1 RHA, SS 13, Exercise Sheet 7

Due June 12. 2013.

Exercise 1:

Let $g : \mathbf{R}^d \to [0, \infty]$ be a radial, nonincreasing and integrable function, and let $f \geq 0$ be locally integrable. Prove that, for every $x \in \mathbf{R}^d$,

$$f * g(x) \leq \|g\|_1 \cdot Mf(x),$$

where M denotes the Hardy-Littlewood maximal function.

Exercise 2:

Combine Exercise 1 with the strong L^p bounds ($1 < p < \infty$) for the Hardy-Littlewood maximal function to give an alternative proof of the Hardy-Littlewood-Sobolev inequality.

Hint: break up the integral in two parts like we did in class:

$$f * |\cdot|^{\alpha-d}(x) = \int_{|y|<R} f(x-y)|y|^{\alpha-d}dy + \int_{|y|\geq R} f(x-y)|y|^{\alpha-d}dy.$$

Use Exercise 1 for the first part, Hölder's inequality for the second part and then optimize in R.

Exercise 3:

Consider the sphere $S^{d-1} \subset \mathbf{R}^d$ equipped with surface measure σ. Let $E \subset S^{d-1}$ be a measurable subset and let $R \geq 1$.

1. Prove that

   (1) $$\int_{S^{d-1}} |\widehat{\chi_E}(\xi)|^2 d\sigma(\xi) \lesssim R|E| + R^{-\frac{d-1}{2}}|E|^2.$$

   Hint: Decompose $d\sigma = d\sigma_1 + d\sigma_2$, where $\widehat{d\sigma_1}(x) = \widehat{d\sigma}(x)(1 - \phi(x/R))$ for some appropriate $\phi \in C_0^\infty(\mathbf{R}^d)$. Estimate $\|\widehat{d\sigma_1}\|_\infty$ and $\|d\sigma_2\|_\infty$.

2. Let $p := \frac{2(d+1)}{d-1}$. Optimize the estimate (1) in R to obtain

   $$\left\| \widehat{\chi_E}\Big|_{S^{d-1}} \right\|_{L^2(\sigma)} \leq \|\chi_E\|_{L^{p'}(\mathbf{R}^d)}.$$

   (Note that this is the endpoint Tomas-Stein inequality for characteristic functions.)