long as the map is conformal. Furthermore, Condition G2 holds after we observe the curve up to a fixed time or a random time and then erase the observed initial part by conformally mapping the complement back to reference domain.

5.5 Interfaces in statistical physics and Condition G2

In this section, we prove (or in some cases survey the proof) that the interfaces in the following models satisfy Condition G2:

- Fortuin–Kasteleyn model with the parameter value $q = 2$, a.k.a. FK Ising, at criticality on the square lattice or on a isoradial graph

- Site percolation at criticality on the triangular lattice

- Harmonic explorer on the hexagonal lattice
- Loop-erased random walk on the square lattice.

It turns out that Condition G2 fails for uniform spanning tree (UST) but it is possible to extend this method to the case of UST Peano curve. This is to appear in [4].

5.5.1 FK Ising

Suppose that $G = (V(G), E(G))$ is a finite graph, which is allowed to be a multigraph, that is, more than one edge can connect a pair of vertices. For any $q > 0$ and $p \in (0,1)$, define a probability measure on $\{0,1\}^{E(G)}$ by

$$\mu_G^{p,q}(\omega) = \frac{1}{Z_G^{q,p}} \left(\frac{p}{1-p} \right)^{|\omega|} q^{k(\omega)} \tag{10}$$

where $|\omega| = \sum_{e \in E(G)} \omega(e)$, $k(\omega)$ is the number of connected components in the graph $(V(G), \omega)$ and $Z_G^{p,q}$ is the normalizing constant making the measure a probability measure. This random edge configuration is called the *Fortuin-Kasteleyn model* (FK) or the *random cluster model*.

A fundamental property of the FK models is the FKG inequality which holds for $q \geq 1$ — it says that two increasing events are positively associated. It is well known that the FK model with parameter q is connected to the Potts model with parameter q. Here we are interested in the model connected to the Ising model and hence we mainly focus to the case $q = 2$ which is called *FK Ising* (model).

Denote $O(U)$ the event that there is a open crossing of a 4-admissible domain U. The following proposition was proved in [3] and it is the main ingredient used to prove that FK Ising satisfies Condition G2.

Proposition 4.5. *Let $U_n = h_n \hat{U}_n$ be a sequence domain such that the sequence of reals $h_n \searrow 0$ and $\hat{U}_n$ is a sequence of 4-admissible domains. If the sequence U_n converges to a quadrilateral (U, a, b, c, d) in the Carathéodory sense as $n \to \infty$, then $\mathbb{P}_n[O(\hat{U}_n)]$ converges to a value $s \in [0,1]$. If (U, a, b, c, d) is non-degenerate then $0 < s < 1$. Here $\mathbb{P}_n$ is the probability measure $\mu_{\hat{U}_n,P}^{p_{\mathrm{sd}},2}$ where P is a fixed partitioning of the set $\{1,2\}$.*

In [2] it was derived using the main theorem of [5] that the interface in FK Ising model converges to $\mathrm{SLE}_{16/3}$.

5.5.2 Percolation

The percolation measure on the whole triangular lattice with a parameter $p \in [0,1]$ is the probability measure $\mu_{\mathbb{T}}^p$ on $\{\text{open}, \text{closed}\}^{\mathbb{T}}$ such that independently each vertex is open with probability p and closed with probability $1-p$. The independence property of the percolation measure gives a consistent way to define the measure on any subset of $\mathbb{T}$ by restricting the measure to that set. The well-known critical value of p is $p_c = 1/2$.

The proof of the fact that the collection $(\mathbb{P}_U : U \text{ admissible})$ satisfies Condition G2 can be easily derived from Russo–Seymour–Welsh theory (RSW) which proves the existence of $q > 0$ such that for any n

$$\mu_{\mathbb{T}}^{p_c}(\exists \text{ open path inside } A(0;n,3n) \text{ separating } 0 \text{ from } \infty) \geq q. \tag{11}$$

5.5.3 Harmonic explorer

The result that the harmonic explorer satisfies Condition G2 appears already in [7]. In [5] just a survey of that proof is given.

5.5.4 Chordal loop-erased random walk

The loop-erased random walk is one of the random curves proved to be conformally invariant. In [6], the radial loop-erased random walk between an interior point and a boundary point was considered. In [5] the chordal loop-erased random walk between two boundary points is treated. Condition G2 is slightly harder to verify in this case. Namely, the natural extension of Condition G2 to the radial case can be verified in the same way, except that Proposition 4.11 is not necessary, and it is done in [6].

Let $(X_t)_{t=0,1,\ldots}$ be a simple random walk (SRW) on the lattice $\mathbb{Z}^2$ and P_x its law so that $P_x(X_0 = x) = 1$. Consider a bounded, simply connected domain $U \subset \mathbb{C}$ whose boundary ∂U is a path in $\mathbb{Z}^2$. Call the corresponding graph G, i.e., G consists of vertices $\overline{U} \cap \mathbb{Z}^2$ and the edges which stay in U (except that the end points may be in ∂U). Let V be the set of vertices and $\partial V := V \cap \partial U$. When $X_0 = x \in \partial V$ condition SRW on $X_1 \in U$. For any $X_0 = x \in V$ define T to be the hitting time of the boundary, i.e., $T = \inf\{t \geq 1 : X_t \in \partial V\}$.

For $a \in V$ and $b \in \partial V$ define $P_{a \to b} = P_{a \to b}^U$ to be the law of $(X_t)_{t=0,1,2,\ldots,T}$ with $X_0 = a$ conditioned on $X_T = b$. If $(X_t)_{t=0,1,2,\ldots,T}$ distributed according to $P_{a \to b}^U$ then the process $(Y_t)_{t=0,1,2,\ldots,T'}$, which is obtained from (X_t) by erasing all loops in chronological order, is called *loop-erased random walk* (LERW) from a to b in U. Denote its law by $\mathbb{P}^{U,a,b}$.

It is shown in [5] that the collection $\{\mathbb{P}^{U,a,b} : (U,a,b)\}$ of chordal LERWs satisfies Condition C2 (one of conformal reformulations of Condition G2), where U runs over all simply connected domains as above and $\{a,b\} \subset \partial U$.

References

[1] M. Aizenman and A. Burchard. *Hölder regularity and dimension bounds for random curves. Duke Math. J.*, 99(3):419–453, 1999.

[2] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, and S. Smirnov. *Convergence of Ising interfaces to Schramm's SLEs. Preprint, 2013 arXiv:1312.0533v2*

[3] D. Chelkak and S. Smirnov. *Universality in the 2D Ising model and conformal invariance of fermionic observables. Invent. Math.*, 189(3):515–580, 2012.

[4] A. Kemppainen. *On convergence of uniform spanning tree interface to* SLE_8 *In preparation, 2013.*

[5] A. Kemppainen and S. Smirnov. *Random curves, scaling limits and Loewner evolutions. Preprint, 2012 arXiv:1212.6215v2*

[6] G. F. Lawler, O. Schramm, and W. Werner. *Conformal invariance of planar loop-erased random walks and uniform spanning trees. Ann. Probab.*, 32(1B):939–995, 2004.

[7] O. Schramm and S. Sheffield. *Harmonic explorer and its convergence to* SLE_4. *Ann. Probab.*, 33(6):2127–2148, 2005.

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6 Basic properties of SLE

after S. Rohde and O. Schramm [3]
A summary written by Jianping Jiang

Abstract

We give a short summary of the basic properties of SLE in the seminal paper by Rohde and Schramm[3]. We discuss the existence of SLE paths, phases for SLE and dimension of the SLE paths.

6.1 Introduction

Let B_t be a standard Brownian motion on $\mathbb{R}$ with $B_0 = 0$. For $\kappa \geq 0$ let $\xi(t) = \sqrt{\kappa} B_t$ and for each $z \in \overline{\mathbb{H}} \setminus \{0\}$ let $g_t(z)$ be the solution of the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z. \tag{1}$$

Let $\tau(z) := \inf\{t \geq 0 : g_t(z) - \xi(t) = 0\}$, and set

$$K_t := \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}.$$

The parameterized collection of maps $\{g_t : t \geq 0\}$ is called *chordal* SLE_κ. The sets K_t are the *hulls* of the SLE. It is easy to verify that for any $t \geq 0$ the map $g_t : H_t \to \mathbb{H}$ is conformal where H_t is the unbounded component of $\mathbb{H} \setminus K_t$.

Note that there are other versions of SLE. For example, the radial SLE_κ, whole-plane SLE_κ and more generally SLE_κ in doubly connected domains (see [4]).

In this paper, we give a short summary of the basic properties of SLE. We will follow the seminal paper by Rhode and Schramm [3]. We will focus ourselves on the chordal SLE_κ, but those properties are also valid for other versions of SLE_κ we mentioned above.

Let $f_t(z) := g_t^{-1}(z)$ and $\hat{f}_t(z) := f_t(z+\xi(t))$. The *trace* γ of chordal SLE_κ is defined by

$$\gamma(t) := \lim_{z \to 0, z \in \mathbb{H}} \hat{f}_t(z).$$

If the limit does not exist, let $\gamma(t)$ denote the set of all limit points.

The first property we will discuss is the existence of the trace γ and moreover $\gamma(t)$ is a continuous function of t.

Theorem 1. *For every $t \geq 0$, the limit*

$$\gamma(t) := \lim_{z \to 0, z \in \mathbb{H}} \hat{f}_t(z) \tag{2}$$

exists, $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a continuous path, and H_t is the unbounded component of $\mathbb{H} \setminus \gamma([0, \infty))$.

Remark 2. *Rohde and Schramm proved the above theorem for $\kappa \neq 8$, the case when $\kappa = 8$ is proved in [2].*

The second property is about the phases of SLE_κ.

Theorem 3. 1. *If $0 \leq \kappa \leq 4$, the SLE_κ trace γ is a.s. a simple path and $\gamma[0, \infty) \subseteq \mathbb{H} \cup \{0\}$.*

2. *If $4 < \kappa < 8$, then $\cup_{t>0} K_t = \overline{\mathbb{H}}$ but $\gamma([0, \infty)) \cap \mathbb{H} \neq \mathbb{H}$.*

3. *If $\kappa \geq 8$, then γ is a space filling curve, i.e., $\gamma([0, \infty)) = \overline{\mathbb{H}}$.*

The third property is about the dimension of the SLE_κ trace.

Theorem 4. *The Hausdorff dimension of the SLE_κ trace is a.s* $\min\{1 + \kappa/8, 2\}$.

Remark 5. *Rohde and Schramm proved the upper bound of the Hausdorff dimension of SLE_κ trace is* $\min\{1 + \kappa/8, 2\}$. *The lower bound is proved by Beffara [1].*

6.2 Existence of the curve

In this section, we will sketch the proof of Theorem 1. Our goal is to prove the existence and continuous in t of the limit in (2). The follow lemma says that it is enough to prove the limit exists along one particular direction in the complex plane.

Lemma 6. *Let g_t be the corresponding solution of* (1). *Suppose $\beta(t) := \lim_{y \downarrow 0} \hat{f}_t(iy)$ exists for all $t \in [0, \infty)$ and is continuous, then Theorem 1 holds.*

The proof of Lemma 6 is not hard, so we skip the proof. So our main concern is to prove $\beta(t) = \lim_{y \downarrow 0} \hat{f}_t(iy)$ exists for all $t \in [0, \infty)$ and is continuous, and this follows form the following theorem.

Theorem 7. *Define $H(y,t) := \hat{f}_t(iy)$ where $y > 0, t \geq 0$. If $\kappa \neq 8$, then a.s. $H(y,t)$ extends continuously to $[0,\infty) \times [0,\infty)$.*

By the scale-invariance of chordal SLE, i.e., $(t,z) \mapsto \alpha^{-1/2} g_{\alpha t}(\sqrt{\alpha} z)$ has the distribution as the process $(t,z) \mapsto g_t(z)$ for any $\alpha > 0$, it is enough to show that $H(y,t)$ extends continuously to $[0,1] \times [0,1]$. We need some upper bounds for $|\hat{f}_t'|$.

Lemma 8. *Let $b \in [0, 1+\kappa/4)$, $a := 2b + \kappa b(1-b)/2$ and $\lambda := 4b + \kappa b(1-2b)/2$. There is a constant $C(\kappa, b)$, depending only on κ and b, such that the following estimate holds for all $t \in [0,1], y, \delta \in (0,1]$ and $x \in \mathbb{R}$.*

$$P(|\hat{f}_t'(x+iy)| \geq \delta y^{-1}) \leq C(\kappa, b)(1 + x^2/y^2)^b (y/\delta)^\lambda \vartheta(\delta, a-\lambda),$$

where

$$\vartheta(\delta, a-\lambda) = \begin{cases} \delta^{\lambda-a}, & a-\lambda > 0 \\ 1 + |\ln \delta|, & a-\lambda = 0 \\ 1, & a-\lambda < 0. \end{cases}$$

The proof of this lemma will be delayed until we finish the proof of Theorem 7.

Proof of Theorem 7. Given $j, k \in \mathbb{N} \cup \{0\}$ with $k < 2^{2j}$, let $R(j,k)$ be the rectangle

$$R(j,k) := [2^{-j-1}, 2^{-j}] \times [k2^{-2j}, (k+1)2^{-2j}],$$

and set

$$d(j,k) := \text{diam} H(R(j,k)).$$

One can show that

$$d(j,k) \sim 2^{-j} N \max\{|\hat{f}_{\hat{t}_n}'(i2^{-j})| : n = 0, 1, \cdots, N\},$$

where N is an integer valued random variable satisfying $P(N > m) \leq \rho^m$ for some $0 < \rho < 1$ and $\hat{t}_i \in [k2^{-2j}, (k+1)2^{-2j}]$. Then Lemma 8 implies for some $0 < \sigma < (\lambda - 2)/\max\{a, \lambda\}$

$$P(d(j,k) > 2^{-j\sigma}) \leq O(1) 2^{-2j} 2^{-\varepsilon j}, \tag{3}$$

where $\varepsilon = \varepsilon(\kappa) > 0$. Note that (3) fails for $\kappa = 8$ since the maximum of λ is achieved when $b = (8+\kappa)/(4\kappa)$, and the maximum is $(8+\kappa)^2/(16\kappa)$ which is greater than 2 if $\kappa \neq 8$ but equals 2 if $\kappa = 8$. (3) implies

$$\sum_{j=0}^{\infty} \sum_{k=0}^{2^{2j}-1} P(d(j,k) > 2^{-j\sigma}) < \infty.$$

So Borel-Cantelli lemma gives $d(j,k) \leq C(w)2^{-j\sigma}$, which completes the proof of the theorem. □

The last piece of the proof of Theorem 1 is the proof of Lemma 8. We need some new notations. For any $z \in \mathbb{H}$ and $u \in \mathbb{R}$, define the random time change

$$T_u(z) := \sup\{t \in \mathbb{R} : \operatorname{Im}(g_t(z)) \geq e^u\}.$$

Let $\hat{z} = \hat{x} + i\hat{y}$ be fixed, $z(u) := g_{T_u(\hat{z})}(\hat{z}) - \xi(T_u(\hat{z})) := x(u) + iy(u)$ and

$$\psi(u) := \frac{\hat{y}}{y(u)} |g'_{T_u(\hat{z})}(\hat{z})|.$$

Theorem 9. *Assume $\hat{y} \neq 1$ and set $\nu := -sign(\ln \hat{y})$. Let $b \in \mathbb{R}$. Define a and λ by*

$$a := 2b + \nu\kappa b(1-b)/2, \lambda := 4b + \nu\kappa b(1-2b)/2.$$

Then

$$\hat{y}^a E[(1 + x(0)^2)^b |g'_{T_0(\hat{z})}(\hat{z})|^a] = (1 + (\hat{x}/\hat{y})^2)^b \hat{y}^\lambda := F(\hat{z}).$$

Proof. It is easy to see $du = -2|z(u)|^2 dt$. So

$$\hat{B}(u) := -\sqrt{2/\kappa} \int_{t=0}^{T_u} |z(u)|^{-1} d\xi$$

is a Brownian motion (with respect to u). Set $M(u) := \psi(u)^a F(z(u))$. Then Ito's formula gives

$$dM(u) = \sqrt{2\kappa} M(u) \frac{bx}{\sqrt{x^2 + y^2}} d\hat{B}.$$

Thus M is a local martingale, some extra work can show that M is a martingale, and thus

$$\psi(\ln \hat{y})^a F(z(\ln \hat{y})) = E[\psi(0)^a F(z(0))],$$

which is exactly what we need to show. □

Proof of Lemma 8. The key idea for the proof is for $u_1 := \ln \operatorname{Im}(g_{-t}(x+iy))$, we have

$$\left| \frac{g'_{-t}(z)}{g'_{T_0(z)}(z)} \right| \leq e^{|u_1|},$$

which follows from $\left|\partial_u \ln |g'_t(z)|\right| \leq 1$. Note that u_1 is bounded for any $0 \leq t \leq 1, 0 < y \leq 1$. The rest of the proof using Theorem 7 and Chebyshev's inequality. □

6.3 Phases

Set $Y_z(t) := \frac{g_t(z)-\xi(t)}{\sqrt{\kappa}}$ for $z \in \overline{\mathbb{H}}$ and $t \geq 0$. Then

$$dY_z(t) = \frac{2/\kappa}{Y_z(t)} dt + dB_t, \tag{4}$$

where $B_t = -\xi(t)/\sqrt{\kappa}$ is a standard Brownian motion. When $z \in \mathbb{R}$, (4) is a Bessel process. Theorem 3 follows from the basic theory of Bessel process, .i.e,

Theorem 10. *1. If $\kappa \leq 4$, then w.p.1 $\tau_x = \infty$ for all $x > 0$.*

2. If $\kappa > 4$, then w.p.1 $\tau_x < \infty$ for all $x > 0$.

3. If $\kappa \geq 8$, then w.p.1 $\tau_x < \tau_y$ for all $0 < x < y$.

4. If $4 < \kappa < 8$ and $0 < x < y$, then $P(\tau_x = \tau_y) > 0$.

We refer the reader to the original paper [3] or Zinsmeister's lecture notes [5] for the detailed proof.

References

[1] Beffara, V., *The dimension of the SLE curves.* Ann. Probab. **36**(4) (2008), 1421-1452.

[2] Lawler, G., Schramm, O. and Werner, W., *Conformal invariance of planar loop-erased random walks and uniform spanning trees.* Ann. Probab. **32**(1B) (2004), 939-995.

[3] Rohde, S. and Schramm, O., *Basic properties of SLE.* Ann. Math. **161** (2005), 883-924.

[4] Zhan, D., *Stochastic Loewner evolution in doubly connected domains.* Probab. Theory Related Fields **129**(3) (2004), 340-380.

[5] Zinsmeister, M., *Stochastic L´owner Evolution.* http://www.univ-orleans.fr/MAPMO/membres/zins/SLE/saigon.pdf

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7 Loewner Curvature

after Joan Lind and Steffen Rohde
A summary written by Kyle Kinneberg

Abstract

We summarize the work of Lind and Rohde in [3] that develops a theory of curvature for sufficiently smooth curves in $\mathbb{H}$. This is done via comparison with self-similar curves in the Loewner framework. It is shown that appropriate upper bounds on the curvature guarantee that the curve is simple.

7.1 Introduction

The Loewner equation provides a canonical way to encode simple curves in the upper half-plane, $\mathbb{H}$, by a continuous 1-dimensional function. If $\gamma\colon [0,T) \to \overline{\mathbb{H}}$ is continuous with $\gamma(0) \in \mathbb{R}$ and $\gamma(0,T) \subset \mathbb{H}$, then after re-parameterization, the hydrodynamic conformal maps $g_t\colon \mathbb{H}\backslash\gamma(0,t] \to \mathbb{H}$ have the form $g_t(z) = z + \frac{2t}{z} + O(\frac{1}{z^2})$ as $z \to \infty$, and they solve the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z, \tag{1}$$

where $\lambda(t) = g_t(\gamma(t)) \in \mathbb{R}$ varies continuously in t.

This process can be reversed. If $\lambda(t)$ is continuous on $[0,T)$, then the solutions to (1) are hydrodynamic conformal maps $g_t\colon \mathbb{H}\backslash K_t \to \mathbb{H}$, where K_t is an increasing family of half-plane hulls. When is the family K_t generated by a simple curve, γ, in the sense that $K_t = \gamma(0,t]$ for all t?

Theorem 1 (Marshall–Rohde [4], Lind [2])**.** *If $||\lambda||_{1/2} < 4$, then λ generates a simple curve meeting $\mathbb{R}$ non-tangentially.*

Here, $||\lambda||_{1/2} = \sup_{s\neq t} |\lambda(s) - \lambda(t)|/\sqrt{|s-t|}$ is the Lip(1/2)-norm. Our goal is to approach the problem above by developing a notion of curvature for which appropriate bounds guarantee that the curve is simple.

We will need some basic properties for the chordal Loewner equation. Let $\lambda(t)$, $t \in [0,T)$ be continuous with corresponding hulls K_t. For $r > 0$, the family rK_{t/r^2} is generated by $r\lambda(t/r^2)$, with $t \in [0, r^2T)$; for $x \in \mathbb{R}$, the family

$K_t + x$ is generated by $\lambda(t) + x$, with $t \in [0, T)$; and for fixed $0 \leq t_0 < T$, the family $g_{t_0}(K_{t_0+t})$ is generated by $\lambda(t_0 + t)$, with $t \in [0, T - t_0)$. We will call these, respectively, the *scaling, translation, and concatenation properties.*

We will also make use of the following fact, proved by Earle and Epstein [1]. If $\gamma\colon (0,T) \to \mathbb{H}$ is a simple curve that is $C^n(0,T)$, then its half-plane capacity parametrization and the corresponding driving function are $C^{n-1}(0,\tau)$, where $\tau = \text{hcap}(\gamma)$.

The idea behind Loewner curvature is relatively straightforward. First, we use the Loewner framework to identify a family of "self-similar" curves in $\mathbb{H}$. By conformal mapping, we obtain a family of self-similar curves in any Jordan domain Ω. These will be the curves of constant curvature. For $\gamma\colon (0,T) \to \mathbb{H}$, we will then define $LC_\gamma(t)$ to be the curvature of the unique self-similar curve in $\mathbb{H}\backslash\gamma(0,t]$ that "best-fits" γ at $\gamma(t)$.

7.2 Identifying self-similar curves

Let $\Omega \subset \mathbb{C}$ be a Jordan domain with distinct boundary points $a, b \in \partial\Omega$. For $0 < T \leq \infty$, we use the notation $\gamma\colon (0,T) \to (\Omega, a, b)$ if $\gamma[0,T)$ is a simple curve with $\gamma(0,T) \subset \Omega$ and $\gamma(0) = a$. Notice that we *do not* assume that γ is parameterized by half-plane capacity. Also, the point b is irrelevant in this notation. However, it is very important in the following definition.

Definition 2. *We say $\gamma\colon (0,T) \to (\Omega, a, b)$ is self-similar if $\gamma \in C^3(0,T)$ and for each $t \in (0,T)$, there is a conformal map $\phi_t\colon \Omega\backslash\gamma(0,t] \to \Omega$, with $\phi_t(\gamma(t)) = a$ and $\phi_t(b) = b$, such that $\phi_t(\gamma(t,T)) \subset \gamma$.*

Note that sub-curves of self-similar curves are self-similar. We use $S(\Omega, a, b)$ to denote the collection of *maximal* self-similar curves in (Ω, a, b).

The triple (Ω, a, b) that one should have in mind is $(\mathbb{H}, 0, \infty)$. In order to identify the self-similar curves in (Ω, a, b) it suffices to find those in $(\mathbb{H}, 0, \infty)$ and then map them to Ω by a conformal map sending 0 to a and ∞ to b.

Proposition 3. *The curves in $S(\mathbb{H}, 0, \infty)$ are precisely those whose driving function, λ, is one of the following:*

$$0, \quad ct, \quad c\sqrt{\tau} - c\sqrt{\tau - t}, \quad c\sqrt{\tau + t} - c\sqrt{\tau}, \quad \textit{where } c \neq 0 \textit{ and } \tau > 0.$$

These curves are uniquely determined by the parameters $\lambda'(0)$ and $\lambda''(0)$.

Proof. We may assume that γ is parameterized by half-plane capacity, though it is now guaranteed only to be $C^2(0,T)$. Let $\lambda(t)$ be the driving term for γ, which is also $C^2(0,T)$, and let $g_t\colon \mathbb{H}\backslash\gamma(0,t] \to \mathbb{H}$ be the corresponding maps. Let $\phi_t\colon \mathbb{H}\backslash\gamma(0,t] \to \mathbb{H}$ come from self-similarity of γ. Then $[g_t - \lambda(t)] \circ \phi_t^{-1}$ is a conformal map of $\mathbb{H}$ fixing 0 and ∞, so $\phi_t = r_t[g_t - \lambda(t)]$ for some $r_t > 0$. By self-similarity,

$$r_t[g_t(\gamma(t,T)) - \lambda(t)] \subset \gamma. \tag{2}$$

The scaling, translation, and concatenation properties show that the driving term for the curve on the left is $r_t[\lambda(t + s/r_t^2) - \lambda(t)]$, so that

$$r_t[\lambda(t + s/r_t^2) - \lambda(t)] = \lambda(s), \quad \text{for } s \text{ small enough.}$$

The left-hand side is twice differentiable at $s = 0$, so the right-hand side is as well. Taking derivatives gives $\lambda'(0) = \lambda'(t)/r_t$ and $\lambda''(0) = \lambda''(t)/r_t^3$, which is valid for each $0 < t < T$. If $\lambda'(0) = 0$, then λ is constant and so is identically 0. Similarly, if $\lambda''(0) = 0$, then λ is linear and thus of the form ct. Otherwise, λ' and λ'' are non-zero, and λ satisfies

$$\frac{\lambda'(t)^3}{\lambda''(t)} = \frac{\lambda'(0)^3}{\lambda''(0)},$$

so that λ' is a solution to the equation $u'/u^3 = \text{const}$. Solving this and integrating shows that λ must be one of the other two listed functions.

Conversely, a straightforward computation shows that, for each of the four types of driving functions in the list, every $t \in (0,T)$ has a corresponding $r_t > 0$ such that $r_t[\lambda(t + s/r_t^2) - \lambda(t)] = \lambda(s)$ for small s. If $\gamma\colon (0,T) \to (\mathbb{H}, 0, \infty)$ is the curve generated by λ, the scaling, translation, and concatenation properties ensure that (2) holds. Thus, γ is self-similar. The maximal intervals on which these driving functions can be defined are $[0,\infty)$ for the functions 0, ct, and $c\sqrt{\tau + t} - c\sqrt{\tau}$; and $[0,\tau)$ for $c\sqrt{\tau} - c\sqrt{\tau - t}$. $\square$

7.3 Curve-fitting and Loewner curvature

Fix $\gamma\colon (0,T) \to (\Omega, a, b)$. We say that a curve $\gamma_* \in S(\Omega, a, b)$ is *best-fitting* to γ at a if, under the conformal transformation $\phi\colon (\Omega, a, b) \to (\mathbb{H}, 0, \infty)$, either the curves $\phi(\gamma)$ and $\phi(\gamma_*)$ agree, in their half-plane capacity parameterizations, up to 2nd order in t near 0, or both have $\lambda'(0) = 0 = \lambda_*'(0)$. If γ is smooth enough, best-fitting self-similar curves always exist and are unique.

Proposition 4. *If $\gamma\colon (0,T) \to (\Omega, a, b)$ is $C^3(0,T)$, then for each $0 < t < T$, there is a unique best-fitting curve $\gamma_* \in S(\Omega \backslash \gamma(0,t], \gamma(t), b)$ to $\gamma(t,T)$ at $\gamma(t)$.*

Proof. We can assume that $\gamma\colon (0,T) \to (\mathbb{H}, 0, \infty)$ is parameterized by half-plane capacity, and so $\gamma \in C^2(0,T)$. Let $\lambda \in C^2(0,T)$ be its driving term. As $g_t - \lambda(t)$ maps $(\mathbb{H}\backslash\gamma(0,t], \gamma(t), \infty)$ conformally to $(\mathbb{H}, 0, \infty)$, it suffices to find a best-fitting curve to $g_t(\gamma(t,T)) - \lambda(t)$ at 0. The driving function for this mapped curve is $\lambda(t+s) - \lambda(t)$ for $0 \le s < T - t$, which is twice differentiable at $s = 0$. Thus, it suffices in general to find a best-fitting curve for $\gamma\colon (0,T) \to (\mathbb{H}, 0, \infty)$ at 0, assuming that the corresponding driving term λ is twice differentiable at $t = 0$.

To this end, choose the (unique) driving term λ_* from the list in Proposition 3 that has $\lambda_*'(0) = \lambda'(0)$ and $\lambda_*''(0) = \lambda''(0)$ if $\lambda'(0) \neq 0$, and choose $\lambda_* \equiv 0$ otherwise. In the latter case, the curve γ_* generated by λ_* is, by definition, best-fitting to γ at 0. Thus, we may assume that $\lambda'(0) \neq 0$. Taylor's theorem then ensures that $|\lambda(t) - \lambda_*(t)| = o(t^2)$. Using [5, Theorem 3.3], we have

$$\sup_{0 \le t < \varepsilon} |\gamma(t) - \gamma_*(t)| \le C \sup_{0 \le t < \varepsilon} |\lambda(t) - \lambda_*(t)|$$

for ε small, so $|\gamma(t) - \gamma_*(t)| = o(t^2)$ as well.

One can also verify, by direct computation, that

$$\gamma_*(t) = 2i\sqrt{t} + at - i(a^2/8)t^{3/2} + bt^2 + o(t^2), \quad \text{for } t \text{ near } 0,$$

where $a = (2/3)\lambda_*'(0)$ and $b = (4/15)\lambda_*''(0) + (1/135)\lambda_*'(0)^3$. Thus, γ has the same expansion, with the same values of a and b, near $t = 0$. □

We can now define Loewner curvature. First, for the curves in $S(\mathbb{H}, 0, \infty)$: if γ has driving term 0, ct, $c\sqrt{\tau} - c\sqrt{\tau - t}$, or $c\sqrt{\tau + t} - c\sqrt{\tau}$ with $c \neq 0$ and $\tau > 0$, then $LC_\gamma \equiv 0$, ∞, $c^2/2$, $-c^2/2$ (respectively) is constant. For curves in $S(\Omega, a, b)$, we define LC_γ by conformal invariance. For all other curves, $LC_\gamma(t)$ is defined by comparison, as follows.

Definition 5. *If $\gamma\colon (0,T) \to (\Omega, a, b)$ is $C^3(0,T)$ and $0 < t < T$, then $LC_\gamma(t) = LC_{\gamma_*}$, where $\gamma_* \in S(\Omega\backslash\gamma(0,t], \gamma(t), b)$ is the best-fitting curve to $\gamma(t,T)$ at $\gamma(t)$.*

The proof of Proposition 4, along with a computation for the driving functions in Proposition 3, gives the following corollary.

Corollary 6. *If $\gamma\colon (0,T) \to (\mathbb{H}, 0, \infty)$ is $C^3(0,T)$ and is parameterized by half-plane capacity, then $LC_\gamma(t) = \lambda'(t)^3/\lambda''(t)$, where λ is the driving function for γ. Here, we declare $0/0$ to equal 0.*

7.4 Simple curves from LC bounds

Our goal in this section is to prove the following result.

Theorem 7. *Let $\gamma\colon (0,T) \to (\Omega, a, b)$ be $C^3(0,T)$. If $LC_\gamma(t) \le c < 8$ for all $0 < t < T$, then γ extends continuously to $[0,T]$ with $\gamma[0,T]$ a simple curve in $\Omega \cup \{a, b\}$.*

The bulk of the proof consists in the following proposition.

Proposition 8. *If $\gamma\colon (0,T) \to (\mathbb{H}, 0, \infty)$ is $C^3(0,T)$ and has $0 < LC_\gamma(t) \le c$, then the corresponding driving function λ has $||\lambda||_{1/2} \le \sqrt{2c}$.*

Proof. Let $0 < T' < T$. It suffices to show the desired bound on $[0, T']$. The bound on Loewner curvature implies that $0 < \lambda'(t)^3/\lambda''(t) \le c$. Let $\sigma = \lambda'$ so we have the inequality $0 < \sigma^3 \le c\sigma'$. A computation shows that the non-negative solutions to $\sigma^3 = c\sigma'$ are of the form $\sigma_{c,B}(t) = \sqrt{c/2}(B-t)^{-1/2}$ for some $B > 0$. We claim that $\sigma \le \sigma_{c,T'}$.

If not, there is $0 \le t_0 < T'$ such that $\sigma(t_0) > \sigma_{c,T'}(t_0)$. Along with

$$\frac{1}{2\sigma(t_0)^2} - \frac{1}{2\sigma(t)^2} = \int_{t_0}^{t} \frac{\sigma'}{\sigma^3} \ge \int_{t_0}^{t} \frac{1}{c} = \frac{1}{2\sigma_{c,T'}(t_0)^2} - \frac{1}{2\sigma_{c,T'}(t)^2},$$

this implies that $\sigma(t) > \sigma_{c,T'}(t)$ for all $t_0 \le t \le T'$. But $\sigma_{c,T'}(t) \to \infty$ as $t \nearrow T'$, so that $\sigma(t) \to \infty$ as well. This contradicts the fact that $\lambda \in C^1(0,T)$.

As $\sigma < \sigma_{c,T'}$, we have

$$\begin{aligned}\lambda(t) - \lambda(s) = \int_s^t \sigma(u)du &\le \int_s^t \sigma_{c,T'}(u)du = 2\sqrt{c/2}\cdot\left(\sqrt{T'-s} - \sqrt{T'-t}\right)\\ &\le \sqrt{2c}\cdot\sqrt{t-s}\end{aligned}$$

for all $0 \le s \le t \le T'$, as desired. □

Proof of Theorem 7. It suffices to treat $(\Omega, a, b) = (\mathbb{H}, 0, \infty)$. Let λ be the driving term for γ, so that $LC_\gamma(t) = \lambda'(t)^3/\lambda''(t)$.

As a first case, assume that there is $t_0 \in (0,T)$ for which $LC_\gamma(t_0) > 0$. We claim that $LC_\gamma(t) > 0$ for $t \in [t_0, T)$. Indeed, the fact that $LC_\gamma < \infty$

implies that $\lambda''(t) = 0$ only if $\lambda'(t) = 0$. This cannot happen on $[t_0, T)$, as λ' and λ'' have the same sign at t_0. Thus, we can apply Proposition 8 to the curve $g_{t_0}(\gamma(t_0, T))$ whose driving function, $t \mapsto \lambda(t + t_0)$, has Lip(1/2)-norm at most $\sqrt{2c} < 4$. This function therefore extends continuously to $[0, T - t_0]$, and Theorem 1 ensures that $g_{t_0}(\gamma(t_0, T])$ is simple. Consequently, γ extends continuously to $[0, T]$ and is simple.

In the other case, $LC_\gamma(t) = \lambda'(t)^3/\lambda''(t) \leq 0$ for all t. Consequently, λ' and λ'' always have opposite sign, or are zero. This implies that λ' is bounded on each interval $[t_0, T)$, for $0 < t_0 < T$. In particular, λ extends continuously to $[0, T]$ and is Lipschitz on $[t_0, T]$. Its local Lip(1/2)-norm is very small on $[t_0, T]$, so again by Theorem 1, the curve $g_{t_0}(\gamma(t_0, T])$ is simple. □

7.5 A comparison theorem

One can also use LC bounds to compare two curves when one of them is self-similar. We need some notation. For $c > 0$, let $\lambda_c(t) = c^2 - c\sqrt{c^2 - t}$ and $\Lambda_c(t) = c\sqrt{c^2 + t} - c^2$, and let γ_c and Γ_c denote, respectively, the corresponding curves. Notice that $\lambda_c'(0) = 1/2 = \Lambda_c'(0)$. Also, let $T_c = c^2$ if $c \geq 4$; otherwise, let T_c be the first time that the tangent to γ_c points downward.

Theorem 9. *Let $\gamma\colon (0, T) \to (\mathbb{H}, 0, \infty)$ be generated by $\lambda \in C^2[0, T)$ with $\lambda'(0) = 1/2$.*

(i) *If $0 < LC_\gamma(t) \leq c^2/2$, then $\gamma[0, T)$ is below $\gamma_c[0, T_c]$.*

(ii) *If $c^2/2 \leq LC_\gamma(t) < \infty$, then $\gamma_c[0, T_c]$ is below $\gamma[0, T)$.*

(iii) *If $-\infty < LC_\gamma(t) \leq -c^2/2$, then $\gamma[0, T)$ is below $\Gamma_c[0, \infty)$.*

(iv) *If $-c^2/2 \leq LC_\gamma(t) < 0$, then $\Gamma_c[0, \infty)$ is below $\gamma[0, T)$.*

Here, we say that γ_1 *is below* γ_2 if, when parameterized by height h, we have $\mathrm{Re}(\gamma_1(h)) \geq \mathrm{Re}(\gamma_2(h))$ for small h, and the curves never cross.

Proof idea. Let γ_1 and γ_2 denote the curves under consideration so that $LC_{\gamma_1} \leq LC_{\gamma_2}$. The first step is to prove that the base of γ_1 lies to the right of the base of γ_2. This is done by comparing the expansions for γ_1 and γ_2 that we found in the proof of Proposition 4.

The second step is more involved. For ease, we use γ_* to denote either γ_c or Γ_c. Notice that $\lambda_*'(t)$ is either monotonically increasing from 1/2 to ∞ or decreasing from 1/2 to 0. As λ' is similarly either increasing or decreasing, we can make the time change $s = s(t) = (\lambda_*')^{-1}(\lambda'(t))$ to get $\lambda_*'(s) = \lambda'(t)$. Consider the Loewner flows associated to λ and λ_*. Namely, for $z_0 \in \mathbb{H}$, let $z_t = g_t(z_0) - \lambda(t)$, and for $w_0 \in \mathbb{H}$, let $w_t = g^*_{s(t)}(w_0) - \lambda_*(s(t))$. The Loewner equation gives $\partial_t z_t = (2/z_t) - \lambda'(t)$ and

$$\partial_t w_t = \frac{2s'(t)}{w_t} - \lambda_*'(s)s'(t) = \left(\frac{2}{w_t} - \lambda'(t)\right) s'(t).$$

Notice that if $w_t = z_t$, then the direction of these flows at time t is the same. The goal is to show that this implies persistence of "staying above/below $g^*_{s(t)}(\gamma_*)$." Namely, if z_{t_0} is above/below $g^*_{s(t_0)}(\gamma_*)$, then z_t is above/below $g^*_{s(t)}(\gamma_*)$ for all $t_0 \leq t < T$. This is done by proving that the LC bounds imply either $|s'(t)| \leq 1$ or $|s'(t)| \geq 1$ (depending on the case), so that one of the flows moves faster than the other for all time. □

References

[1] Earle, C.J. and Epstein, A. *Quasiconformal variation of slit domains.* Proc. Amer. Math. Soc. **129** (2001), no. 11, 3363–3372.

[2] Lind, J. *A sharp condition for the Loewner equation to generate slits.* Ann. Acad. Sci. Fenn. Math. **30** (2005), no. 1, 143–158.

[3] Lind, J. and Rohde, S. *Loewner curvature.* preprint (2014), `arXiv:1406.2607v1`.

[4] Marshall, D. and Rohde, S. *The Loewner differential equation and slit mappings.* J. Amer. Math. Soc. **18** (2005), no. 4, 763–778.

[5] Wong, C. *Smoothness of Loewner slits.* Trans. Amer. Math. Soc. **366** (2014), no. 3, 1475–1496.

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8 Quasiconformal Variation of Slit Domains

after Clifford J. Earle and Adam Lawrence Epstein [1]
A summary written by Kirill Lazebnik

Abstract

We use quasiconformal variations to study Riemann mappings onto variable single slit domains when the slit is the tail of a Jordan arc with a real analytic parametrization. The results show that the function κ in Löwner's differential equation is real analytic.

8.1 Introduction

We let Ω be a simply connected region in $\mathbb{C}$ that contains the origin, and $f : [0,T) \to \Omega$, with $0 < T \leq \infty$ a parametrization of a Jordan arc Γ in Ω. (In particular f is a homeomorphism onto Γ). We will assume the following... First, f is regular (differentiable everywhere with non-zero derivative). Second, $\Gamma \subset \Omega$ is closed. Third, $0 \notin \Gamma$. Then by topological considerations we can prove that:

Proposition 1. *If $t_n \to T$, then for some subsequence t_{n_j} of t_n we have $f(t_{n_j})$ converges to some point in $\partial\Omega$. (the 'end' of the arc Γ tends to the boundary of Ω). Each arc $\Gamma_t := f([t,T))$ is closed in Ω, and each region $\Omega_t := \Omega - \Gamma_t$ is simply connected.*

Some examples of such arcs/domains include $\Omega = \mathbb{C}, \Gamma = \{x + i\sin x : x \geq 1\}$ and $\Omega = \text{Re}z < 1, \Gamma = \{1 - t^{-1} + it\sin t : t \geq 1\}$. We establish the following lemma which we will use in several proofs. It allows us to deform a disc in the complex plane by a nice function, controlling where the center of the disc is sent. The function η taken below is usually constructed in say, the theory of partitions of unity on a smooth real manifold, or as a standard mollifier.

Lemma 2. *Given $t_0 \in \mathbb{C}, r > 0$, let $D(t_0, r) \subset \mathbb{C}$ be the open disk with center t_o and radius r. Let η be a compactly supported real-valued C^∞ function on $D(t_0, r)$ such that $\eta(t_0) = 1$, and let M be the maximum of $2\left|\frac{\partial \eta}{\partial t}\right|$. For any complex number $\lambda < 1/M$ the function:*

$$\psi_\lambda(t) = t + \lambda\eta(t), \quad t \in D(t_0, r)$$

Is a bi-Lipschitz, quasiconformal C^∞ diffeomorphism of $D(t_0, r)$ onto itself, and its Beltrami coefficient μ_λ satisfies the following identity:

$$\mu_\lambda = \frac{\partial\psi_\lambda/\partial\bar{t}}{\partial\psi_\lambda/\partial t} = \frac{\lambda\partial\eta/\partial\bar{t}}{1+\lambda\partial\eta/\partial t} = \lambda\frac{\partial\eta}{\partial\bar{t}}\sum_{n=0}^{\infty}\left(-\lambda\frac{\partial\eta}{\partial t}\right)^n$$

In addition ψ_λ is the identity in a neighborhood of the boundary of $D(t_0, r)$, and when $t_0, \lambda \in \mathbb{R}$ then ψ_λ maps $D(t_0, r) \cap \mathbb{R}$ onto itself.

Proof: This is a nice, straightforward proof. I regretfully leave it out to meet length requirements.

8.2 A theorem on a conformal radius function

Through the rest of our discussion, we will assume the parametrization f is in fact real analytic. (The original paper by Clifford, Earle explores weaker assumptions on f.) Recall that the conformal radius of the region Ω_t at 0 is the size $|g'(0)|$ where g is any Riemann map from the unit disk $\triangle$ to Ω_t fixing 0. (It is easy to see this is well-defined.) This defines for us a map $R : [0, T) \to \mathbb{R}$ sending any $t \in [0, T)$ to the conformal radius of Ω_t at 0. Our first theorem is that this map is in fact real analytic:

Theorem 3. *The conformal radius function $R(t)$ of Ω_t at the origin is a real analytic function of t in the interval $[0, T)$.*

The idea of the proof is to construct (using our lemma) a holomorphic motion of $\partial\Omega_{t_0}$ to get to $\partial\Omega_{t_0+\varepsilon}$ for small ε. The conclusions of the theorem then follow by a result of Rodin [2]. Rodin's result in turn relies on a form of the powerful λ-lemma. For convenience we have recopied Rodin's theorem (but not the proof) in the appendix and changed notation a bit from the original paper to conform with ours.

Proof: We fix $t_0 \in [0, T)$. We abbreviate $D = D(t_0, r)$. Now we choose a small enough r so that several things happen:

1. $t_0 + r < T$
2. There is a conformal map $\tilde{f} : D(t_0, r) \to \Omega$ agreeing with f on their common domain $D \cap [0, T)$.
3. $0 \notin \tilde{f}(D)$.
4. $\Gamma \cap \tilde{f}(D) = f(D \cap [0, T))$

The ability to choose r so that 2. happens comes from the fact that f is real analytic at t_0 and so we can define$\tilde{f}$ by the same power series but with complex variables. Conformality is an important consequence of the fact that $f'(t_0) \neq 0$. 1,3,4 are topological considerations.

Now we define our holomorphic motion $\phi : D(0, 1/M) \times \partial\Omega_{t_0} \to \mathbb{C}$ (using ψ_λ from our lemma):

$$\phi(\lambda, w) = \begin{cases} \tilde{f} \circ \psi_\lambda \circ \tilde{f}^{-1}(w) & : w \in \partial\Omega_{t_0} \cap \tilde{f}(D) \\ w & : w \notin \partial\Omega_{t_0} \cap \tilde{f}(D) \end{cases}$$

It is easy to check this is in fact a holomorphic motion. The most important observation to make is that, if $\lambda \in D(0, 1/M) \cap \mathbb{R}$ and $t_0 + \lambda \geq 0$, then $\phi_\lambda(\partial\Omega_{t_0}) = \partial\Omega_{t_0+\lambda}$ and so the region $\phi_\lambda(\partial\Omega_{t_0})$ bounds is exactly $\Omega_{t_0+\lambda}$. The fact that $\phi_\lambda(\partial\Omega_{t_0}) = \partial\Omega_{t_0+\lambda}$ can be proven by looking back at our function ψ_λ. Now we apply Rodin's theorem. We let Φ_λ denote the normalized Riemann mapping of $\triangle$ onto $\Omega_{t_0+\lambda}$, so that:

$$R(t_0 + \lambda) = \Phi'_\lambda(0) \text{ if } \lambda \in D(0, 1/M) \cap \mathbb{R} \text{ and } t_0 + \lambda \geq 0$$

Rodin's theorem says exactly that for some $\varepsilon < 1/M$ we know that $(\lambda, z) \to \Phi_\lambda(z)$ is real analytic on $D(0, \varepsilon) \times \triangle$ in the sense already described. In the same sense $\lambda \to \Phi'_\lambda(0)$ is real analytic on $D(0, \varepsilon)$. It follows by our various definitions of real analytic and the equation relating R, Φ that $R(t)$ is real analytic.

8.3 A theorem regarding the dependence on t of Riemann maps from the unit disc to Ω_t

Our next theorem has two parts. The first proves, as in our previous theorem, that a certain function is real analytic. The second gives a differentiation formula which will lead to the proof that the κ in Löwner's differential equation is real analytic. As before, 'real analytic' is defined in the statement of Rodin's theorem.

Theorem 4. *For $t \in [0, T)$, let $z \to h(z, t)$ be the Riemann mapping of $\triangle$ onto Ω_t such that $h(0, t) = 0$, $h(1, t) = f(t)$. Then the mapping $h : \triangle \times [0, T) \to \mathbb{C}$ is real analytic. Moreover we have real-valued, real analytic functions α, β on $[0, T)$ with α positive so that the following differentiation formula holds:*

$$\frac{\partial h}{\partial t}(z,t) = z\frac{\partial h}{\partial z}(z,t)\left[\alpha(t)\frac{1+z}{1-z} + i\beta(t)\right] \text{ for all } (z,t) \in \triangle \times [0,T)$$

The idea of the proof is to use quasiconformal maps to construct a formula relating $h(\cdot, t_0)$ and $h(\cdot, t)$ for t close to t_0. One then uses this formula to deduce both the real analyticity of h and the differentiation formula. The proof uses a theorem from a classical paper by Ahlfors and Bers [3] that says if one has a family of beltrami coefficients that depend real analytically on a parameter, then so do the normalized solutions to the beltrami equation.

Proof: Let $(z_0, t_0) \in \triangle \times [0,T)$ and denote $h_0 = h(\cdot, t_0)$. Let $r > 0$ and $\tilde{f}$ conformal on D as in the proof of our previous theorem. We specify additionally that $h_0(z_0) \notin \tilde{f}(D)$. We let $\varepsilon = 1/M$ and define quasiconformal maps ϕ_λ, W_λ for $\lambda \in (-\varepsilon, \varepsilon)$ as follows:

$$\phi_\lambda(w) = \begin{cases} \tilde{f} \circ \psi_\lambda \circ \tilde{f}^{-1}(w) & : w \in \tilde{f}(D) \\ w & : w \notin \tilde{f}(D) \end{cases}$$

It is easy to see that ϕ_λ is quasiconformal (since ψ_λ is). It is important also to note that $\phi_\lambda(\Omega_{t_0}) = \Omega_{t_0+\lambda}$ if $t_0 + \lambda \geq 0$. Next we let E be the compact support of the function η from our lemma, and $V := \mathbb{C} - \tilde{f}(E)$. It follows that $V, \tilde{f}(D)$ cover $\mathbb{C}$ and that ϕ_λ is the identity on V. Notice that the beltrami coefficients ν_λ of ϕ_λ depend real analytically on λ:

$$\nu_\lambda(z) = \chi_{\tilde{f}(D)}(z) \cdot \frac{\overline{\tilde{f}'(\tilde{f}^{-1}(z))}}{\tilde{f}'(\tilde{f}^{-1}(z))} \cdot \lambda \frac{\partial \eta}{\partial \bar{t}} \sum_{n=0}^{\infty} \left(-\lambda \frac{\partial \eta}{\partial t}\right)^n$$

Next we define W_λ a quasiconformal self-map of $\triangle$ by applying the measurable Riemann mapping theorem to the beltrami coefficient:

$$\sigma_\lambda(z) = v_\lambda(h_0(z)) \frac{\overline{h_0'(z)}}{h_0'(z)}, \quad z \in \triangle$$

By reflection we can extend W_λ to a quasiconformal map of the whole plane, and we specify that this map fixes $0, 1$. We can compute then that W_λ, $\phi_\lambda \circ h_0$ have the same beltrami coefficient on $\triangle$ and so then $\phi_\lambda \circ h_0 \circ W_\lambda^{-1}$ is conformal. It follows since that $\phi_\lambda \circ h_0 \circ W_\lambda^{-1} : \triangle \to \Omega_{t_0+\lambda}$ is a conformal homeomorphism fixing 0 and 1 that $\phi_\lambda \circ h_0 \circ W_\lambda^{-1} = h(\cdot, t_0 + \lambda)$ as long as

$t_0 + \lambda \geq 0$. We have thus established the following formula from which the rest of the proof relies on:

$$\phi_\lambda \circ h_0(z) = h(W_\lambda(z), t_0 + \lambda) \text{ if } t_0 + \lambda \geq 0 \text{ and } \lambda \in (-\varepsilon, \varepsilon)$$

One uses this formula to establish the real analyticity of h at (z_0, t_0). Similarly one differentiates this formula and uses some standard complex analysis to deduce the formula in the conclusion of our theorem. Details are deferred so that we have space to present the last of our results.

8.4 Löwner's Equation

The differentiation formula of our previous result reduces to traditional Löwner form under a change of parametrization of Γ and the normalization of the Riemann mapping.

Theorem 5. *Let $a(t)$ be a real analytic antiderivative for $\alpha(t)$ on $[0,T)$ normalized by $a(0) = 0$, and let $[0,\hat{T})$ be the image of $[0,T)$ under a. For each $\tau \in [0,\hat{T})$ let $z \to g(z,\tau)$ be the Riemann mapping of $\triangle$ onto $\Omega_{a^{-1}(\tau)}$ normalized by $g(0,\tau) = 0$ and $\frac{\partial g}{\partial z}(0,\tau) > 0$. Then g is a real analytic function on $\triangle \times [0,\hat{T})$ and there is a real analytic function κ on $[0,\hat{T})$ so that $|\kappa(\tau)| = 1$, $g(1/\kappa(\tau),\tau) = f(a^{-1}(\tau))$, and*

$$\frac{\partial g}{\partial \tau}(z,\tau) = z\frac{\partial g}{\partial z}(z,t)\frac{1+z\kappa(\tau)}{1-z\kappa(\tau)} \textit{ for all } (z,\tau) \in \triangle \times [0,\hat{T})$$

Proof: We define $\hat{h}(z,\tau) = h(z, a^{-1}(\tau))$ on $\triangle \times [0,\hat{T})$. Our previous result together with a chain rule computation show that:

$$\frac{\partial \hat{h}}{\partial \tau}(z,\tau) = z\frac{\partial \hat{h}}{\partial z}(z,\tau)\left[\frac{1+z}{1-z} + i\hat{\beta}(\tau)\right] \text{ for all } (z,\tau) \in \triangle \times [0,\hat{T})$$

Where $\hat{\beta}$ is the function on $[0,\hat{T})$ so that $\hat{\beta}(a(t))a'(t) = \beta(t)$ for $t \in [0,T)$. Since $g(\cdot,0), \hat{h}(\cdot,0)$ fix the origin and are Riemann maps onto the same region, we have the formula $g(z,0) = \hat{h}(e^{-ib_0}z, 0)$ for $z \in \triangle$. Let $b(\tau)$ be a real analytic antiderivative for $\hat{\beta}$ on $[0,\hat{T})$ so that $b(0) = b_0$. Next we define the real analytic function $\hat{g}(z,\tau) = \hat{h}(e^{-ib(\tau)}z,\tau)$ on $\triangle \times [0,\hat{T})$. One deduces from this definition that in fact $\hat{g} = g$ and by a chain rule computation:

$$\frac{\partial g}{\partial \tau}(z,\tau) = z\frac{\partial g}{\partial z}(z,\tau)\frac{1+e^{-ib(\tau)z}}{1-e^{-ib(\tau)z}} \text{ if } (z,\tau) \in \triangle \times [0,\hat{T})$$

So that the theorem follows with $\kappa(\tau) := e^{-ib(\tau)}$.

8.5 Appendix

Here is Rodin's result on which our first theorem relies:

Theorem 6. *Let Ω_{t_0} be a simply connected proper subregion of $\mathbb{C}$, let $r > 0$, $0 \in \Omega_{t_0}$ and let*

$$\phi : D(0, 1/M) \times \partial\Omega_{t_0} \to \bar{\mathbb{C}}$$

be a holomorphic motion of $\partial\Omega_{t_0}$. Then there is an $\varepsilon > 0$ such that, for $|\lambda| < \varepsilon$, $\phi(\lambda, \partial\Omega_{t_0})$ is the boundary of a simply connected region Ω_λ containing 0. We also have that the Riemann mapping function:

$$\Phi_\lambda : \{|w| < 1\} \to \Omega_\lambda$$

normalized by $\Phi_\lambda(0) = 0, \Phi'_\lambda(0) > 0$ depends real analytically on $Re\lambda$, $Im\lambda$ in the following sense: If $|\lambda_0| < \varepsilon$ then there is a $\delta_0 > 0$ so that $\Phi_\lambda(w)$ is a power series in $Re(\lambda - \lambda_0)$, $Im(\lambda - \lambda_0)$, w which converges in

$$\{-\delta_0 < Re(\lambda - \lambda_0) < \delta_0\} \times \{-\delta_0 < Im(\lambda - \lambda_0) < \delta_0\} \times \{|w| < 1\}$$

References

[1] Earle, Clifford and Epstein, Lawrence, *Quasiconformal Variation of Slit Domains* Proceedings of AMS **129** (2001)

[2] Rodin, Burton, *Behavior of the Riemann Mapping Function under Complex Analytic Deformations of the Domain* Complex Variables **5** (1986)

[3] Ahlfors, Lars and Bers, Lipman *Riemann's Mapping Theorem for Variable Metrics* Annals of Mathematics **72** (1960)

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9 Second Phase Transition of the Deterministic Loewner Equation

after J. Lind and S. Rohde [3]
A summary written by Jhih-Huang Li

Abstract

We are interested in the hulls of the deterministic Loewner equation driven by functions $\lambda \in Lip(1/2)$, where the norm $\|\lambda\|_{1/2}$ would play a similar role as κ in SLE process. We will show a result about its phase transition.

9.1 Introduction

It is known that the famous Schramm-Loewner Evolution SLE_κ process exhibits 2 phase transitions when κ varies. More precisely, for $\kappa \leq 4$, the traces are simple curves, while for $\kappa > 4$, they touch themselves. For $\kappa < 8$, the curves have empty interior, while for $\kappa \geq 8$, these are space-filling.

We could establish an analogy for the processes driven by deterministic $Lip(1/2)$ functions. When $\|\lambda\|_{1/2} < 4$, it is known that the trace is simple ([1], [4]). And here we show a result for the second phase transition.

Theorem 1. *If λ is a $Lip(1/2)$ driving function generating a curve with non-empty interior, then $\|\lambda\|_{1/2} > 4.0001$.*

To get this result, it is important to require the curve to have non-empty interior. If it were only dense in the upper half plane, Theorem 1 would be no longer true, as shown by the following Theorem.

Theorem 2. *Let (z_n) be a sequence of points in $\mathbb{H}$. Then there exists a trace γ going through these points having $Lip(1/2)$ norm at most 4.*

It is possible that $Lip(1/2)$ driving functions generate only curves with empty interior, in which case Theorem 1 would not be interesting. A criterion allows us to say, given a curve, whether its driving function has $Lip(1/2)$ norm. In particular, the Hilbert space-filling curve (cf. Figure 1) falls into this category and its $Lip(1/2)$ norm is greater than 4.0001 according to Theorem 1.

9.2 Reminder: some geometric properties

To make the summary self-containing, we would like to remind some basic properties related to the Loewner equation.

Let us take (K_t), the family of hulls generated by the driving term $\lambda(t)$, then we have the following 4 geometric properties.

1. Scaling: for $r > 0$, the scaled hulls (rK_{t/r^2}) are driven by $r\lambda(t/r^2)$.

2. Translation: for $x \in \mathbb{R}$, the translated hulls (K_t+x) are driven by $t \mapsto \lambda(t)+x$.

3. Reflection: The reflection of hulls with respect to the imaginary axis $R_I(K_t)$ are driven by $t \mapsto -\lambda(t)$.

4. Concatenation: for fixed T, the mapped hulls $(g_T(K_{T+t}))$ are driven by $t \mapsto \lambda(T+t)$.

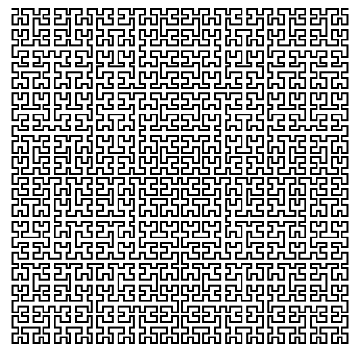

Figure 1: Hilbert space-filling curve, drawn with Lindenmayer system of tikz package.

9.3 An important example: Theorem 2

First of all, we would like to contruct a dense curve whose driving function has $Lip(1/2)$ norm smaller or equal to 4.

Let (z_n) be a sequence of points in $\mathbb{H}$. The idea would be to construct the curve piece by piece then glue them all together in such a way that the norm of the driving function stays small.

Preliminary: given $x \in \mathbb{R}$ and $z \in \mathbb{H}$, we will contruct a $Lip(1/2)$ driving function with norm at most 4 generating a simple curve from x to z in $\mathbb{H} \cup \{x\}$.

- If $\Re z = x$, then take $\lambda(x) \equiv x$ on an apporpriate time interval.

Construction: we will construct by induction a family of driving functions $\lambda_n : [0, T_n] \to \mathbb{R}$, $\|\lambda_n\|_{1/2} \leq 4$ such that the curve generated contains $z_1, \ldots, z_n$. We want this family to be compatible, meaning that λ_{n+1} restricted to $[0, T_n]$ is λ_n.

- Otherwise, let us consider $\lambda(t) = 4\sqrt{1-t}$, whose $Lip(1/2)$ norm is 4. The curve generated by λ is shown in Figure 2. (For more details, see [2].) It is a simple curve from 4 to 2 and for all $\theta \in (\pi/2, \pi)$, each ray $\{4 + re^{i\theta}, r > 0\}$ intersects the curve exactly once. So by scaling, reflecting (cf. subsection 9.2) and taking an appropriate time interval, we have a cuve from x to z.

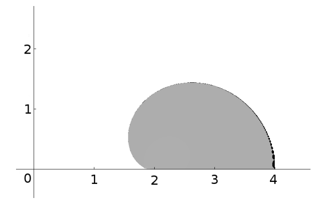

Figure 2:

- If z_{n+1} is already in the trace, we have nothing to do. Set $T_{n+1} = T_n$.
- Otherwise, set $\lambda_{n+1}(t) \equiv \lambda_n(T_n)$ for all $t \in [T_n, T_n + \tau_n]$ where τ_n is to be determined later. (View from g_{T_n}, the trace is vertical.) We then construct $\hat{\lambda}$ on $[0, \sigma_n]$, driving function given in the first step generating a trace from $x_n = \lambda_n(T_n)$ to $w_n = g_{T_n+\tau_n}(z_{n+1})$. We put all these together and set $T_{n+1} = T_n + \tau_n + \sigma_n$ and $\lambda_{n+1}(t) = \hat{\lambda}(t - (T_n + \tau_n))$ for $t \in [T_n + \tau_n, T_{n+1}]$. Since λ_{n+1} restricted to $[0, T_n]$ and $[T_n + \tau_n, T_{n+1}]$ has $Lip(1/2)$ norm at most 4, it remains to show that, by choosing τ_n carefully, the $Lip(1/2)$ norm stays at most 4 for λ_{n+1}.

By scaling property mentionned in Subsection 9.2, we can show that this can be satisfied for large enough τ_n. The proof is thus completed.

Remark 3. *If we want the points (z_n) to be visited in the order, we may modify slightly the previous construction.*

9.4 Proof of the main Theorem

9.4.1 How the points in $\mathbb{R}$ are captured?

First of all, we would like to understand how the points in $\mathbb{R}$ are captured by the driving function. We say that a point $x \in \mathbb{R}\backslash\{\lambda(0)\}$ is *captured* by λ at time t if $g_t(x) = \lambda(x)$. By scaling, we can assume for now that this takes place at time $t = 1$, the case of general t will be given later (Lemma 6).

We take $\lambda : [0, 1] \to \mathbb{R}$, whose $Lip(1/2)$ norm is C and satisfying $\lambda(1) = 0$ and $\lambda(t) > 0$ for all $t \in [0, 1)$. We will consider a point $x < \lambda(0)$ captured by λ at $t = 1$. First of all, we make the time change $s = -\ln(1-t)$, or

$t = 1 - e^{-s}$. We then define for $s \in [0, \infty)$,

$$G_s(z) := \frac{g_t(z)}{\sqrt{1-t}} \text{ and } \sigma(s) := \frac{\lambda(t)}{\sqrt{1-t}} \leq C.$$

We then rewrite the Loewner equation as

$$\frac{\partial}{\partial s} G_s = \frac{2}{G_s - \sigma(s)} + \frac{G_s}{2} = -\frac{1}{2} \frac{G_s^2 - \sigma(s) G_s + 4}{\sigma(s) - G_s} \text{ , } G_0(z) = z \qquad (1)$$

and we say G_s is generated by σ.

We write $x_s = G_s(x)$ the solution to (1). Since we assume that x is captured only at time $t = 1$, x_s is well defined on $[0, \infty)$.

If $\sigma(s) < 4$ for all s, then x_s is decreasing if $x < \sigma(0) = \lambda(0)$.

If $\sigma(s) \geq 4$ for all s, we can factorize the numerator of (1) :

$$\frac{\partial}{\partial s} x_s = -\frac{1}{2} \frac{(x_s - A_s)(x_s - B_s)}{\sigma(s) - x_s}$$

where $\sigma(s) > A_s > B_s$ and are functions of $\sigma(s)$. To make A_s and B_s well defined for all s, we take $A_s = B_s = 2$ whenever $\sigma(s) < 4$.

A special case: $\lambda(t) = C\sqrt{1-t}$, and $\sigma(s) \equiv C$. In consequence, all the points in $[A_s, \sigma(s)]$ are captured by λ at time $t = 1$ because x_s decreases towards A_s.

The next Lemma says that in a more general case, if $x < \lambda(0)$, we encounter a similar situation: x_s decreases towards A_s until it reaches a small neighborhood of $[B_s, A_s]$ in which it stays forever.

Lemma 4. *Let $\varepsilon \in (0, 1/2)$. Suppose that $\|\lambda\|_{1/2} \leq 4 + 2\varepsilon$ and that $x < \lambda(0)$ captured by λ at $t = 1$. Then there exist a time $S_0 < \infty$ and an interval I containing $[B_s, A_s]$ of length $5\sqrt{\varepsilon}$ so that $x_s \in I$ for $s \geq S_0$.*

Proof. The proof is actually based on finding bounds to estimate the decreasing speed, which is quite similar to what we have done. It is omitted here. □

Since x_s should stay in the interval I, if we have a nearer look to the partial differential equation, we can see that σ cannot be bounded from above. More precisely, see the following Lemma.

Lemma 5. *Let $\varepsilon \in (0, 1/2)$ and $M \in (0, 4)$. Suppose that $\|\lambda\|_{1/2} \leq 4 + 2\varepsilon$ and $x < \lambda(0)$ captured at time T. Let S_0 be given previously. Then there exists $\Delta < \infty$ so that is $\sigma < M$ on $[s_1, s_2]$ with $s_1 > S_0$, then $s_2 - s_1 \leq \Delta$. We may take $\Delta = \frac{10\sqrt{\varepsilon}}{4-M}$.*

Proof. The technique is quite similar, we only give important steps and formulae here. We assume that $\sigma(s) < M$ on the time interval $[s_1, s_2]$, therefore we can get a lower bound for the decreasing speed of x_s :

$$-\frac{\partial}{\partial s} x_s \geq \frac{4 - M}{2},$$

which implies that x_s cannot stay in I, an interval of length $5\sqrt{\varepsilon}$ for a time longer than $\Delta = \frac{10\sqrt{\varepsilon}}{4-M}$. □

Here is the last lemma, combining Lemma 4 and Lemma 5 without the time change.

Lemma 6 (without time change)**.** *Let $\varepsilon \in (0, 1/2)$ and $M \in (0, 4)$. Suppose that $\|\lambda\|_{1/2} \leq 4 + 2\varepsilon$ and that $x < \lambda(0)$ captured at time T. Then there exist $S_0, \Delta < \infty$ (depending only on ε and M) so that for all $s \geq S_0$, we can find a time $t \in [(1 - e^{-s})T, (1 - e^{-(s+\Delta)})T]$ such that $|\lambda(T) - \lambda(t)| \geq M\sqrt{T - t}$. Moreover, we may take $\Delta = \frac{10\sqrt{\varepsilon}}{4-M}$.*

9.4.2 Proof of Theorem 1

Now, we are able to show the Theorem 1.

Proof. Take $\lambda \in Lip(1/2)$ generating γ with non-empty interior. We assume that $\|\lambda\|_{1/2} \leq 4 + 2\varepsilon$ and want to get a contradiction when ε is too small.

By Baire category theorem, there is a T such that $\gamma[0, T]$ has non-empty interior. We notice as well that if $\gamma(t_0) \in \text{Int}(\gamma)$, then $\lambda(t_0) = g_{t_0}(\gamma(t_0)) \in \text{Int}(g_{t_0}(\gamma))$ in $\overline{\mathbb{H}}$. We can, by replacing $\lambda(t)$ with $\lambda(t + t_0) - \lambda(t_0)$ and scaling, assume that there is an interval $I \subset K_T \cap \mathbb{R}^+$ where K is the hull at time T.

The points in I are captured at different time and there are uncountably many of them, so there exist $T_1 < T_2$ in I such that $T_2 - T_1 \leq e^{-2S_0}T_2$, where S_0 is given earlier. (Note that S_0 depends only on M and ε whose values will be specified later.)

Take $\Delta = \frac{10\sqrt{\varepsilon}}{4-M}$ as before and consider $I_2 = [(1 - e^{-s})T_2, (1 - e^{-(s+\Delta)})T_2]$ where s is chosen so that T_1 is the midpoint of I_2. Then we define $I_1 =$

$I_2 - (T_2 - T_1) = [(1-e^{-s'})T_2, (1-e^{-(s'+\Delta)})T_2]$, an interval of the same length but shifted. We can easily check that both s and s' are greater than S_0 by our hypothesis. By Lemma 6, there exist $t_1 \in I_1$ and $t_2 \in I_2$ such that

$$\lambda(T_2) - \lambda(t_2) \geq M\sqrt{T_2 - t_2} \text{ and } \lambda(T_1) - \lambda(t_1) \geq M\sqrt{T_1 - t_1}.$$

Our goal is to conclude by contradiction by taking appropriate M and ε. In particular, we would like to have

$$\lambda(T_2) - \lambda(t_1) > (4 + 2\varepsilon)\sqrt{T_2 - t_1}.$$

We write

$$\begin{aligned}\lambda(T_2) - \lambda(t_1) &= (\lambda(T_2) - \lambda(t_2)) + (\lambda(t_2) - \lambda(T_1)) + (\lambda(T_1) - \lambda(t_1)) \\ &\geq M\sqrt{T_2 - t_2} - (4 + 2\varepsilon)\sqrt{|t_2 - T_1|} + M\sqrt{T_1 - t_1},\end{aligned}$$

so we just need to show that the last formula is greater than $(4+2\varepsilon)\sqrt{T_2 - t_1}$. This inequality can be satisfied by taking $M = 3.5$ and $\varepsilon = 0.00005$. □

References

[1] J. Lind, *A sharp condition for the Loewner equation to generate slits.* Ann. Acad. Sci. Fenn. Math. **30** (2005), 143–158.

[2] J. Lind, D.E. Marshall and S. Rohde, *Collisions and Spirals of Loewner Traces.* Duke Math. J. **154** (2010), 527–573.

[3] J. Lind and S. Rohde, *Spacefilling Curves and Phases of the Loewner Equation.* Comment. Math. Helv. **62** (1987), no. 1, 18–37.

[4] D.E. Marshal and S. Rohde, *Convergence of a variant of the Zipper algorithm for conformal mapping.* J. Amer. Math. Soc. **18** (2005), 763–778.

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10 Distributional limits of Riemannian manifolds and graphs with sublinear genus growth

A summary written by Zhiqiang Li

Abstract

We discuss a recent work of H. Namazi, P. Pankka, and J. Souto on distributional limits of certain sequences of Riemannian manifolds and applications to distributional limits of certain sequences of graphs of sublinear genus growth, which somewhat improves I. Benjamini and O. Schramm's original result on the recurrence of the simple random walk on limits of planar graphs of uniformly bounded valence.

10.1 Introduction

I. Benjamini and O. Schramm proved in their seminal paper [1] the recurrence of the simple random walk on limits of planar graphs. We will discuss a recent work of H. Namazi, P. Pankka, and J. Souto [2] on related results in the context of certain Riemannian manifolds and certain non-planar graphs.

Recall the following main theorem of [1].

Theorem 1 (I. Benjamini and O. Schramm [1])**.** *Let (G, o) be a distributional limit of rooted random unbiased finite planar graphs G_j with uniformly bounded valence. Then with probability one G is recurrent.*

We say a (rooted) graph is *recurrent* if the simple random walk on it is recurrent. In the same spirit, one of the results of [2] is the following.

Theorem 2. *Let $\{G_i\}_{i \in \mathbb{N}}$ be a sequence of finite graphs with uniformly bounded valence, with $|G| \longrightarrow +\infty$, and with sublinear genus growth. If $\{G_i\}_{i \in \mathbb{N}}$ converges in distribution to λ, then the set of recurrent rooted graphs (G, p) has full λ-measure.*

Here a graph G is considered as a metric space equipped with the graph metric, and $|G|$ denotes the cardinality of the set $V(G)$ of vertices of G. We now define formally what exactly it means to say that $\{G_i\}_{i \in \mathbb{N}}$ converges in distribution to λ, which is similar to the corresponding concept in [1].

Let $\mathcal{H}$ be the space of all isometry classes of pointed metric spaces endowed with the pointed Gromov-Hausdorff topology. Recall that two pointed metric spaces (X, x, d_X) and (Y, y, d_Y) are close to each other in the pointed Gromov-Hausdorff topology if for $R > 0$ large and $\varepsilon > 0$ small, there exists $L \in \mathbb{R}$ close to 1 and two discrete subsets $U \subseteq X$ and $V \subseteq Y$ with $x \in X$, $Y \in Y$,

$$B_{(X,d_X)}(x, R) \subseteq \bigcup_{u \in U} B_{(X,d_X)}(u, \varepsilon), \qquad B_{(Y,d_Y)}(y, R) \subseteq \bigcup_{v \in V} B_{(Y,d_Y)}(v, \varepsilon),$$

and an L-bi-Lipschitz map $(U, d_X) \to (V, d_Y)$ mapping x to y.

Given a sequence $\{G_i\}_{i\in\mathbb{N}}$ of finite graphs with uniformly bounded valence. For each i, consider the map $\tau_i \colon V(G_i) \to \mathcal{H}$ such that $\tau_i(v) = (G_i, v)$ for $v \in V(G)$. Let $\mu_i = \frac{1}{|G_i|} \sum_{v \in V(G)} \delta_v$ be the probability measure which gives equal weight to each vertex. Then we say $\{G_i\}_{i\in\mathbb{N}}$ converges in distribution to a measure λ on $\mathcal{H}$ if the push-forward $(\tau_i)_* \mu_i$ of μ_i converges to λ in the weak*-topology as i tends to $+\infty$.

In [2], Theorem 2 is actually a consequence of one of the main results in the context of Riemannian manifold. For a Riemannian manifold M, we denote κ_M, $\mathrm{inj}(M)$, $g(M)$, and vol_M the sectional curvature, the injectivity radius, the genus, and the Lebesgue measure induced by the Riemannian metric of M, respectively.

Theorem 3. *Let $\varepsilon > 0$, and $\{M_i\}_{i\in\mathbb{N}}$ be a sequence of closed Riemannian surfaces with $|\kappa_{M_i}| \leq 1$ and $\mathrm{inj}(M_i) > \varepsilon$ for all $i \in \mathbb{N}$. Suppose that $\{M_i\}_{i\in\mathbb{N}}$ converges in distribution to λ and that*

$$\lim_{i\to+\infty} \frac{g(M_i) + 1}{\mathrm{vol}_{M_i}(M_i)} = 0.$$

Then the set of $(X, x) \in \mathcal{H}$ such that X is a Riemannian surface conformally equivalent to $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$ has full λ-measure.

Similar to the definition above, we say that $\{M_i\}_{i\in\mathbb{N}}$ converges in distribution to a measure λ on $\mathcal{H}$ if $(\tau_i)_* \left(\frac{\mathrm{vol}_{M_i}}{\mathrm{vol}_{M_i}(M_i)} \right)$ converges to λ in the weak*-topology, where $\tau_i \colon M_i \to \mathcal{H}$ is the map with $\tau_i(x) = (M_i, x)$ for $x \in M_i$.

10.2 Technical results

In this section, we summarize some technical results that are needed to prove Theorem 3.

As experts would expect, we need a form of the Benjamini-Schramm lemma in the context of Riemannian manifold.

Recall that if (X, d) is a metric space and $C \subseteq X$ a finite set of points, then the *isolation radius* $\rho_{X,C}(w)$ of a point $w \in C$ is its minimal distance to a different point in C:

$$\rho_{X,C} = \min_{z \in C,\, z \neq w} d(w, z).$$

Give $\delta \in (0, 1)$ and $s > 0$, we say that $w \in C$ is (δ, s)-*supported* if

$$\min_{p \in X} \left| C \cap \left(B_{(X,d)}(w, \delta^{-1}\rho_{X,C}(w)\right) \setminus B_{(X,d)}(p, \delta\rho_{X,C}(w)) \right| \geq s.$$

Lemma 4 (I. Benjamini and O. Schramm [1])**.** *For every $d \in \mathbb{N}$ and every $\delta \in (0, 1)$, there exists a constant $c(d, \delta)$ such that for every finite subset C of $\mathbb{R}^d$ and every $s \geq 2$ the set of (d, δ)-supported points in C has cardinality at most $c(d, \delta)\frac{|C|}{s}$.*

For Riemannian manifold, we need the following form.

Lemma 5. *Let $k \in \mathbb{N}$ be a positive integer, M be a compact d-manifold, $\{U_i\}_{i \in I}$ an open covering of $M = \bigcup U_i$ with multiplicity k, and $\phi_i \colon U_i \to \mathbb{R}^d$ an embedding for each $i \in I$, where I is a finite index set.*

For each $\delta \in (0, 1)$, each finite subset C of M, and each $s \geq 2$, the set of $x \in C$ for which there exists $i \in I$ with $x \in U_i$ and such that $\phi_i(x)$ is (δ, s)-supported in $\phi_i(C \cap U_i)$ has cardinality at most $c(d, \delta)\frac{k|C|}{s}$, where $c(d, \delta)$ is the constant in Lemma 4.

Recall that the *multiplicity* of a covering $\{U_i\}_{i \in I}$ of M is

$$\sup\{|\{i \in I \,:\, x \in U_i\}| \,:\, x \in M\}.$$

In order to apply Lemma 5 in the context of Riemannian surfaces, we need to be able to construct coverings in a controlled way.

Lemma 6. *There exists $k \in \mathbb{N}$ and $\delta > 0$ such that for each orientable Riemannian surface M with constant curvature $\kappa_M \equiv -1, 0, 1$ (and with $\mathrm{vol}_M(M) = 1$ if M is a torus), there exists an open covering $\{U_i\}_{i \in I}$ with the following properties:*

(i) The cover has at most multiplicity k.

(ii) For every $x \in M$ there is $i \in I$ such that $B_M(x, \delta) \subseteq U_i$.

(iii) Each U_i admits a conformal embedding into $\mathbb{C}$.

We also need some properties of quasi-conformal maps on Riemannian manifolds. Let M and N be Riemannian manifolds. A homeomorphism $f\colon M \to N$ is *quasi-conformal* if there exists $H \in [1, +\infty)$ so that for each $x \in M$,

$$\limsup_{r \to 0} \frac{\max\limits_{d(x,y)=r} d(f(x), f(y))}{\min\limits_{d(u,v)=r} d(f(u), f(v))} \leq H.$$

Equivalently, f is in the local Sobolev space $W^{1,n}_{\mathrm{loc}}(M, N)$ of mappings $M \to N$ and there exists a constant $K \in [1, +\infty)$ so that $||\mathrm{d}f_x||^n \leq K \det(\mathrm{d}f_x)$ for almost every $x \in M$, in which case we call f a K-quasi-conformal map.

Proposition 7. *For all d, K, and κ, there exists $C(d, K, \kappa)$ and $\varepsilon_0(d, K, \kappa)$ such that the following holds:*

If $f\colon M' \to M$ is a K-quasi-conformal homeomorphism between two Riemannian d-manifolds satisfying $|\kappa_M| \leq \kappa$, $|\kappa_{M'}| \leq 1$, and $\mathrm{inj}(M') \geq 10$, then for each 1-net $\mathcal{N} \subseteq M'$, each $R \geq 1$, and each $\varepsilon \in (0, \varepsilon_0(d, K, \kappa))$, we have

$$|\{p \in \mathcal{N} \,:\, \mathrm{diam}_M(f(B_{M'}(p, R))) > \varepsilon\}| \leq C(d, K, \kappa)\, \mathrm{vol}_M(M) R^d \varepsilon^{-d} e^{(d-1)R}.$$

A r-net $\mathcal{N}$ of a metric space (X, d) is a subset of X with $d(x, y) \geq r$ for each pair of distinct points $x, y \in \mathcal{N}$.

The following compactness property of quasi-conformal maps may seem familiar.

Lemma 8. *Fix d and suppose that X is a complete Riemannian d-manifold with $|\kappa_X| \leq 1$ and $\mathrm{inj}(X) > 0$. Let $\Omega \subseteq X$ be a domain, and M be a closed Riemannian d-manifold.*

Suppose that $\mathcal{F}$ is a family of K-quasi-conformal embeddings $\Omega \to M$ so that for some uniform $\delta > 0$ and for every $f \in \mathcal{F}$, there exist points $p_f, q_f \in M \setminus f(X)$ with $d(p_f, q_f) \geq \delta$. Then $\mathcal{F}$ is equicontinuous.

The following corollary is used in the proof of Theorem 3.

Corollary 9. *Fix d and suppose that X is a complete Riemannian d-manifold and that $\Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2 \subseteq \cdots$ is an exhaustion of X by bounded domains.*

Then a sequence $f_i\colon \Omega_i \to \mathbb{R}^d$ of K-quasi-conformal embeddings has a subsequence converging to a K-quasi-conformal embedding $f\colon X \to \mathbb{R}^d$ if there exist $x, y \in X$ so that the sequences $\{f_i(x)\}_{i\in\mathbb{N}}$ and $\{f_i(y)\}_{i\in\mathbb{N}}$ converge in $\mathbb{R}^d$ to different points.

The following well-known $C^{1,1}$-compactness theorem of Gromov is important in the proof of Theorem 3.

Theorem 10 (M. Gromov)**.** *Fix d and ε, and suppose that $\{(M_i, p_i)7\}_{i\in\mathbb{N}}$ is a sequence of pointed Riemannian d-manifolds satisfying $|\kappa_{M_i} \leq 1$ and $\mathrm{inj}(M_i, p_i) \geq \varepsilon$ for $i \in \mathbb{N}$. If $\{(M_i, p_i)7\}_{i\in\mathbb{N}}$ converges in $\mathbb{H}$ to a pointed metric space (X, x), then X is a smooth manifold endowed with a $C^{1,1}$-Riemannian metric and (M_i, p_i) converges to (X, x) in the $C^{1,\alpha}$-topology for all $\alpha \in (0, 1)$.*

Recall that a sequence $\{(M_i, p_i)\}_{i\in\mathbb{N}}$ of pointed Riemannian manifolds converges in the $C^{1,\alpha}$-topology to a pointed Riemannian manifold (N, p) if for each $R > 0$, there exists a domain $\Omega \subseteq N$ containing $B_N(p, R)$ and a sequence of maps $f\colon (\Omega, p) \to (M_i, p_i)$ such that the pulled-back metrics converges in the $C^{1,\alpha}$-topology on tensors on Ω to the restriction to Ω of the metric of N.

10.3 Generalizations to higher dimensions

Theorem 3 has a higher-dimensional version, which is proved before Theorem 3 is established in [2], with an analogous but more technical proof.

Denote by $\mathcal{Q}$ the set of all Riemannian manifolds M' with $|\kappa_{M'}| \leq 1$ such that there exists a K-quasi-conformal homeomorphism $f\colon M' \to M$.

Theorem 11. *Fix $K \geq 1$ and $d \geq 3$, and suppose that M is a closed Riemannian d-manifold. Let $\{M_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathcal{Q}(M, K)$ which converges in distribution to λ. If $\mathrm{vol}_{M_i}(M_i) \longrightarrow +\infty$, then the set of $(X, x) \in \mathcal{H}$ such that X is a Riemannian manifold K-quasi-conformally equivalent to $\mathbb{R}^d$ or $\mathbb{R}^d \setminus \{0\}$ has full α-measure.*

References

[1] Benjamini, I. and Schramm, O., *Recurrence of distributional limits of finite planar graphs.* Electron. J. Probab. **6** (2001), no. 23, (electronic);

[2] Namazi, H., Pankka, P. and Souto, J., *Distributional limits of Riemannian manifolds and graphs with sublinear genus growth.* Geom. Funct. Anal. **1** (2014), pp. 322–359.

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11 Backward SLE and the symmetry of the welding

after S. Rhode and D. Zhan [2]
A summary written by Benjamin Mackey

Abstract

We introduce the backward chordal SLE_κ process and its associated conformal welding. We then use the welding to prove reversability of backward chordal SLE_κ for $\kappa < 4$.

11.1 Introduction

The goal of this paper is to show that the **conformal welding** induced by backward chordal SLE_κ satisfies a certain symmetry.

$$\text{SLE}_\kappa : \partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \text{ Backward SLE}_\kappa : \partial f_t(z) = \frac{-2}{f_t(z) - \sqrt{\kappa} B_t}$$

The particulars of the welding will be defined later in Section 11.3.3. The main theorem is

Theorem 1 (Theorem 1.1, [2])**.** *If $\kappa < 4$ and ϕ is a backward chordal SLE_κ welding, then $h \circ \phi \circ h$ has the same distribution as ϕ.*

A consequence of this theorem is that backward SLE_κ is **reversible** for $\kappa < 4$. That is, the trace is invariant under the map $h(z) = -1/z$. For $\kappa \leq 8$, we know that chordal SLE_κ is **reversible** (For $\kappa \leq 4$, [3]. For $\kappa \in (4, 8]$, [1]).

Theorem 2 (Theorem 1.2, [2])**.** *Let $\kappa < 4$, let β be a "normalized global backward chordal SLE_κ trace," and suppose $h(z) = -1/z$. Then the random sets $\beta \backslash \{0\}$ and $h(\beta \backslash \{0\})$ have the same distribution.*

Some tools will need to be developed first. We will need to discuss hulls in $\mathbb{H}$ and hulls in $\mathbb{D}$, along with associated lifts of symmetric conformal maps. We will then discuss backward Loewner equations and their traces in generality. We will also need to discuss backward SLE with force points and their commutativity properties. We will then discuss proofs of the above theorems.

11.2 Extension of Conformal Maps

11.2.1 Definitions

A set $K \subset \mathbb{H}$ is called an **$\mathbb{H}$-hull** if K is bounded, K is relatively closed in $\mathbb{H}$, and $\mathbb{H}\backslash K$ is simply connected. There is a unique surjective conformal map $g_K : \mathbb{H}\backslash K \to \mathbb{H}$ such that $g_K(z) = z + c/z + O(1/z^2)$ as $z \to \infty$. Define the half plane capacity of K by $\text{hcap}(K) = c$. Let $f_K = g_K^{-1}$.

A set $K \subset \mathbb{D}$ is called a **$\mathbb{D}$-hull** if K is relatively closed in $\mathbb{D}$, $0 \notin K$, and $\mathbb{D}\backslash K$ is simply connected. By the Riemann mapping theorem, there is a unique surjective conformal map $g_k : \mathbb{D}\backslash K \to \mathbb{D}$ so that $g_K(0) = 0$ and $g_K'(0) > 0$. Define the **$\mathbb{D}$-capacity** of K to be $\text{dcap}(K) = \log g_K'(0)$. Most results in this section which are true for $\mathbb{H}$-hulls are also true for $\mathbb{D}$-hulls with minor modifications, so they will be omitted.

Let K_1, K_2 be $\mathbb{H}$-hulls. If $K_1 \subset K_2$, define the **quotient hull** K_2/K_1 by $K_2/K_1 = g_{K_1}(K_2\backslash K_1)$. We say $K_1 \prec K_2$ if there is a hull $K \subset K_2$ so that $K_2/K = K_1$. This hull K is unique and is denoted by $K_2 : K_1$.

Let $I_\mathbb{R}$ be the conjugation map, $I_\mathbb{R}(z) = \overline{z}$. Let $B_K = \overline{K} \cap \mathbb{R}$ be the **base** of K, and define the **double** of K by $\hat{K} = K \cup I_\mathbb{R}(K) \cup B_K$. The map g_K extends to $\hat{\mathbb{C}}\backslash\hat{K}$ with image $\hat{\mathbb{C}}\backslash S_K$ for some compact interval $S_K \subset \mathbb{R}$ called the **support** of K.

A set $S \subset \hat{\mathbb{C}}$ is called **$\mathbb{R}$-symmetric** if $I_\mathbb{R}(S) = S$. A map W defined on an $\mathbb{R}$-symmetric domain Ω is called **$\mathbb{R}$-symmetric** if $\overline{W(\overline{z})} = W(z)$ and $W(\Omega \cap \mathbb{H}) \subset \mathbb{H}$.

11.2.2 Lifting and Collapsing

Let Ω be an $\mathbb{R}$-symmetric domain, and K and $\mathbb{H}$-hull.

(a) If $S_K \subset \Omega$, let $\Omega^K = \hat{K} \cup f_K(\Omega\backslash S_K)$ be the **lift** of Ω via K.

(b) If $\hat{K} \subset \Omega$, let $\Omega_K = S_K \cup g_k(\Omega\backslash\hat{K})$ be the **collapse** of Ω via K.

(c) If $\hat{K} \subset \Omega$, and $W : \Omega \to \mathbb{C}$ is an $\mathbb{R}$-symmetric conformal map so that $\infty \notin W(\hat{K})$, define the **collapse** of W via K to be the conformal map $W_K : \Omega_K \to \mathbb{C}$ given by

$$W_K = g_{W(K)} \circ W \circ f_K.$$

So we can collapse a conformal map. Can we lift it?

Theorem 3 (Theorem 2.12, [2])**.** *Let W be an $\mathbb{R}$-symmetric conformal map defined in an $\mathbb{R}$-symmetric domain Ω. Let K be an $\mathbb{H}$-hull so that $S_K \subset \Omega$ and $\infty \notin W(S_K)$. Then there exists a unique $\mathbb{R}$-symmetric conformal map $V : \Omega^K \to \mathbb{C}$ so that $V_K = W$.*

The map V obtained in Theorem 3 is called the **lift** of W via K, and is denoted by W^K. We denote the $\mathbb{H}$-hull $W^K(K)$ by $W^*(K)$. A similar result holds for $\mathbb{D}$-hulls, with $\mathbb{R}$-symmetry replaced by $\mathbb{T}$-symmetry.

11.3 Backward Loewner Equations

11.3.1 Chordal backward Loewner equations

Let $\lambda \in C[0, T)$. The **backward chordal Loewner equation** driven by λ is

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \lambda(t)}, \text{ and } f_0(z) = z. \tag{1}$$

Each f_t is a conformal map defined on $\mathbb{H}$ with image contained in $\mathbb{H}$. Let $L_t = \mathbb{H} \backslash f_t(\mathbb{H})$ denote the backward chordal hulls driven by λ. How do we know automatically that they are hulls? We can appeal to a more general Loewner equation which connects the backward and forward Loewner equations. For $t_1, t_2 < T$, consider a family of functions which give the maximal solution to the ODE

$$\partial_{t_2} f_{t_2,t_1}(z) = \frac{-2}{f_{t_2,t_1}(z) - \lambda(t_2)}, \text{ and } f_{t_1,t_1}(z) = z. \tag{2}$$

A quick inspection of this differential equation reveals the following facts:

(a) $f_t = f_{t,0}$ for every $t > 0$.

(b) $f_{t_3,t_2} \circ f_{t_2,t_1}$ is a restriction of f_{t_3,t_1}. In particular, $f_{t_2,t_1} = f_{t_1,t_2}^{-1}$.

(c) Fix $t_0 \in [0, T)$. Then $(f_{t_0+t,t_0})_{0 \le t < T - t_0}$ are the backward Loewner maps driven by $\lambda(t_0 + t)$.

(d) Fix $t_0 \in [0, T)$. Then $(f_{t_0-t,t_0})_{0 \le t < t_0}$ are the forward Loewner maps driven by $\lambda(t_0 - t)$.

Define $L_{t_2,t_1} = \mathbb{H} \backslash f_{t_2,t_1}(\mathbb{H})$, which are all $\mathbb{H}$-hulls by the forward theory, and $f_{t_2,t_1} = L_{t_2,t_1}$. Also, $L_{t,0} = L_t$.

Lemma 4 (Lemma 3.2, Lemma 3.3 [2]). *For $0 \leq t < T$, L_t is an $\mathbb{H}$-hull with $hcap L_t = 2t$, and $f_t = f_{L_t}$. If $0 \leq t_1 < t_2 < T$, then $L_{t_1} \prec L_{t_2}$ and $S_{L_{t_1}} \subset S_{L_{t_2}}$. For any fixed $t_0, (L_{t_0} : L_{t_0 - t})_{0 \leq t < t_0}$ are chordal Loewner hulls for $\lambda(t_0 - t)$.*

When $\lambda(t) = \sqrt{\kappa} B_t$ for a Brownian motion B_t, the resulting process is the **backward chordal SLE$_\kappa$** process. There are also backward radial and backward covering Loewner equations which play an important role in this paper, but they have been omitted from this summary.

11.3.2 Traces

Note that we do not have $L_{t_1} \subset L_{t_2}$ for $t_1 \subset t_2$, so this family of hulls is not increasing in the sense that forward chordal Loewner hulls define an increasing family. This makes defining "a" trace more complicated, so we must take a different approach.

Suppose for all t_0, the forward Loewner equation driven by $\lambda(t_0 - t)$ generates a forward chordal trace, which we will denote $\beta_{t_0}(t_0 - t)$. Then λ generates a backward chordal trace β_{t_0} with $L_{t_2, t_1} = \beta_{t_2}[t_1, t_2]$ for each t.

Since $L_t = \beta_t[0, t]$ for each t, we get that $f_{t_2, t_1}(\beta_{t_1}(t)) = \beta_{t_2}(t)$. Note, the path flows from the tip down to the base, and grows from the base rather than from the tip as in the foward case.

We have shown how the backward Loewner equation can create a family of curves, but what we want is some form of global trace for the entire process. We use the following Lemma to construct such a trace.

Lemma 5 (Lemma 3.6 [2]). *There exists a family of conformal maps $(F_{T,t})_{t<T}$ defined on $\mathbb{H}$ so that*

$$F_{T, t_1} = F_{T, t_2} \circ f_{t_2, t_1}$$

in $\mathbb{H}$ for all $t_1 \leq t_2$.

Using this lemma, we can define a trace β by $\beta(t) = F_{T, t_0}(\beta_{t_0}(t))$. Define $F_T = F_{T,t} \circ f_t$. Then we can normalize this family by requiring

$$F_T(\lambda(0)) = \lambda(0), \text{ and } F_T(\lambda(0) + i) = \lambda(0) + i. \tag{3}$$

With this, the curve β depends only on λ and is called the **normalized backward chordal trace** driven by λ. It satisfies $\beta(0) = \lambda(0)$ and $\lambda(0) + i \notin \beta$.

Since we know that forward SLE$_\kappa$ always generates a trace, we know that backward chordal SLE$_\kappa$ generates traces and a normalized global trace β.

11.3.3 Weldings

Let β be an $\mathbb{H}$-simple curve. Then β is an $\mathbb{H}$-hull with B_β =point $\in \mathbb{R}$ and $S_\beta \subset \mathbb{R}$ a compact interval. Then f_β can be extended continuously to $\overline{\mathbb{H}}$ with $f_\beta(S_\beta) = \beta$. The endpoints of S_β both go to B_β. There is a unique $z_\beta \in S_\beta$ which is sent to the tip of β, and this point divides S_β into two components, each mapped continuously and bijectively onto β by f_β. This induces the involution ϕ_β of $S_\beta : S_\beta \to S_\beta$ which satisfies $y = \phi_\beta(x) \implies f_\beta(x) = f_\beta(y)$. The map ϕ_β is the **welding** induced by β.

Let $(L_t)_{t<T}$ be backward chordal Loewner hulls generated by λ which generates traces $L_t = \beta_t[0,t]$ and has support $S_t = S_{L_t}$. Then the family of supports is increasing and we can consider $S_T := \cup_{t<T} S_t$. By the properties of the general Loewner maps, we can see that if $t_1 < t_2$, then $\phi_{t_2|S_{t_1}} = \phi_{t_1}$. Thus, there is a unique involution $\phi : S_T \to S_T$ so that $\phi(x) = y$ if and only if $f_t(x) = f_t(y)$ for some $t < T$. The map ϕ is the **welding induced by the process**. If F_T is the normaliized map used to define the global trace, then $F_T(x) = F_T(\phi(x))$ for every $x \in S_T$. In the case of a backward chordal SLE_κ, $S_T = \mathbb{R}$. Welding is well preserved by conformal mappings.

Proposition 6. *Suppose (L_t) is backward chordal Loewner hulls which are all $\mathbb{H}$-simple curves and that W is an $\mathbb{R}$-symmetric map. Suppose that ϕ is the welding induced by this process. Then $W^*(L_t)$ is a family of backward Loewner hulls which are all simple curves. Let ϕ_W denote the welding induced by the new family. Then*

$$\phi_W = W \circ \phi \circ W^{-1}.$$

11.4 Backward SLE with force points

Let $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n$, and suppose $x_0, q_1, \dots, q_n \in \mathbb{R}$ so that $x_0 \neq q_k$ for each k. Let $\lambda(t)$ and $f_t^\lambda(z)$ solve the B_t-adapted SDEs:

$$d\lambda(t) = \sqrt{\kappa} dB_t + \sum_{k=1}^{n} \frac{-\rho_k}{\lambda(t) - f_t^\lambda(q_k)} dt, \text{ and } \lambda(0) = x_0, \tag{4}$$

where f_t^λ is the backward chordal Loewner map driven by λ. The process driven by λ is called a **backward chordal SLE**$(\kappa; \rho)$ **process started from** x_0 **with force points** $(q_1, \dots, q_n)$. We write this as the process is started from $(x_1; q_1, \dots, q_n)$.

Definition 7. *Let $\kappa_1, \kappa_2 > 0$ and $\rho_1, \rho_2 \in \mathbb{R}$. Let $z_1, z_2 \in \mathbb{R}$ be distinct. We say that a backward chordal SLE(κ_1, ρ_1) process started from $(z_1; z_2)$* ***commutes*** *with a backward SLE(κ_2, ρ_2) process from (z_2, z_1) if there is a coupling of the two processes $\{L_j(t) : 0 \leq t < T_j\}$ so that*

(a) Each $L_j(t)$ os a complete backward chordal SLE(κ_j, ρ_j) process started from (z_j, z_k),

(b) If $\bar{t}_k < T_k$ is any stopping time with respect to $\mathcal{F}_t^k$, the sigma field generated by $L_k(t)$, then

$$\{(f_k(\bar{t}_k, \cdot)^*(L_j(t)) : 0 \leq t < T_j(\bar{t}_k)\}$$

has the distribution of a partial backward chordal SLE(κ_j, ρ_j) process started from $(f_k(\bar{t}_k, z_j); \lambda_k(\bar{t}_k))$,

where $f_k(\bar{t}_k, \cdot) = f_{L_k(\bar{t}_k)}$ and $T_j(\bar{t}_k) = \sup\{t > 0 : S_{L_k(\bar{t}_k)} \cap S_{L_j(t)} = \emptyset\}$.

Theorem 8 (Theorem 5.2 and 6.1, [2])**.** *For any $\kappa > 0$, any backward chordal(resp. radial) SLE$(\kappa; -\kappa - 6)$ process started from $(z_1; z_2)$ commutes with any backward chordal(resp. radial) SLE$(\kappa, -\kappa - 6)$ process started from $(z_2; z_1)$. In the radial case, both processes a.s. induce the same welding ψ.*

The proof of this theorem is the heart of the paper. Stochastic calculus is used to craft a weight used to create local couplings for the process. To create a global coupling, the techinique developed in [3] is used to create local couplings μ_n on an appropriate dense subset of hulls. The global coupling is the limit of μ_n in an appropriate topology.

11.5 Weldings and reversability of backward SLE

Proof of Theorem 1. Let $L_1(t), L_2(t)$ be two backward radial SLE$(\kappa, -\kappa - 6)$ process which commute, and therefore induce the same welding ψ by Theorem 8. Let $W_j : \mathbb{D} \to \mathbb{H}$ be appropriate Möbius transformations so that $W_2 = h \circ W_1$. By Corollary 4.8 in [2], $K_j(t) := W_j^*(L_j(t))$ is a backward chordal SLE$_\kappa$ after a time change which induces a welding ϕ_j for each j. Then we know that ϕ, ϕ_1, and ϕ_2 all have the same distribution. By Proposition 6, we have that $\phi_j = W_j \circ \psi \circ W_j^{-1}$ for each j. Therefore, $\phi_2 = W_2 \circ \psi \circ W_2^{-1} = h \circ (W_1 \circ \psi \circ W_1^{-1}) \circ h = h \circ \phi_1 \circ h$. □

We can show how the symmetry of the welding is used to prove the reversability of backward chordal SLE.

Proof idea for Theorem 2: First, let ϕ_1 and ϕ_2 be two backward chordal SLE$_\kappa$ weldings with corresponding global traces β_1 and β_2. Recall that to construct the traces, we created conformal maps $F_j : \mathbb{H} \to \mathbb{C}\backslash\beta$ with $F_j(0) = 0$ and $F_j(i) = i$. By construction, this map satisfied $F_j(x) = F_j(\phi_j(x))$ for each $x \in \mathbb{R}$. By Theorem 2, we can assume that ϕ_1 and ϕ_2 are coupled so that $\phi_2 = h \circ \phi_1 \circ h$. It suffices to show that $h(\beta_2\backslash\{0\}) = \beta_1\backslash\{0\}$. Let $G = h \circ F_2 \circ h \circ F_1^{-1} : \mathbb{C}\backslash\beta_1 \to \mathbb{C}$ be conformal. We show that G extends continuously to $\beta_1\backslash\{0\}$ and satisfies $G(\beta_1\backslash\{0\}) = h(\beta_2\backslash\{0\})$, and then it is shown that G is the identity map.

□

References

[1] Miller, Jason and Sheffield, Scott. *Imaginary geometry III: reversibility of SLE$_\kappa$ for $\kappa \in (4, 8)$*. Arxiv, http://arxiv.org/abs/1201.1498

[2] Rhode, Steffon and Zhan, Dapeng. *Backward SLE and the symmetry of the welding*. Arxiv, http://arxiv.org/abs/1307.2532

[3] Zhan, Dapeng. *Reversability of Chordal SLE*. Ann. Probab. Volume 36, Number 4 (2008), 1472-1494.

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12 Random conformal weldings

after K. Astala, P. Jones, A. Kupiainen and E. Saksman [2, 3]
A summary written by Miika Nikula

Abstract

Define a homeomorphism $\phi : \mathbb{T} \to \mathbb{T}$ by formally setting

$$\tau(dx) = \exp\left(\beta \mathrm{GFF}|_{\mathbb{T}}(e^{2\pi i x})\right) dx,$$

where $\beta > 0$ is a parameter and $\mathrm{GFF}|_{\mathbb{T}}$ is the two-dimensional Gaussian free field restricted to the unit circle, and taking $\phi(e^{2\pi i x}) = \exp\left(2\pi i \frac{\tau([0,x))}{\tau([0,1))}\right)$. The conformal welding problem for the random homeomorphism ϕ is shown to almost surely have a unique solution, a random Hölder continuous planar Jordan curve.

12.1 Overview

Formally, the two-dimensional Gaussian free field is a Gaussian process X on the plane with the covariance structure

$$\mathbb{E}X(z)X(w) = \log \frac{1}{|z-w|}, \quad z, w \in \mathbb{C}.$$

Due to the logarithmic singularity of the covariance kernel, X is a random distribution rather than a random pointwise function. Given a parameter $\beta^2 < 2$, in Section 12.2 we see how to give a precise meaning for a random measure $\tau(dx) \propto \exp\left(\beta X(e^{2\pi i x})\right) dx$ and thus define a random circle homeomorphism $\phi_\beta(e^{2\pi i x}) = \phi(e^{2\pi i x}) = \exp\left(2\pi i \frac{\tau([0,x))}{\tau([0,1))}\right)$.

The main theorem in [2, 3] is the solution of the conformal welding problem for the random homeomorphism ϕ, thus associating a planar Jordan curve to ϕ. It is easiest to explain conformal welding concretely through its inverse process. Let Γ be a given Jordan curve and write $\mathbb{C} = \Omega^+ \cup \Gamma \cup \Omega^-$ where $\Omega+$ and Ω^- are disjoint domains. Denoting the exterior disk by $\mathbb{D}_\infty = \mathbb{C} \setminus \overline{\mathbb{D}}$, by the Riemann mapping theorem there exist conformal maps $f_+ : \mathbb{D} \to \Omega^+$ and $f_- : \mathbb{D}_\infty \to \Omega^-$. Since $\partial\Omega^+ = \partial\Omega^- = \Gamma$ is a Jordan curve, by Carathéodory's extension theorem the maps f_+ and f_- may be extended to continuous homeomorphisms $\overline{\mathbb{D}} \to \Omega \cup \Gamma$ and $\overline{\mathbb{D}_\infty} \to \Omega^- \cup \Gamma$ respectively (the extensions are still denoted by f_+ and f_-). We may thus form the composition $\phi = f_+^{-1}|_\Gamma \circ f_-|_\mathbb{T} : \mathbb{T} \to \mathbb{T}$, which is a homeomorphism by construction. The ϕ obtained this way is called the welding homeomorphism of the Jordan curve Γ. In conformal welding the problem is to recover the curve Γ that corresponds to a given homeomorphism ϕ. Generally both existence and uniqueness of a solution are nontrivial.

With this preparation, the main theorem may be stated as follows.

Theorem 1. *Let $\beta^2 < 2$. Then the welding problem associated to $\phi = \phi_\beta$ almost surely has a solution. Explicitly, almost surely there exists a domain Ω_+ bounded by a Jordan curve Γ and conformal maps $f_+ : \mathbb{D} \to \Omega_+$, $f_- : \mathbb{D}_\infty \to \mathbb{C} \setminus \bar{\Omega}_+$ for which $f_+^{-1}|_\Gamma \circ f_-|_\mathbb{T} = \phi$. Moreover, the Jordan curve Γ is unique (up to Möbius transformations) and Hölder continuous.*

This theorem is further elaborated on by proving that almost surely, the welding curve depends continuously on the parameter β. More precisely, almost surely for a fixed realization of the free field one may consider a parametric family of curves Γ_β that depends continuously on $\beta^2 < 2$.

12.2 A white noise decomposition of the Gaussian free field

The first technical issue is to give a precise meaning to the exponential of the two-dimensional Gaussian free field on the circle. Following Bacry and Muzy [4], the restriction of the free field onto the circle is represented by a white noise decomposition, bringing a transparent geometric interpretation for the correlation structure of the field.

It simplifies the notation to make the identification $\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \cong [0,1)$ and define a periodic Gaussian process on $\mathbb{R}$. Let $\lambda(dx\,dy) = \frac{dx\,dy}{y^2}$ denote the hyperbolic area measure on the upper half plane and let w denote the white noise on $[0,1) \times \mathbb{R}_+$ with the control measure λ. This means that $w = (w(A))$ is a centered Gaussian process indexed by those Borel sets contained in $[0,1) \times \mathbb{R}_+$ that satisfy $\lambda(A) < \infty$, whose covariance structure is given by

$$\mathbb{E}w(A)w(B) = \lambda(A \cap B) \quad \text{for } A, B \subset [0,1) \times \mathbb{R}_+ \text{ s.t. } \lambda(A), \lambda(B) < \infty.$$

To define a periodic white noise on the upper half-plane, first extend $w(A) = w\left(A \cap ([0,1) \times \mathbb{R}_+)\right)$ and then set $W(A) = \sum_{n \in \mathbb{Z}} w(A + n)$ for Borel sets

$A \subset \mathbb{H}$ such that $\lambda(A) < \infty$. Define the set

$$H = \left\{ (x, y) \in \mathbb{H} : -\frac{1}{2} < x < \frac{1}{2} \text{ and } y > \frac{2}{\pi} \tan |\pi x| \right\}.$$

By computation, formally we have

$$\mathbb{E} W(H + x) W(H' + x) = 2 \log 2 + \log \frac{1}{2 \sin \pi |x - x'|} \quad \text{for all } x, x' \in \mathbb{R}. \quad (1)$$

The hyperbolic area of the set H is itself not finite, so to get well-defined processes in the usual sense one regularizes by defining

$$H_\varepsilon(x) = W\left((H + x) \cap \{y \geq \varepsilon\}\right)$$

for $x \in \mathbb{R}$ and $\varepsilon \in (0, 1)$. As $\varepsilon \searrow 0$ the covariance of the process H_ε converges to that given by (1).

From (1) we have

$$\lim_{\varepsilon \searrow 0} \mathbb{E} H_\varepsilon(x) H_\varepsilon(x') = 2 \log 2 + \log \frac{1}{|e^{2\pi i x} - e^{2\pi i x'}|}.$$

Up to a random scalar, the limit of the processes $(H_\varepsilon(x))_{x \in \mathbb{R}}$ as $\varepsilon \searrow 0$ is the restriction of the Gaussian free field on the circle $\mathbb{T} \cong [0, 1)$ in the sense that is has the same covariance structure. While the convergence also holds in the sense of distributions, the exponential may be defined without reference to the distributional limit by setting, for a parameter $\beta > 0$,

$$\tau_{\beta,\varepsilon}(dx) = \tau_\varepsilon(dx) = e^{\beta H_\varepsilon(x) - \frac{\beta^2}{2} \mathbb{E} H_\varepsilon(x)^2} dx$$

and taking the limit $\varepsilon \searrow 0$. From the construction it follows that for any given $x \in \mathbb{R}$ the density $e^{\beta H_\varepsilon(x) - \frac{\beta^2}{2} \mathbb{E} H_\varepsilon(x)^2}$ is a martingale. By the martingale convergence theorem, the almost sure weak limit

$$\tau(dx) = \lim_{\varepsilon \searrow 0} \tau_\varepsilon(dx)$$

exists.

This kind of construction of a random measure is known as Gaussian multiplicative chaos. Kahane [6] and later work on the measures have established the following basic properties of the random limit measure $\tau = \tau_\beta$.

Theorem 2. *1. For $\beta^2 < 2$, almost surely τ_β satisfies $\tau_\beta(I) > 0$ for any interval $I \subset \mathbb{R}$. Further $\mathbb{E}\tau(I) = |I|$.*

2. For $\beta^2 < 2$ and $-\infty < p < 2/\beta^2$, $\mathbb{E}\tau_\beta(I)^p < \infty$ for any interval $I \subset \mathbb{R}$.

3. For $\beta^2 < 2$, the measure τ_β almost surely has no atoms.

4. For $\beta^2 \geq 2$, $\tau_\beta \equiv 0$ almost surely.

The integral of the exponentiated two-dimensional Gaussian free field restricted to the unit circle, i.e. the homeomorphism for which the welding problem is to be solved, is defined as

$$h_\beta(x) = h(x) = \frac{\int_0^x \tau(dx)}{\tau([0,1))} \quad \text{for } x \in \mathbb{R}. \tag{2}$$

By the properties of τ given in Theorem 2, $h : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism which is periodic in the sense that $h(x+1) = h(x) + 1$ for all $x \in \mathbb{R}$. Explicitly on the circle, the corresponding homeomorphism is

$$\phi_\beta(e^{2\pi i x}) = \phi(e^{2\pi i x}) = \exp\left(2\pi i h(x)\right). \tag{3}$$

12.3 Welding problem and the Beltrami equation

The theory of the Beltrami equation provides an effective tool for solving the welding problem. To briefly explain this approach, let $\phi : \mathbb{T} \to \mathbb{T}$ be a homeomorphism. Suppose ϕ may be extended to a homeomorphism $f : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ which is locally quasiconformal, i.e. satisfies $f \in W^{1,1}_{loc}$ and the Beltrami equation

$$f_{\bar{z}}(z) = \mu(z) f_z(z) \quad \text{for a.e. } z \in \mathbb{D} \tag{4}$$

with $\sup_{z\in K} |\mu(z)| < 1$ for all compact $K \subset \mathbb{D}$. Further suppose one may find a homeomorphic solution $F : \mathbb{C} \to \mathbb{C}$ to the equation

$$F_{\bar{z}}(z) = \chi_{\mathbb{D}}(z)\mu(z) f_z(z) \quad \text{for a.e. } z \in \mathbb{C}, \tag{5}$$

where the dilatation μ is the same as in (4). Since F is a homeomorphism, the image $\Gamma = F(\mathbb{T})$ is a Jordan curve and moreover by (5) the restriction $f_- = F|_{\mathbb{D}_\infty}$ is a conformal mapping of $\mathbb{D}_\infty$ onto $\Omega_- = F(\mathbb{D}_\infty)$.

The existence of a solution to the welding problem is thus assured once one finds a conformal map of $\mathbb{D}$ onto $\Omega_+ = F(\mathbb{D})$. One of the fundamental results of the theory of quasiconformal mappings is that in the case $||\mu||_\infty < 1$ the solutions to (4) are unique up to postcomposition with a conformal mapping. Thus in this case, since both f and F solve (4), one may define $f_+ = F \circ f^{-1}$ and obtain the desired conformal map $f_+ : \mathbb{D} \to \Omega_+$. But by the assumption that f is locally quasiconformal and the fact that conformality is a local property, the same definition in fact gives a conformal mapping also in the general case.

The method of using the Beltrami equation thus reduces the conformal welding problem to finding the locally quasiconformal extension $f : \mathbb{D} \to \mathbb{D}$ and finding the homeomorphic solution F to the equation (5).

12.4 Beurling–Ahlfors extension

The extension of the random homeomorphism $\phi : \mathbb{T} \to \mathbb{T}$ as constructed in (3) is effected by the Beurling–Ahlfors extension. The formula is simpler to write down on the upper half-plane $\mathbb{H}$. By the conformal map $z \mapsto \frac{1}{2\pi i} \log z$ we obtain a one-to-one correspondence between homeomorphisms of the circle and homeomorphisms $h : \mathbb{R} \to \mathbb{R}$ that satisfy $h(x+1) = h(x)+1$. For $x \in \mathbb{R}$ and $0 < y < 1$ such a homeomorphism h is extended as

$$\Phi(x+iy) = \begin{cases} \frac{1}{2}\int_0^1 \big(h(x+ty)+h(x-ty)\big)dt + i\int_0^1 \big(h(x+ty)-h(x-ty)\big)dt, & 0 < y < 1 \\ x+iy+(2-y)c_0, & 1 \le y < 2 \\ x+iy, & 2 \le y \end{cases}$$

with $c_0 = \int_0^1 h(t)dt - \frac{1}{2}$. It is easy to check that this definition gives a continuously differentiable homeomorphism $\Phi : \mathbb{H} \to \mathbb{H}$. The desired extension of ϕ is thus obtained as

$$f(z) = \exp\left(2\pi i \Phi\left(\frac{1}{2\pi i}\log z\right)\right). \tag{6}$$

The essential property of the Beurling–Ahlfors extension is that it admits the following estimate for the distortion function of the extension in terms of the original boundary homeomorphism. Consider the Whitney decomposition $\{C_I\}$ of $\mathbb{H}$ close to the boundary. Explicitly, let $\mathcal{D}$ denote the dyadic subintervals of $[0,1]$ and denote $C_I = I \times [|I|, 2|I|]$. Suppose h is obtained from a measure τ as in (2). Then for $I \in \mathcal{D}$ the distortion function satisfies

$$K_\Phi(z) \le CK_\tau(I) = C\sum_{J,J'} \frac{\tau(J)}{\tau(J')} \quad \text{for all } z \in C_I, \tag{7}$$

where the sum runs over dyadic intervals of length $2^{-4}|I|$ that are contained in I or its dyadic neighbors of the same length.

12.5 Lehto's method

The existence of a solution to the Beltrami equation (5) is proved by using a condition due to Lehto [7]. For a function $K \in L^1_{loc}(\mathbb{C})$ such that $K(z) \geq 1$ for a.e. $z \in \mathbb{C}$, define the Lehto integral by

$$L(z, r, R) = L_K(z, r, R) = \int_r^R \frac{1}{\int_0^{2\pi} K(z + \rho e^{i\theta}) d\theta} \frac{d\rho}{\rho}.$$

In geometric terms, the importance of Lehto integrals is that they may be used to control the distortion of annuli under a locally quasiconformal mapping F with the given distortion function K. The bound eventually used to obtain Hölder continuity for the random welding curves is

$$\operatorname{diam}\big(F(B(z, r))\big) \leq 16 \exp\left(-2\pi^2 L_K(z, r, R)\right) \operatorname{diam}\big(F(B(z, R))\big). \tag{8}$$

It is this kind of geometric control of the distortion of annuli that leads to the following theorem of Lehto, a condition for the existence of a homeomorphic solution to the Beltrami equation.

Theorem 3. *Let $\mu : \mathbb{C} \to \mathbb{C}$ have compact support and satisfy $|\mu(z)| < 1$ for almost every $z \in \mathbb{C}$. Suppose the associated distortion function $K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$ satisfies $K \in L^1_{loc}(\mathbb{C})$ and*

$$\lim_{r \searrow 0} L_K(z, r, 1) = \infty \quad \text{for a.e. } z \in \mathbb{C}.$$

Then the Beltrami equation

$$F_{\bar{z}}(z) = \mu(z) F_z(z) \quad \text{for a.e. } z \in \mathbb{C}$$

admits a homeomorphic $W^{1,1}_{loc}(\mathbb{C})$-solution F.

Remark. The assumptions of Lehto's theorem are too weak guarantee the uniqueness of the solution.

12.6 Distortion estimates for the random mappings

The technical core of the article [3] consists of obtaining strong enough probabilistic estimates for the distortion function K_f of the Beurling–Ahlfors extension f (recall (6)) of the random homeomorphism ϕ obtained by integration of the Gaussian free field on the circle. The main estimate is the following theorem.

Theorem 4. *The Lehto integrals of the Beurling–Ahlfors extension f of the random homeomorphism $\phi : \mathbb{T} \to \mathbb{T}$ satisfy, for $z_0 \in \mathbb{T}$,*

$$\mathbb{P}\left(L_{K_f}(z_0, 2^{-N}, 1) \leq \delta N\right) \leq 2^{-(1+b)N}$$

for some $b > 0$, for all N large enough.

By (7) the proof of this estimate reduces to estimating the measure τ, which is in turn effected through the white noise decomposition. Essentially, the white noise decomposition gives a multi-scale decomposition of the measure τ, which allows one to decompose the Lehto integral to a sum of nearly independent terms.

12.7 Existence and uniqueness of the random weldings

From Theorem 4 one would like to show that almost surely, for $z \in \mathbb{T}$ one has

$$L_{K_f}(z, r, 1) \geq a \log \frac{1}{r} \quad \text{for } r \in (0, 1)$$

for some (random) constant $a > 0$. Theorem 3 would then give the existence of the solution to the Beltrami equation (5) and thus, by the argument of Section 12.3, the existence part of Theorem 1. On its own Theorem 4 implies slightly less than this. By a Borel–Cantelli argument one obtains the almost sure existence of a set $\{z_{n,k} : n \in \mathbb{N}, k = 1, 2, \ldots, N_n\}$, where $N_n \simeq 2^{(1+b/2)n}$, such that

$$L_{K_f}(z_{n,k}, 2^{-n}, 1) \geq a \log \frac{1}{2^{-n}} \quad \text{for all } n, k.$$

Carefully retracing the steps in the proof of Theorem 4 (as given e.g. in [1]), this turns out to be enough to prove the existence part of Theorem 1. Applying the estimate (8) also gives the claim on Hölder continuity.

Uniqueness of the welding curve is proven by appealing to conformal removability of Hölder continuous Jordan curves, which is a result due to Jones and Smirnov [5]. Suppose there are two solutions to the welding problem, i.e. two pairs of conformal maps f_+, f_- and g_+, g_- that solve the welding problem for the homeomorphism ϕ. One may then define the map

$$\Psi(z) = \begin{cases} g_+ \circ f_+^{-1}(z) & , z \in f_+(\mathbb{D}) \\ g_- \circ f_-^{-1}(z) & , z \in f_-(\mathbb{D}_\infty) \end{cases}$$

and observe that it is a well-defined homeomorphism $\mathbb{C} \to \mathbb{C}$ which is conformal off $f_+(\mathbb{T}) = f_-(\mathbb{T})$. As the Jordan curve $f_+(\mathbb{T})$ is Hölder continuous by the proven parts of Theorem 1, the theorem of Jones and Smirnov implies that Ψ is a conformal homeomorphism of $\mathbb{C} \to \mathbb{C}$ i.e. a Möbius transformation. This is the desired uniqueness statement.

References

[1] K. Astala, T. Iwaniec, G. Martin: *Elliptic partial differential equations and quasiconformal mappings in the plane.* Princeton University Press (2009)

[2] K. Astala, P. Jones, A. Kupiainen, E. Saksman: *Random curves by conformal welding.* Comptes Rendus Mathematique, 348(5–6), p. 257–262. (2010)

[3] K. Astala, P. Jones, A. Kupiainen, E. Saksman: *Random conformal weldings.* Acta Mathematica, 207(2), p. 203–254. (2012)

[4] E. Bacry, F. Muzy: *Log-infinitely divisible multifractal processes.* Communications in Mathematical Physics, 236(3), p. 449–475. (2003)

[5] P. Jones, S. Smirnov: *Removability theorems for Sobolev functions and quasiconformal maps.* Arkiv för Matematik, 38, p. 263–279. (2000)

[6] J.-P. Kahane: *Sur le chaos multiplicatif.* Annales de sciences mathématiques du Québec, 9(2), p. 105–150. (1985)

[7] O. Lehto: *Homeomorphisms with a given dilatation.* Proceedings of the Fifteenth Scandinavian Congress, p. 58–73. (1968)

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13 Stochastic Loewner evolution, part 1

after M. Zinsmeister [1] (chapters 1 – 3)
A summary written by Eveliina Peltola

Abstract

We give an introduction to the Loewner differential equation, originally discovered by Charles Loewner[1] in 1923 while studying the Bieberbach conjecture. Solutions of this equation, called Loewner chains, arise as well in the theory of stochastic Loewner evolutions (SLEs), invented by Oded Schramm[2] in 1999.

13.1 Preliminaries in complex analysis

In this section, we introduce the half–plane capacity which can be thought of as a way of measuring the size of a compact subset of the upper half plane.

A subset $\Omega \subset \mathbb{C}$ is called a *domain* if it is nonempty, open and connected. If also $\hat{\mathbb{C}} \setminus \Omega$ is connected, Ω is *simply connected*. For example, the upper half plane $\mathbb{H}$ and the open unit disc $\mathbb{D}$ are simply connected domains. We call a holomorphic bijection $f : \Omega \to f(\Omega)$ a *conformal isomorphism*. Recall that, by the Riemann mapping theorem, there exists a conformal isomorphism between any two proper simply connected subdomains of $\mathbb{C}$.

Let us now investigate subsets of $\mathbb{H}$. A bounded set $K \subset \mathbb{H}$ is called a compact *hull* if $K = \overline{K} \cap \mathbb{H}$ and the complement $\mathbb{H} \setminus K$ is simply connected. A crucial property of hulls is the following, proved in [1], for instance.

Lemma 1. *Let $K \subset \mathbb{H}$ be a compact hull. Then there exists a unique conformal isomorphism $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that $\lim_{z\to\infty} \Big(g_K(z) - z \Big) = 0$.*

The map g_K is called the *hydrodynamically normalized* conformal isomorphism associated to K. The *half–plane capacity* of K is defined by

$$\mathrm{hcap}(K) := \lim_{z\to\infty} \Big(g_K(z) - z \Big) z.$$

One can show that $\mathrm{hcap}(K) > 0$, unless $K = \emptyset$, in which case $\mathrm{hcap}(\emptyset) = 0$. We also have a bunch of nice properties [1], summarized in the next lemma.

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[1] Loewner, C., *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I.* Math. Ann., **89** (1923), pp. 103–121.

[2] Schramm, O., *Scaling limits of loop-erased random walks and uniform spanning trees.* Israel J. Math., **118** (2000), pp. 221–288.

Lemma 2. *The conformal map* $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ *and the half–plane capacity* $\mathrm{hcap}(K)$ *satisfy the following properties.*

(a) Monotonicity: *If* $K \subset K' \subset \mathbb{H}$ *are two compact hulls, then*

$$\mathrm{hcap}(K') = \mathrm{hcap}(K) + \mathrm{hcap}(g_K(K' \setminus K)).$$

In particular, the map $\cdot \mapsto \mathrm{hcap}(\cdot)$ *is strictly increasing.*

(b) Continuity: *If* $\varepsilon > 0$ *and* $K \subset K' \subset \mathbb{H}$ *are two compact hulls such that* $\mathrm{dist}(z, K) < \varepsilon$ *for any* $z \in \partial K'$*, then there exists* $C \geq 0$ *such that*

$$\mathrm{hcap}(K') \leq \mathrm{hcap}(K) + C\varepsilon^{1/3}\mathrm{diam}(K').$$

(c) Homogeneity: $\mathrm{hcap}(\cdot)$ *is invariant under translations of* $\mathbb{H}$*, and for any* $\lambda > 0$*, under the scaling* $z \mapsto \lambda z$ *we have* $\mathrm{hcap}(\lambda\, \cdot) = \lambda^2\, \mathrm{hcap}(\cdot)$.

(d) For any $z \in \mathbb{H} \setminus K$*, we have* $|g_K(z) - z| \leq 3\,\mathrm{rad}(K)$.

(e) There exists $c \geq 0$ *so that for any* $|z| \geq \mathrm{rad}(K)$ *we have*

$$|z - g_K(z) + \frac{\mathrm{hcap}(K)}{z}| \leq c\,\frac{\mathrm{rad}(K)\,\mathrm{hcap}(K)}{|z|^2}.$$

13.2 Loewner differential equation

In this section, we introduce Loewner chains and the Loewner equation. In general, a Loewner chain in a simply connected domain Ω is a collection of conformal maps $(g_t)_{t\geq 0}$ – or equivalently, a growing family of compact hulls $(K_t)_{t\geq 0}$ in Ω. The characterizing property of Loewner chains is that, for a fixed z, the maps $t \mapsto g_t(z)$ satisfy the Loewner differential equation. The explicit form of the equation depends on Ω, but it has nice conformal transformation properties, presented in [2]. We may thus let Ω to be $\mathbb{H}$ or $\mathbb{D}$.

13.2.1 Chordal Loewner equation

For simplicity, let us first consider slit domains which are complements of hulls $K_t = \gamma(0, t]$ for an injective curve γ in the upper half plane $\mathbb{H}$. In the *chordal* case, the starting and end points of γ lie on the boundary of $\mathbb{H}$, whereas, if instead the end point of γ is an interior point, the process is called *radial*. In 13.2.2, we shall briefly describe radial Loewner chains in $\mathbb{D}$.

We will also later generalize the special case of slit domains to apply for curves with self–touchings, as well as more general simply connected domains arising from families of hulls which satisfy a certain local growth condition.

Slit domains. Let $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ be a continuous injective curve such that $\gamma(0) = 0$, $\gamma(0,\infty) \subset \mathbb{H}$ and $\lim_{t\to\infty} \gamma(t) = \infty$. Denote by $K_t = \gamma(0,t]$ and $\mathbb{H}_t = \mathbb{H} \setminus K_t$. The sets K_t are compact hulls generated by the curve γ and they are growing the sense that $K_s \subset K_t$ for $s < t$. Let $g_t = g_{K_t} : \mathbb{H}_t \to \mathbb{H}$ be the conformal map given by Lemma 1, so that around $z = \infty$,

$$g_t(z) = z + \frac{b(t)}{z} + \mathcal{O}(|z|^{-2}), \tag{1}$$

where $b(t) = \mathrm{hcap}(K_t) > 0$. By Lemma 2(a & b), b is continuous and strictly increasing in $[0,\infty)$. In particular, by a time reparametrization, we may assume that $b \in C^1[0,\infty)$. In the literature, the choice $b(t) = 2t$ is often used.

Theorem 3. *For any $t \in [0,\infty)$, the limit $U_t := \lim_{z_n \to \gamma(t)} g_t(z_n)$ along any sequence $(z_n)_{n\in\mathbb{N}}$ converging to $\gamma(t)$ in $\mathbb{H}_t$ exists and it defines a continuous function $t \mapsto U_t$ in $[0,\infty)$. Moreover, for any $z \in \mathbb{H}_t$, the process $t \mapsto g_t(z)$ satisfies the* chordal Loewner differential equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{\frac{d}{dt} b(t)}{g_t(z) - U_t} \tag{2}$$

with the initial condition $g_0(z) = z$. The solution is defined up to the lifetime $T_z = \inf\{t \geq 0 \mid z \in K_t\} \in [0,\infty]$.

Proof. By the general theory of boundary behaviour of conformal maps, since $\partial\mathbb{H}_t$ is locally connected, $g_t^{-1} : \mathbb{H} \to \mathbb{H}_t$ extends continuously to $\overline{\mathbb{H}}$, and since $\gamma(t)$ is not a cut point of $\partial\mathbb{H}_t$, there exists $U_t \in \mathbb{R}$ such that $g_t^{-1}(U_t) = \gamma(t)$. By techniques related to prime ends, the limit U_t can be seen to exist [3].

Using Lemma 2(d) for the hull $K_{t,t+\delta} = g_t(\gamma(t,t+\delta])$ and the observation that $g_{K_{t,t+\delta}} \circ g_t = g_{t+\delta}$, we have

$$|g_{t+\delta}(z) - g_t(z)| \;=\; |g_{K_{t,t+\delta}}(g_t(z)) - g_t(z)| \;\leq\; 3\,\mathrm{rad}(K_{t,t+\delta})$$

uniformly for all $z \in \mathbb{H}_t$. By continuity of γ and the Beurling estimate, one can show [2] that $\mathrm{rad}(K_{t,t+\delta}) \leq c\,\varphi(\delta)$ for an increasing function φ such that $\lim_{\delta\to 0+} \varphi(\delta) = 0$. Continuity of $t \mapsto U_t$ follows from taking $z \to \gamma(t)$.

Similarly, $g_{K_{t,t+\delta} - U_t}(g_t(z) - U_t) = g_{t+\delta}(z) - U_t$ and, by Lemma 2(a & c), $\mathrm{hcap}(K_{t,t+\delta} - U_t) = b(t+\delta) - b(t)$. Thus, we obtain for $\delta > 0$ small enough

$$g_{t+\delta}(z) - g_t(z) = \frac{b(t+\delta) - b(t)}{g_t(z) - U_t} + \varphi(\delta)\,(b(t+\delta) - b(t))\,\mathcal{O}\Big(|g_t(z) - U_t|^{-2}\Big),$$

by 2(e). Dividing both sides by δ and taking the limit $\delta \to 0$ gives (2). □

General Loewner chains. Let now $t \mapsto U_t$ be a given continuous real valued function on $[0, \infty)$ and $b \in C^1[0, \infty)$ an increasing function. By the theory of ODEs, (2) has a unique solution $t \mapsto g_t(z)$ with $g_0(z) = z$ which is called a *chordal Loewner chain* associated to the *driving function* U.

We shall next describe what kind of a local growth property is needed to construct general Loewner chains. A family $(K_t)_{t \geq 0}$ of compact hulls is called *right continuous* at $t \in [0, \infty)$ if $\bigcap_{\delta > 0} \overline{g_t(K_{t+\delta} \setminus K_t)} = \xi_t \in \mathbb{R}$ is a single point. If $(K_t)_{t \geq 0}$ is right continuous at all $t \in [0, \infty)$, we define its *driving function* by $t \mapsto \xi_t$. If, moreover, ξ is continuous and $\mathrm{hcap}(K_t)$ is increasing and belongs to $C^1[0, \infty)$, we say that $(K_t)_{t \geq 0}$ is a *continuously growing family of hulls*. The following theorem states that continuously growing hulls correspond one–to–one to Loewner chains with continuous driving functions.

Theorem 4. *Let $(g_t)_{t \geq 0}$ be a chordal Loewner chain. Then $z \mapsto g_t(z)$ is the unique hydrodynamically normalized conformal isomorphism from the domain $\mathbb{H}_t := \{z \in \mathbb{H} \mid T_z > t\}$ onto $\mathbb{H}$, with the expansion* (1) *around $z = \infty$. Moreover, the sets $K_t := \mathbb{H} \setminus \mathbb{H}_t$ form a continuously growing family of hulls.*

Conversely, let $(K_t)_{t \geq 0}$ be a continuously growing family of hulls. Then the hydrodynamically normalized conformal maps $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ define a chordal Loewner chain with $b(t) = \mathrm{hcap}(K_t)$ and the driving function ξ.

Proof. Let $(g_t)_{t \geq 0}$ be a chordal Loewner chain. Using (2), we can solve

$$\Im\mathfrak{m}(g_t(z)) = \Im\mathfrak{m}(z) \; \exp\Big\{ - \int_0^t ds \; \frac{\frac{d}{ds} b(s)}{|g_s(z) - U_s|^2} \Big\} \; > \; 0$$

which shows in particular that $g_t(\mathbb{H}_t) \subset \mathbb{H}$. Similarly, for $z, w \in \mathbb{H}_t$ we have $g_t(z) - g_t(w) = (z - w) \; \exp\Big\{ - \int_0^t ds \; \frac{\frac{d}{ds} b(s)}{(g_s(z) - U_s)(g_s(w) - U_s)} \Big\}$ and hence,

$$\frac{d}{dz} g_t(z) = \lim_{w \to z} \frac{g_t(z) - g_t(w)}{z - w} = \exp\Big\{ - \int_0^t ds \; \frac{\frac{d}{ds} b(s)}{(g_s(z) - U_s)^2} \Big\}.$$

This shows that $z \mapsto g_t(z)$ is holomorphic and injective in $\mathbb{H}_t$. To show that $g_t(\mathbb{H}_t) = \mathbb{H}$, we use the *reverse Loewner equation*, defined for $0 \leq s \leq t$ by

$$\frac{\partial}{\partial s} h_s(w) = \frac{-\frac{d}{ds} b(s)}{h_s(w) - U_{t-s}} \tag{3}$$

with the initial condition $h_0(w) = w$. This equation admits a unique solution for all $w \in \mathbb{H}$ because the map $s \mapsto \mathfrak{Im}(h_s(w)) > 0$ is strictly increasing, by a similar calculation as above. Now, $g_s(z) = h_{t-s}(w)$ is a solution of (2) with the initial condition $h_t(w) = z$, so that by uniqueness, $g_t(z) = h_0(w) = w$.

Let now $t \leq \tau := \inf\{0 \leq s < T_z \mid |g_s(z) - z| = R\} \wedge T$ and suppose $|z - U_0| \geq 4\left(\sup_{0\leq s\leq T}|U_s - U_0| \vee \sqrt{\frac{T}{2}\sup_{0\leq s\leq T}|\frac{d}{ds}b(s)|}\right)$. Then we have $|g_t(z) - U_t| \geq |z - U_0| - |U_0 - U_t| - |g_t(z) - z| \geq 2R$ and hence by (2),

$$|g_t(z) - z| \leq \int_0^t ds \frac{|\frac{d}{ds}b(s)|}{|g_s(z) - U_s|} \leq \frac{t}{2R} \sup_{0\leq s\leq t}\Big|\frac{d}{ds}b(s)\Big|. \tag{4}$$

Now $T = \tau$ because if $\tau < T$, equation (4) gives a contradiction. In particular, it follows that $K_T \subset B(U_0, 4R)$ since by the above, $g_t(z)$ is bounded away from U_t for all $t \leq T$ and $|z - U_0| \geq 4R$. Thus, K_T is a compact hull.

The same estimate applies to show that for any $t \geq 0$ and $\delta > 0$ we have

$$g_t(K_{t+\delta} \setminus K_t) \subset B\Big(U_t, 4\max\Big\{\sup_{0\leq s\leq\delta}|U_{t+s} - U_t|, \sqrt{\frac{\delta}{2}\sup_{0\leq s\leq\delta}\Big|\frac{d}{ds}b(s)\Big|}\Big\}\Big)$$

because the map $g_{g_t(K_{t+\delta}\setminus K_t)}$ is a Loewner chain associated to the driving function $\delta \mapsto U_{t+\delta}$ (this is a kind of a Markovian property). Taking $\delta \to 0$ we obtain right continuity of the hulls $(K_t)_{t\geq 0}$, by continuity of U.

The expansion (1) can be proved by the uniqueness part of Lemma 1, observing that when $z \to \infty$, taking also $T, R \to \infty$ we obtain $|g_t(z) - z| \to 0$.

The proof of the converse is a straightforward generalization of the proof of Theorem 3. For more details, the reader may look at [1] or [2]. We also recommend the lecture notes [3], which have a slightly different approach. □

If we relax the condition of continuity of ξ, we still obtain a solution of (2) but the Loewner chain will be branching. For hulls generated by curves with self–intersections, there is no branching but "jumps" in discontinuities of ξ. However, curves having only self–touchings generate continuously growing families of hulls – such processes occur in particular in the theory of SLEs. In turn, Lévy–Loewner evolution (LLE) is a branching generalization of SLE.

In view of the case of slit domains, it is natural to ask whether the hulls of a Loewner chain are generated by a curve γ, in the sense that for each $t \geq 0$, the domain $\mathbb{H}_t$ of g_t is given by the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$. In [1] it is proved that this is the case exactly when $\overline{K_t}$ are locally connected for each $t \geq 0$. In particular, not all Loewner chains are generated by curves.

13.2.2 Radial Loewner equation

In this section, we outline the main results concerning the radial Loewner equation in $\mathbb{D}$, in analogy with the chordal case – see [1] or [2] for details.

As in the chordal case, for a continuous injective curve γ from 1 to 0 in $\mathbb{D}$, there exists a unique conformal isomorphism $g_t : \mathbb{D} \setminus \gamma(0,t] \to \mathbb{D}$ such that

$$g_t(z) = e^{b(t)} z + \mathcal{O}(|z|^2)$$

around $z = 0$. The coefficient $e^{b(t)}$ is the *logarithmic capacity* of K_t. With the customary parametrization $b(t) = t$, the *radial Loewner equation* reads

$$\frac{\partial}{\partial t} g_t(z) = -g_t(z) \, \frac{g_t(z) + e^{\mathrm{i}U_t}}{g_t(z) - e^{\mathrm{i}U_t}}, \tag{5}$$

in analogy with Theorem 3. One can also show that a local growth condition for a family of hulls in $\mathbb{D}$ corresponds to continuity of the driving function U (Theorem 4). Similarly to (3), the *reverse radial equation* is defined by

$$\frac{\partial}{\partial s} h_s(w) = h_s(w) \, \frac{h_s(w) + e^{\mathrm{i}U_{t-s}}}{h_s(w) - e^{\mathrm{i}U_{t-s}}}$$

for $0 \leq s \leq t$ and the solution with the initial condition $h_0(w) = w$ is a well defined map $h_s : \mathbb{D} \to \mathbb{D}_s$ in the whole unit disc.

Finally, we mention that the Loewner equation can also be defined in the whole complex plane, describing conformal maps associated to hulls evolving from the origin towards infinity. There is no essential difference between this and the radial case – the whole plane equation is (5), started from $t = -\infty$.

References

[1] Zinsmeister, M., *Stochastic Löwner evolution.* Lectures held at Warsaw University (2006);

[2] Lawler, G., *Conformally invariant processes in the plane.* American Mathematical Society (2005);

[3] Kemppainen, A., *Schramm–Loewner evolution.* Lectures held at University of Helsinki (2011).

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14 Conformal welding and Koebe's theorem.

Christopher J. Bishop [1]
A summary written by Larissa Richards.

Abstract

We will present several results stating that every orientation preserving homeomorphism of the circle is "almost" a conformal welding in some precise way. In addition, we will prove one, possibly two, of these results. As the proofs rely on applying Koebe's circle domain theorem, we will demonstrate the usefulness of this approach by proving the classical result that every quasisymmetric map is a conformal welding.

14.1 Introduction

Consider glueing two disks in $\mathbb{C}$ by an orientation preserving homeomorphism between their boundaries. By the celebrated Moore's theorem, the resulting surface is a topological copy of the Riemann sphere along with an embedded curve in it. Can we make it a conformal sphere? The answer is not always in the affirmative. If it is, then the glueing homeomorphism is called a *conformal welding*. We can also approach this question from another direction. Let $\mathbb{D}$ be the open unit disk, $\mathbb{D}^* = S^2 \backslash \bar{\mathbb{D}}$ and $\mathbb{T} = \partial\mathbb{D} = \partial\mathbb{D}^*$ be the unit circle. Consider a closed Jordan curve Γ. By the Jordan curve theorem, $S^2 \backslash \Gamma = \Omega \cup \Omega^*$ where Ω is the bounded component and Ω^* is the unbounded component. Applying the Riemann mapping theorem to the interior, Ω, and exterior, Ω^*, of Γ, we have conformal maps $f : \mathbb{D} \to \Omega$, $g : \mathbb{D}^* \to \Omega^*$ which extend continuously to their boundary $\mathbb{T}$. Thus, they induce a homeomorphism $h = g \circ f^{-1}$ of the unit circle to itself. Any homeomorphism arising in this manner is a *conformal welding*. For a given curve, h is unique up to a Mobius transformation.

A natural question arises: given a circle homeomorphism h, can we construct a closed Jordan curve Γ of the plane? In other words, is the map $\Gamma \to h$ of closed curves to circle homeomorphisms (up to Mobius transformations) $1-1$? Is it onto? For general curves, it is well known that it is neither $1-1$ nor onto. That is, we know that not every orientation preserving homeomorphism of the circle to itself is a conformal welding.

Example 1 (Circle homeomorphism which is NOT a conformal welding).
Let K be the closure of the graph of $\sin(\frac{1}{x})$. We know that the graph of $\sin(\frac{1}{x})$ oscillates more and more frequently between $y = \pm i$, that is, over the small interval

$$\frac{1}{2\pi(n+1)} \leq x \leq \frac{1}{2\pi n}$$

it goes through an entire wave. This divides the plane into a pair of simply connected domains Ω, Ω^*. Let $F : \Omega \to \mathbb{D}$ and $G : \Omega^* \to \mathbb{D}^*$ be the corresponding conformal maps. These maps extend continuously to $\mathbb{T}$ except at one point where the radial limit both exist and equal 0. Then $h = G \circ F^{-1} : \mathbb{T} \to \mathbb{T}$ is well-defined, continuous, and $1-1$ except at one point $\{b\}$. Thus h is a homeomorphism of the circle.
Claim. *h is not a conformal welding.*
Suppose $h = g^{-1} \circ f$ for some closed Jordan curve Γ. As conformality is preserved through composition, $f \circ F$ and $g \circ G$ are conformal off K and continuous except on the line segment $[-i, i]$. Then as $\sin(\frac{1}{x})$ is an analytic curve $f \circ F$ and $g \circ G$ are conformal on $\hat{\mathbb{C}} \backslash [-i, i]$. By Morera's theorem, they extend to conformal maps on $\hat{\mathbb{C}} \backslash \{b\}$. This is a contradiction to Liouville's theorem.

Throughout our talk, we will discuss several results stating that the map $\Gamma \to h$ of closed curves to circle homeomorphisms is almost onto. We will show that every circle homeomorphism is close to a conformal welding in some precise way. Our proofs will rely on Koebe's circle domain theorem.

Theorem 2 (Koebe's Circle Domain Theorem).
Any finitely connected plane domain can be conformally mapped to a domain whose boundary components are all circles and points.

We will demonstrate the usefulness of Koebe's theorem to conformal weldings by presenting an almost elementary, geometric proof of the classical theorem of conformal weldings.

Theorem 3 (Classical Theorem of Conformal Weldings).
Every quasisymmetric map is a conformal welding.

A homeomorphism h is quasisymmetric if there is an $M < \infty$ such that

$$\frac{1}{M} \leq \frac{|h(I)|}{|h(J)|} \leq M$$

for any adjacent arcs $I, J \subset \mathbb{T}$ of equal length.

14.2 Main Results

Definition 4. *Let $E \subset \mathbb{T}$. Suppose $f : \mathbb{D} \to \Omega,\ g : \mathbb{D}^* \to \Omega^*$ are conformal maps onto disjoint domains such that f has radial limits on E, g has radial limits on $h(E)$, and $h = g^{-1} \circ f$ on E. Then h is a generalized conformal welding.*

A homeomorphism being a generalized conformal welding is a weaker condition than being a conformal welding. For instance, consider the setting in Example 1 that is the closure of the graph $\sin(\frac{1}{x})$ denoted by K. Take $E = \mathbb{T}\backslash\{1\}$ We saw that K divides the plane into two simply connected domains Ω, Ω^* and gives corresponding conformal maps $F : \Omega \to \mathbb{D},\ G : \Omega^* \to \mathbb{D}^*$ which induces a circle homeomorphism $h = G \circ F^{-1}$. This h is indeed a generalized conformal welding everywhere on $\mathbb{T}\backslash\{1\}$. However, we showed that it is not a conformal welding.

Theorem 5 ("almost" onto)**.**
Given any orientation preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ and any $\varepsilon > 0$, there is a set $E \subset \mathbb{T}$ with $|E| + |h(E)| < \varepsilon$ and a conformal welding homeomorphism $H : \mathbb{T} \to \mathbb{T}$ such that $h(x) = H(x)$ for all $x \in \mathbb{T}\backslash E$. In particular, every such h is a generalized welding on a set E with Lebesgue measure as close to 1 as we wish.

In other words, every orientation preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ agrees with a conformal welding homeomorphism H, except on a set of small Lebesgue measure. The proof of Bishop's "almost" onto theorem relies on two main results.

Theorem 6. *Any orientation preserving homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is a generalized conformal welding on $\mathbb{T}\backslash F$ where $F = F_1 \cup F_2$ and both F_1 and $h(F_2)$ have log capacity zero.*

We will notice that this theorem gives us no information when h is log-singular, i.e. there is a Borel set E such that both E and $h(\mathbb{T}\backslash E)$ have zero log capacity. That is, in the set-up of the theorem F_1 and $h(F_2)$ have zero log capacity by definition of h being log-singular. Hence, the result is redundant. However, with a different approach, we can show that such a map is indeed a conformal welding but in this case our homeomorphism h is not well-behaved.

Definition 7. *A closed Jordan curve γ is flexible if:*

1. *For all closed Jordan curves γ' and for all $\varepsilon > 0$, there exists a homeomorphism $H : S^2 \to S^2$ such that H is conformal off γ and maps γ to within ε of γ' in the Hausdorff metric.*

2. *Given z_1, z_2 in each component of $S^2\backslash\gamma$ and w_1, w_2 in each component of $S^2\backslash\gamma'$. We can choose H as in 1. such that $H(z_1) = w_1$ and $H(z_2) = w_2$.*

Notice that γ and $H(\gamma)$ generate the same conformal welding homeomorphism h. These h's are dense in the set of all closed curves (in the Hausdorff metric). Hence, we have nonuniqness of closed Jordan curves γ.

Theorem 8 (not $1-1$)**.**
Suppose h is an orientation preserving homeomorphism of the circle. Then h is a conformal welding of a flexible curve if and only if it is log-singular.

Therefore, we know that h is a conformal welding if it is either "good" enough (quasisymmetric) by the classical theorem of conformal weldings or "wild" enough (log-singular) by the above theorem. As we have seen, generalized conformal welding is a weaker condition than conformal welding. Hence, in order to use Theorem 6 and Theorem 8 to prove Bishop's "almost" onto theorem, we need to extend our homeomorphism from a generalized conformal welding to a conformal welding.

Theorem 9. *Suppose $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ are conformal maps onto disjoint Jordan domains and let $E = f^{-1}(\partial\Omega\cap\partial\Omega^*)$. On E define $h = g^{-1}\circ f$. Then h can be extended from E to a conformal welding homeomorphism of $\mathbb{T}$ to itself.*

The idea behind proving Theorem 9 is to use Koebe's circle domain theorem to build a conformal welding. Given a set of equidistributed points $\{x_k\}_1^n$ along the unit circle $\mathbb{T}$. Instead of considering a circle homeomorphism, suppose that h is a homeomorphism from $\mathbb{T}$ to $2\mathbb{T} = \{z : |z| = 2\}$. Connect the points $x_k \in \mathbb{T}$ and $h(x_k) \in 2\mathbb{T}$ by a smooth curve γ_k in the annulus $A = \{z : 1 < |z| < 2\}$. For instance, we can take γ_k to be the hyperbolic geodesic in A. Define the domain $\Omega_{n,\varepsilon}$ to be the union of $\mathbb{D}$, $2\mathbb{D}^*$, and an ε-neighbourhood of each γ_k where ε is small enough such that these neighbourhoods are pairwise disjoint. By Koebe's circle domain theorem, we can conformally map our domain $\Omega_{n,\varepsilon}$ to a domain whose complementary components are all disks.

Definition 10. *A circle chain $\mathcal{C}$ is a finite union of closed disks $\{D_k\}_1^n$ in $\mathbb{R}^2$ which have pairwise disjoint interiors such that:*

- *D_k is tangent to D_{k+1} for $k = 1, \cdots, n-1$*
- *D_n is tangent to D_1*
- *and there are no other tangencies.*

Notice that a circle chain divides the plane into two disjoint Jordan domains Ω, Ω^* which are the bounded and unbounded components, respectively. This induces a pair of Riemann maps (f, g) where $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ called a circle chain pair. By taking $\varepsilon \to 0$ in our domain given by Koebe's theorem, we obtain a circle chain. Hence, it divides the plane into two domains Ω_n and Ω_n^* inducing a circle chain pair (f_n, g_n). Suppose that there exists an $R < \infty$ such that the circle chain is contained in the annulus $\{z \ : \ 1 \leq |z| \leq R\}$ independently of n. Then as we take n to infinity, at most $(\frac{R}{\varepsilon})^2$ disks remain larger than ε. That is, most of the disks in our circle chain collapse into points implying that $|f_n(x) - g_n(h(x))| \to 0$ for all $x \in \mathbb{T}$ except on a set of countably many points. In order to show that it is possible to find such an R, we need to place another assumption on our homeomorphism h.

Theorem 11. *Suppose $h : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism which is not log-singular. Then there are sequences of conformal maps $\{f_n\}$ on $\mathbb{D}$ and $\{g_n\}$ on $\mathbb{D}^*$ such that*

1. *$f_n(0) = 0, \ g_n(\infty) = \infty$.*
2. *$\Omega_n = f_n(\mathbb{D})$ and $\Omega_n^* = g_n(\mathbb{D}^*)$ are disjoint Jordan domains.*
3. *There is an $R < \infty$ such that $S^2 \backslash (\Omega_n \cup \Omega_n^*) \subset \{z \ : \ 1 \leq |z| \leq R\}$ independently of n.*
4. *There is a countable set $E \subset \mathbb{T}$ such that $\lim_{n\to\infty} |f_n(x) - g_n(h(x))| = 0$ for all $x \in \mathbb{T} \backslash E$.*

We can extend Theorem 11 to prove Theorem 9 using extremal length methods. Although we may want to go directly from Theorem 11 to Theorem 9, we cannot pass the sequences of functions $\{f_n, g_n\}$ in Theorem 11 to their limits. The problem is that we can have functions such that the $\lim f_n(x) =$

$f(x)$, $\lim g_n(x) = g(x)$, and $f_n(x) = g_n(h(x))$ for all n but still have that $f(x) \neq g(x)$. Thus, in the proof of Theorem 9, we pass to a subsequence such that $f_n \to g$ and $g_n \to g$ converge uniformly on compact sets. Unfortunately, we may not get convergence of boundary values off a set of zero log capacity and so we still need to use properties of our maps to complete the proof.

14.3 Plan of Talk

The breakdown of our talk is as follows:

- Discuss several results stating that the map $\Gamma \to h$ of closed curves to circle homeomorphisms is almost onto in some precise way.
- Demonstrate the usefulness of applying Koebe's circle domain theorem to conformal weldings through a new proof of the classical theorem of conformal weldings.
- Prove Theorem 11.
- If time permits, prove Theorem 9.

References

[1] Bishop,Christopher J. *Conformal welding and Koebe's theorem.* Annals of Mathematics **166** (2007), pp. 613-656.

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15 Recurrence on planar graph limits

after O. Gurevich and A. Nachmias
A summary written by Martin Tassy

Abstract

We prove that any distributional limit of finite planar graphs in which the degree of the root has an exponential tail is almost surely recurrent. As a corollary, we obtain that the uniform infinite planar triangulation and quadrangulation (UIPT and UIPQ) are almost surely recurrent. We also show that in any bounded degree, finite planar graph the probability that the simple random walk started at a uniform random vertex avoids its initial location for T steps is at most $\frac{C}{\log T}$.

15.1 Introduction

A *distributional limit* of finite graphs G_n is a random rooted infinite graph (U, ρ) such that eighborhoods of G_n around a random vertex converge in distribution to neighborhoods of U around ρ. As such the random infinite planar triangulation can be defined as the distributional limit of uniform triangulation of the sphere with n vertices. The purpose of this article from Asaf Nachmias and Ori Gurel-Gurevich is to prove that this limiting random graph is almost surely recurrent that is a random walk starting from ρ would almost surely come back to the root. The Paper proves proves in fact more a general theorem

Theorem 1. *Let (U, ρ) be a distributional limit of planar graphs such that the degree of ρ has an exponential tail. Then U is almost surely recurrent.*

The result on UIPT is a direct consequence of this theorem since in this case the root has exponential tail [see unifrom infinite planar triangulation] Another noteworthy consequence of this theorem is that uniform infinite planar quadriangulation (UIPQ) is also recurrent.

The first step of the proof is to show the recurrence in the case of distributional limit of planar graph with bounded degrees. In fact we prove a sharper result which gives us a upper bound at which the probability to leave the root growth. This is a consequence of the following theorem for finite graphs. If G is a finite graph and we consider a random walk $(X_t)_t \geq 0$ we can define $\phi(T,G)$ the probability that $X_t \neq X_0$ for all $t = 1, \ldots, T$. For any $D \geq 1$ define

$$\phi_D(T) = \sup\left\{\phi(T,G) : G \text{ is planar with degrees bounded by } D\right\}$$

then we have:

Theorem 2. *For any $D \geq 1$ there exists $C < \infty$ such that for any $T \geq 2$*

$$\phi_D(T) \leq \frac{C}{\log T}.$$

The second step of the proof of Theorem 1 is to expand the recurrence to graphs with exponential tails. In order to do that we introduce a graph transformation called the *star-tree transform*.

15.2 Preliminaries

15.2.1 Electrical networks

Many results on random walks on graph can be translated in terms of electrical networks. In this section we will present the one that are useful for our demonstration.

Let $G = (V, E)$ be a finite graph with non-negative edge weights $\{c_e\}_{e \in E}$. We call these weights conductances and their inverses, $R_e = c_e^{-1}$, are called resistances (by convention $0^{-1} = \infty$). For any two vertices $a \neq z$ define the effective resistance $R_{eff}(a \leftarrow z; \{R_e\})$ between a and z as the minimum energy $\mathcal{E}(\theta) = \sum_{e \in E} R_e[\theta(e)]^2$ of any unit flow θ from a to z. The unit flow attaining this minimum is called the unit current flow. We write $R_{eff}(a \leftarrow z)$ when all the conductances are 1.

Given two disjoint sets of vertices A and Z, the effective resistance $R_{eff}(A \leftarrow Z; \{R_e\})$ between A and Z is the effective resistance between the two corresponding vertices in the graph obtained from G by contracting the sets A and Z into single vertices and retaining the same resistances on the remaining edges. If either A or Z are empty sets, then $R_{eff}(A \leftarrow Z; \{R_e\}) = \infty$. To define effective resistances on infinite graphs, we will only compute effective resistances between disjoint sets A and Z such that $V \setminus (A \cup Z)$ is finite. When G is infinite we define the effective resistance from a to ∞ as

$$R_{eff}(a \leftarrow \infty; \{R_e\}) = \lim_{n \to \infty} R_{eff}(a \leftarrow G \setminus B_n; \{R_e\}),$$

where $\{B_n\}$ is any sequence of finite vertex sets which exhaust G. This limit does not depend on the choice of exhausting sequence.

For a function $g : V \to \mathbb{R}$, the Dirichlet energy is defined as

$$\mathcal{E}(g) = \sum_{e=(x,y)\in E} c_e \big[g(x) - g(y)\big]^2 .$$

We will use the dual definition of effective resistance, that is, the discrete Dirichlet principle (see Exercise 2.13 of [3]) stating that

$$\frac{1}{R_{eff}(A \leftarrow Z; \{R_e\})} = \min\big\{\mathcal{E}(g) \,:\, g : V \to \mathbb{R}, g_{|A} = 0, g_{|Z} = 1\big\} . \tag{1}$$

Consider the network random walk $(X_n)_{n\geq 0}$ on G with transition probabilities $p(x,y) = c_{(x,y)}[\sum_{y:(x,y)\in E} c_{(x,y)}]^{-1}$ and write $\mathbf{P}_x$ for the probability measure of a network random walk started at $X_0 = x$. Write τ for the stopping time $\tau = \min\{n \geq 1 : X_n \in \{a, z\}\}$. It is classical (stemming from the fact that the minimizer of (1) is the unique harmonic function with the corresponding boundary values, see [3]) that

$$R_{eff}(a \leftarrow z; \{R_e\}) = \frac{1}{\mathbf{P}_a(X_\tau = z) \displaystyle\sum_{y:(a,y)\in E} c_{(a,y)}} . \tag{2}$$

This gives a useful electrical interpretation of recurrence. An infinite network $(G; \{R_e\})$ is recurrent if and only if $R_{eff}(a \leftarrow \infty; \{R_e\}) = \infty$.

It is not too hard to see that this implies the following two useful criteria for recurrence/transience. First, an infinite graph is G is recurrent if and only if for some vertex a there exists $c > 0$ such that for any integer $m \geq 0$ there exists a finite vertex set B such that

$$R_{eff}(B_G(a,m) \leftarrow G \setminus B; \{R_e\}) \geq c , \tag{3}$$

Secondly, a network is recurrent if and only if there exists a unit flow from some vertex a to ∞ with finite energy.

Finally, we will use the following bound.

Lemma 3. *Let $G = (V,E)$ be a finite network with resistances $\{R_e\}$ and two vertices a and z. Let $A \subset V$ such that $a \in A$ and $z \notin A$ and define $R_e^A = R_e$ for each edge e that has both endpoints in A and $R_e^A = \infty$ otherwise. Then*

$$R_{eff}(a \leftarrow z; \{R_e\}) \leq R_{eff}(A \leftarrow z; \{R_e\}) + \max_{v\in A} R_{eff}(a \leftarrow v; \{R_e^A\}) .$$

15.2.2 Circle packings

Another key concept for the demonstration is theory of circle packing [3]. A circle packing is a collection of circles in the plane with disjoint interiors. The tangency graph of a circle packing is a planar graph $G = (V, E)$ in which the vertex set V is the set of circles and two circles are neighbors if they are tangent in the packing. The degree of a circle in the packing is its degree in the tangency graph. The Koebe-Andreev-Thurston Circle Packing Theorem asserts that for any finite planar graph $G = (V, E)$ there exists a circle packing in the plane which has tangency graph isomorphic to G. Furthermore, if G is a triangulation, then this packing is unique up to Möbius transformations of the plane and reflections along lines. The Ring Lemma assets that if a circle C is completely surrounded by D other circles $C_0, \ldots, C_{D-1}$ (that is, C_i is tangent to $C_{i+1 \text{ mod D}}$ and to C), then the ratio r/r_i between the radius of C and C_i is bounded above by a constant depending only on D. Thus, in a circle packing of a bounded degree triangulation (every inner circle is completely surrounded) the ratio of radii of every two tangent circles is bounded above and below by a constant depending only on D, with the possible exception of the three boundary circles.

15.3 Demonstration

In this section we give the different steps which lead to the proof of Theorem 1 and Theorem 2.

Given a circle packing $P = \{C_v : v \in G\}$ of a graph $G = (V, E)$ and given a domain $D \subset \mathbb{R}^2$ we write $V_D \subset V$ for the set of vertices such that their corresponding circles have centers in D. We also write $B_{\text{euc}}(p, r)$ for the Euclidean ball of radius r around p. The first step of our demonstration is to prove the folowing lemma which gives a lower bound on the resistance between two euclidian ball around the root which ration is greator than 1

Lemma 4. *Let $P = \{C_v : v \in G\}$ be a circle packing of a finite graph $G = (V, E)$ such that the ratio of radii of two tangent circles is bounded by K. Then for any $\alpha > 1$ there exists $c = c(K, \alpha) > 0$ such that for all $r > 0$ and all $p \in \mathbb{R}^2$*

$$R_{\text{eff}}\big(V_{B_{\text{euc}}(p,r)} \leftrightarrow V_{\mathbb{R}^2 \setminus B_{\text{euc}}(p,\alpha r)}\big) \geq c\,,$$

provided that both sets $V_{B_{\text{euc}}(p,r)}$ and $V_{\mathbb{R}^2 \setminus B_{\text{euc}}(p,\alpha r)}$ are nonempty.

Proof. The proof works by building a function f on the vertices of the graph whose total energy is bounded by a constant $\square$

As a corollary of this lemma we can show that a lower bound for the growth rate of the effective resistance.

Lemma 5. *Let P be a finite circle packing in $\mathbb{R}^2$ such that the ratio of radii of two tangent circles is bounded by K and such that there exists a circle in P entirely contained in $B_{\mathrm{euc}}(0,1)$. Then there exists a constant $c = c(K) > 0$ such that for all radii $r \geq 2$ we have*

$$R_{\mathrm{eff}}\big(V_{B_{\mathrm{euc}}(0,1)} \leftrightarrow V_{\mathbb{R}^2 \setminus B_{\mathrm{euc}}(0,r)}\big) \geq c \log r \,,$$

provided that $V_{\mathbb{R}^2 \setminus B_{\mathrm{euc}}(0,r)}$ is nonempty.

Proof. This is a consequence of Lemma 4 and the series law for electrical networks. $\square$

The next result which we will admit in this lecture is an upper bound for the probability of the set $V_{B_{\mathrm{euc}}(0,r) \setminus B_{\mathrm{euc}}(p,r^{-1})}$ to be large.

Lemma 6. *Let G be a finite planar triangulation and $P = \{C_v : v \in G\}$ be an arbitrary circle packing of G. Let ρ be a random uniform vertex of G and let $\widehat{P} = \{\widehat{C}_v : v \in G\}$ be the circle packing obtained from P by translating and dilating so that $\widehat{C}_\rho$ has radius 1 and is centered around the origin. Then there exists a universal constant $A > 0$ such that for any $r \geq 2$ and any $s \geq 2$*

$$\mathbf{P}\big(\forall p \in \mathbb{R}^2 \;\; |V_{B_{\mathrm{euc}}(0,r) \setminus B_{\mathrm{euc}}(p,r^{-1})}| \geq s\big) \leq \frac{A r^2 \log r}{s} \,.$$

We can now prove the following Lemma

Lemma 7. *Let $G = (V, E)$ be a finite planar graph with degrees at most D and let ρ be a random uniform vertex. Then there exists $c = c(D) > 0$ such that for all $k \geq 1$*

$$\mathbf{P}\Big(\exists B \subset V \text{ with } |B| \leq c^{-1} k R_{\mathrm{eff}}(\rho \leftrightarrow V \setminus B) \geq c \log k\Big) \geq 1 - c^{-1} k^{-1/3} \log k \,,$$

where we interpret $R_{\mathrm{eff}}(\rho \leftrightarrow V \setminus B) = \infty$ when $B = V$.

Proof. If G is a triangulation we can always dilate and translate our circle packing to obtain the circle packing $\widehat{P}$ of Lemma 6. We can now consider separetely the two cases $|V_{B_{\text{euc}}(p,r^{-1})}| \leq 1$ and $|V_{B_{\text{euc}}(p,r^{-1})}| > 1$. In the first case we set $B = V_{B_{\text{euc}}(0,r)}$ and use Lemma 4, in the second case we set $B = V_{B_{\text{euc}}(0,r) \setminus B_{\text{euc}}(p,r^{-1})}$ and use Lemma 4 again. To extend our result to the case where G is no more a triangulation but a finite graph with bounded degree, we use a graph transformation described in [2] and proves that this does not changes the result. □

If we set $k = T$ then Theorem 2 becomes a Corollary of 6. Now we consider the events

$$\mathcal{A}_k = \left\{ \exists B \subset U \text{ with } |B| \leq c^{-1} k R_{\text{eff}}(\rho \leftrightarrow U \setminus B) \geq c \log k \right\}.$$

and apply Borel cantelli we obtain the following result for graphs with bounded degrees.

Theorem 8. *Let (U, ρ) be the distributional limit of finite planar graphs of bounded degree. Then (U, ρ) almost surely satisfies the following. There exists $c > 0$ such that for any $k \geq 0$ there exists a finite set $B_k \subset U$ with $|B_k| \leq c^{-1}k$ and*

$$R_{\text{eff}}(\rho \leftrightarrow U \setminus B_k) \geq c \log k \,.$$

The rest of the lecture is dedicated to extend Theorem 8 to graphs with exponential tails. In order do that we will consider the following transformation

15.3.1 The star-tree transform

Let G be a graph. We define the *star-tree* transform G^* of G as the graph of maximal degree at most 3 obtained by the following operations

1. We subdivide each edge e of G by adding a new vertex w_e of degree 2. Denote the resulting intermediate graph by G'.

2. Replace each vertex v of G and its incident edges in G' by a balanced binary tree T_v with $\deg(v)$ leaves which we identify with v's neighbors in G'. When G is planar we choose this identification so as to preserve planarity, otherwise, this is an arbitrary identification. We denote by w_v the root of T_v. Denote the resulting graph by G^*.

As mentionned in the introduction we prove that the star tree transform has he following three properties

1. Each vertix is of degree at most 3
2. If G^* is recurrent then G is recurrent
3. (U^*, ρ^*) is the distributional limit of (G_n^*, ρ_n^*)

Using those we show how to complete the proof of Theorem 1.

References

[1] O. Angel, O.Schramm., *Uniform infinite planar triangulations* Comm. Math. Phys., 241 (2003), pp. 191–213.

[2] Benjamini, Itai *Recurrence of distributional limits of finite planar graphs* Electronic Journal of Probability, 97 (2001),no. 23, 13 pp. (electronic)

[3] R. Lyons with Y. Peres *Introduction to circle packing* Cambridge University Press, 2008

[4] Stephenson, Kenneth *Recurrence of distributional limits of finite planar graphs* Cambridge University Press, 2005

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16 The Loewner equation and Lipschitz graphs

after S. Rohde, H. Tran, and M. Zinsmeister [1]
A summary written by Ömer Faruk Tekin

Abstract

We give an elementary proof of the fact that chordal Loewner equation when fed with a function of Hölder-1/2 norm less than 4 generates simple curves. Our work is based on the analysis of the ODE of the upward flow generated by the chordal Loewner equation.

16.1 Introduction

Let $\gamma : (0, T] \to \mathbb{H}$ be a simple curve with $\gamma_0 = 0$. By reparametrizing γ, there exists a unique conformal map $g_t : \mathbb{H} \setminus \gamma[0, t] \to \mathbb{H}$ with the following normalization:

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \text{ as } z \to \infty.$$

Then, the function $t \mapsto g_t(z)$ satisfies *(downward) chordal Loewner equation:*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \quad g_0(z) = z, \tag{1}$$

where λ is a contiuous function on the real line with $g_t(\gamma_t) = \lambda_t$.

Conversely, one can start with a continuous function λ_t defined on some interval $[0, T]$, and consider the following initial value problem for each $z \in \mathbb{H}$:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \quad g(0, z) = z,$$

to obtain a t-parametrized family of simply connected subdomains of the upper half plane. Specifically, one can define

$$T_z = \sup\{s \in [0, T] : g(t, z) \text{ exists on } [0, s)\},$$
$$H_t = \{z \in \mathbb{H} : T_z > t\}.$$

Then, H_t is a simply connected subdomain of $\mathbb{H}$, and $g_t(\cdot) = g(t, \cdot)$ is the unique conformal map mapping H_t onto $\mathbb{H}$, with the normalization

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \text{ as } z \to \infty.$$

The *driving function* λ is said to *generate a curve* if there exists a curve γ such that H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$.

This note verifies the following theorem by studying the quantitative behaviour of the ODE (1):

Theorem 1. *If the driving function λ has Hölder-1/2 norm less than 4, then the chordal Loewner equation generates a simple curve γ.*

16.2 Preliminaries

16.2.1 Criteria for simple curve generation

We will employ the following theorem (See Theorem 4.1 in [4], Proposition 2.19 in [3] and Proposition 3.11 in [2]) to prove Theorem 1:

Theorem 2. *Let λ_t be the driving function for the Loewner chain (g_t). Then, one of the following statements guarantee that the Loewner chain generates a curve*

1. *The limit*

$$\lim_{y \to 0^+} g_t^{-1}(\lambda_t + iy)$$

exists, and is continuous in t.

2. *The quantity*

$$v(t, \varepsilon) := \int_0^\varepsilon |(g_t^{-1})'(\lambda_t + iy)| dy$$

converges to zero as $\varepsilon \to 0$, uniformly in t.

Note: We will prove the second statement above throughout this note.

16.2.2 ODE associated to the upward flow

For convenience, we will work with the *upward Loewner equation:*

$$\partial_t f_t(z) = -\frac{2}{f_t(z) - \xi_t}, \quad f_0(z) = z, \tag{2}$$

for $z \in \mathbb{H}$, and ξ is a continuous function on the real line. The imaginary part of $f_t(z)$ is strictly increasing in t, so the solution to (2) exists for all $t \geq 0$. Moreover, if $(g_s)_{0 \leq s \leq t}$ is the solution to the downward Loewner equation with driving term λ, then the solution $(f_s)_{0 \leq s \leq t}$ to the upward equation with $\xi_s = \lambda_{t-s}$ satisfy

$$f_t(z) = g_t^{-1}(z).$$

The *(upward) flow* z_t starting at $z \in \mathbb{H}$ will be defined by

$$z_t = x_t + iy_t := f_t(z) - \xi_t,$$

so that we will have the system

$$\partial_t(x_t + \xi_t) = \frac{-2x_t}{x_t^2 + y_t^2}, \tag{3}$$

$$\partial_t(y_t) = \frac{2y_t}{x_t^2 + y_t^2}. \tag{4}$$

16.2.3 Computation of f_t' in terms of the flow

Since

$$f_t'(z) = e^{\log f_t'(z)} = e^{\int_0^t \partial_s \log f_s'(z) ds}$$

and

$$\partial_s \log f_s'(z) = \frac{\partial_s f_s'(z)}{f_s'(z)},$$

we compute

$$|f_t'(z)| = \exp\left(2\int_0^t \frac{x_s^2 - y_s^2}{(x_s^2 + y_s^2)^2} ds\right). \tag{5}$$

16.3 Proof of Theorem 1

16.3.1 Cone condition

Notice that the differential equation (4) implies

$$\partial_t y_t^2 = \frac{4y_t^2}{x_t^2 + y_t^2} = \frac{4}{1 + \left(\frac{x_t}{y_t}\right)^2}. \tag{6}$$

Therefore, it is crucial to analyze the quantity $\frac{|x_t|}{y_t}$, and hence the cones

$$A_c = \{x + iy : |x| \le cy\}.$$

Whenever $z_t \in A_c$, by trivial estimates, (6) implies

$$\frac{4t}{1 + c^2} + y_0^2 \le y_t^2 \le y_0^2 + 4t. \tag{7}$$

Our main theorem for this section is the following, which verifies that z_t indeed lies inside A_c for some c:

Theorem 3. *Let ξ be a function with Hölder-1/2 norm $\sigma < 4$. Consider the flow z_t with $z_0 = iy$, and $z_t = f_t(z_0 + \xi_0) - \xi_t$. Then, there exists a constant c_σ such that z_t stays in the cone A_{c_σ}. Moreover, $c_\sigma \le \sigma/\sqrt{4 - \sigma^2}$ for $\sigma < 2$.*

Note: A sketch of the proof of Theorem 3 is given in appendix.

16.3.2 Estimating $v(t, \varepsilon)$

By (5), (7) and Theorem 3, we compute

$$|f_t'(\xi_0 + iy)| \le \exp \int_0^t \frac{c_\sigma^2 - 1}{c_\sigma^2 + 1} \frac{2ds}{x_s^2 + y_s^2} \le \left(\frac{y_t}{y}\right)^{\frac{c^2-1}{c^2+1}} \le (4t + y^2)^{\frac{1-\alpha}{2}} y^{\alpha - 1}, \tag{8}$$

where $\alpha = \min\{1 - \frac{c^2-1}{c^2+1}, 1\} \in (0, 1]$.

Now, let λ be the driving term for the Loewner chain g_s, with Hölder-1/2 norm less than 4. Let $\xi_s = \lambda_{t-s}$ be the driving term for the upward Loewner chain f_s, so that $g_t^{-1} = f_t$. Hence by (8)

$$|(g_t^{-1})'(\lambda_t + iy)| = |f_t'(\xi_0 + iy)| \le (4t + y^2)^{\frac{1-\alpha}{2}} y^{\alpha - 1}.$$

Integrating the above estimate we obtain

$$v(t,\varepsilon) = \int_0^\varepsilon |(g_t^{-1})'(\lambda_t + iy)|dy \le \frac{(4t+\varepsilon^2)^{\frac{1-\alpha}{2}}}{\alpha}\varepsilon^\alpha.$$

The above estimate yields that $v(t,\varepsilon)$ converges to zero as $\varepsilon \to 0$, uniformly in t. Hence by Theorem 2, the Loewner chain generate a curve, proving Theorem 1.

16.4 Appendix

16.4.1 A useful lemma

Lemma 4. *Let ξ be an arbitrary continuous function.*

1. *If $x_s \ge 0$ for all $0 \le s \le t$, then $x_t \le x_0 + \xi_0 - \xi_t$.*
2. *In general, $|x_t| \le |x_0| + M_{0,t}^\xi$, where $M_{0,t}^\xi = \sup\{|\xi_r - \xi_t| : r \in [0,t]\}$.*

The above lemma establishes that Hölder norm of x is controlled by the Hölder norm of ξ as long as x does not switch its sign.

16.4.2 Sketch of Proof of Theorem 3

We are interested in estimates of the form

$$|x_t| \le A\sqrt{t}, \tag{9}$$

$$y_t \ge B\sqrt{t}, \tag{10}$$

as (9)-(10) would imply the cone condition

$$\frac{|x_t|}{y_t} \le \frac{A\sqrt{t}}{B\sqrt{t}} = \frac{A}{B}.$$

By Lemma 4, (9) is satisfied with $A = \sigma$, where σ is the Hölder-1/2 norm of ξ. To obtain (10), we would like to prove that y_t^2/t is bounded. Note that

$$\partial_t \frac{y_t^2}{t} = \frac{2ty_t\dot{y}_t - y_t^2}{t^2} = \frac{y_t^2}{t^2}\left(\frac{4t}{x_t^2+y_t^2} - 1\right).$$

Therefore, at the critical points of y_t^2/t, we have

$$x_t^2 + y_t^2 = 4t.$$

Hence, if $\sigma < 2$, as $|x_t| \leq \sigma\sqrt{t}$, we get $y_t \geq \sqrt{4-\sigma^2}\sqrt{t}$. Therefore, the harder part is to push the bound on σ to $2 < \sigma < 4$. In [1], the authors verify (9) for some $A < 2$, via a bootstrapping argument for large values of t, so that the above critical value analysis is still applicable.

References

[1] Rohde, S., Tran, H. and Zinsmeister, M., *The Loewner equation and Lipschitz graphs.* preprint, 2013;

[2] Viklund, F. and Lawler, G. *Optimal Hölder exponent for the SLE path.*, Duke Mathematical Journal **159** (2011), no. 3, 351–383;

[3] Lawler, G. *Schramm-Loewner evolution* IAS/Park City Mathematical Series, AMS (2009), 231–295;

[4] Rohde, S., Schramm O., *Basic properties of SLE* Ann. Math., **161** (2005), 883–924.

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