1 NLPDE I, WS 12/13, Exercise Sheet 7

Due Dec 4 in tutorial session. This sheet has 4 exercises

Exercise 1:

Recall the definition of the real Hardy space $H^1(\mathbf{R}^n)$: An atom a is an L^∞ function $a : \mathbf{R}^n \to \mathbf{R}$ with $\int a = 0$, a is supported on some cube Q, and $\|a\|_\infty \le |Q|^{-1}$. A Hardy space function f is an L^1 function which is a L^1 convergent linear combination of the form

$$\sum_i c_i a_i$$

with a_i and atom and c_i real coefficients with $\sum_i |c_i| < \infty$. The Hardy space norm of f is the infimum of the quantity $\sum_i |c_i| < \infty$ over all possible representations of the function f as above.

The dyadic Hardy space H^1_d with respect to the standard dyadic grid is defined similarly, except that the cube Q in the definition of an atom has to be a dyadic cube.

Consider the dyadic maximal function (note the absolute value signs are outside the integral)

$$M_d f(x) = \sup_Q \frac{1}{|Q|} \left| \int_Q f(y)\, dy \right|$$

where the supremum is taken over all dyadic cubes Q in the standard grid which contain the point y.

Prove that

$$\|M_d f\|_1 \le C \|f\|_{H^1_d}$$

for some constant C, and prove that there does not exist a constant C such that

$$\|M_d f\|_1 \le C \|f\|_{H^1}$$

(Hence H^1 and H^1_d are not the same space.)

Exercise 2

Let f be in $H^1(\mathbf{R})$ and consider the harmonic extension F of f to the upper half plane defined with the aid of the Poisson kernel. Prove that if we define for $x \in \mathbf{R}$

$$Mf(x) := \sup_{y \in \mathbf{R}} |F(x + iy)|$$

then

$$\|Mf\|_1 \le C \|f\|_{H^1}$$

for some constant C. (Compare with previous exercise.)

Exercise 3:

Let γ be a smooth compactly supported function $\gamma : \mathbf{R} \to \mathbf{R}$ and consider the curve $z : \mathbf{R} \to \mathbf{C}$ in the complex plane defined by $z(x) = x + i\gamma(x)$. For a smooth compactly supported function $f : \mathbf{R} \to \mathbf{C}$ define the Cauchy integral along the curve as

$$Cf(x) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\mathbf{R}} \frac{f(z(y))}{z(y) - x - i(\gamma(x) + \epsilon)} z'(y) dy$$

Prove that the limit exists in each of the following three senses: L^∞, L^2, and L^1.

Exercise 4:

As surveyed in Coifman-Jones-Semmes' paper, C as above extends to a bounded operator in $L^2(\mathbf{R})$. Prove that C is a projection in the sense $C(Cf) = Cf$. Prove that it is not an orthogonal projection (i.e. not self adjoint in $L^2(\mathbf{R})$) unless γ is constant 0.