1 RHA, SS 13, Exercise Sheet 11

Due June 10. 2013.

Exercise 1:

Let L be a finite collection of lines in $\mathbf{R}^3$ and consider the set J of points (joints) which are the intersection of three linearly independent lines of L.

Let J' be a subset of J with the property that every line of L which meets one point in J' meets at least m points of J'. Prove that

$$|J'| \geq Cm^3 \ .$$

Hint: consider a polynomial of degree less than m that vanishes on all points of J'. Show that this polynomial has to vanish by considering all first and then higher order partial derivatives at the points of J'. Conclude the lower bound on $|J'|$.

Exercise 2 :

Use Exercise 1 to show that

$$|J| \leq C|L|^{3/2}$$

in the setting of that theorem (possibly different constant C).

Hint: Choose a line on L which meets few joints (use Exercise 1 to estimate how many). Then iterate and do careful bookkeping until no lines are left.

Exercise 3:

Consider the following special case of an inequality shown in the lecture:

$$\int \prod_{j=1}^{2} (\int e^{-\pi(x-v_j t)^2} \, d\mu_j(v_j))^{p_j} \, dx \leq \prod_{j=1}^{2} \|\mu_j\|^{p_j}$$

where $t \geq 0$, and for $j = 1, 2$ we have that $p_j > 0$, and μ_j is a compactly supported nonnegative measure on $\mathbf{R}$.

1. Prove from scratch the case $p_1 = p_2 = 1$, carefully check the constants in the inequality, do not just refer to the lecture.
2. Deduce from the case $p_1 = p_2 = 1/2$ the Cauchy Schwartz inequality in $L^2(\mathbf{R})$. (Hint: scale the measure μ_j and consider a limit)