1 RHA, SS 13, Exercise Sheet 4

Due May 8. 2014.

Exercise 1:

For a set $S \subset R^n$ define the ϵ neighborhood of S to be $N_\epsilon(S) = \{x : \mathrm{dist}(x, S) \leq \epsilon\}$

Define the upper Minkowski dimension of S to be

$$\limsup_{\epsilon \to 0}(n - \frac{\log |N_\epsilon(S)|}{\log \epsilon})$$

and the lower Minkowski dimension to be

$$\liminf_{\epsilon \to 0}(n - \frac{\log |N_\epsilon(S)|}{\log \epsilon})$$

If upper and lower Minkowski dimension coincide, we call the common value the Minkowski dimension.

Construct a set whose upper an lower Minkowski dimension are not equal.

Exercise 2 :

Let C be the middle half Cantor set on $[0, 1]$, that is $C = \bigcap_{n \in \mathbf{N}} C_n$ where $C_0 = [0, 1]$ and

$$C_{n+1} := \{x : 4x \in C_n\} \cup \{x : 4x - 3 \in C_n\}$$

Consider two copies of this set in $\mathbf{R}^2$, namely

$$B_0 = \{(-1 + 2x, 0), x \in C\}$$

$$B_1 = \{(-1/2 + x, 1), x \in C\}$$

Let B be the union of all line segments from a point in B_0 to a point in B_1.

1. Prove that B is compact.
2. Prove that B contains a unit line segment in every direction which has angle less than 45 degrees from the vertical direction.

Exercise 3:

Prove that every compact set in $\mathbf{R}^2$ that contains a unit line segment in every direction has Minkowski dimension 2. Hint: evaluate a Kakey maximal function at appropriate far distance of the set.