1 RHA, SS 13, Exercise Sheet 10

Due July 3. 2013.

Exercise 1:

Let $d \geq 2$, and take $1 < p < \infty$ with the following property: if $\{e_j\} \subset S^{d-1}$ is a maximal δ-separated set, and if $\{r_j\}$ is any sequence of nonnegative numbers for which $\sum_j r_j^{p'} \leq \delta^{-(d-1)}$, then for any choice of centers $\{a_j\} \subset \mathbf{R}^d$, one has

$$\Big\| \sum_j r_j \chi_{T^\delta_{e_j}(a_j)} \Big\|_{p'} \leq A.$$

Prove the following bound on the Kakeya maximal function:

$$\|f^*_\delta\|_{L^p(S^{d-1})} \lesssim A\|f\|_p.$$

Hint: Use the duality between ℓ^p and $\ell^{p'}$ to establish the existence of a sequence $\{r_j\}$ such that $\sum_j r_j^{p'} = \delta^{-(d-1)}$ and

$$\|f^*_\delta\|_p \lesssim \delta^{d-1} \sum_j r_j |f^*_\delta(e_j)|.$$

Exercise 2:

Let $\gamma : I \to \mathbf{R}^2$ be a smooth curve with $\gamma' \neq 0$ and $\gamma'' \neq 0$ on some finite interval $I \subset \mathbf{R}$. Let $4 < q \leq \infty$ and $3p' \leq q$. Prove that

$$\Big\| \int_I e^{i\gamma(t)\cdot\xi} f(t)dt \Big\|_{L^q_\xi(\mathbf{R}^2)} \lesssim_{p,q} \|f\|_{L^p(I)}$$

for any $f \in L^p(I)$.

Hint: Change variables **very** carefully and establish a Hausdorff-Young $L^r \to L^{r'}$ bound with $r' = q/2 > 2$. Then use fractional integration.

Exercise 3:

Let A be a convex body (i.e. a nonempty, convex, open and bounded set) in $\mathbf{R}^d$. Show that the cross-sectional area

$$\mathfrak{S}(x_d) := |\{x' \in \mathbf{R}^{d-1} : (x', x_d) \in A\}|$$

is a log-concave function of x_d i.e.

$$\mathfrak{S}((1-\theta)x_d + \theta y_d) \geq \mathfrak{S}(x_d)^{1-\theta}\mathfrak{S}(y_d)^\theta$$

for every $\theta \in [0, 1]$ and $x_d, y_d \in \mathbf{R}$.