1 RHA, SS 13, Exercise Sheet 2

Due April 24 2013.

Exercise 1 :

a) Let $1 < p < \infty$ and assume $T : L^p(\mathbf{R}^n) \to L^1(\mathbf{R}^n) \oplus L^\infty(\mathbf{R}^n)$ is a linear operator satisfying for all $f \in L^p(\mathbf{R})$

$$\|Tf\|_\infty \leq \|f\|_\infty$$

(which is trivial in case the RHS is infinite) and for all $\lambda > 0$

$$|\{x : |Tf(x)| \geq \lambda\}| \leq \lambda^{-1}\|f\|_1$$

Prove that there is a constant C_p such that for all $f \in L^p(\mathbf{R}^n)$ we have

$$\|f\|_p \leq C_p\|f\|_p$$

Hint: Use

$$\|g\|_p^p = p\int_0^\infty \lambda^{p-1}|\{x : |g(x)| > \lambda\}|\, d\lambda\ ,$$

cut functions where they are large and where they are small, and use Chebysheff.

b) Do the same except with $p < 2$ and $\|Tf\|_2 \leq \|f\|_2$ instead of the estimate $\|Tf\|_\infty \leq \|f\|_\infty$

c) Similarly, Assume $1 < p < 2$ and $1/p + 1/q = 1$ and assume

$$\|Tf\|_\infty \leq \|f\|_1$$

and

$$\|Tf\|_2 \leq \|f\|_2$$

Then prove

$$\|Tf\|_q \leq \|f\|_p$$

Exercise 2:

Prove Khintchine's inequality in the following form:

On the interval $[0, 1]$ consider for $n = 0, 1, 2, \ldots$ the function

$$r_n(x) := \text{sign}(\sin(2\pi 2^n x))$$

which take values -1, 1 (and 0 on a set of measure 0).

Then for every $0 < p < \infty$ there is a constant C_p such that for any N and any tuple $(a_0, a_1, \ldots, a_N)$ we have

$$C_p^{-1}(\sum_{n=0}^{N}|a_n|^2)^{1/2} \leq \|\sum_{n=0}^{N} a_n r_n\|_p \leq C_p(\sum_{n=0}^{N}|a_n|^2)^{1/2}$$

Exercise 3:

Let $1 < p < 2 < q < \infty$ such that $1/p + 1/q = 1$. Prove that there does not exists a constant C such that

$$\|\widehat{f}\|_p \leq C\|f\|_q$$

for every $f \in C_0^\infty(\mathbf{R}^n)$

Hint: take a random linear combination of translated and/or modulated Gaussians and use Khintchine.