1 NLPDE I, WS 12/13, Exercise Sheet 5

Due Nov 13 in tutorial session. This sheet has 4 exercises

Exercise 1:

Let ϕ be a smooth compactly supported function in $\mathbf{R}^n$.

Prove that for every x

$$\lim_{\epsilon \to 0} \int_{\mathbf{R}^n \setminus B_\epsilon(x)} (f(y) - f(x))|y-x|^{-n-1}\, dy$$

exists.

Exercise 2:

Define f to be an α- Hölder atom on the cube Q (in $\mathbf{R}^n$) if

1. for all x, y we have $|f(x) - f(y)| \le |x-y|^\alpha$
2. f is supported on Q.
3. $\int f(x)\, dx = 0$

Define an atomic C^α function f to be a countable linear combination

$$\sum_i a_i f_i$$

where f_i is an α-Hölder atom on a cube Q_i and a_i are real numbers such that $\sum_i |a_i||Q_i|$ is finite.

Prove that the above linear combination converges almost everywhere.

Prove or disprove that for every $\epsilon > 0$ there is a set E_ϵ of measure at most ϵ and a constant C_ϵ such that outside the set E_ϵ the function $f(x) = \sum_i a_i f_i(x)$ is defined and satisfies a Hölder condition

$$|f(x) - f(y)| \le C_\epsilon |x-y|^\alpha$$

for x, y not in the set E_ϵ.

Exercise 3:

Define the atomic C^α norm $\|f\|_{C^\alpha}$ to be the infimum of the quantity $\sum_i |a_i||Q_i|$ over all representations of f as in the previous exercise (two such representations are considered to represent the same function if they coincide almost everywhere).

Prove that for $0 < \alpha < \alpha + s < 1$ convolution with the function $g(x) = |x|^{-n+s}$ maps atomic C_α Hölder atom to atomic $C_{\alpha+s}$ with

$$\|g * f\|_{C_{\alpha+s}} \le C\|f\|_{C_\alpha}$$

for some constant C possibly depending on s, α.

(Hint: First define $g * f$ on the dense subset of finite linear combinations of atoms, prove the desired estimate there, and then extend the map to all of atomic C^α by density arguments.)

Exercise 4:

A centered Gaussian is a function of the form $g_s(x) = e^{-s|x|^2}$ for some parameter $s > 0$.

1. Calculate $\int_{\mathbf{R}^n} g_s(x)\,dx$
2. Calculate the convolution of two centered Gaussians with parameters s, t
3. Determine for which c, α and β we have

$$c|x|^\beta = \int_0^\infty \lambda^\alpha e^{-\lambda|x|^2}\,d\lambda$$

   for some convergent integral. Your expression should be explicit using the Gamma function defined by

$$\Gamma(c) = \int_0^\infty e^{-t} t^{c-1}\,dt$$

   for $c > 0$.

4. Define I^s to be the convolution operator with $|x|^{-n+s}$ in $\mathbf{R}^n$. Use all of the above to find an expression for the constant in $I^s I^t = cI^{s+t}$ whenever $0 < s, t$ and $s + t < 1$.