# 1 RHA, SS 13, Exercise Sheet 6

**Due June 05. 2013.**

## Exercise 1:

Construct for arbitarrily large $N$ finite sets $A, B$ of cardinality at most $N$ in an abelian group, and construct $G \subset A \times B$ such that $\{a + b : (a, b) \in G\}$ has cardinality at most $N$ and the map $(a, b) \to a - b$ is injective on $G$, such that the cardinality of $G$ is at least $N^{1.6}$

Hint: Consider an example where $A$ and $B$ are the subset $\{0, 1, 3\}$ of cardinality 3 in the group of integers. Then take large Cartesian products of this set with itself.

## Exercise 2 :

Let $f$ be an analytic function in the strip of complex numbers with real part between 0 and 1 and assume $f$ has continuous extension to the closed strip. Assume the modulus of the function is bounded by $a$ on the line $\Re(z) = 0$ and by $b$ on the line $\Re(z) = 1$. Assume the function satisfies

$$|f(z)| \leq C + |z|^n$$

for some constants $C$ and $n$ and all $z$ in the strip.

Prove that

$$|f(z)| \leq a^{1-\Re(z)} b^{\Re(z)}$$

for all $z$ in the strip.

Hint: First consider the case $a = b = 1$ and apply the maximum principle to the function $f$ times an appropriate exponential function restricted to the upper half of the strip. To pass from special case to general $a, b$ multiply the function with another exponential function.

## Exercise 3:

Let $T$ be a linear operator mapping $L^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$ and $L^\infty(\mathbf{R}^n)$ to $L^\infty(\mathbf{R}^n)$.

For $E_i$ and $F_j$ two finite collection of pairwise disjoint measurable sets on $\mathbf{R}^n$, complex numbers $c_i, d_j$ of modulus one and real numbers $u_i, v_j$ consider

$$f(s) := \left( T(\sum_i u_i^s c_i 1_{E_i}), \sum_j v_j^{1-s} d_j 1_{F_j}) \right) ,$$

the bracket denoting integration of the product of the two functions. Show hat this function $f$ is analytic in $s$.

Carefully apply the previous exercise to show that $T$ has a bounded extension from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ and give a good bound on the operator norm.